Quantum-like Representation of Macroscopic Configurations

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Abstract

The aim of this paper is to apply a contextual probabilistic model (in the spirit of Mackey, Gudder, Ballentine) to represent and to generalize some results of quantum logic about possible macroscopic quantumlike (QL) behaviour. The crucial point is that our model provides QLrepresentation of macroscopic configurations in terms of complex probability amplitudes – wave functions of such configurations. Thus, instead of the language of propositions which is common in quatum logic, we use the language of wave functions which is common in the conventional presentation of QM. We propose a quantum-like representation algorithm, QLRA, which maps probabilistic data of any origin in complex (or even hyperbolic) Hilbert space. On the one hand, this paper clarifyes some questions in foundations of QM, since some rather mystical quantum features are illustrated on the basis of behavior of macroscopic systems. On the other hand, the approach developed in this paper may be used e.g. in biology, sociology, or psychology. Our example of QL-representation of hidden macroscopic configurations can find natural applications in those domains of science.

Keywords: contextual probabilistic model, quantum-like representation algorithm, macroscopic quantum-like systems

1 Introduction

One should sharply distinguish QM as a physical theory and the mathematical formalism of QM. In the same way as one should distinguish classical Newtonian mechanics and its mathematical formalism. Nobody is surprised that the differential and integral calculi which are basic in Newtonian mechanics can be fruitfully applied in other domains of science. Unfortunately, the situation with the mathematical formalism of QM is essentially more complicated – some purely mathematical features of QM are identified with features of quantum physical systems. Although

already Nils Bohr pointed out [1], see also [2], [3], to the possibility to apply the mathematical formalism of QM outside of physics, prejudice based on the identification of mathematics and physics still survives (but cf. e.g. Accardi, Aerts, Ballentine, De Muynck, Grib et al., Gudder, Gustafson, Landé, Mackey [4]–[19] and also [20]–[22]). One can point out just to a few applications outside of physics. Here we discuss not reductionist models in that the quantum description appears as a consequence of the evident fact that any physical system, even living (for example, the brain, see e.g. [23], [24]), is composed of quantum particles, but really the possibility to use the mathematical formalism of QM without direct coupling with quantum physics, see e.g. [5], [25], [26], [10]–[13].

We remark that importance of mentioned separation between quantum physics and quantum mathematics has been already well recognized in quantum logic, see e.g. Mackey [19] or Beltrametti and Cassinelli [27]. In particular, an exiting possibility to apply quantum mathematics to macroscopic systems is not surprising for quantum logicians. However, one could not see visible results of diffussion of this quantum logic knowledge into real quantum physics. There are a few reasons for this, in particular, psychological ones. It seems that the main problem is that the majority of physicists think that QM is not about new logic, but new physics. Thus the massage of Birkhof and von Neumann[28], as well as Bohr [1] who discussed a possibility to reduce quantum particularities to elaboration of new "quantum language", was practically ignored in quantum physics.

We point out that quantum logic is closely interrelated with quantum probability which is a caculus of complex probability amplitudes and self-adjoint operators (in contrast to classical Kolmogorovian probability theory which is a calculus of measures and measurable functions, random variables). Roughly speaking quantum logic emphasizes the observational part quantum formalism, the calculus of propositions [27] representing results of quantum observations. The complex probability amplitude (the wave function) does not belong to the main field of interest of quantum logicians.¹ On the other hand, the wave function is the basic object of practical quantum physics [29], [30].

Recently I developed so called contextual probability theory [31] which was inspired essentially by quantum logic and quantum probability, especially Mackey's approach [19].² The main distinguishing feature of the theory of contextual probabilities is a possibility to derive the complex probability amplitude, the wave function, from probabilistic data. Such an algorithm for mapping of probabilistic data into the complex probabil-

¹Personally I am not quantum logician. Thus my interpretation may be wrong from the internal viewpoint of quantum logic. But I think that it has right as rather general external opinion.

