Quantum Algorithms for Evaluating Min-Max Trees

Richard Cleve*† Dmitry Gavinsky* David L. Yeung*

Abstract

We present a bounded-error quantum algorithm for evaluating Min-Max trees with $N^{\frac{1}{2}+o(1)}$ queries, where N is the size of the tree and where the allowable queries are comparisons of the form $[x_j < x_k]$. This is close to tight, since there is a known quantum lower bound of $\Omega(N^{\frac{1}{2}})$.

A Min-Max tree is a tree whose internal nodes are *minimum* and *maximum* gates, at alternating levels, and whose leaves are values from some underlying ordered set. The size N of such a tree the number of its leaves, whose values are referred to as $x_1, \ldots, x_N$. The value of a Min-Max tree is the value of its root, a function of $x_1, \ldots, x_N$. In the *input value* query model, queries explicitly access the values of the leaves. In the *comparison* query model, the values of $x_1, \ldots, x_N$ are not directly accessible; rather, queries are comparisons of the form $[x_j < x_k]$. In this latter model, the appropriate output is any $j \in \{1, \ldots, N\}$ such that x_j is the value of the tree.

Note that, when the ordered set is $\{0, 1\}$, a Min-Max tree reduces to an And-Or tree. This implies that Barnum and Saks's lower bound of $\Omega(N^{\frac{1}{2}})$ [2] for the quantum query complexity of And-Or trees applies to Min-Max trees.

Recent results initiated by Farhi *et al.* have shown that quantum algorithms can evaluate all And-Or trees with order $N^{\frac{1}{2}+o(1)}$ queries [8, 5, 6, 1]. We show that these results carry over to Min-Max trees in both the input value model and the comparison model.

Let $W(N)$ be the query complexity for And-Or trees of size N. We show that Min-Max trees can be evaluated with $O(W(N)\log(N))$ queries in both the input value model and the comparison model. Our algorithm combines the results on And-Or trees in Refs. [1, 6] with the lemma below and Grover's search algorithm [9].

Lemma 1 *Let $\mathcal{T}$ be a* Min-Max *tree with inputs $x_1, x_2, \ldots, x_N$. Let $\mathcal{T}^v$ be an* And-Or *tree with identical structure to $\mathcal{T}$, but with* And *and* Or *gates in place of* Min *and* Max *gates (respectively), and with the k^{th} input assigned to* 1 *if and only if $x_k \geq v$. Then* $\text{value}(\mathcal{T}^v) = 1$ *if and only if* $\text{value}(\mathcal{T}) \geq v$.

*David R. Cheriton School of Computer Science and Institute for Quantum Computing, University of Waterloo.

†Perimeter Institute, Waterloo, Ontario, Canada.

Lemma 1 is easy to prove by induction. It implies that, if the underlying ordered set is a numerical range of size $N^{O(1)}$, then the tree can be evaluated in $\log(N)$ stages by a simple binary search. Each stage can be implemented with $O(W(N)\log\log(N))$ queries, which reflects the cost of evaluating an AND-OR tree amplified so that its error probability is $O(1/\log(N))$. The result is an $O(W(N)\log(N)\log\log(N))$ query algorithm.

A complication arises in performing such a binary search in the comparison model, where it is not possible to directly compute the midpoint of an interval like $[x_j, x_k]$. Problems can also arise in the input value model when the numerical range is too large: the binary search may not converge in a logarithmic number of steps. For this reason, we avoid the standard binary search approach where a midpoint is chosen as a pivot. Instead, we take a *random* input value among those that that lie within a current interval as our pivot. What is noteworthy about this simple approach is that *it does not work efficiently in the classical case*: given an interval $[x_j, x_k]$, finding an interior point is as hard as searching, which can cost $\Omega(N)$ queries to do even once [3]. In the setting of *quantum* algorithms, we can utilize Grover's search algorithm [9, 4] which costs $O(\sqrt{N})$.

As an aside, we note that there is a classical reduction from MIN-MAX trees to AND-OR trees that yields an $O(N^{0.753})$ query algorithm for balanced MIN-MAX trees [10]. We can use that reduction with an $N^{\frac{1}{2}}$ query *quantum* algorithm for balanced AND-OR trees; however, the resulting algorithm for MIN-MAX costs $\Omega(N^{0.58})$. Our alternate approach yields exponent $\frac{1}{2} + o(1)$ and is not restricted to balanced trees.

What follows is a description of our algorithm with the analysis of its error. For convenience, let $\perp$ and $\top$ be such that $x_\perp < x_j$ and $x_\top > x_j$ for any $j \in \{1, \ldots, N\}$ and let c be a constant.

