Bell's Inequalities: Foundations and Quantum Communication

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Abstract For individual events quantum mechanics makes only probabilistic predictions. Can one go beyond quantum mechanics in this respect? This question has been a subject of debate and research since the early days of the theory. Efforts to construct deeper, realistic, level of physical description, in which individual systems have, like in classical physics, preexisting properties revealed by measurements are known as hidden-variable programs. Demonstrations that a hidden-variable program necessarily requires outcomes of certain experiments to disagree with the predictions of quantum theory are called "no-go theorems". The Bell theorem excludes local hidden variable theories. The Kochen-Specker theorem excludes noncontextual hidden variable theories. In local hidden-variable theories faster-that-light-influences are forbidden, thus the results for a given measurement (actual, or just potentially possible) are independent of the settings of other measurement devices which are at space-like separation. In noncontextual hidden-variable theories the predetermined results of a (degenerate) observable are independent of any other observables that are measured jointly with it.

It is a fundamental doctrine of quantum information science that quantum communication and quantum computation outperforms their classical counterparts. If this is to be true, some fundamental quantum characteristics must be behind better-than-classical performance of information processing tasks. This chapter aims at establishing connections between certain quantum information protocols and foundational issues in quantum theory. After a brief discusion of the most common misinterpretations of Bell's theorem and a discussion of what its real me aning is, it

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will be demonstrated *how quantum contextuality and violations of local realism can be used as useful resources* in quantum information applications. In any case, the readers should bear in mind that this chapter is not a review of the literature of the subject, but rather a quick introduction.

1 Introduction

Which quantum states are useful for quantum information processing? All non-separable states? Only distillable non-separable states? Only those which violate constraints imposed by local realism? Entanglement is the most distinct feature of quantum physics with respect to the classical world [1]. On one hand, entangled states violate Bell inequalities, and thus rule out local realistic explanation of quantum mechanics. On the other hand, they enable certain communication and computation tasks to have an efficiency not achievable by the laws of classical physics. Intuition suggests that these two aspects, the fundamental one, and the one associated with applications, are intimately linked. It is natural to assume that the quantum states which allow the no-go theorems of quantum theory, such as Kochen-Specker, Bell's or Greenberger-Horne-Zeilinger theorem should also be useful for quantum information processing. If this were not true, one might expect that the efficiency of quantum information protocols could be simulatable by classical, essentially local realistic or noncontextual models, and thus achievable already via classical means. This intuitive reasoning is supported by the results of, for example, Acin *et. al* [2]: violation of a Bell's inequality is a criterium for the security of quantum key distribution protocols. Also it was shown that violation of Bell's inequalities by a quantum state implies that pure-state entanglement can be distilled from it [3] and that Bell's inequalities are related to optimal solutions of quantum state targeting [4]. In this overview we will give other examples that demonstrate the strong link between fundamental features of quantum states and their applicabilities in quantum information protocols, such as in quantum communication complexity problems, quantum random access, or certain quantum games.

2 Quantum predictions for two qubits systems

To set the stage for our story let us first describe two-qubits systems in full detail.

We shall present predictions for all possible local yes-no experiments on two spin-1/2 systems(in modern terminology, qubits) for all possible quantum states, i.e. from the pure maximally entangled singlet state (or the Bohm-EPR state), via factorizable (i.e. non-entangled) states, up to any mixed state. This will enable us to reveal the distinguishing traits of the quantum predictions for entangled states of the simplest possible compound quantum system. The formalism can be applied to any system consisting of two subsystems, such that each of them is described by a two dimensional Hilbert space. We choose the spin-$1/2$ convention to simplify the description.

2.1 Pure states

An important tool simplifying the analysis of the pure states of two subsystems is the so-called Schmidt decomposition.

2.1.1 Schmidt decomposition

For any nonfactorizable (i.e., entangled) pure state, $|\psi\rangle$ of *pair* of quantum subsystems, one described by a Hilbert space of dimension N, the other by space of dimension M, $N \leq M$, it is always possible to find preferred bases, one basis for the first system, another one for the second, such that the state becomes a sum of bi-orthogonal terms, i.e.

$$|\,\psi\rangle = \sum_{i=1}^{N} c_i\,|\,a_i\rangle_1\,|\,b_i\rangle_2 \tag{1}$$

with ${}_n\langle x_i|x_j\rangle_n = \delta_{ij}$, for $x = a,b$ and $n = 1,2$. It is important to stress that the appropriate single subsystem bases, here $|\,a_i\rangle_1$ and $\left|\,b_j\right\rangle_2$, depend upon the state that we want to Schmidt-decompose.

The ability to Schmidt decompose the state is equivalent to a well known fact form matrix algebra, that any $N \times M$ matrix $\hat{\mathbf{A}}$ can be always put into a diagonal form $\hat{\mathbf{D}}$, by applying a pair of unitary transformations: $\sum_{j=1}^{N}\sum_{k=1}^{M} U_{ij}A_{jk}U_{kl} = D_l\delta_{il}$.

The interpretation of the above formula could be put as follows. If the quantum pure state of two systems is non-factorizable, then there exist a pair of local observables (for system 1 with eigenstates $|a_i\rangle$, and for system 2 with eigenstates $|b_i\rangle$) such that the results of their measurement are perfectly correlated.

The method of Schmidt decomposition allows one to put every pure normalized state of two spins into

$$|\psi\rangle = \cos\alpha/2\,|+\rangle_1\,|+\rangle_2 + \sin\alpha/2\,|-\rangle_1\,|-\rangle_2\,. \tag{2}$$

Schmidt decomposition generally allows the coefficients to be real. This is achievable via trivial phase transformations of the preferred bases.

2.2 Arbitrary states

Systems can be in mixed states. Such states describe situations in which there does not exist any *nondegenerate* observable for which measurement result is deterministic. This is the case when the system can be with various probabilities $P(x) \geq 0$ in some non-equivalent states $|\psi(x)\rangle$, with $\sum_x P(x) = 1$. Mixed states are represented by self adjoint density non-negative operators $\rho = \sum_x P(x)|\psi(x)\rangle\langle\psi(x)|$. As $\mathrm{Tr}|\psi(x)\rangle\langle\psi(x)| = 1$ one has $\mathrm{Tr}\rho = 1$.

Let us present in detail properties of mixed states of the two spin-$1/2$ systems. Any self adjoint operator for one spin-$1/2$ particle is a linear combination of the Pauli matrices σ_i, $i = 1,2,3$ and the identity operator, $\sigma_0 = \mathbf{1}$, with *real* coefficients. Thus, any self adjoint operator in the tensor product of the two spin-$1/2$ Hilbert spaces, must be a real linear combination of all possible products of the operators $\sigma^1_\mu \sigma^2_\nu$, where the Greek indices run from 0 to 3, and the superscripts denote the particle. As the trace of σ_i is zero we arrive at the following form of the general density operator for two spin $1/2$ systems:

$$\rho = \frac{1}{4}\left(\sigma_0^{(1)}\sigma_0^{(2)} + \mathbf{r}\cdot\sigma^{(\mathbf{1})}\sigma_{\mathbf{0}}^{(\mathbf{2})} + \sigma_{\mathbf{0}}^{(\mathbf{1})}\mathbf{s}\cdot\sigma^{(\mathbf{2})} + \sum_{m,n=1}^{3} T_{nm}\sigma_n^{(1)}\sigma_m^{(2)}\right), \tag{3}$$

where, $\mathbf{r}$, $\mathbf{s}$ are real three dimensional vectors and $\mathbf{r}\cdot\sigma \equiv \sum_{i=1}^3 r_i\sigma_i$. We shall use the tensor product symbol $\otimes$ only sparingly, only whenever it is deemed necessary. The condition $\mathrm{Tr}\rho = 1$ is satisfied thanks to the first term.

Since the average of any real variable which can have only two values $+1$ and -1 cannot be larger than 1 and less than -1, the real coefficients T_{mn} satisfy relations

$$-1 \leq T_{mn} = \mathrm{Tr}\rho\,\sigma_n^{(1)}\sigma_m^{(2)} \leq 1, \tag{4}$$

and they form a matrix which will be denoted by $\hat{\mathbf{T}}$. One also has

$$-1 \leq r_n = \mathrm{Tr}\rho\,\sigma_n^{(1)} \leq 1, \tag{5}$$

and

$$-1 \leq s_m = \mathrm{Tr}\rho\,\sigma_m^{(2)} \leq 1. \tag{6}$$

2.2.1 Reduced density matrices for subsystems

A reduced density matrix represents the local state of a compound system. If we have two subsystems, then the average of any observable which pertains to the first system only, i.e. of the form $A\otimes\mathbf{1}$, where $\mathbf{1}$ is the identity operation for system 2, can be expressed as follows $\mathrm{Tr}_{12}(A\otimes\mathbf{1}\rho) = \mathrm{Tr}_1[A(\mathrm{Tr}_2\rho)]$. Here Tr_i represents a trace with respect to system i. As trace is a basis independent notion, one can always choose a factorizable basis, and therefore split the trace calculation into two stages.

The reduced one particle matrices for spins $1/2$, are of the following form:

$$\rho_1 \equiv \mathrm{Tr}_2\rho = \frac{1}{2}(\mathbf{1} + \mathbf{r}\cdot\sigma^{(\mathbf{1})}), \tag{7}$$

$$\rho_2 \equiv \mathrm{Tr}_1\rho = \frac{1}{2}(\mathbf{1} + \mathbf{s}\cdot\sigma^{(\mathbf{2})}). \tag{8}$$

with $\mathbf{r}$ and $\mathbf{s}$ the two local Bloch vectors of the spins.

Let us denote the eigenvectors of the spin projection along direction $\mathbf{a}$ of the first spin as: $|\,\psi(\pm1,\mathbf{a})\rangle_1$. They are defined by the relation

$$\mathbf{a}\cdot\sigma^{(\mathbf{1})}\,|\,\psi(\pm\mathbf{1},\mathbf{a})\rangle_{\mathbf{1}} = \pm\mathbf{1}\,|\,\psi(\pm\mathbf{1},\mathbf{a})\rangle_{\mathbf{1}}\,, \tag{9}$$

where $\mathbf{a}$ is a real vector of unit length (i.e. $\mathbf{a}\cdot\sigma^{\mathbf{1}}$ is a Pauli operator in the direction of $\mathbf{a}$). The probability of a measurement of this Pauli observable to give a result ±1 is given by

$$P(\pm1|\mathbf{a})_1 = \mathrm{Tr}_1\rho_1\pi^{(1)}_{(\mathbf{a},\pm\mathbf{1})} = \frac{1}{2}(1\pm\mathbf{a}\cdot\mathbf{r}), \tag{10}$$

and it is positive for arbitrary $\mathbf{a}$, if and only if, the norm of $\mathbf{r}$ satisfies

$$|\mathbf{r}| \leq \mathbf{1}. \tag{11}$$

Here $\pi^{(1)}_{(\mathbf{a},\pm\mathbf{1})}$ is the projector $|\,\psi(\pm1,\mathbf{a})\rangle_{11}\,\langle\psi(\pm1,\mathbf{a})|$.

2.3 Local measurements on two spins

The probabilities for local measurements to give the result $l=\pm1$ for particle 1 and the result $m=\pm1$ for particle 2, under specified local settings, $\mathbf{a}$ and $\mathbf{b}$ respectively, of the quantization axes are given by:

$$P(l,m|\mathbf{a},\mathbf{b})_{1,2} = \mathrm{Tr}\rho\,\pi^{(1)}_{(\mathbf{a},l)}\pi^{(2)}_{(\mathbf{b},m)} = \frac{1}{4}\left(1+l\mathbf{a}\cdot\mathbf{r}+m\mathbf{b}\cdot\mathbf{s}+lm\mathbf{a}\cdot\hat{\mathbf{T}}\mathbf{b}\right), \tag{12}$$

where $\hat{\mathbf{T}}\mathbf{b}$ denotes the transformation of the column vector $\mathbf{b}$ by the matrix $\hat{\mathbf{T}}$ (we treat here Euclidean vectors as column matrices).

One can simplify all these relations by performing suitable local unitary transformations upon each of the subsystems, i.e. via factorizable unitary operators $U^{(1)}U^{(2)}$. It is well known that any unitary operation upon a spin $1/2$ is equivalent to a three dimensional rotation in the space of Bloch vectors. In other words, for any real vector $\mathbf{w}$

$$U(\hat{\mathbf{O}})\mathbf{w}\cdot\sigma U(\hat{\mathbf{O}})^{\dagger} = (\hat{\mathbf{O}}\mathbf{w})\cdot\sigma, \tag{13}$$

where $\hat{\mathbf{O}}$ is the orthogonal matrix of the rotation. If the density matrix is subjected to such a transformations on either spins subsystem, i.e. to the $U^1(\hat{\mathbf{O}}_1)U^2(\hat{\mathbf{O}}_2)$ transformation, the parameters $\mathbf{r},\mathbf{s}$ and $\hat{\mathbf{T}}$ transform themselves as follows

$$\begin{aligned}\mathbf{r}' &= \hat{\mathbf{O}}_1\mathbf{r},\\ \mathbf{s}' &= \hat{\mathbf{O}}_2\mathbf{s},\\ \hat{\mathbf{T}}' &= \hat{\mathbf{O}}_1\hat{\mathbf{T}}\hat{\mathbf{O}}_2^{\mathbf{T}}.\end{aligned} \tag{14}$$

Thus, for an arbitrary state, we can always choose such factorizable unitary transformation that the corresponding rotations (i.e. orthogonal transformations) will diagonalize the correlation tensor (matrix) $\hat{\mathbf{T}}$. This can be seen as another application of Schmidt's decomposition, this time in case of second rank tensors.

The physical interpretation of the above is that one can always choose two (local) systems of coordinates, one for the first particle, the other for the second particle, in such a way that the $\hat{\mathbf{T}}$ matrix will be diagonal.