²I was lucky to meet George Mackey at the Congress of Quantum Structures Association in Castiglioncello, Italy, 1992. Our conversations on the probabilistic structure of QM were the starting point of my further studies on contextual probability. I also was lucky to speak shortly with Andrei Nikolaevich Kolmogorov (when he submitted my paper to Doklady Akademii Nauk USSR). I was surprised that personally he was not satisfied by his axiomatics of probability theory [32]. Later (after his death) his former students Albert Shiryaev and Alexander Bulinskii explained me that contextuality of probabilities and the impossibility to play the whole game with a single Kolmogorov space was evident for Kolmogorov [33].

ity amplitude was proposed in [31], quantum-like representation algorithm – QLRA. This algorithm also generates representation of observables (in fact, to fixed "reference observables") by self-adjoint operators. Thus by contextual probability theory the mathematical quantum structure is not fundamental. It appears as a special representation of probabilistic data. The main distinguishing feature of the QL-representation is ignorance by details about system's behaviour which are not approachable by an external observer. This is a consistent way to proceed within incomplete description of system's behavior.³

The aim of this paper is to use our contextual probabilistic model, the $V\ddot{a}xj\ddot{o}$ model, to represent and to generalize some results of quantum logic on macroscopic quantum-like (QL) behaviour in terms of complex probability amplitudes. On the one hand, it may be intersting for physicists, since some rather mystical quantum features will be illustrated on the basis of behavior of macroscopic systems. On the other hand, the approach developed in this paper may be used e.g. in biology, sociology, or psychology. Our example of QL-representation of hidden macroscopic configurations can find natural applications in these domains of science.

The basic example which we would like to generalize in the contextual probabilistic framework is well known in quantum logic. This is "firefly in the box". It was proposed by Foulis who wanted to show that a macroscopic system, firefly, can exhbit a QL-behavior which can be naturally represented in terms of quantum logics. First time this example was published in Cohen's book [34], a detailed presentation can be found in Foulis' paper [35], see also Svozil [36]. Later "firefly in the box" was generalized to a so called generalized urn's model, by Wright [37] (psychologist).

From the viewpoint of quantum logic such examples illustrate the following problem. For a given quantum logic one wants to find a Boolean algebra such that by ignoring some elements of this algebra one obtains the original quantum logic. I would formulate this problem in the following way: "To quantum (and more general QL) structures through ignorance of some information about underlying classical Boolean algebras." We shall use two lessons of previous studies in quantum logic: a) essentially quantum structures (lattices of quantum projectors) can be obtained from purely classical Boolean models; b) not all quantum structures have underlying classical Boolean models.

Similar lessons we have from studies on contextual probability:

- a). The QLRA can be applied to classical probabilistic data (which can be described by the Kolmogorov model). The result will be nontrivial: Born's rule, interference of probabilities, representation of Kolmogorovian random variables by self-adjoint operators. Thus all basic quantum structures are present in the classical probabilistic models, but in a latent form.
- b). The quantum probabilistic structure could not be completely reproduced on the basis of a single Kolmogorov probability space (by using Gudder's terminology one must consider a *probability manifold* with the atlas consisting of a few Kolmogorovian charts).

³Consistency is an extremely important feature of quantum and QL representations of probabilistic data. Of course, one may try cut off data ocationaly, but such a data-processing would (soon or later) induce chaos.

Regarding b) we point out to one very important difference between quantum logic and contextual probability theory. According to the latter even in the two dimensional case an undelying classical model does not exist. By applying QLRA to probabilistic data obtained on the basis of a single Kolmogorov probability space we are not able to get all pure quantum states and all pairs of noncommutative observables. To show this, we use an analogue of Bell's inequality for transition probabilities, see [38] and appendix.