QUANTUM MIN-MAX TREE EVALUATION

1. Let $\gamma \leftarrow \perp$ and $\delta \leftarrow \top$, and initialize the stack.

2. Repeat the following steps for $c\log(N)$ iterations, then go to Step 3:

   (a) Find a random pivot:
   Call the quantum search subroutine to find a random pivot index j with $x_\gamma < x_j < x_\delta$. If no value is found, go to Step 2(c).

   (b) Refine the search:
   Call the AND-OR tree subroutine to check if $\text{value}(\mathcal{T}) < x_j$. If so, let $\delta \leftarrow j$; otherwise, let $\gamma \leftarrow j$.

   (c) Backtrack if out of range:
   Call the AND-OR subroutine to check if $x_\gamma \le \text{value}(\mathcal{T}) < x_\delta$. If so, push (γ, δ) onto the stack. Otherwise, pop (γ, δ) off the stack. (If the stack is empty, let $\gamma \leftarrow \perp$ and $\delta \leftarrow \top$.)

3. Return γ as an index corresponding to the value of the MIN-MAX tree.

Clearly, the algorithm makes $O(W(N)\log(N))$ queries. We claim the following.

Theorem 1 *The algorithm returns the value of the* MIN-MAX *tree with probability at least* $\frac{2}{3}$.

To prove Theorem 1, we must consider the progress made by the random choices of pivots as well as the error probabilities of the subroutines for AND-OR and the searches (each errs with constant probability).

To begin with, assume that the subroutines for AND-OR and search never err (thus, $x_\gamma \leq \text{value}(\mathcal{T}) < x_\delta$ at all times). Under this assumption, the progress of the algorithm is determined by how quickly the subinterval converges. Once no value in Step 2(a) is found the algorithm has *converged* (with $x_\gamma = \text{value}(\mathcal{T})$) and can go to Step 3 and terminate (however it is harmless to perform more iterations before doing this).

Let $C(m)$ denote the expected number of iterations of the algorithm until it converges, assuming that m of its inputs are within its current range.

Then, for $m > 1$, $C(m)$ satisfies the recurrence

$$C(m) \leq \frac{2}{m}\left(\sum_{k=\lfloor m/2 \rfloor}^{m-1} C(k)\right) + 1. \tag{1}$$

This can be seen by assuming that the pivot is uniformly selected among all m possible positions within the subinterval and that value($\mathcal{T}$) always lies in the larger side of the pivot. It is straightforward to verify that the recurrence implies $C(m) \in O(\log(m))$. Therefore, the expected number of iterations of Step 2 made by the algorithm before $x_\gamma = \text{value}(\mathcal{T})$, under the assumption that the subroutines never err, is $O(\log(N))$. By the Markov bound, $O(\log(N))$ iterations suffice to obtain error probability less than any particular constant.

We now consider the fact that the subroutines for AND-OR and searching can fail. First, note that, by incurring a multiplicative factor of only $O(\log\log(N))$, each call to the AND-OR and search algorithm can be amplified so that its error probability is $O(1/\log(N))$. This results in an $O(W(N)\log(N)\log\log(N))$ algorithm for MIN-MAX.

These amplification costs are not necessary in our algorithm, since it can cope with a constant fraction of errors in subroutine calls. To see why this is so, let ε be the probability that one or more subroutines err during one iteration of Step 2 of the algorithm. The algorithm begins some $O(\log(N))$ steps away from reaching a *good* state — of the form (γ', δ) such that $x_{\gamma'} = \text{value}(\mathcal{T})$. Before reaching a good state, an "incorrect" step for the algorithm places value($\mathcal{T}$) outside the search interval, and a "correct" step either narrows the search interval or backtracks from a previous error. After reaching a good state, a "correct" step pushes a pair of the form (γ', δ) onto the stack and an "incorrect" step pops it off. In each iteration, the algorithm takes a correct step with probability at least $1 - \varepsilon$ and an incorrect step with probability at most ε. Therefore, with all but exponentially small probability, the number of correct steps minus the number of incorrect ones after $c\log(N)$ iterations is at least $\frac{c}{2}\log(N)$. For suitably large c this means that, with constant probability, when the algorithm terminates, $x_\gamma = \text{value}(\mathcal{T})$ (typically with many copies of pairs of the form (γ', δ) on the top of its stack).

Finally, we note that, in game-playing contexts, it is useful to determine optimal moves. This corresponds to finding the subtree of a MIN-MAX tree that attains its value. If the leaf values $x_1, \ldots, x_N$ are distinct, this is easily deduced from value($\mathcal{T}$). Otherwise, one can use a slightly modified version of the minimum/maximum finding algorithm in Ref. [7] to find the appropriate subtree.

Acknowledgements

We would like to thank Peter van Beek, Peter Høyer and Pascal Poupart for helpful discussions. This research was supported in part by Canada's NSERC, CIAR, MITACS, QuantumWorks, and the U.S. ARO/DTO.

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