Let us note that one can decompose the two spin density matrix into:

$$\rho = \rho_1 \otimes \rho_2 + \frac{1}{4} \sum_{m,n=1}^{3} C_{nm} \sigma_n^1 \otimes \sigma_m^2, \tag{15}$$

i.e., it is a sum of the product of the two reduced density matrices and a term $\hat{\mathbf{C}} = \hat{\mathbf{T}} - \mathbf{r}\mathbf{s}^{\mathbf{T}}$ which is responsible for correlation effects.

Any density operator satisfies the inequality $\frac{1}{d} < \mathrm{Tr}\rho^2 \leq 1$, where d is the dimension of the Hilbert space in which it acts, i. e. of the system it describes. The value of $\mathrm{Tr}\rho^2$ is a measure of the purity of the quantum state. It is equal to 1 only for single dimensional projectors, i.e. the pure states. In the studied case one must have

$$|\mathbf{r}|^2 + |\mathbf{s}|^2 + ||\hat{\mathbf{T}}||^2 \leq 3. \tag{16}$$

For pure states, represented by Schmidt decomposition (2), $\hat{\mathbf{T}}$ is diagonal with entries $T_{xx} = -\sin\alpha$, $T_{yy} = \sin\alpha$ and $T_{zz} = 1$, whereas $\mathbf{r} = \mathbf{s}$, and their z component is non-zero: $s_z = m_z = \cos\alpha$. Thus in case of a maximally entangled states $\hat{\mathbf{T}}$ has only diagonal entries equal to $+1$ and -1. In the case of the singlet state,

$$|\psi\rangle = \frac{1}{\sqrt{2}} \left(|+\rangle_1 |-\rangle_2 - |-\rangle_1 |+\rangle_2 \right), \tag{17}$$

which can be obtained from eq. (2), by putting $\alpha = -\frac{\pi}{2}$ and rotating one of the subsystems such that $|+\rangle$ and $|-\rangle$ interchange (This is equivalent to a 180 degrees rotation with respect to the axis x; See above (14)), the diagonal elements of the correlation tensor are all -1.

3 Einstein-Podolsky-Rosen Experiment

In their seminal 1935 paper [5] entitled *”Can quantum-mechanical description of physical reality be considered complete?”* Einstein, Podolsky and Rosen (EPR) consider quantum systems consisting of two particles such that, while neither position nor momentum of either particle is well defined, both the difference of their positions and the sum of their momenta are both precisely defined. It then follows that measurement of either position or momentum performed on, say, particle 1 immediately implies for particle 2 a precise position or momentum respectively even when the two particles are separated by arbitrary distances without any actual interaction between them.

We shall present the EPR argumentation for incompleteness of quantum mechanics in the language of spins $1/2$. This has been done by Bohm in 1952. A two qubit example of an EPR state is the singlet state (17). Properties of a singlet can be inferred without mathematical considerations given above. This is a state of zero total spin. Thus measurements of the same component of the two spins must always give opposite values - this is simply the conservation of angular momentum at work. In terms of the language od Pauli matrices the product of the local results is then always -1. We have (infinitely many) *prefect (anti-)correlations*. We assume that the two spins are very far away, but nevertheless in the singlet state.

After the translation into the Bohm's example EPR argument runs as follows. Here are their premises:

1. *Perfect correlations* If whatever spin components of particles 1 and 2, then with certainly the outcomes will be found to be perfectly anti-correlated.
2. *Locality*: "Since at the time of measurements the two systems no longer interact, no real change can take place in the second system in consequence of anything that may be done to the first system."
3. *Reality*: "If, without in any way disturbing a system, we can predict with certainty (i.e., with probability equal to unity) the value of a physical quantity, then there exists an element of physical reality corresponding to this physical quantity."
4. *Completeness*: "Every element of the physical reality must have a counterpart in the [complete] physical theory."

In contrast to the last three premises which, thought they are quite plausible, are still indications of a certain philosophical viewpoint, the first premise is a statement about a well established property of a singlet state.

The EPR argument is as follows. Because of the perfect anti-correlations (1.), we can predict with certainty the result of measuring either x component or y component of spin of particle 2 by previously choosing to measure the same quantity of particle 1. By locality (2.), the measurement of particle 1 cannot cause any real change in particle 2. This implies that by the premise (3.), both the x *and* the y components of spin of particle 2 are elements of reality. This is also the case for particle 1 by a parallel argument where particle 1 and 2 interchange their roles. Yet, (according to Heisenberg's uncertainty principle) there is no quantum state of a single spin in which both x and y spin components have definite values. Therefore, by premise (4.) quantum mechanics cannot be a complete theory.

In his answer [6], published in the same year and under the same title as of the EPR paper, Bohr criticized the EPR concept of "reality" as assuming the systems having intrinsic properties independently of whether they are observed or not and he argued for "the necessity of a final renunciation of the classical ideal of causality and a radical revision of our attitude towards the problem of physical reality." Bohr pointed out that the wording of the criterion of physical reality (3.) proposed by EPR contains an ambiguity with respect to the expression "without in any way disturbing the system". And, while, as Bohr wrote, there is "no question of mechanical disturbance of the system”, there is ”the question of *an influence on the very conditions which define the possible types of predictions regarding the future behavior of the system.*” Bohr thus pointed out that the results of quantum measurements, in contrast to these of classical measurements, depend on the complete experimental arrangement (context), which can even be non-local as in the EPR case. Before any measurement is performed only the correlations between the spin components of two particles, but not spin components of individual particles are defined. The x or y component (but never both) of an individual particle becomes defined only when the respective observable of the distant particle is measured.

Perhaps the most clear way to see how strongly the philosophical viewpoints of EPR and Bohr differ is in their visions of the future development of quantum physics. While EPR wrote: ”We believe that such [complete] a theory is possible”, Bohr’s opinion is that (his) complementarity ”provides room for new physical law, the coexistence of which might at first sight appear irreconcilable with the basic principles of science.”

4 Bell’s theorem

Bell’s theorem can be thought of as a disproof of the validity of EPR ideas. Elements of physical reality cannot be an internally consistent notion. A broader interpretation of this result is that a local and realistic description of nature, at the fundamental level, is untenable. Further consequences are that there exist quantum processes which cannot be medelled by any classical ones, not necessarily physical processes, but also some classical computer simulations with a communication constraint. This opened the possibility of development of quantum communication.

We shall present now a derivation of Bell’s inequalities. The stress will be put on clarification of the underlying assumptions. These will be presented in the most reduced form.

4.1 Thought experiment

At two measuring stations A and B, which are far away from each other, two characters Alice and Bob observe simultaneous flashy appearances of numbers $+1$ or -1 at the displays of their local devices (or the monitoring computers). The flashes appear in perfect coincidence (with respect to a certain reference frame). In the middle between the stations is something that they call “source”. When it is absent, or switched off, the numbers ± 1’s do not appear at the displays. The activated source always causes two flashes, one at A, one at B. They appear slightly after a relativistic retardation time with respect to the activation of the source, never before. Thus there is enough “evidence” for Alice and Bob that the source causes the flashes. The devices at the stations have a knob which can be put in two positions:

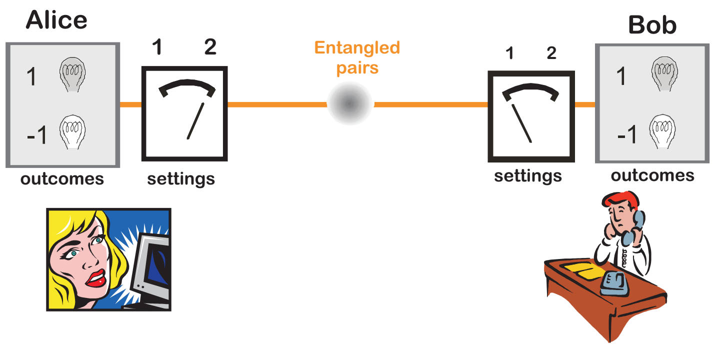


Fig. 1 Test of Bell's inequalities. Alice and Bob are two separated parties who share entangled particles. Each of them is free to choose two measurement settings 1 and 2 and they observe flashes in their detection station which indicate one of the two possible measurement outcomes +1 or -1.

$m = 1$ or 2 at A station, and $n = 1$ or 2 at B. Local procedures used to generate random choices of local knob positions are equivalent to *independent, fair coin tosses*. Thus, each of the four possible values of the pair n,m are equally likely, i.e. the probability $P(n,m) = P(n)P(m) = \frac{1}{4}$. The "tosses", and knob settings, are made at random times, and often enough, so that the information on these is never available at the source during its activation periods (the tosses and settings cannot have a causal influence on the workings of source). The local measurement data (setting, result, moment of measurement) are stored and very many runs of the experiment are performed.

4.1.1 Assumptions leading to Bell's inequalities

A concise *local realistic* description of such an experiment would use the following assumptions [7]:

1. We assume *realism*, which is any logically self-consistent model that allows one to use *eight* variables in the theoretical description of the experiment: $A_{m,n}$, $B_{n,m}$, where $n,m = 1,2$. The variable $A_{m,n}$ gives the value, ± 1, which could be obtained at station A, if the knob settings, at A and B, were at positions n,m, respectively. Similarly, $B_{n,m}$ plays the same role for station B, under the same settings. This is equivalent to the assumption that a joint (non-negative, properly normalized) probability distribution of these variables, $P(A_{1,1}, A_{1,2}, A_{2,1}, A_{2,2}; B_{1,1}, B_{1,2}, B_{2,1} B_{2,2})$, is always allowed to exist.[1]
2. The assumption of *locality* does not allow influences to go outside the light cone.

[1] Note, that no hidden variables appear, beyond these eight. However, given a (possibly stochastic) hidden variables theory, one will be able to define our eight variables as (possibly random) functions of the variables in that theory.

3. Alice and Bob are free to choose their settings "at the whim". This the *freedom, or "free will"*, often only a tacit assumption [8]. A less provocative version of this assumption: *There exists stochastic processes which could be used to choose the values of the local settings of the devices which are independent of the workings of the source, that is they neither influence it or are influenced by it.* By the previous assumptions the events of activation of the source and of the choice and fixing of the local settings must be space-like separated.

Note that when setting labels m, n are sent to the measurement devices, they will likely cause some unintended disturbance: by these assumptions *any disturbance at A, as far as it influences the outcome at A, is not related to the coin toss nor to the potential outcomes at B, and vice versa*.

Note further, that $A_{n,m}$ and $B_{n,m}$ are not necessarily actual properties of the systems. The only thing that is assumed it that there is a theoretical description which allows one to use these all *eight* values.

4.1.2 First consequences

Let us write down the immediate consequences of these assumptions:

- By *locality*: for all n, m:

$$A_{m,n} = A_m, \quad B_{n,m} = B_n \tag{18}$$

  That is, the outcome which would appear at A does not depend on which setting might be chosen at B, and vice versa. *Thus* $P(A_{1,1}, \ldots, B_{2,2})$ *can be reduced to* $P(A_1, A_2, B_1, B_2)$.
- By *freedom*

$$(n, m) \text{ is statistically independent of } (A_1, A_2, B_1, B_2). \tag{19}$$

Thus, the *overall* probability distributions for potential settings and potential outcomes satisfy

$$P(n, m, A_1, A_2, B_1, B_2) = P(n, m)p(A_1, A_2, B_1, B_2) \tag{20}$$

The choice of settings in the two randomizes, A and B, is causally separated from the local realistic mechanism, which produces the potential outcomes.

4.1.3 Lemma: Bell's inequality

The probabilities, Pr, of the four logical propositions, $A_n = B_m$, satisfy

$$\Pr\{A_1 = B_2\} - \Pr\{A_1 = B_1\} - \Pr\{A_2 = B_1\} - \Pr\{A_2 = B_2\} \ \leq \ 0. \tag{21}$$

Proof: only four, or two, or none of the propositions, in the left hand side of the inequality can be true, thus (21). QED.

Now, if the observation settings are totally random (dictated by "coin tosses"), $P(n,m) = \frac{1}{4}$. Then, according to all our assumptions

$$P(A_n = B_m|n,m) = P(n,m)\,\mathrm{Pr}\{A_n = B_n\} = \frac{1}{4}\,\mathrm{Pr}\{A_n = B_m\}. \tag{22}$$

Therefore, we have a Bell inequality: under the *conjunction* of the assumptions for the *experimentally accessible* probabilities one has

$$P(A_1 = B_2 \mid 1,2) - P(A_1 = B_1 \mid 1,1) - P(A_2 = B_1 \mid 2,1) - P(A_2 = B_2 \mid 2,2) \;\leq\; 0. \tag{23}$$

This is the well-known Clauser-Horne-Shimony-Holt (CHSH) inequality [9].

4.2 The Bell theorem

Quantum mechanics predicts for some experiments satisfying all the features of the thought experiment the left hand side of inequality (23) to be as high as $\sqrt{2}-1$, which is larger that the local realistic bound 0. *Hence, one has Bell's theorem [10]: if quantum mechanics holds, local realism, defined by the full set of the above assumptions, is untenable.* But, how does nature behave – according to local realism or quantum mechanics? It seems that we are approaching the moment, in which one could have as perfect as possible laboratory realization of the thought experiment (locality loophole was closed in [11, 12], detection loophole in [13] and in recent experiment measurement settings were space-like separated from the photon pair emission [14]). Hence local realistic approach to description of physical phenomena is close to be shown untenable too.