In general, our contextual model is based on the frequency definition of probability which was formalized by R. von Mises [39] (this formalization was simplified and justified in [40]). By using frequency probabilities we can reproduce completely the pobabilistic structure of QM. However, the contextual statistical model is not reduced to the quantum probabilistic model. Besides ordinary trigonometric cos-interference it predicts hypebolic cosh-interference. Corresponding contexts are repsented not by complex, but hyperbolic probability amplitude, i.e., in an analogue of Hilbert space, but over the algebra of hypebolic numbers, z = x + jy, $j^2 = +1, x, y \in \mathbb{R}$, see [31].

We can mention some consequences of our QL-representation of macroscoipic configurations for foundations of quantum physics. All distinguishing features of the quantum probabilistic behavior can be modeled by using macroscopic systems. For such macroscopic models the QL-description is not complete. Thus hidden variables exist, but they could not be observed on the basis of available observables. Those observables which we (external observers) could use are too fuzzy, cf. [41]. Nevertheless, a kind of Einstein's demon can observe behavior of hidden variables. Since our examples are macroscopic, such Einstein's demon can be a macroscopic observer. Classical probability describes models in that measurements of complementary observables are not mutually disturbing. As we remarked, such models do not cover completely QM.⁵ On the other hand, by using models with mutual distrbance and the frequency approach to probability we can reconstruct QM in the realistic framework. We also discuss "flyrealization" of the EPR-Bohm experiment. Since flyes are macroscopic systems, realism could not be questioned. Possible explanations of violation of Bell's inequality are nonlocality [42], unfair sampling [43], [44], [45], ensemble nonreproducibility [46], [40], [47], [48]. For macroscopic systems the latter two possibilities are essentially more natural than the first one.

Of course, we understand well that our *fly-methaphor* can not be used for derivation of crucial consequences about microscopic quantum systems, such as photons and electrons. It might be that similarities in mathematical description are just occational. Nevertheless, these similarities are really astonishing.

⁴This demon is similar to Maxwell's demon who might (in principle) violate the principles of thermodynamics. Einstein's demon might violate principles of the Copenhagen interpretation of QM, in particular, the principle of complementarity.

⁵To show this, we use an analogue of Bell's inequality which we obtained for transition probabilities, see appendix.



Figure 1: Internal description.



Figure 2: The a-observable.

2 Firefly in the box

We recall the well known example [35] of QL-behavior. We modify its presentation by emphasizing its probabilistic structure. Let us consider a box which is divided into four sub-boxes. These small boxes which are denoted by $\omega_1, \omega_2, \omega_3, \omega_4$ provides internal description. These elements are avalaible for Einstein's demon, but they are not avalaible for some external observable.

We consider the Kolmogorov probability space: $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$, the algebra of all finite subsets \mathcal{F} of Ω and a probability measure determined by probabilities $\mathbf{P}(\omega_i) = p_i$, where $0 < p_i < 1, p_1 + ... + p_4 = 1$.

We now consider two different disjoint partitions of the set Ω :

$$C_{\alpha_1} = \{\omega_1, \omega_2\}, C_{\alpha_2} = \{\omega_3, \omega_4\},$$

 $C_{\beta_1} = \{\omega_1, \omega_4\}, C_{\beta_2} = \{\omega_2, \omega_3\}.$

We can obtain such partitions by dividing the box: a) into two equal parts by the vertical line: the left-hand part gives C_{α_1} and the right-hand part C_{α_2} ; b) into two equal parts by the horizontal line: the top part gives C_{β_1} and the bottom part C_{β_2} .

We introduce two random variables corresponding to these partitions: $\xi_a(\omega) = \alpha_i$, if $\omega \in C_{\alpha_i}$ and $\xi_b(\omega) = \beta_i \in \text{if } \omega \in C_{\beta_i}$. Here α_i and β_i are arbitrary labels. Suppose now that the external observer is able to measure only these two variables, denote the corresponding observables by the symbols a and b. We remark that there exist other random variables, they are avalable for Einstein's demon, but not for the external observer. Roughly speaking elements ω_j are not visible for the latter observer. They are "hidden variables."

