Geometry and Physics

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Dedicated to Stephan Luckhaus,
with respect and gratitude for his critical mind

The aim of physics is to write down the Hamiltonian of the universe. The rest is mathematics.

Mathematics wants to discover and investigate universal structures. Which of them are realized in nature is left to physics.

Preface

Perhaps, this is a bad book. As a mathematician, you will not find a systematic theory with complete proofs, and, even worse, the standards of rigor established for mathematical writing will not always be maintained. As a physicist, you will not find coherent computational schemes for arriving at predictions.

Perhaps even worse, this book is seriously incomplete. Not only does it fall short of a coherent and complete theory of the physical forces, simply because such a theory does not yet exist, but it also leaves out many aspects of what is already known and established.

This book results from my fascination with the ideas of theoretical high energy physics that may offer us a glimpse at the ultimate layer of reality and with the mathematical concepts, in particular the geometric ones, underlying these ideas.

Mathematics has three main subfields: analysis, geometry and algebra. Analysis is about the continuum and limits, and in its modern form, it is concerned with quantitative estimates establishing the convergence of asymptotic expansions, infinite series, approximation schemes and, more abstractly, the existence of objects defined in infinite-dimensional spaces, by differential equations, variational principles, or other schemes. In fact, one of the fundamental differences between modern physics and mathematics is that physicists usually are satisfied with linearizations and formal expansions, whereas mathematicians should be concerned with the global, nonlinear aspects and prove the convergence of those asymptotic expansions. In this book, such analytical aspects are usually suppressed. Many results have been established through the dedicated effort of generations of mathematicians, in particular by those among them calling themselves mathematical physicists. A systematic presentation of those results would require a much longer book than the present one. Worse, in many cases, computations accepted in the physics literature remain at a formal level and have not yet been justified by such an analytical scheme. A particular issue is the relationship between Euclidean and Minkowski signatures. Clearly, relativity theory, and more generally, relativistic quantum field theory require us to work in Lorentzian spaces, that is, ones with an indefinite metric, and the corresponding partial differential equations are of hyperbolic type. The mathematical theory, however, is easier and much better established for Riemannian manifolds, that is, for spaces with positive definite metrics, and for elliptic partial differential equations.

In the physics literature, therefore, one often carries through the computations in the latter situation and appeals to a principle of analytic continuation, called Wick rotation, that formally extends the formulae to the Lorentzian case. The analytical justification of this principle is often doubtful, owing, for example, to the profound difference between nonlinear elliptic and hyperbolic partial differential equations. Again, this issue is not systematically addressed here.

Algebra is about the formalism of discrete objects satisfying certain axiomatic rules, and here there is much less conflict between mathematics and physics. In many instances, there is an alternative between an algebraic and a geometric approach. The present book is essentially about the latter, geometric, approach. Geometry is about qualitative, global structures, and it has been a remarkable trend in recent decades that some physicists, in particular those considering themselves as mathematical physicists (in contrast to the mathematicians using the same name who, as mentioned, are more concerned with the analytical aspects), have employed global geometric concepts with much success. At the same time, mathematicians working in geometry and algebra have realized that some of the physical concepts equip them with structures that are at the same time rich and tightly constrained and thereby afford powerful tools for probing old and new questions in global geometry.

The aim of the present book is to present some basic aspects of this powerful interplay between physics and geometry that should serve for a deeper understanding of either of them. We try to introduce the important concepts and ideas, but as mentioned, the present book neither is completely systematic nor analytically rigorous. In particular, we describe many mathematical concepts and structures, but for the proofs of the fundamental results, we usually refer to other sources. This keeps the book reasonably short and perhaps also aids its coherence. – For a much more systematic and comprehensive presentation of the fundamental theories of high-energy physics in mathematical terms, I wish to refer to the forthcoming 6-volume treatise [111] of my colleague Eberhard Zeidler.

As you will know, the fundamental problem of contemporary theoretical physics[1] is the unification of the physical forces in a single, encompassing, coherent "Theory of Everything". This focus on a single problem makes theoretical physics more coherent, and perhaps sometimes also more dynamic, than mathematics that traditionally is subdivided into many fields with their own themes and problems. In turn, however, mathematics seems to be more uniform in terms of methodological standards than physics, and so, among its practioners, there seems to be a greater sense of community and unity.

Returning to the physical forces, there are the electromagnetic, weak and strong interactions on one hand and gravity on the other. For the first three, quantum field theory and its extensions have developed a reasonably convincing, and also rather successful unified framework. The latter, gravity, however, more stubbornly resists such attempts at unification. Approaches to bridge this gap come from both sides. Superstring theory is the champion of the quantum camp, ever since the appearance

[1] More precisely, we are concerned here with *high-energy* theoretical physics. Other fields, like solid-state or statistical physics, have their own important problems.

of the monograph [50] of Green, Schwarz and Witten, but many people from the gravity camp seem unconvinced[2] and propose other schemes. Here, in particular Ashtekar's program should be mentioned (see e.g. [92]). The different approaches to quantum gravity are described and compared in [74]. A basic source of the difficulties that these two camps are having with each other is that quantum theory does not have an ontology, at least according to the majority view and in the hands of its practioners. It is solely concerned with systematic relations between observations, but not with any underlying reality, that is, with laws, but not with structures. General relativity, in contrast, is concerned with the structure of space–time. Its practioners often consider such ideas as extra dimensions, or worse, tunneling between parallel universes, that are readily proposed by string theorists, as too fanciful flights of the imagination, as some kind of condensed metaphysics, rather than as honest, experimentally verifiable physics. Mathematicians seem to have fewer difficulties with this, as they are concerned with structures that are typically believed to constitute some higher form of 'Platonic' reality than our everyday experience. In the present book, I approach things from the quantum rather than from the relativity side, not because of any commitment at a philosophical level, but rather because this at present offers the more exciting mathematical perspectives. However, this is not meant to deny that general relativity and its modern extensions also lead to deep mathematical structures and challenging mathematical problems.

While I have been trained as a mathematician and therefore naturally view things from a structural, mathematical rather than from a computational, physical perspective, nevertheless I often find the physicists' approach more insightful and more to the point than the mathematicians' one. Therefore, in this book, the two perspectives are relatively freely mixed, even though the mathematical one remains the dominant one. Hopefully, this will also serve to make the book accessible to people with either background. In particular, also the two topics, geometry and physics, are interwoven rather than separated. For instance, as a consequence, general relativity is discussed within the geometry part rather than the physics one, because within the structure of this book, it fits into the geometry chapter more naturally.

In any case, in mathematics, there is more of a tradition of explaining theoretical concepts, and good examples of mathematical exposition can provide the reader with conceptual insights instead of just a heap of formulae. Physicists seem to make fewer attempts in this direction. I have tried to follow the mathematical style in this regard.

I have assembled a representative (but perhaps personally biased) bibliography, but I have made no attempt at a systematic and comprehensive one. In the age of the Arxiv and googlescholar, such a scholarly enterprise seems to have lost its usefulness. In any case, I am more interested in the formal structure of the theory than in its historical development. Therefore, the (rather few) historical claims in this book should be taken with caution, as I have not checked the history systematically or carefully.

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[2] For an eloquent criticism, see for example Penrose [85].

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