4.2.1 The assumptions as a communication complexity problem

Assume that we heave two programmers P_k, where $k = 1,2$, each possessing an enormously powerful computer. They share certain joint classical information strings of arbitrary lengths and/or some computer programs. All these will be collectively denoted as λ. But, once they both posses λ, no communication whatsoever between them is allowed. After this initial stage, each one of them gets from a Referee a one bit random number $x_k \in \{0,1\}$, known only to him/her (P_1 knows only x_1, P_2 knows only x_2). The *individual* task of each of them is to produce, via whatever computational program, a one bit number $I_k(x_k,\lambda)$, and communicate only this one bit to a Referee, who just compares the received bits. There is no restriction on the form and complication of the *possibly stochastic* functions I_k, or any actions taken to define the values, but any communication between the partners is absolutely not allowed. The *joint* task of the partners is to devise a computer code which under the constraints listed above, and without any cheating, allows to have after very many repetitions of the procedures (each starting with establishing a new shared λ) the following functional dependence of the probability that their bits sent back to the Referee are equal:

$$P\{I_1(x_1) = I_2(x_2)\} = \frac{1}{2} + \frac{1}{2}\cos\left[-\pi/4 + (\pi/2)(x_1 + x_2)\right]. \tag{24}$$

This is a variant of communication complexity problems. The current task is absolutely impossible to achieve with the classical means at their disposal, and without communication. Simply because whatever is the protocol

$$\Pr\{I_1(1) = I_2(1)\} - \Pr\{I_1(0) = I_2(0)\} - \Pr\{I_1(1) = I_2(0)\} - \Pr\{I_1(0) = I_2(1)\} \leq 0. \tag{25}$$

whereas, the value of this expression in quantum strategy P_Q can be as high as $\sqrt{2} - 1$. If the programmers use entanglement as resource and receive their respective qubits from an entangled pair (e.g. singlet) during the communication stages (when λ is established), one can obtain on average P_Q. Instead of computing, the partners make a local measurement on their qubits. They measure Pauli observables $\mathbf{n} \cdot \sigma$, where $||\mathbf{n}|| = 1$. Since the probability for them to get identical results, r_1, r_2, for observation directions $\mathbf{n}_1, \mathbf{n}_2$ is

$$P_Q\{r_1 = r_2 | \mathbf{n}_1, \mathbf{n}_2\} = \frac{1}{2} - \frac{1}{2}\mathbf{n}_1 \cdot \mathbf{n}_2, \tag{26}$$

for suitably chosen $\mathbf{n}_1(x_1), \mathbf{n}_2(x_2)$ they get values of P_Q equal to those in (24). The messages sent back to the Referee encode the local results of measurements of $\mathbf{n_1} \cdot \sigma \otimes \mathbf{n_2} \cdot \sigma$, and the local measurement directions are suitably chosen as functions of x_1 and x_2. We will come back to the relation between Bell's inequalities and quantum communication complexity problems in more details in Sec. 6.

4.2.2 Philosophy or physics? Which assumptions?

The assumptions behind Bell inequalities are often criticized as being "philosophical". If one reminds oneself on Mach's influence on Einstein, philosophical discussions related to physics may be very fruitful.

For those who are, however, still skeptical one can argue as follows. The whole (relativistic) classical theory of physics is realistic (and local). Thus we have an important exemplary realization of the postulates of local realism. Philosophical propositions could be defined as those which *are not* observationally or experimentally falsifiable at the given moment of the development of human knowledge, or in pure mathematical theory are not logically derivable. Therefore, the *conjunction* of all assumptions of Bell inequalities is not a philosophical statement, as it is *testable* both experimentally and logically (within, known at the moment, mathematical formulation of fundamental laws of physics). Thus, Bell's theorem removed the question of possibility of local realistic description from the realm of philosophy. Now this is just a question of a good experiment.

The other criticism is formulated in the following way. Bell inequalities can be derived using a single assumption of existence of joint probability distribution for the observables involved in them, or that the probability calculus of the experimental propositions involved in the inequalities is of Kolmogorovian nature, and nothing more. But if we want to apply these assumptions to the thought experiment we stumble on the following question: *does the joint probability take into account full experimental context or not?*. The experimental context is in our case (at least) the full state of the settings (m,n). Thus if we use the same notation as above for the realistic values, this time applied to the possible results of measurements of observables, initially we can assume existence of only $p(A_{1,1},A_{1,2},A_{2,1},A_{2,2};B_{1,1},B_{1,2},B_{2,1},B_{2,2})$. Note that such a probability could be e.g. factorizable into $\prod_{n,m} P(A_{n,m},B_{n,m})$. That is one could in such a case have different probability distributions pertaining to different experimental contexts (which can even be defined through the choice of measurement settings in space-like separated laboratories!)

Let us discuss this from the quantum mechanical point of view, only because such considerations have a nice formal description within this theory, familiar to all physicists. Two observables, say $\hat{A}_1 \otimes \hat{B}_1$ and $\hat{A}_2 \otimes \hat{B}_2$, as well as other possible pairs are functions of two different *maximal* observables for the whole system (which are non-degenerate by definition). If one denotes such a maximal observable linked with $\hat{A}_m \otimes \hat{B}_n$ by $\hat{M}_{m,n}$ and its eigenvalues by $M_{m,n}$ the existence of the aforementioned joint probability is equivalent to the existence of a $p(M_{1,1},M_{1,2},M_{2,1},M_{2,2})$ in form of a proper probability distribution. Only if one assumes additionally context independence, this can be reduced to the question of existence of (non-negative) probabilities $P(A_1,A_2,B_1,B_2)$, where A_m and B_n are eigenvalues of $\hat{A}_m \otimes \mathbf{1}$ and $\mathbf{1} \otimes \hat{B}_n$, where it turn $\mathbf{1}$ is the unit operator for the given subsystem. While context independence is physically doubtful, when the measurements are not spatially separated, and thus one can have mutual causal dependence, it is well justified for spatially separated measurements. I.e., *locality* enters our reasoning, whether we like it or not. Of course one cannot derive any Bell inequality of the usual type if the random choice of settings is not independent of the distribution of A_1,A_2,B_1,B_2, that is without (20).

There is yet another challenge to the set of assumptions presented above. It is often claimed, that realism can be derived, once one considers the fact that maximally entangled quantum systems reveal perfect correlations, and one additionally assumes locality. Therefore it would seem that the only basic assumption behind Bell inequalities is locality, with the other auxiliary ones of freedom. Such a claim is based on the ideas of EPR, who conjectured that one can introduce "elements of reality" of a remote system, provided this system is perfectly correlated with another system. To show the fallacy of such a hope, let us now discuss three particle correlations, in the case of which consideration of just few "elements of reality" reveals that they are a logically inconsistent notion. Therefore, they cannot be a starting point for deriving a self-consistent realistic theory. The three particle reasoning is used here because of its beauty and simplicity, not because one cannot reach a similar conclusion for two particle correlations.

4.3 Bell's theorem without inequalities: three entangled particles or more

As the simplest example, take a Greenberger-Horne-Zeilinger [15] (GHZ) state of $N = 3$ particles (fig.2):

$$|\mathrm{GHZ}\rangle = \frac{1}{\sqrt{2}}\left(|a\rangle|b\rangle|c\rangle + |a'\rangle|b'\rangle|c'\rangle\right) \tag{27}$$

where $\langle x|x'\rangle = 0$ ($x = a,b,c$, and kets denoted by one letter pertain to one of the particles). The observers, Alice, Bob and Cecil measure the observables: $\hat{A}(\phi_A)$, $\hat{B}(\phi_B)$, $\hat{C}(\phi_C)$, defined by

$$\hat{X}(\phi_X) = |+,\phi_X\rangle\langle+,\phi_X| - |-,\phi_X\rangle\langle-,\phi_X| \tag{28}$$

where

$$|\pm,\phi_X\rangle = \frac{1}{\sqrt{2}}\left(\pm i|x'\rangle + \exp(i\phi_X)|x\rangle\right). \tag{29}$$

and $\hat{X} = \hat{A},\hat{B},\hat{C}$. The quantum prediction for the expectation value of the product of the three local observables is given by

$$E(\phi_A,\phi_B,\phi_c) = \langle\mathrm{GHZ}|\hat{A}(\phi_A)\hat{B}(\phi_B)\hat{C}(\phi_C)|\mathrm{GHZ}\rangle = \sin(\phi_A+\phi_B+\phi_c). \tag{30}$$

Therefore, if $\phi_A+\phi_B+\phi_c = \pi/2 + k\pi$, quantum mechanics predicts perfect correlations. For example, for $\phi_A = \pi/2$, $\phi_B = 0$ and $\phi_c = 0$, whatever may be the results of local measurements of the observables, for say the particles belonging to the i-th triple represented by the quantum state $|\mathrm{GHZ}\rangle$, their product must be unity. In a local realistic theory one would have

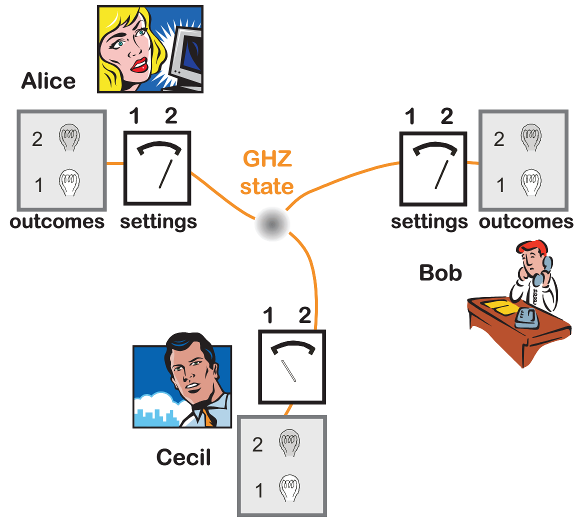
$$A^i(\pi/2)B^i(0)C^i(0) = 1, \tag{31}$$

where $X^i(\phi)$, $X = A,B$ or C is the local realistic value of a local measurement of the observable $\hat{X}(\phi)$ that *would have been* obtained for the i-th particle triple if the setting of the measuring device is ϕ. By locality $X^i(\phi)$ depends solely on the local parameter. The eq. (31) indicates that we can predict with certainty the result of measuring the observable pertaining to one of the particles (say c) by choosing to measure suitable observables for the other two. Hence the value $X^i(\phi)$ are EPR elements of reality.

However, if the local apparatus settings are different one *would have had*, e.g.

$$A^i(0)B^i(0)C^i(\pi/2) = 1, \tag{32}$$

Fig. 2 Test of the GHZ theorem. Alice, Bob and Cecil are three separated parties who share three entangled particles in the GHZ state. Each of them are free to choose between two measurement settings 1 and 2 and they observe flashes in their detection station which indicate one of the two possible measurement outcomes +1 or -1.



$$A^i(0)B^i(\pi/2)C^i(0) = 1, \tag{33}$$
$$A^i(\pi/2)B^i(\pi/2)C^i(\pi/2) = -1. \tag{34}$$

Yet, the four statements (31-34) are inconsistent within local realism. Since $X^i(\phi) = \pm 1$, if one multiples side by side the eqs. (31-34), the result is

$$1 = -1. \tag{35}$$

This shows that the mere concept of existence of "elements of physical reality" as introduced by EPR is in a contradiction with quantum mechanical predictions. We have a "Bell's theorem without inequalities" [15].

Some people still claim that EPR correlations together with the assumption of locality allow one to derive realism. The above example clearly shows that such a realism would allow one to infer that $1 = -1$.

4.4 Implications of Bell's theorem

Violations of Bell's inequalities imply that the underlying *conjunction of assumptions of realism, locality and "free will"* is not valid, and *nothing more*.

It is often said that the violations indicate "(quantum) non-locality". However if one wants *non-locality* to be *the* implication, one has to assume "free will" and realism. But this is only at this moment a philosophical choice (it seems that there is no way to falsify it). *It is not a necessary condition for violations of Bell's inequalities.*

The theorem of Bell shows that even a local inherently probabilistic hidden-variable theory cannot agree with all predictions of quantum theory (we base our considerations on $p(A_1, A_2, B_1, B_2)$ without assuming its actual structure, or whether the distribution for a single run is essentially deterministic, all we require is a joint "co-existence" of the variables $A_1, ..., B_2$ in a *theoretical* description). Therefore the above statements cover theories that treat probabilities as irreducible, and for which one can define $p(A_1, A_2, B_1, B_2)$. Such theories contradict quantum predictions. This, for some authors indicates that nature is non-local. While the mere existence of Bohm's model [16] demonstrates that non-local hidden-variables are a logically valid option, we now know that there are plausible models, such as Leggett's crypto-nonlocal hidden-variable model [17], that are in disagreement with both quantum predictions and experiment [18]. But, perhaps more importantly, if one is ready to consider inherently probabilistic theories, then there is no immediate reason to require the existence of (non-negative and normalized) probabilities $p(A_{1,1} ..., B_{2,2})$. Violation of this condition on realism, together with locality, which allows one to reduce the distribution to $p(A_1, ..., B_2)$, is not in a *direct* conflict with the theory of relativity, as it does not necessarily imply the possibility of signalling superluminally. To the contrary, quantum correlations cannot be used for direct communication between Alice to Bob, but still violate Bell's inequalities. It is therefore legitimate to consider quantum theory as a probability theory subject to, or even derivable from more general principles, such as non-signaling condition [19, 20] or information theoretical principles [21, 22].

Note that complementarity, inherent in quantum formalism[2], completely contradicts the form of realism defined above. So why quantum-non-locality?

To put it short, Bell's theorem does not imply *any* property of quantum mechanics. It just tells what it is not.

5 All Bell's inequalities for two possible settings on each side

We shall now present a general method of deriving *all* standard Bell inequalities (that is Bell's inequalities involving two-outcome measurements and with two settings per observer). Although these will not be spelled out explicitly, all the assumptions discussed above are behind the algebraic manipulations leading to the inequalities. We present in detail a derivation for two-observer problem, because the generalization to more observers is, surprisingly, obvious.