Such a probabilistic model can be illustrated by the following example [35]. Let us consider a firefly in the box. It has definite position in space. The firefly position can be seen by Einstein's demon living inside this box.

⁶For example, $\xi(\omega) = +1$, $\omega = \omega_1, \omega_2, \omega_3$, and $\xi(\omega) = -1$, $\omega = \omega_4$.



Figure 3: The b-observable

Now we consider an external observer who has only two possibilities to observe the firefly in the box:

- 1) to open a small window at the point a which is located in such a way (the bold dot in the middle of the bottom side of the box, Figure 2) that it is possible to determine only either the firefly is in the section C_{α_1} or in the section C_{α_2} of the box;
- 2) to open a small window at the point b which is located in such a way (the bold dot in the middle of the right-hand side of the box, Figure 3) that it is possible to determine only either the firefly is in the section C_{β_1} or in the section C_{β_2} of the box.

In the first case such an external observer can determine in which part, C_{α_1} or C_{α_2} , the firefly is located. In the second case he can only determine in which part, C_{β_1} or C_{β_2} , the firefly is located. But he is not able to look into both windows simultaneously. In such a situation the observables a and b are the only source of information about the firefly ("reference observables"). The Kolmogorov description is meaningless for the external observer (although it is present in the latent form), but it is very useful for Eistein's demon.

Can one apply in such a situation the QL-description? Can we construct the wave function of the firefly in the box? Can we represent observables (in fact, classical random variables) a and b by self-adjoint operators? The answers are to be positive.

3 Contextual probability

A general statistical model for observables based on the contextual viewpoint to probability will be presented. It will be shown that classical as well as quantum probabilistic models can be obtained as particular cases of our general contextual model, the Växjö model, [31]. As was mentioned in introduction, I was inspired by Mackey's program: To deduce the probabilistic formalism of quantum mechanics starting with a system of natural probabilistic axioms. We reduced essentially the number of axioms (Mackey had 8 axioms and we have only two axioms). But the main difference between Mackey's model and the Växjö model is that Makey should postulate the complex Hilbert space structure, but in our model it is derived from our two axioms. Moreover, representations of the Växjö model are not reduced to the conventional, classical and quantum ones. Our model also implies hyperbolic cosh-interference that induces "hyperbolic quantum mechanics" [31].

A physical, biological, social, mental, genetic, economic, or financial context C is a complex of corresponding conditions. Contexts are fundamental elements of any contextual probabilistic model. Thus construction of any model M should be started with fixing the collection of contexts of this model. Denote the collection of contexts by the symbol C (so the family of contexts C is determined by the model M under consideration). In the mathematical formalism C is an abstract set (of "labels" of contexts).

We remark that in some models it is possible to construct a settheoretic representation of contexts – as some family of subsets of a set Ω . For example, Ω can be the set of all possible parameters (e.g., physical, or mental, or economic) of the model. However, in general we do not assume the possibility to construct a set-theoretic representation of contexts.

Another fundamental element of any contextual probabilistic model M is a set of observables \mathcal{O} : each observable $a \in \mathcal{O}$ can be measured under each complex of conditions $C \in \mathcal{C}$. For an observable $a \in \mathcal{O}$, we denote the set of its possible values ("spectrum") by the symbol X_a .

We do not assume that all these observables can be measured simultaneously. To simplify considerations, we shall consider only discrete observables and, moreover, all concrete investigations will be performed for dichotomous observables.

Axiom 1: For any observable $a \in \mathcal{O}$ and its value $\alpha \in X_a$, there are defined contexts, say C_{α} , corresponding to α -selections: if we perform a measurement of the observable a under the complex of physical conditions C_{α} , then we obtain the value $a = \alpha$ with probability 1. We assume that the set of contexts \mathcal{C} contains C_{α} -selection contexts for all observables $a \in \mathcal{O}$ and $\alpha \in X_a$.