Consider pairs of particles (say, photons) simultaneously emitted in well defined opposite directions. After some time the photons arrive at two very distant measuring devices A and B operated by Alice and Bob. Alice, chooses to measure either observable $\hat{A}_1$ or $\hat{A}_2$, and Bob either $\hat{B}_1$ or $\hat{B}_2$. The hypothetical results that they may get for the j-th pair of photons are A_1^j and A_2^j, for Alice's two possible choices, and B_1^j and B_1^j, for Bob's. The numerical values of these results ($+1$ or -1) are defined by the two eigenvalues of the observables.

[2] Which can be mathematically expressed as non-existence of joint probabilities for non-commuting, i.e. non-commeasurable, observables.

Since, always either $|B_1^j - B_2^j| = 2$ and $|B_1^j + B_2^j| = 0$, or $|B_1^j - B_2^j| = 0$ and $|B_1^j + B_2^j| = 2$, with a similar property of Alice's hypothetical results the following relation holds

$$|A_1^j \pm A_2^j| \cdot |B_1^j \pm B_2^j| = 0 \tag{36}$$

for all possible sign choices within (36) except one, for which one has 4. Therefore

$$\sum_{k,l=0}^{1} |(A_1^j + (-1)^k A_2^j)(B_1^j + (-1)^l B_2^j)| = 4, \tag{37}$$

or equivalently one has the set of identities

$$\sum_{s_1,s_2=-1}^{1} S(s_1,s_2)[(A_1^j + s_1 A_2^j)(B_1^j + s_2 B_2^j)] = \pm 4, \tag{38}$$

with any $S(s_1,s_2) = \pm 1$. There are $2^{2^2} = 16$ such S functions.

Imagine now that N pairs of photons are emitted, pair by pair (N is sufficiently large, such that $\sqrt{1/N} \ll 1$). The average value of the products of the local values is given by

$$E(A_n, B_m) = \frac{1}{N} \sum_{j=1}^{N} A_n^j B_m^j, \tag{39}$$

where $n, m = 1, 2$.

Therefore after averaging, the following single Bell-type inequality emerges:

$$\sum_{k,l=0}^{1} |E(A_1,B_1) + (-1)^l E(A_1,B_2) + (-1)^k E(A_2,B_1) + (-1)^{k+l} E(A_2,B_2)| \leq 4, \tag{40}$$

or equivalently a series of inequalities:

$$\sum_{s_1,s_2=-1}^{1} S(s_1,s_2)[E(A_1,B_1) + s_2 E(A_1,B_2) + s_1 E(A_2,B_1) + s_1 s_2 E(A_2,B_2)] \leq 4. \tag{41}$$

As the choice of measurement settings is assumed to be statistically independent of the working of the source, i.e of the distribution of A_1's, A_2's, B_1's and B_2's, the averages $E(A_n, B_m)$ cannot differ much, for high N, from the *actually observed* ones in the subsets of runs for which the given pair of settings was selected.

5.1 Completeness of the inequalities

The inequalities form a complete set. That is, they define the faces of the convex polytope formed out of all possible local realistic models for the given set of measurements. Whenever local realistic model exists inequality (40) is satisfied by its predictions. To prove the sufficiency of condition (40) we construct a local realistic model for any correlation functions which satisfy it, i.e. we are interested in the local realistic models for $E^{LR}_{k_1k_2}$ such that they fully agree with the measured correlations $E(k_1,k_2)$ for all possible observables $k_1,k_2 = 1,2$.

One can introduce $\hat{E}$ which is a "tensor" or matrix built out of E_{ij}, with $i,j = 1,2$. If all its components can be derived from local realism, one must have

$$\hat{E}_{LR} = \sum_{\mathbf{A},\mathbf{B}=-1}^{1} P(\mathbf{A},\mathbf{B})\mathbf{A}\otimes\mathbf{B}, \tag{42}$$

with $\mathbf{A} = (A_1(\mathbf{n}_1), s_1A_1(\mathbf{n}_2))$, $\mathbf{B} = (A_2(\mathbf{n}_1), s_2A_2(\mathbf{n}_2))$, where $s_1,s_2 \in \{-1,1\}$ and nonnegative normalized probabilities $P(\mathbf{A},\mathbf{B})$.

Let us ascribe for fixed s_1,s_2, a hidden probability that $A_j(\mathbf{n}_1) = s_jA_j(\mathbf{n}_2)$ (with $j = 1,2$) in the form familiar from Eq. (40):

$$P(s_1,s_2) = \frac{1}{4}\Big|\sum_{k_2,k_2=1}^{2} s_1^{k_1-1}s_2^{k_2-1}E(k_1,k_2)\Big|. \tag{43}$$

Obviously these probabilities are positive. However they sum up to identity only if inequality (40) is saturated, otherwise there is a "probability deficit", ΔP. This deficit can be compensated without affecting correlation functions.

First we construct the following structure, which is indeed the local realistic model of the set of correlation functions if the inequality is saturated:

$$\sum_{s_1,s_2=-1}^{1} \Sigma(s_1,s_2)P(s_1,s_2)(1,s_1)\otimes(1,s_2), \tag{44}$$

where $\Sigma(s_1,s_2)$ is the sign of the expression within the modulus in Eq. (43).

Now if $\Delta P > 0$, we add a "tail" to this expression given by:

$$\frac{\Delta P}{16}\sum_{A_1=-1}^{1}\sum_{A_2=-1}^{1}\sum_{B_1=-1}^{1}\sum_{B_2=-1}^{1}(A_1,A_2)\otimes(B_1,B_2). \tag{45}$$

This "tail" does not contribute to the values of the correlation functions, because it represents the fully random noise. The sum of (44) is a valid local realistic model for $\hat{E} = (E(1,1),E(1,2),E(2,1),E(2,2))$. The sole role of the "tail" is to make all hidden probabilities to add up to 1.

To give the reader some intuitive grounds for the actual form of, and the completeness of the derived inequalities, we shall now give some remarks. The gist is that the consecutive terms in the inequalities are just expansion coefficients of the tensor $\hat{E}$ in terms of a complete orthogonal sequence of basis tensors. Thus the expansion coefficients represent the tensors in a one-to-one way.

In the four dimensional real space where both $\hat{E}_{LR}$ and $\hat{E}$ are defined one can find an orthonormal basis set $\hat{S}_{s_1 s_2} = \frac{1}{2}(1,s_1)\otimes(1,s_2)$. Within these definitions the hidden probabilities acquire a simple form:

$$P(s_1,s_2) = \frac{1}{2}|\hat{S}_{s_1 s_2}\cdot\hat{E}|, \tag{46}$$

where the dot denotes the scalar product in $\mathbf{R}^4$. Now the local realistic correlations, $\hat{E}_{LR}$, can be expressed as:

$$\hat{E}_{LR} = \sum_{s_1,s_2=-1}^{1} |\hat{S}_{s_1 s_2}\cdot\hat{E}|\Sigma(s_1,s_2)\hat{S}_{s_1 s_2}. \tag{47}$$

The modulus of any number $|x|$ can be split into $|x| = x\,\mathrm{sign}(x)$, and we can always demand the product $A_1(\mathbf{n}_1)A_2(\mathbf{n}_1)$ to have the same sign as the expression inside the modulus. Thus we have:

$$\hat{E} = \sum_{s_1,s_2=-1}^{1} (\hat{S}_{s_1 s_2}\cdot\hat{E})\hat{S}_{s_1 s_2}. \tag{48}$$

The expression in the bracket is the coefficient of tensor $\hat{E}$ in the basis $\hat{S}_{s_1 s_2}$. These coefficients are then summed over the same basis vectors, therefore the last equality appears.

5.2 Two-qubit states that violate the inequalities

A general two qubit state can be put in the following concise form

$$\hat{\rho} = \frac{1}{4}\sum_{\mu,\nu=0}^{3} T_{\mu\nu}(\hat{\sigma}_\mu^1\otimes\hat{\sigma}_\nu^2). \tag{49}$$

The two qubit correlation function for measurements of spin 1 along direction $\mathbf{n}(1)$ and of spin 2 along $\mathbf{n}(2)$ is given by

$$E_{QM}(\mathbf{n}(1),\mathbf{n}(2)) = \mathrm{Tr}\left[\hat{\rho}\left(\mathbf{n}(1)\cdot\hat{\sigma}^1\otimes\mathbf{n}(2)\cdot\hat{\sigma}^2\right)\right], \tag{50}$$

and it reads

$$E_{QM}(\mathbf{n}(1),\mathbf{n}(2)) = \sum_{i,j=1}^{3} T_{ij}n(1)_i n(2)_j. \tag{51}$$

Two particle correlations are fully defined once one knows the components of T_{ij}, $i,j=1,2,3$, of the tensor $\hat{\mathbf{T}}$. Equation (51) can be put into a more convenient form:

$$E_{QM}(\mathbf{n}(1),\mathbf{n}(2)) = \hat{\mathbf{T}} \bullet \mathbf{n}(1) \otimes \mathbf{n}(2), \tag{52}$$

where "$\bullet$" is the scalar product in the space of tensors, which in turn is isomorphic with $\mathbf{R}^3 \otimes \mathbf{R}^3$.

Quantum correlation $E_{QM}(\mathbf{n}(1),\mathbf{n}(2))$ can be described by a local realistic model if, and only if, for *any* choice of the settings $\mathbf{n}(1)^{k_1}$ and $\mathbf{n}(2)^{k_2}$, where $k_1,k_2=1,2$, one has

$$\frac{1}{4}\sum_{k,l=1}^{2}\left|\hat{\mathbf{T}} \bullet [\mathbf{n}(1)^1 + (-1)^k \mathbf{n}(1)^2] \otimes [\mathbf{n}(2)^1 + (-1)^l \mathbf{n}(2)^2)]\right| \le 1. \tag{53}$$

Since there always exist two mutually orthogonal unit vectors $\mathbf{a}(x)^1$ and $\mathbf{a}(x)^2$ such that

$$\mathbf{n}(x)^1 + (-1)^k \mathbf{n}(x)^2 = 2\alpha(x)_k \mathbf{a}(x)^k \text{ with } k=1,2 \tag{54}$$

and with $\alpha(x)_1 = \cos\theta(x)$, $\alpha(x)_2 = \sin\theta(x)$, one obtains

$$\sum_{k,l=1}^{2}\left|\alpha(1)_k \alpha(2)_l \hat{\mathbf{T}} \bullet \mathbf{a}(1)^k \otimes \mathbf{a}(2)^l\right| \le 1. \tag{55}$$

Note that $\hat{\mathbf{T}} \bullet \mathbf{a}(1)^k \otimes \mathbf{a}(2)^l$ is a component of the tensor $\hat{\mathbf{T}}$ after a transformation of the local coordinate systems of each of the particles into such ones where the two first basis vectors are $\mathbf{a}(x)^1$ and $\mathbf{a}(x)^2$. We shall denote such transformed components again by T_{kl}.

The necessary and sufficient condition for a two-qubit correlation to be described within a local realistic model is that in any plane of observations for each particle (defined by the two observation directions) one must have

$$\sum_{k,l=1}^{2} |\alpha(1)_k \alpha(2)_l T_{kl}| \le 1. \tag{56}$$

for arbitrary $\alpha(1)_k$, $\alpha(2)_l$.

Using the Cauchy inequality one obtains

$$\sum_{k,l=1}^{2} |\alpha(1)_k \alpha(2)_l T_{kl}| \le \sqrt{\sum_{k,l=1}^{2} T_{kl}^2}. \tag{57}$$

Therefore, if

$$\sum_{k,l=1}^{2} T_{kl}^2 \le 1 \tag{58}$$

for any set of local coordinate systems, the two particle correlation functions of the form of (51) can be understood within the local realism (in a two settings per observer experiment).

This condition is both necessary and sufficient.

5.2.1 Sufficient condition for violation of the inequality

The full set of inequalities is derivable from the identity (38) where we put non-factorable sign function $S(s_1,s_2)=\frac{1}{2}(1+s_1)+(1-s_1)s_2$. In this case one obtains the CHSH inequality in its standard form:

$$\left|\langle (A_1+A_2)B_1+(A_1-A_2)B_2\rangle_{avg}\right|\leq 2, \tag{59}$$

where $\langle ...\rangle_{arg}$ denotes average. All other non-trivial inequalities are obtainable by all possible sign changes $X_k\rightarrow -X_k$ (with $k=1,2$ and $X=A,B$). It is easy to see that factorizable sign functions, such as e.g. $S(s_1,s_2)=s_1s_2$, lead to trivial inequalities $|E(A_n,B_m)|\leq 1$. As noted above the quantum correlation function $E_Q(\mathbf{a}_k,\mathbf{b}_l)$ is given by the scalar product of the correlation tensor $\hat{\mathbf{T}}$ with the tensor product of the local measurement settings represented by unit vectors $\mathbf{a}_k\otimes\mathbf{b}_l$, i.e. $E_Q(\mathbf{a}_k,\mathbf{b}_l)=(\mathbf{a}_k\otimes\mathbf{b}_l)\cdot\hat{\mathbf{T}}$. Thus, the condition for a quantum state endowed with the correlation tensor $\hat{\mathbf{T}}$ to satisfy the inequality (59), is that for all directions $\mathbf{a}_1,\mathbf{a}_2,\mathbf{b}_1,\mathbf{b}_2$ one has

$$\left|\left[(\frac{\mathbf{a}_1+\mathbf{a}_2}{2})\otimes\mathbf{b}_1+(\frac{\mathbf{a}_1-\mathbf{a}_2}{2})\otimes\mathbf{b}_2\right]\cdot\hat{\mathbf{T}}\right|\leq 1, \tag{60}$$

where both sides of (59) were divided by 2.

Next notice that $\mathbf{A}_\pm=\frac{1}{2}(\mathbf{a}_1\pm\mathbf{a}_2)$ satisfy the following relations: $\mathbf{A}_+\cdot\mathbf{A}_-=0$ and $||\mathbf{A}_+||^2+||\mathbf{A}_-||^2=1$. Thus $\mathbf{A}_++\mathbf{A}_-$ is a unit vector, and $\mathbf{A}_\pm$ represent its decomposition into two orthogonal vectors. If one introduces unit vectors $\mathbf{a}_\pm$ such that $\mathbf{A}_\pm=a_\pm\mathbf{a}_\pm$, one has $a_+^2+a_-^2=1$. Thus one can put inequality (60) into the following form:

$$|\hat{\mathbf{S}}\cdot\hat{\mathbf{T}}|\leq 1, \tag{61}$$

where $\hat{\mathbf{S}}=a_+\mathbf{a}_+\otimes\mathbf{b}_1+a_-\mathbf{a}_-\otimes\mathbf{b}_2$. Note that since $\mathbf{a}_+\cdot\mathbf{a}_-=0$, one has $\hat{\mathbf{S}}\cdot\hat{\mathbf{S}}=1$, i.e. $\hat{\mathbf{S}}$ is a tensor of unit norm. Any tensor of unit norm, $\hat{\mathbf{U}}$, has the following Schmidt decomposition $\hat{\mathbf{U}}=\lambda_1\mathbf{v}_1\otimes\mathbf{w}_1+\lambda_2\mathbf{v}_2\otimes\mathbf{w}_2$, where $\mathbf{v}_i\cdot\mathbf{v}_j=\delta_{ij},\mathbf{w}_i\cdot\mathbf{w}_j=\delta_{ij}$ and $\lambda_1^2+\lambda_2^2=1$. The (complete) freedom of the choice of the measurement directions $\mathbf{b}_1$ and $\mathbf{b}_2$, allow one by choosing $\mathbf{b}_2$ orthogonal to $\mathbf{b}_1$ to put $\hat{\mathbf{S}}$ in the form isomorphic with $\hat{\mathbf{U}}$, and the freedom of choice of $\mathbf{a}_1$ and $\mathbf{a}_2$ allows $\mathbf{A}_+$ and $\mathbf{A}_-$ to be arbitrary orthogonal unit vectors, and $\mathbf{a}_+$ and $\mathbf{a}_-$ to be also arbitrary. Thus $\hat{\mathbf{S}}$ can be equal to any unit tensor. To get the maximum of the left hand side of (60), we Schmidt decompose the correlation tensor, and take two terms of the decomposition which have the largest coefficients. In this way we get a tensor $\hat{\mathrm{T}}'$, of Schmidt rank two. We put $\hat{\mathrm{S}}=\frac{1}{||\hat{\mathrm{T}}'||}\hat{\mathrm{T}}'$, and the maximum is $||\hat{\mathrm{T}}'||=\sqrt{\hat{\mathrm{T}}'\cdot\hat{\mathrm{T}}'}$. Thus, in other words,

$$\max \left[\sum_{k,l=1}^{2} T_{kl}^2 \right] \leq 1 \tag{62}$$

is the necessary and sufficient condition for the inequality (40) to hold, provided the maximization is taken over all local coordinate systems of two observers. The condition is equivalent to the necessary and sufficient condition of Horodeccy Family [27] for violation of the CHSH inequality.

5.3 Bell's inequalities for N particles

Let us consider a Bell inequality test with N observers. Each of them chooses between two possible observables, determined by local parameters $\mathbf{n}_1(j)$ and $\mathbf{n}_2(j)$, where $j = 1,...,N$. Local realism implies existence of two numbers A_1^j and A_2^j, each taking values +1 or -1, which describe the predetermined result of a measurement by the j-th observer for the two observables. The following algebraic identity holds:

$$\sum_{s_1,...,s_N=-1}^{1} S(s_1,...,s_N) \prod_{j=1}^{N} [A_1^j + s_j A_2^j] = \pm 2^N, \tag{63}$$

where $S(s_1,...,s_N)$ is an arbitrary "sign" function, i.e. $S(s_1,...,s_N) = \pm 1$. It is a straightforward generalization of the one for two observers as given in (41). The correlation function is the average over many runs of the experiment $E_{k_1,...,k_N} = \langle \prod_{j=1}^{N} A_{k_j}^j \rangle_{avg}$ with $k_1,...k_N \in \{1,2\}$. After averaging (63) over the ensemble of the runs one obtains the Bell inequalities [3]

$$\left| \sum_{s_1,...,s_N=-1}^{1} S(s_1,...,s_N) \sum_{k_1,...,k_N=1}^{2} s_1^{k_1-1} ... s_N^{k_N-1} E_{k_1,...,k_N} \right| \leq 2^N. \tag{64}$$

Since there are 2^{2^N} different functions S, the above inequality represents a set of 2^{2^N} Bell inequalities.

All these boil down to just one inequality (!):

$$\sum_{s_1,...,s_N=-1}^{1} \left| \sum_{k_1,...,k_N=1}^{2} s_1^{k_1-1} ... s_N^{k_N-1} E_{k_1,...,k_N} \right| \leq 2^N, \tag{65}$$

The proof of this fact is trivial exercise with the use of the property that either $|X| = 1$ or $|X| = -1$, where X is a real number. This inequality was derived independently in Refs [24] and [25]. The presented derivation follows mainly Ref. [26].

[3] This set of inequalities is a sufficient and necessary condition for the correlation functions entering them to have a local realistic model. Compare it to the two particle case.

5.4 N-qubit correlations

A general N-qubit state can be put in the form

$$\hat{\rho} = \frac{1}{2^N} \sum_{\mu_1,\cdots,\mu_N=0}^{3} T_{\mu_1\cdots\mu_N} (\otimes_{k=1}^{N} \hat{\sigma}_{\mu_K}^{k}). \tag{66}$$

Thus, the N qubit correlation function has the following structure

$$E_{QM}(\mathbf{n}(1), \mathbf{n}(2), ..., \mathbf{n}(N)) = \hat{\mathbf{T}} \bullet \mathbf{n}(1) \otimes \mathbf{n}(2) ... \otimes \mathbf{n}(N), \tag{67}$$

where $\hat{\mathbf{T}}$ stands for an N index tensor, with components $T_{k_1...k_N}$, where $k_i = 1,2,3$. The necessary and sufficient condition for a description of the correlation function within local realism in the general case reads

$$\sum_{k_1,k_2...,k_N=1}^{2} \left| \alpha(1)_{k_1} \alpha(2)_{k_2} ... \alpha(N)_{k_N} T_{k_1 k_2 ... k_N} \right| \leq 1. \tag{68}$$

for any possible choice of local coordinate systems for individual particles. Again if

$$\sum_{k_1,...,k_N=1}^{2} T_{k_1...k_N}^2 \leq 1 \tag{69}$$

for any set of local coordinate systems, the N-qubit correlation function can be described by a local realistic model. The proof of these fact are generalizations of the ones presented earlier pertaining to two particles. The sufficient condition for violation of the general Bell's inequality for N particles by a general state of N qubits can be found in Ref. [26].

5.5 Concluding remarks

The inequalities presented above represent the full set of standard "tight" Bell's inequalities for an arbitrary number of parties. Any non tight inequality is weaker than tight ones. Such Bell's inequalities can be used to detect entanglement, not as efficiently as entanglement general witnesses. However, they have the advantage over the witnesses that they are systems-independent. They detect entanglement no matter what is the actual Hilbert space that describes the subsystems.

As we shall show below the Bell inequalities analyzed above also show that the entanglement violating them is directly applicable in some quantum informational protocols that beat any classical ones of the same kind. This will be shown via an explicit construction of such protocols.

6 Quantum reduction of communication complexity

In his review paper entitled ”Quantum Communication Complexity (A Survey)” Brassard [28] posed a question: *”Can entanglement be used to save on classical communication?”* He continued that there are good reasons to believe at first that the answer to the question is negative. Holevo’s theorem [29] states that no more than n bits of classical information can be communicated between parties by the transmission of n qubits regardless of the coding scheme as long as no entanglement is shared between parties. If the communicating parties share prior entanglement, twice as much classical information can be transmitted (this is so called ”superdense coding” [30]), but no more. It is thus reasonable to expect that even if the parties share entanglement no savings in communication can be achieved beyond that of the superdense coding ($2n$ bits per n qubits transmitted).

It is also well known that entanglement alone cannot be used for communication. Local operations performed on any subsystem of an entangled composite system cannot have any observable effect on any other subsystem; otherwise it could be exploited to communicate faster than light. One would thus intuitively conclude that entanglement is useless for saving communication. Brassard, however, concluded *”... all the intuition in this paragraph is wrong.*”

The topic of classical communication complexity was introduced and first studied by Andrew Yao in 1979 [31]. A typical communication complexity problem can be formulated as follows. Let Alice and Bob be two separated parties who receive some input data of which they know only their own data and not the data of the partner. Alice receives an input string x and Bob an input string y and the goal is for both of them to determine the value of a certain function $f(x,y)$. Before they start the protocol Alice and Bob are even *allowed to share (classically correlated) random strings* or any other data, which might improve the success of the protocols. They are allowed to process their data locally in whatever way. The obvious method to achieve the goal is for Alice to communicate x to Bob, which allows him to compute $f(x,y)$. Once obtained, Bob can then communicate the value $f(x,y)$ back to Alice. It is the topic of communication complexity to address the questions: *Could there be more efficient solutions for some functions* $f(x,y)$*? What are these functions?*

A trivial example that there could be more efficient solutions then the obvious one given above is a constant function $f(x,y)=c$, where c is a constant. Obviously here Alice and Bob do not need to communicate at all, as they can simply take c for the value of the function. However there are functions for which the only obvious solution is optimal, that is only transmission of x to Bob warrants that he reaches the correct result. For instance, it is shown that n bits of communication are necessary and sufficient for Bob to decide whether or not Alice’s n-bit input is the same as his one [28, 32].

Generally one might distinguish the following two types of communication complexity problems:

1. What is the minimal amount of communication (minimal number of bits) required for the parties to determine the value of the function with certainty?

2. What is the highest possible probability for the parties to arrive at the correct value for the function if only a *restricted* amount of communication is allowed?

Here we will consider only the second class of problems. Note that in this case one does not insist on the correct value of the function to be obtained with certainty. While an error in computing the function is allowed, the parties try to compute it correctly with as high probability as possible.

From the perspective of the physics of quantum information processing the natural questions is: *Are there communication complexity tasks for which the parties could increase the success in solving the problem if they share prior entanglement?* In their original paper Cleve and Buhrman [33] showed that entanglement can indeed be used to save classical communication. They showed that to solve a certain three-party problem with certainty the parties need to broadcast at least 4 bits of information, in a classical protocol, whereas in the quantum protocol (with entanglement shared) it is sufficient for them to broadcast only 3 bits of information. This was the first example of a communication complexity problem that could be solved with higher success than it is be possible with any classical protocol. Subsequently, Buhrman, Cleve and van Dam [34] found a two-party problem that can be solved with a probability of success exceeding 85% and 2 bits of information communicated if prior shared entanglement is available, whereas the probability of success in a classical protocol could not exceed 75% with the same amount of communication.

The first problem whose quantum solution requires significantly smaller amount of communication compared to classical solutions was discovered by Buhrman, van Dam, Høyer and Tapp [35]. They considered a k-party task which requires roughly $k \ln k$ bits of communication in a classical protocol, and exactly k bits of classical communication if the parties are allowed to share prior entanglement. The quantum protocol of Ref. [34] is based on the violation of the CHSH inequality by two-qubit maximally entangled state. Similarly, the quantum protocols of multi-party problems [34, 33, 35] are based on an application of the GHZ-type argument against local realism for multi-qubit maximally entangled states. Galvao [36] has shown an equivalence between the CHSH and GHZ tests for three particles and the two- and three-party quantum protocols of Ref. [34], respectively. In a series of papers [37, 38, 39, 40] it was shown that entanglement violating a Bell inequality can always be exploited to find a better-than-any-classical solution to some communication complexity problems. In this brief overview we mainly follow the approach introduced in these papers. The approach has been further developed and applied in Ref. [41, 42] (See also Ref. [43]).

6.1 The problem and its optimal classical solution

Imagine several spatially separated partners, P_1 to P_N, each of whom has some data known to him/her only, denoted here as X_i, with $i = 1,\ldots,N$. They face a joint task: to compute the value of a function $T(X_1,\ldots,X_N)$. This function depends on all data. Obviously they can get the value of T by sending all their data to partner P_N, who does the calculation and announces the result. But are there ways to reduce the amount of communicated bits, i.e. to reduce the communication complexity of the problem?

Assume that every partner P_k receives a two bit string $X_k = (z_k, x_k)$ where $z_k, x_k \in \{0,1\}$. We shall consider specific task functions which have the following form

$$T = f(x_1, ..., x_N)(-1)^{\sum_{k=1}^{N} z_k},$$

where $f \in \{0,1\}$ the sum in the exponent is modulo 2. The partners know also the probability distribution ("promise") of the bit strings ("inputs"). There are two constraints on the problem. Firstly, we shall consider only distributions, which are completely random with respect to z_k's, that is a class of the form $p(X_1, ..., X_N) = 2^{-N} p'(x_1, ..., x_N)$. Secondly, communication between the partners is restricted to $N-1$ bits. Assume that we ask the last partner to give his/her answer $A(X_1, ..., X_N)$, equal to ± 1, to the question what is functional value $T(X_1, ..., X_N)$ in each run for the given set of inputs $X_1, ..., X_N$.

For simplicity, we shall introduce now $y_k = (-1)^{z_k}$, $y_k \in \{-1,1\}$. We shall use y_k as a synonym of z_k. Since T is proportional to $\prod_k y_k$, the final answer A is completely random if it does not depend on *every* y_k. Thus, information on z_k's from all $N-1$ partners must somehow reach P_N. Therefore the only communication "trees" which might lead to a success are those in which each P_k sends only a one-bit message $m_k \in \{0,1\}$. Again we introduce: $e_k = (-1)^{m_k}$, $e_k \in \{-1,1\}$, and will treat is as synonym of m_k.