For example, let a be the observable corresponding to some question: a=+ (the answer "yes") and a=- (the answer "no"). Then the C_+ -selection context is the selection of those participants of the experiment who answering "yes" to this question; in the same way we define the C_- -selection context. By Axiom 1 these contexts are well defined. We point out that in principle a participant of this experiment might not want to reply at all to this question or she might change her mind immediately after her answer. By Axiom 1 such possibilities are excluded. By the same axiom both C_+ and C_- -contexts belong to the system of contexts under consideration.

Axiom 2: There are defined contextual (conditional) probabilities $p_C^a(\alpha) \equiv \mathbf{P}(a = \alpha | C)$ for any context $C \in \mathcal{C}$ and any observable $a \in O$.

Thus, for any context $C \in \mathcal{C}$ and any observable $a \in O$, there is defined the probability to observe the fixed value $a = \alpha$ under the complex of conditions C.

Especially important role will be played by "transition probabilities" $p^{a|b}(\alpha|\beta) \equiv \mathbf{P}(a=\alpha|C_{\beta}), a, b \in \mathcal{O}, \alpha \in X_a, \beta \in X_b$, where C_{β} is the $[b=\beta]$ -selection context. By axiom 2 for any context $C \in \mathcal{C}$, there is defined the set of probabilities: $\{p_C^a: a \in \mathcal{O}\}$. We complete this probabilistic data for the context C by transition probabilities. The corresponding collection of data $D(\mathcal{O}, C)$ consists of contextual probabilities: $p^{a|b}(\alpha|\beta), p_C^b(\beta), p^{b|a}(\beta|\alpha), p_C^a(\alpha)...$, where $a, b, ... \in \mathcal{O}$. Finally, we denote

the family of probabilistic data $D(\mathcal{O}, C)$ for all contexts $C \in \mathcal{C}$ by the symbol $\mathcal{D}(\mathcal{O}, \mathcal{C}) (\equiv \bigcup_{C \in \mathcal{C}} D(\mathcal{O}, C))$.

Definition 1. (Växjö Model) A contextual probabilistic model of reality is a triple $M = (\mathcal{C}, \mathcal{O}, \mathcal{D}(\mathcal{O}, \mathcal{C}))$, where \mathcal{C} is a set of contexts and \mathcal{O} is a set of observables which satisfy to axioms 1,2, and $\mathcal{D}(\mathcal{O}, \mathcal{C})$ is probabilistic data about contexts \mathcal{C} obtained with the aid of observables belonging \mathcal{O} .

We call observables belonging the set $\mathcal{O} \equiv \mathcal{O}(M)$ reference of observables. Inside of a model M observables belonging to the set \mathcal{O} give the only possible references about a context $C \in \mathcal{C}$. In the definition of the Växjö Model we speak about "reality." In our approach it is reality of contexts.

In what follows we shall consider Växjö models with two dichotomous reference observables.

4 Frequency definition of probabilities

The definition of probability has not yet been specified. In this paper we shall use the frequency definition of probability as the limit of frequencies in a long series of trials, von Mises' approach, [39], [40]. We are aware that this approach was criticized a lot in mathematical literature. However, the main critique was directed against von Mises' definition of randomness. If one is not interested in randomness, but only in frequencies of trials, then the frequency approach is well established, see [40].

We consider a set of reference observables $\mathcal{O} = \{a, b\}$ consisting of two observables a and b. We denotes the sets of values ("spectra") of the reference observables by symbols X_a and X_b , respectively.

Let C be some context. In a series of observations of b (which can be infinite in a mathematical model) we obtain a sequence of values of b: $x \equiv x(b|C) = (x_1, x_2, ..., x_N, ...), \quad x_j \in X_b$. In a series of observations of a we obtain a sequence of values of $a: y \equiv y(a|C) = (y_1, y_2, ..., y_N, ...), \quad y_j \in X_a$. We suppose that the *principle of the statistical stabilization* for relative frequencies [39], [40] holds. This means that the frequency probabilities are well defined: $p_C^b(\beta) = \lim_{N \to \infty} \nu_N(\beta; x), \quad \beta \in X_b; \quad p_C^a(\alpha) = \lim_{N \to \infty} \nu_N(\alpha; y), \quad \alpha \in X_a$. Here $\nu_N(\beta; x)$ and $\nu_N(\alpha; y)$ are frequencies of observations of values $b = \beta$ and $a = \alpha$, respectively (under the complex of conditions C).

Remark. (On the notions of collective and S-sequence) R. von Mises considered in his theory two principles: a) the principle of the statistical stabilization for relative frequencies; b) the principle of randomness. A sequence of observations for which both principle hold was called a collective, [39]. However, it seems that the validity of the principle of statistical stabilization is often enough for applications. Here we shall use just the convergence of frequencies to probabilities. An analog of von Mises' theory for sequences of observations which satisfy the principle of statistical stabilization was developed in [40]; we call such sequences S-sequences.

Everywhere in this paper it will be assumed that sequences of observations are S-sequences, cf. [40] (so we are not interested in the validity

of the principle of randomness for sequences of observations, but only in existence of the limits of relative frequencies).

Let $C_{\alpha}, \alpha \in X_a$, be contexts corresponding to α -filtrations, see Axiom 1. By observation of b under the context C_{α} we obtain a sequence: $x^{\alpha} \equiv x(b|C_{\alpha}) = (x_1, x_2, ..., x_N, ...), \quad x_j \in X_b$. It is also assumed that for sequences of observations $x^{\alpha}, \alpha \in X_a$, the principle of statistical stabilization for relative frequencies holds true and the frequency probabilities are well defined: $p^{b|a}(\beta|\alpha) = \lim_{N \to \infty} \nu_N(\beta; x^{\alpha}), \quad \beta \in X_b$. Here $\nu_N(\beta; x^{\alpha}), \alpha \in X_a$, are frequencies of observations of value $b = \beta$ under the complex of conditions C_{α} . We can repeat all previous considerations by changing b|a-conditioning to a|b-conditioning. There can be defined probabilities $p^{a|b}(\alpha|\beta)$.

5 Quantum-like representation algorithm – QLRA

In [31] we derived the following formula for interference of probabilities:

$$p_C^b(\beta) = \sum_{\alpha} p_C^a(\alpha) p^{b|a}(\beta|\alpha) + 2\lambda(\beta|\alpha, C) \sqrt{\prod_{\alpha} p_C^a(\alpha) p^{b|a}(\beta|\alpha)}, \quad (1)$$

where the coefficient of interference

$$\lambda(\beta|a,C) = \frac{p_C^b(\beta) - \sum_{\alpha} p_C^a(\alpha) p^{b|a}(\beta|\alpha)}{2\sqrt{\prod_{\alpha} p_C^a(\alpha) p^{b|a}(\beta|\alpha)}}.$$
 (2)

A similar representation we have for the a-probabilities. Such interference formulas are valid for any collection of contextual probabilistic data satisfying the conditions:

R1). Observables a and b are symmetrically conditioned⁷:

$$p^{b|a}(\beta|\alpha) = p^{a|b}(\alpha|\beta).$$

R2). Observables a and b are mutually nondegenerate⁸:

$$p^{a|b}(\alpha|\beta) > 0, \ p^{b|a}(\beta|\alpha) > 0.$$

R2a). Context C is nondegenerate with respect to both observables a and b:

$$p_C^b(\beta) > 0, \ p_C^a(\alpha) > 0.$$

Suppose that also the following conditions hold:

⁷This condition will induce symmetry of the scalar product

 $^{^8{\}rm This}$ condition will induce noncommutativity of operators \hat{a} and \hat{b} representing these observables.