The average success of a communication protocol can be measured with the following fidelity function

$$F = \sum_{X_1,...,X_N} p(X_1, ...X_N) T(X_1, ...X_N) A(X_1, ...X_N), \tag{70}$$

or equivalently

$$F = \frac{1}{2^N} \sum_{x_1,...,x_N=0}^{1} p'(x_1, ..., x_N) f(x_1, ..., x_N) \sum_{y_1,...y_N=-1}^{1} \prod_{k=1}^{N} y_k A(x_1, ..., x_N; y_1, ..., y_N). \tag{71}$$

The probability of success is $P = (1+F)/2$.

The first steps of a derivation of the reduced form of the fidelity function for an optimal classical protocol will now be presented (the reader may reconstruct the other steps or consult references [38, 39]). In a classical protocol the answer A of the partner P_N can depend on the local input y_N, x_N, and messages, $e_{i_1}, ..., e_{i_l}$, received *directly* from a subset of l partners $P_{i_1}, ..., P_{i_l}$:

$$A = A(x_N, y_N, e_{i_1}, ..., e_{i_l}). \tag{72}$$

Let us fix x_N, and treat A as a function A_{x_N} of the remaining $l+1$ dichotomic variables

$$y_N, e_{i_1}, ..., e_{i_l}.$$

That is, we treat now x_N as a fixed index. All such functions can be thought of as 2^{l+1} dimensional vectors, because the values of each such a function form a sequence of the length equal to the number of elements in the domain. In the 2^{l+1} dimensional space containing such functions one has *an orthogonal basis* given by

$$V_{jj_1\ldots j_l}(y_N, e_{i_1}, \ldots, e_{i_l}) = y_N^j \prod_{k=1}^{l} e_{i_k}^{j_k}, \tag{73}$$

where $j, j_1, \ldots, j_l \in \{0,1\}$. Thus, one can expand $A(x_N, y_N, e_{i_1}, \ldots, e_{i_l})$ with respect to this basis and the expansion coefficients read

$$c_{jj_1\ldots j_l}(x_N) = \frac{1}{2^{l+1}} \sum_{y_N, e_{i_1}, \ldots, e_{i_l}=-1}^{1} A(x_N, y_N, e_{i_1}, \ldots, e_{i_l}) V_{jj_1\ldots j_l}(y_N, e_{i_1}, \ldots, e_{i_l}). \tag{74}$$

Since $|A| = |V_{jj_1,\ldots j_l}| = 1$, one has $|c_{jj_1\ldots j_l}(x_N)| \leq 1$. We put the expansion into the expression for F and obtain

$$F = \frac{1}{2^N} \sum_{x_1,\ldots,x_N=0}^{1} g(x_1, \ldots, x_N) \sum_{y_1,\ldots,y_N=-1}^{1} \prod_{h=1}^{N} y_h \left[\sum_{j, j_1, \ldots j_l=0}^{1} c_{jj_1\ldots j_l}(x_N) y_N^j \prod_{k=1}^{l} e_{i_k}^{j_k} \right], \tag{75}$$

where $g(x_1, \ldots, x_N) \equiv f(x_1, \ldots, x_N) p'(x_1, \ldots, x_N)$. Because $\sum_{y_N=-1}^{1} y_N y_N^0 = 0$, and $\sum_{y_k=-1}^{1} y_k e_k^0 = 0$, only the term with all $j, j_1, \ldots, j_l$ equal to unity can give a non-zero contribution to F. Thus, A in F can be replaced by

$$A' = y_N c_N(x_N) \prod_{k=1}^{l} e_{i_k}, \tag{76}$$

where $c_N(x_N)$ stands for $c_{11\ldots 1}(x_N)$. Next, notice that, for example, e_{i_1}, can depend only on local data x_{i_1}, y_{i_1} and the messages obtained by P_{i_1} from a subset of partners: $e_{p_1}, \ldots, e_{p_m}$. This set does not contain any of the e_{i_k}'s of the formula (76) above. In analogy with A, the function e_{i_1}, for a fixed x_{i_1}, can be treated as a vector, and thus can be expanded in terms of orthogonal basis functions (of a similar nature as eq. (73)), etc. Again, the expansion coefficients satisfy $|c'_{jj_1\ldots j_m}(x_{i_1})| \leq 1$. If one puts this into A', one obtains a new form of F, which after a trivial summation over y_N and y_{i_1} depends on $c_N(x_N) c_{i_1}(x_i) \prod_{k=2}^{l} e_{i_k}$, where $c_{i_1}(x_i)$ stands for $c'_{11\ldots 1}(x_{i_1})$, and its modulus is again bounded by 1. Note that, y_N and y_{i_1} disappear, as $y_k^2 = 1$.

As each message appears in the product only once, we continue this procedure of expanding those messages which depend on earlier messages, till it halts. The final reduced form of the formula for the fidelity of an optimal protocol reads

$$F = \sum_{x_1,\ldots,x_N=0}^{1} g(x_1, \ldots, x_N) \prod_{n=1}^{N} c_n(x_n), \tag{77}$$

with $|c_n(x_n)| \leq 1$. Since F in eq. (77) is linear in every $c_n(x_n)$, its extrema are at the limiting values $c_n(x_n) = \pm 1$. *In other words, a Bell-like inequality* $|F| \leq Max(F) \equiv B(N)$ *gives the upper fidelity bound*. Note, that the above derivation shows that optimal classical protocols include one in which partners P_1 to P_{N-1} send to P_N one bit messages which encode the value of $e_k = y_k c(x_k)$, where $k = 1, 2, ..., N-1$.

6.2 Quantum solutions

The inequality for F suggests that some problems may have quantum solutions, which surpass any classical ones in their fidelity. Simply one may use an entangled state $|\psi\rangle$ of N qubits that violates the inequality. Send to each of the partners one of the qubits. In a protocol run all N partners make measurements on the local qubits, the settings of which are determined by x_k. They measure a certain qubit observable $\mathbf{n}_k(x_k)\cdot\sigma$. The measurement results $\gamma_k = \pm 1$ are multiplied by y_k, and the partner P_k, for $1 \leq k \leq N-1$, sends a bit message to P_N encoding the value of $m_k = y_k\gamma_k$. The last partner calculates $y_N \gamma_N \prod_{k=1}^{N-1} m_k$, and announces this as A. The average fidelity of such a process is

$$F = \sum_{x_1,\ldots,x_N=0}^{1} g(x_1,\ldots,x_N)\langle\psi| \otimes_{n=1}^{N} (\mathbf{n}_k(x_k)\cdot\sigma_k)|\psi\rangle, \tag{78}$$

and in certain problems can even reach *unity*.

For some tasks the quantum vs. classical fidelity ratio grows *exponentially* with N. This is the case, for example, for the so-called *modulo-4 sum* problem. Each partner receives a two-bit input string $(X_k = 0,1,2,3;\ k = 1,\ldots,N)$. The promise is that X_k's are distributed such that $(\sum_{k=1}^{N} X_k)\text{mod}2 = 0$. The task is[4]: P_N must tell whether the sum modulo-4 of all inputs is 0 or 2.

For this problem the classical fidelity bounds decrease exponentially with N, that is $B(F) \leq 2^{-K+1}$, where $K = N/2$ for even and $K = (N+1)/2$ for odd number of parties. If one uses the N qubit GHZ states: $|\text{GHZ}\rangle = \frac{1}{\sqrt{2}}(|z+,...,z+\rangle + |z-,...,z-\rangle)$, where $|z\pm\rangle$ is the state of spin ± 1 along the z-axis, and suitable pairs of local settings, the associated Bell inequality can be violated maximally. Thus, one has a quantum protocol which always gives the correct answer.

In all quantum protocols considered here entanglement that leads to a violation of Bell's inequality is a resource that allows for better-than-classical efficiency of the protocol. Surprisingly, one can also show a version of a quantum protocol without entanglement [36, 39]. The partners exchange a single qubit, P_k to P_{k+1} and so on, and each of them makes a suitable unitary transformation on it (which depends on z_k and x_k). The partner P_N, who receives the qubit as the last one, additionally performs a dichotomic measurement. The result he/she gets is equal to T. For details, includ-

[4] It can be formulated in terms of a task function $T = 1 - (\sum_{k=1}^{N} X_k)\text{mod}4$. An alternative formulation of the problem reads $f = \cos(\frac{\pi}{2}\sum_{k=1}^{N} X_k)$ with $p' = \frac{1}{2^{-N+1}}|\cos(\frac{\pi}{2}\sum_{k=1}^{N} X_k)|$.

ing an experimental realization see Ref. [39]. The obvious conceptual advantage of such a procedure is that the partners exchange a single qubit, from which due to the Holevo bound [29] one can read out at most one bit of information. In contrast with the protocol involving entanglement, no classical transfer of any information is required, except from the announcement by P_N of his measurement result!

In summary, if one has a pure entangled state of many qubits (this can be generalized to higher-dimensional systems and Bell's inequalities involving more than two measurement settings per observer), then there exist a Bell inequality which is violated by this state. This inequality has some coefficients $g(x_1,...,x_n)$, in front of correlation functions, which can always be renormalized in such a way that

$$\sum_{x_1,...,x_n=0}^{1} |g(x_1,...,x_n)| = 1.$$

The function g can always be interpreted as a product of the dichotomic function $f(x_1,...,x_n) = \frac{g(x_1,...,x_N)}{|g(x_1,...,x_N)|} = \pm 1$ and a probability distribution $p'(x_1,...,x_n) = |g(x_1,...,x_n)|$. Thus we can construct a communication complexity problem that is tailored to a given Bell's inequality, with task function $T = \prod_i^N y_i f$. All this can be extended beyond qubits, see Ref. [37, 40].

As it was shown, for three or more parties, $N \geq 3$, quantum solutions for certain communication complexity problems can achieve probabilities of success of unity. This is not the case for $N = 2$ and the problem based on the CHSH inequality. The maximum quantum value for the left hand side of the CHSH inequality (25) is just $\sqrt{2}-1$. This is much bigger than the Bell bound of 0, but still not the largest possible value, for an arbitrary theory that is not following local realism, which equals to 1. Because the maximum possible violation of the inequality is not attainable by quantum mechanics several questions arise. Is this limit forced by the theory of probability, or by physical laws? We will address this question in the next section, and look what would be the consequences of a maximal logically possible violation of the CHSH inequality.

6.3 *Stronger-than-quantum-correlations*

The Clauser-Horne-Shimony-Holt (CHSH) inequality [9] for local realistic theories gives the upper bound on a certain combination of correlations between two space-like separated experiments. Consider Alice and Bob who independently perform one out of two measurements on their part of the system, such that in total there are four experimental set-ups: $(x,y) = (0,0)$, $(0,1)$, $(1,0)$ or $(1,1)$. For any local hidden variable theory the CHSH inequality must hold. One can put it the following form:

$$\begin{aligned} &p(a=b|x=0,y=0) + p(a=b|x=0,y=0) \\ &+p(a=b|x=0,y=0) + p(a=-b|x=0,y=0) \leq 3, \end{aligned} \tag{79}$$

or equivalently,

$$\sum_{x,y=0,1} p(a \oplus b = x \cdot y) \leq 3. \tag{80}$$

In the latter form we interpret the dichtomic measurement results as of binary values, 0 or 1, and their relations are put as 'modulo 2 sums', denoted here by $\oplus$. One has $0 \oplus 0 = 1 \oplus 1 = 0$ and $0 \oplus 1 = 1$. For example, $p(a = b|x = 0, y = 0)$ is the probability that Alice's and Bob's outcomes are the same when she chooses setting x and he setting y.

As discussed in previous sections quantum mechanical correlations can violate the local realistic bound of inequality (80) and the limit was proven by Cirel'son [44] to be $2+\sqrt{2}$. In Ref. [19] Popescu and Rohrlich asked why quantum mechanics allows a violation of the CHSH inequality with a value of $2+\sqrt{2}$, but not more, though the maximal logically possible value is 4. Would a violation with a value larger than $2+\sqrt{2}$ lead to (superluminal) signaling?. If this were true, then quantum correlations could be understood as maximal allowed correlations respecting non-signaling requirement. This could give us an insight on the origin of quantum correlations, without any use of the Hilbert space formalism.

The non-signaling condition is equivalent to the requirement that the marginals are independent of the partners choice of setting

$$p(a|x,y) \equiv \sum_{b=0,1} p(a,b|x,y) = p(a|x), \tag{81}$$

$$p(a|x,y) \equiv \sum_{b=0,1} p(a,b|x,y) = p(a|x) \tag{82}$$

where $p(a,b|x,y)$ is the joint probability for outcomes a and b to occur given x and y are the choices of measurement settings, respectively and $p(a|x)$ is the probability for outcome a given x is the choice of measurement setting. Popescu and Rohrlich constructed a toy-theory where the correlations reach the maximal algebraic value of 4 for left hand expression of the inequality (79), but are nevertheless not in contradiction with signaling. The probabilities in the toy model are given by

$$\left.\begin{array}{l} p(a=0,b=0|x,y)=\frac{1}{2} \\ p(a=1,b=1|x,y)=\frac{1}{2} \end{array}\right\} \text{ if } xy \in \{00,01,10\},$$

$$\left.\begin{array}{l} p(a=1,b=0|x,y)=\frac{1}{2} \\ p(a=0,b=1|x,y)=\frac{1}{2} \end{array}\right\} \text{ if } xy = 11. \tag{83}$$

Indeed one has

$$\sum_{x,y=0,1} p(a \oplus b = x \cdot y) = 4. \tag{84}$$

Van Dam [45] and independently Cleve considered how plausible are stronger-than-quantum correlations from the point of view of communication complexity, which describes how much communication is needed to evaluate a function with distributed inputs. It was shown that the existence of correlations that maximally violate the CHSH inequality would allow to perform all distributed computations (between two parties) of dichotomic functions with a communication constraint to just one bit. If one is ready to believe that nature should not allow for "easy life" concerning communication problems, this could be a reason why superstrong correlations are indeed not possible.

Instead of superstrong correlations one usually speaks about a "nonlocal box" (NLB) or Popescu-Rohrlich (PR) box, as an imaginary device that takes as inputs x at Alices and y at Bobs side, and outputs a and b at respective sides, such that $a \otimes b = x \cdot y$. Quantum mechanical measurements on a maximally entangled state allow for a success probability of $p = \cos^2 \frac{\pi}{8} = \frac{2+\sqrt{2}}{4} \approx 0.854$ at the game of simulating NLBs. Recently, it was shown that in any "world" in which it is possible to implement an approximation to the NLB, that works correctly with probability greater than $\frac{3+\sqrt{6}}{6} = 90.8\%$, for all distributed computations of dichotomic functions with a one-bit communication constraint, one can find a protocol that gives always the correct values, Ref. [46]. This bound is an improvement over van Dam's one, but still has a gap with respect to the bound imposed by quantum mechanics.

6.3.1 Superstrong correlations trivializes communication complexity

We shall present a proof that availability of a perfect NLB would allow for a solution of a general communication complexity problem for a binary function, with an exchange of a single bit of information. The proof is due to van Dam [45].

Consider a Boolean function $f : \{0,1\}^n \times \{0,1\}$, which has as inputs two n-bit strings $\mathbf{x} = (x_1, ..., x_n)$ and $\mathbf{y} = (y_1, ..., y_n)$. Suppose that Alice receives the $\mathbf{x}$ string and Bob, who is separated from Alice, the $\mathbf{y}$-string, and they are to determine the function value $f(\mathbf{x}, \mathbf{y})$ by communicating as little as possible. They have, however, NLBs as resources.

First, let us notice that any dichotomic function $f(\mathbf{x}, \mathbf{y})$ can be rewritten as a finite summation:

$$f(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{2^n} P_i(\mathbf{x}) Q_i(\mathbf{y}), \tag{85}$$

where $P(\mathbf{x})$ are polynomials in $\mathbf{x} \in \{0,1\}$ and $Q_i(\mathbf{y}) = y_1^{i_1} \cdot ... \cdot y_n^{i_n}$ are monomials in $y_i \in \{0,1\}$ with $i_1, ..., i_n \in \{0,1\}$. Note that the latter ones constitute an orthogonal basis in a 2^n dimensional space. The decomposed function f is treated as a function of y's, while the inputs $x_1, ..., x_n$ are considered as indices numbering functions f. Note that there are 2^n different monomials. Alice can locally compute all the P_i values by herself and likewise Bob can compute all Q_i by himself. These values determine the settings of Alice and Bob that will be chosen in i-th run of the experiment. Note that to this end they need in general exponentially many NLBs. Alice and Bob perform for every $i \in \{1, ..., 2^n\}$ a measurement on the i-th NLB in order to obtain without any communication a collection of bit values a_i and b_i, with the property $a_i \otimes b_i = P_i(\mathbf{x}) Q_i(\mathbf{y})$. Bob can add all his b_i to $\sum_{i=1}^{2^n} b_i$ values without requiring any information from Alice, and he can broadcast this single bit to Alice.

She, on her part, computes the sum of her a_i to $\sum_{i=1}^{2^n} a_i$ and adds Bob's bit to it. The final result

$$\sum_{i=1}^{2^n}(a_i \oplus b_i) = \sum_{i=1}^{2^n} P_i(\mathbf{x})Q_i(\mathbf{y}) = f(\mathbf{x},\mathbf{y}) \tag{86}$$

is the function value. Thus, *superstrong correlations trivialize every communication complexity problem.*

7 The Kochen-Specker Theorem

In previous sections we have seen, that tests of Bell's inequalities are not only theory independent tests of non-classicality, but also have applications in quantum information protocols. Examples are communication complexity problems [38], entanglement detection [47], security of key distribution [2], and quantum state discrimination [48]. Thus entanglement which violates local realism can be seen as a resource for efficient information processing. Can quantum contextuality – the fact that quantum predictions disagree from the ones of non-contextual hidden-variable theories – also be seen as such a resource? We will give an affirmative answer to this question by considering explicit examples of a quantum game.

The Kochen-Specker theorem is a ”no go” theorem that proves a contradiction between predictions of quantum theory and those of *non-contextual* hidden variable theories. It was proved by Bell in 1966 [49] and independently by Kochen and Specker in 1967 [50]. The non-contextual hidden-variable theories are based on the conjecture of the following three assumptions:

1. *Realism*: It is a model that allows one to use all variables $A_m(n)$ in the theoretical description of the experiment, where $A_m(n)$ gives the value of some observable A_m which *could* be obtained if the knob setting were at positions m. The index n describes the entire experimental "context" in which A_m is measured and is operationally defined through the positions of all other knob settings in the experiment, which are used to measure other observables jointly with A_m. All $A_m(n)$'s are treated as perhaps unknown, but still fixed, (real) numbers, or variables for which a proper joint probability distribution can be defined.
2. *Non-contextuality*: The value assigned to an observable $A_m(n)$ of an individual system is independent of the experimental context n in which it is measured, in particular of any properties that are measured *jointly* with that property. This implies that $A_m(n) = A_m$ for all contexts n.
3. "Free will". The experimenter is free to choose the observable and its context. The choices are independent of the actual hidden values of A_m's, etc.

Note that "non-contextuality" implies locality (i.e., non-contextuaily with respect to a remote context), but there is no implication other way round. One might have theories which are local, but locally non-contextual.

It should be stressed that the local realistic and non-contextual theories provide us with predictions which can be tested experimentally, and which can be derived *without making any reference to quantum mechanics* (though many derivations in the literature give exactly the opposite impression). In order to achieve this, it is important to realize that predictions for noncontextual realistic theories can be derived in a completely operational way [53]. For concreteness, imagine that an observer wants to perform a measurement of an observable, say the square, $S^2_{\mathbf{n}}$, of a spin component of a spin-1 particle along a certain direction **n**. There will be an experimental procedure for trying to do this as accurately as possible. We will refer to this procedure by saying that one sets the "control switch" of his/her apparatus to the position **n**. In all experiments that we will discuss only a finite number of different switch positions is required. By definition different switch positions are clearly distinguishable for the observer, and the switch position is all he knows about. Therefore, in an operational sense the measured physical observable is entirely defined by the switch position. From the above definition it is clear that the same switch position can be chosen again and again in the course of an experiment. Notice that in such an approach as described above, it does not matter which observable is "really" measured and to what precision. One just derives general predictions, provided that certain switch positions are chosen.

In the original Kochen-Specker proof [50], the observables that are considered are squares of components of the spin 1 along various directions. Such observables have values 1 or 0, as the components themselves have values 1,0, or -1. The squares of spin components $\hat{S}^2_{\mathbf{n}_1}$, $\hat{S}^2_{\mathbf{n}_2}$ and $\hat{S}^2_{\mathbf{n}_3}$ along any three orthogonal directions $\mathbf{n}_1$, $\mathbf{n}_2$, and $\mathbf{n}_3$ can be measured jointly. Simply, the corresponding quantum operators commute with each other. In the framework of a hidden-variable theory one assigns to an individual system a set of numerical values, say $+1$,0,$+1$,... for the square of spin component along each direction $S^2_{\mathbf{n}_1}$, $S^2_{\mathbf{n}_2}$, $S^2_{\mathbf{n}_3}$,... that can be measured on the system. If any of the observables is chosen to be measured on the individual system, the result of the measurement would be the corresponding value. In a noncontextual hidden variable theory one has to assign to an observable, say $S^2_{\mathbf{n}_1}$, the *same* value independently of whether it is measured in an experimental procedure jointly as a part of some set $\{S^2_{\mathbf{n}_1}, S^2_{\mathbf{n}_2}, S^2_{\mathbf{n}_3}\}$ or of some other set $\{S^2_{\mathbf{n}_1}, S^2_{\mathbf{n}_4}, S^2_{\mathbf{n}_5}\}$ of physical observables, where $\{\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3\}$ and $\{\mathbf{n}_1, \mathbf{n}_4, \mathbf{n}_5\}$ are triads of orthogonal directions. Notice that within quantum theory some of the operators corresponding to the observables from the first set may *not commute* with some corresponding to the observables from the second set.

The squares of spin components along orthogonal directions satisfy

$$\hat{S}^2_{\mathbf{n}_1} + \hat{S}^2_{\mathbf{n}_2} + \hat{S}^2_{\mathbf{n}_3} = s(s+1) = 2. \tag{87}$$

This is *always* so for a particle of spin 1 (s=1). This implies that for every measurement of three squares of mutually orthogonal spin components two of the results will be equal to one, and one of them will be equal to zero. The Kochen-Specker theorem considers a set of triads of orthogonal directions $\{\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3\}$, $\{\mathbf{n}_1, \mathbf{n}_4, \mathbf{n}_5\}$,..., for which at least some of the directions have to appear in several of the triads. The statement of the theorem is that there are sets of directions for which it is not possible to give any assignment of 1's and 0's to the directions consistent with the constraint (87). The original theorem in [50] used 117 vectors, but this has subsequently been reduced to 33 vectors [51] and 18 vectors [52]. Mathematically the contradiction with quantum predictions has its origin in the fact that the classical structure of non-contextual hidden variable theories is represented by commutative algebra, whereas quantum mechanical observables need not be commutative, making it impossible to embed the algebra of these observables in a commutative algebra.

The disproof of noncontextually relies on the assumption that the same value is assigned to a given physical observable, $\hat{S}^2_{\mathbf{n}}$, regardless with which two other observables the experimenter chooses to measure it. In quantum theory the additional observables from one of those sets correspond to operators that do not commute with the operators corresponding to additional observables from the other set. As it was stressed in a masterly review on hidden variable theories by Mermin [54], Bell wrote [49] that "These different possibilities require different experimental arrangements; there is no *a priori* reason to believe that the results ... should be the same. The result of observation may reasonably depend not only on the state of the system (including hidden variables) but also on the complete disposition apparatus." Nevertheless, as Bell himself showed, the disagreement between predictions of quantum mechanics and of the hidden-variables theories can be strengthened if non-contextuality is replaced by a much more compelling assumption of locality. Note that in Bohr's doctrine of the inseparability of the object and the measuring instrument, an observable *is* defined through the entire measurement procedure applied to measure it. Within this doctrine one would not speak about measuring the same observable in different contexts, but rather about measuring entirely different maximal observables, and deriving from it the value of a degenerate observable. Note that Kochen-Specker argument necessarily involves degenerate observables. This is why it does not apply to single qubits.

7.1 A Kochen-Specker Game

We will now consider a quantum game which is based on the Kochen-Specker argument strengthened by the locality condition (See Ref. [55]). We consider a pair of entangled spin 1 particles, which form a singlet state with total spin 0. A formal description of this state is given by

$$|\Psi\rangle = \frac{1}{\sqrt{3}}(|1\rangle_{\mathbf{n}}|-1\rangle_{\mathbf{n}} + |-1\rangle_{\mathbf{n}}|1\rangle_{\mathbf{n}} - |0\rangle_{\mathbf{n}}|0\rangle_{\mathbf{n}}), \tag{88}$$

where, for example, $|1\rangle_{\mathbf{n}}|-1\rangle_{\mathbf{n}}$ is the state of the two particles with spin projection +1 for the first particle and spin projection -1 for the second particle 1 along the same direction $\mathbf{n}$. It is important to note that this state is invariant under a change of the direction $\mathbf{n}$. This implies that if the spin components for the two particles are measured along an arbitrary direction, however the same both sides, the sum of the two local results is always zero. This is a direct consequence of the conservation of angular momentum.

We now present the quantum game introduced in Ref. [56]. The requirement in the proof of the Kochen-Specker theorem can be formulated as the following problem in geometry. There exists an explicit set of vectors $\{\mathbf{n}_1, ..., \mathbf{n}_m\}$ in $\mathbf{R}^3$ that cannot be colored in red (i.e., assign the value 1 to the spin squared component along that direction) or blue (i.e., assign the value 0) such that both of the following conditions hold:

1. For every orthogonal pair of vectors $\mathbf{n}_1$ and $\mathbf{n}_2$, they are not both colored red.
2. For every mutually orthogonal triple of vectors $\mathbf{n}_i$, $\mathbf{n}_j$, and $\mathbf{n}_k$, at least one of them is colored red.

For example, the set of vectors can consist of 117 vectors from the original Kochen-Specker proof [50], 33 vectors from Peres's proof or 18 vectors from Cabello's proof [52].

The Kochen-Specker game employs the above sets of vectors. Consider two separated parties, Alice and Bob. Alice receives a random triple of orthogonal vectors as her input and Bob receives a single vector randomly chosen from the triple as his input. Alice is asked to give a trit indicating which of her three vectors is assigned color 1 (implicitly, the other two vectors are assigned color 0). Bob outputs a bit assigning a color to his vector. The requirement is that Alice and Bob assign the same color to the vector that they receive in common. Nevertheless, it is straightforward to show that the existence of a perfect classical strategy in which Alice and Bob can share classically correlated strings for this game would violate the reasoning used in the Kochen-Specker theorem. On the other hand, there is a perfect quantum strategy using the entangled state (88). If Alice and Bob share two particles in this state, Alice can perform a measurement of squared spin components pertaining to directions $\{\mathbf{n}_i, \mathbf{n}_j, \mathbf{n}_k\}$, which are equal to those of the three input vectors, and Bob measures squared spin component in direction $\mathbf{n}_l$ for his input. Then Bob's measurement will necessarily yield the same answer as the measurement by Alice along the same direction.

Concluding this section we note that quantum contextuality is also closely related to quantum error correction [57], quantum key distribution [58], one-location quantum games [59], and entanglement detection between internal degrees of freedom.

7.2 Temporal Bell's Inequalities (Leggett-Garg Inequalities)

In the last section we will consider one more basic information processing task, random access code problem. It can be solved with a quantum set-up with a higher efficiency than it is classically possible. We will show that the resource for better-than-classical efficiency is a violation of "temporal Bell's inequalities" – the inequalities that are satisfied by temporal correlations of certain class of hidden-variable theories. Instead of considering correlations between measurement results on distantly located physical systems, here we focus on one and the same physical system and analyze correlations between measurement outcomes at different times. The inequalities were first introduced by Leggett and Garg [60] in the context of testing superspositions of macroscopically distinct quantum states. Since our aim here is different, we will look at general assumptions that allows us to derive temporal Bell's inequalities irrespectively of whether the object under consideration is macroscopic or not. This is why our assumptions differ from the original ones of Ref. [60]. Compare also Ref. [65, 66, 67]

We consider the theories which are based on the conjunction of the following four assumptions[5]:

1. *Realism*: It is a model that allows one to use all variables $A_m(t)$ $m = 1, 2, \ldots$ in the theoretical description of the experiment performed at time t, where $A_m(t)$ gives the value of some observable which *could* be obtained if it were measured at time t. All $A_m(t)$'s are treated as perhaps unknown, but still fixed numbers, or variables for which a proper joint probability distribution can be defined.
2. *Non-invasiveness*: The value assigned to an observable $A_m(t_1)$ at time t_1 is independent whether or not a measurement was performed at some earlier time t_0 or which observable $A_n(t_0)$ $n = 1, 2, \ldots$ at that time was measured. In other words, (actual or potential) measurement values $A_m(t_1)$ at time t_1 are *independent* of the measurement settings chosen at earlier times t_0.
3. *Induction*: The standard arrow of time is assumed. In particular, the values $A_m(t_0)$ at earlier times t_0 do not depend on the choices of measurement settings at later times t_1[6].
4. *"Free will"*: The experimenter is free to choose the observable. The choices are independent of the actual hidden values of A's, etc.

Consider an observer and allow her to choose at time t_0 and at some later time t_1 to measure one of two dichotomic observables $A_1(t_i)$ and $A_2(t_i)$, $i \in \{0, 1\}$. The assumptions given above imply existence of numbers for $A_1(t_i)$ and $A_2(t_i)$, each taking values either +1 or -1, which describe the (potential or actual) predetermined result of the measurement. For the temporal correlations in an individual experimental run the following identity holds: $A_1(t_0)[A_1(t_1) - A_2(t_1)] + A_2(t_0)[A_1(t_1) + A_2(t_1)] = \pm 2$. With similar steps as in derivation of the standard Bell's inequalities, one easily obtains:

$$p(A_0A_0 = 1) + p(A_0A_1 = -1) + p(A_1A_0 = 1) + p(A_1A_1 = 1) \leq 3, \qquad (89)$$

where we omit the dependence on time.

An important difference between quantum contextuality and temporal Bell's inequalities is that later can also be tested on single qubits or *two-dimensional* quan-

[5] There is one more difference between the present approach and this of Ref. [60]. While there the observer measures a single observable having a choice between different times of measurement, here at any given time the observer has a choice between two (or more) different measurement settings. One can use both approaches to derive temporal Bell' inequalities.

[6] Note that this already follows from the "'non-invasiveness" when applied symmetrically to both arrows of time.

tum systems. We will now calculate the temporal correlation function for consecutive measurements of a single qubit. Take an arbitrary mixed state of a qubit, written as $\rho = \frac{1}{2}(\mathbf{1} + \mathbf{r} \cdot \sigma)$, where $\mathbf{1}$ is the identity operator, $\hat{\sigma} \equiv (\sigma_{\mathbf{x}}, \sigma_{\mathbf{y}}, \sigma_z)$ are the Pauli operators for three orthogonal directions x, y and z, and $\mathbf{r} \equiv (r_x, r_y, r_z)$ is the Bloch vector with the components $r_i = \mathrm{Tr}(\rho \sigma_{\mathbf{i}})$.

Suppose that the measurement of the observable $\sigma \cdot \mathbf{a}$ is performed at time t_0, followed by the measurement of $\sigma \cdot \mathbf{b}$ at t_1, where $\mathbf{a}$ and $\mathbf{b}$ are directions at which spin is measured. The quantum correlation function is given by $E_{QM}(\mathbf{a}, \mathbf{b}) = \sum_{k,l=\pm 1} k \cdot l \cdot \mathrm{Tr}(\rho \pi_{\mathbf{a},k}) \cdot \mathrm{Tr}(\pi_{\mathbf{a},k} \pi_{\mathbf{b},l})$, where, e.g., $\pi_{\mathbf{a},k}$ is the projector onto the subspace corresponding to the eigenvalue $k = \pm 1$ of the spin along $\mathbf{a}$. Here we use the fact that after the first measurement the state is projected on the new state $\pi_{\mathbf{a},k}$. Therefore, the probability to obtain the result k in the first measurement and l in the second one is given by $\mathrm{Tr}(\rho \pi_{\mathbf{a},k}) \mathrm{Tr}(\pi_{\mathbf{a},k} \pi_{\mathbf{b},l})$. Using $\pi_{\mathbf{a},k} = \frac{1}{2}(\mathbf{1} + k\sigma \cdot \mathbf{a})$ and $\frac{1}{2}\mathrm{Tr}[(\sigma \cdot \mathbf{a})(\sigma \cdot \mathbf{b})] = \mathbf{a} \cdot \mathbf{b}$ one can easily show that the quantum correlation function can simply be written as

$$E_{QM}(\mathbf{a}, \mathbf{b}) = \mathbf{a} \cdot \mathbf{b}. \tag{90}$$

Note that in contrast to the usual correlation function the temporal one (90) does not dependent of the initial state ρ. Note also that a slight modification of our derivation of Eq. (90) can also apply to the cases in which the system evolves between the two measurements following an arbitrary unitary transformation.

The scalar product form of quantum correlations (90) allows for the violation of the temporal Bell inequality and the maximal value of the left-hand side of (89) is achieved for the choice of the measurement settings: $\mathbf{a}_1 = \frac{1}{\sqrt{2}}(\mathbf{b}_1 - \mathbf{b}_2)$, $\mathbf{a}_2 = \frac{1}{\sqrt{2}}(\mathbf{b}_1 + \mathbf{b}_2)$ and is equal to $2 + \sqrt{2}$.

7.3 Quantum Random Access Codes

Random access code is a communication task for two parties, whom we call again Alice and Bob. Alice receives some classical n-bit string known only to her (her local input). She is allowed to send just a one bit message, m, to Bob. Bob is asked to tell the value of the b-th bit of Alice, $b = 1, 2..., n$. However b is known only to him (this is his local input data). The goal is to construct a protocol enabling Bob to tell the value b-th bit of Alice, with as high average probability of success as possible, for a uniformly random distribution of Alices bit-strings, and a uniform distribution of b's. Note that, since Alice does not know in advance which bit Bob is to recover. Thus she has no option to send just this required bit.

If they share a quantum channel then one speaks about a quantum version of the previous problem. Alice is asked to encode her classical n-bit message into 1 qubit (quantum bit) and send it to Bob. He performs some measurement on the received qubit to extract the required bit. In general, the measurement that he uses will depend on which bit he wants to reveal. The idea behind these so-called quantum random access codes already appeared in a paper written circa 1970 and published in 1983 by Stephen Wiesner [63].

We illustrate the concept of random access code with the simplest scheme, in which in a classical framework Alice needs to encode a two-bit string b_0b_1 into a single bit, or into a single qubit in a quantum framework.

In the classical case Alice and Bob need to decide on a protocol defining which bit-valued message is to be sent by Alice, for each of the four possible values of her two-bit string b_0b_1. There are only $2^4 = 16$ different deterministic protocols, thus the probability of success can be evaluated in a straightforward way. The optimal deterministic classical protocols can then be shown to have a probability of success $P_C = 3/4$. For example, if Alice sends one of the two bits, then Bob will reveal this bit with certainty and have probability of $1/2$ to reveal the other one. Since any probabilistic protocol can be represented as a convex combination of the 16 deterministic protocols, the corresponding probability of success for any such probabilistic protocol will be given by the weighted sum of the probabilities of success of the individual deterministic protocols. This implies that the optimal probabilistic protocols can at best be as efficient as the optimal deterministic protocol, which is $3/4$.

Ambainis *et al.* [64] showed that there is a quantum solution of the random access code with probability of success $P_Q = \cos^2(\pi/8) \approx 0.85$. It is realized as follows: depending on her two-bit string b_0b_1, Alice prepares one of the four states $|\psi_{b_0b_1}\rangle$. These states are chosen to be on the equator of the Bloch sphere, separated by equal angles of $\pi/2$ radians (see figure 3). Using the Bloch sphere parametrization $|\psi(\theta,\phi)\rangle = \cos(\theta/2)|0\rangle + \exp(i\phi)\sin(\theta/2)|1\rangle$, the four encoding states are represented as:

$$
\begin{aligned}
|\psi_{00}\rangle &= |\psi(\pi/2,\pi/4)\rangle, \\
|\psi_{01}\rangle &= |\psi(\pi/2,7\pi/4)\rangle, \\
|\psi_{10}\rangle &= |\psi(\pi/2,3\pi/4)\rangle, \\
|\psi_{11}\rangle &= |\psi(\pi/2,5\pi/4)\rangle. \qquad (91)
\end{aligned}
$$

Bob's measurements, which he uses to guess the bits, will depend on which bit he wants to obtain. To guess b_0, he projects the qubit along the x-axis in the Bloch sphere, and to decode b_1 he projects it along the y-axis. He then estimates the bit value to be 0 if the measurement outcome was along the positive direction of the axis and 1 if it was along the negative axis. It can easily be calculated that the probability of successful retrieving of the correct bit value is the same in all cases: $P_Q = \cos^2(\pi/8) \approx 0.85$, which is higher than the optimal probability of success $P_C = 0.75$ of the classical random access code using one bit of communication.

We will now introduce a hidden variable model of the quantum solution to see that the key resource in its efficiency lies in violation of temporal Bell's inequalities. Galvao [61] was the first to point to the relation between violation of Bell's type inequalities and quantum random access codes. See also Ref. [62] for a relation with the parity-oblivious multiplexing.

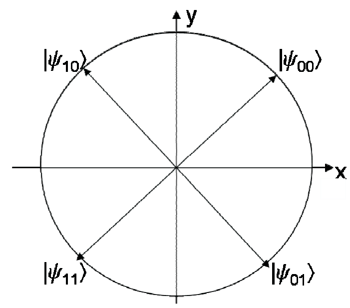


Fig. 3 The set of encoding states and decoding measurements in quantum random access code represented in the $x-y$ plane of the Bloch sphere. Alice prepares one of the four quantum states $\psi_{b_0 b_1}$ to encode two bits $b_0, b_1 \in \{0,1\}$. Depending on which bit Bob wants to reveal he performs either a measurement along the x (to reveal b_0) or along the y axis (to reveal b_1).

A hidden-variable model equivalent to the quantum protocol, which best fits the temporal Bell's inequalities can be put as a description of the following modification of the original quantum protocol. Alice prepares the initial state of her qubit as a completely random state, described by a density matrix proportional to the unit operator, σ_0. Her parity of bit values $b_0 \oplus b_1$ defines a measurement basis, which is used by her to prepare the state to be sent to Bob. Note that the result of the dichotomic measurement in the basis defined by $b_0 \oplus b_1$ is, due to the nature of the initial state, completely random, and totally uncontrollable by Alice. To fix the bit value b_1 (and thus also the value b_0, since the parity is defined by the choice of the measurement basis) on her wish, Alice either leaves the state unchanged, if the result of measurement corresponds to her wish of b_1 or she rotates the state in the $x-y$ plane at 180^o to obtain the orthogonal state, if the result corresponds to $b_1 \oplus 1$. Just a glance at the states involved in the standard quantum protocol shows what are the two complementary (unbiased) bases which define her measurement settings, and which resulting states are linked with which values of $b_0 b_1$. After the measurement the resulting state is sent to Bob, while Alice is in possession of a bit pair $b_0 b_1$, which is perfectly correlated with the qubit state on the way to Bob. That is, we have exactly the same starting point as in the original quantum protocol.

Now, it is obvious that the quantum protocol violates the temporal inequalities, while any hidden variable model of the above procedure, using the four assumptions (1.-4.) behind the temporal inequalities is not violating them. What is important the saturation of the temporal inequalities is equivalent to a probability of success of $3/4$.

The link with temporal Bell's inequalities points onto another advantage of quantum over classical random access codes. Usually, one considers the advantage to be only *resource dependent*. With this we mean that there is an advantage as far as one compares one classical bit with one qubit. Yet, the proof given above shows that quantum strategy has an advantage over *all* hidden variable models respecting (1.-4.), i.e. also those where Alice and Bob use systems with arbitrarily large number of degrees of freedom.

Concluding this section and the Chapter we would like to point onto an interesting research avenue. Here we gave a brief review on the results demonstrating that "no go theorems" for various hidden variable classes of theories, are behind better-than-classical efficiency in many quantum communication protocols. It would be interesting to investigate the link between fundamental features of quantum mechanics and the power of quantum computation. It has been shown that temporal Bell's inequalities distinguish between classical and quantum search (Grover) algorithm [68]. Cluster states – a resource for measurement-based *quantum* computation (also known as "one-way" quantum computation) in which information is processed by a sequence of adaptive single-qubit measurements on the state – are shown to violate Bell's inequalities [69, 70]. Similarly, the CSHS and GHZ problems are shown to be closely related to measurement-based *classical* computation, as does the Popescu-Rohrlich box [71]. These results point on the aforementioned link but we are still far away from understanding what are the key non-classical ingredients that give rise to the enhanced quantum computational power. The question gets even more fascinating after realizing that not only too low [71, 72, 73, 74, 75, 76] but also too much entanglement does not allow powerful quantum computation [77, 78].

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