R3). Coefficients of interference are bounded by one⁹:

$$\Big|\frac{p_C^b(\beta) - \sum_\alpha p_C^a(\alpha) p^{b|a}(\beta|\alpha)}{2\sqrt{\prod_\alpha p_C^a(\alpha) p^{b|a}(\beta|\alpha)}}\Big| \leq 1,$$

$$\Big|\frac{p_C^a(\alpha) - \sum_\beta p_C^b(\beta) p^{a|b}(\alpha|\beta)}{2\sqrt{\prod_\alpha p_C^b(\beta) p^{a|b}(\alpha|\beta)}}\Big| \leq 1,$$

A context C such that R3) holds is called trigonometric, because in this case we have the conventional formula of trigonometric interference:

$$p_C^b(\beta) = \sum_{\alpha} p_C^a(\alpha) p^{b|a}(\beta|\alpha) + 2\cos\theta(\beta|\alpha, C) \sqrt{\prod_{\alpha} p_C^a(\alpha) p^{b|a}(\beta|\alpha)}, \quad (3)$$

where $\lambda(\beta|a,C) = \cos\theta(\beta|a,C)$. Parameters $\theta(\beta|\alpha,C)$ are said to be b|arelative phases with respect to the context C. We defined these phases
purely on the basis of probabilities. We have not started with any linear
space; in contrast we shall define geometry from probability. ¹⁰

We denote the collection of all trigonometric contexts by the symbol C^{tr} .

By using the elementary formula:

$$D = A + B + 2\sqrt{AB}\cos\theta = |\sqrt{A} + e^{i\theta}\sqrt{B}|^2,$$

for real numbers $A,B>0,\theta\in[0,2\pi]$, we can represent the probability $p_C^b(\beta)$ as the square of the complex amplitude (Born's rule):

$$p_C^b(\beta) = |\psi_C(\beta)|^2 . \tag{4}$$

Here

$$\psi(\beta) \equiv \psi_C(\beta) = \sqrt{p_C^a(\alpha_1)p^{b|a}(\beta|\alpha_1)} + e^{i\theta_C(\beta)}\sqrt{p_C^a(\alpha_2)p^{b|a}(\beta|\alpha_2)}, \ \beta \in X_b,$$
(5)

where $\theta_C(\beta) \equiv \theta(\beta|\alpha, C)$.

The formula (5) gives the quantum-like representation algorithm – QLRA. For any trigonometric context C by starting with the probabilistic data – $p_C^b(\beta)$, $p_C^a(\alpha)$, $p^{b|a}(\beta|\alpha)$ – QLRA produces the complex amplitude ψ_C . This algorithm can be used in any domain of science to create the QL-representation of probabilistic data (for a special class of contexts).

We point out that QLRA contains the reference observables as parameters. Hence the complex amplitude give by (5) depends on $a,b:\psi_C\equiv\psi^{b|a}$.

We denote the space of functions: $\varphi: X_b \to \mathbf{C}$ by the symbol $\Phi = \Phi(X_b, \mathbf{C})$. Since $X = \{\beta_1, \beta_2\}$, the Φ is the two dimensional complex linear space. By using QLRA we construct the map

$$J^{b|a}: \mathcal{C}^{\mathrm{tr}} \to \Phi(X, \mathbf{C})$$
 (6)

 $^{^9}$ This condition will induce representation of the context C in the complex Hilbert space. Thus complex numbers appear due to this condition.

¹⁰We remark that conditions R1) and R3) are also nessesary.

which maps contexts (complexes of, e.g., physical conditions) into complex amplitudes. The representation (4) of probability is nothing other than the famous **Born rule.** The complex amplitude $\psi_C(x)$ can be called a **wave function** of the complex of physical conditions (context) C or a (pure) state. We set $e^b_\beta(\cdot) = \delta(\beta - \cdot)$ – Dirac delta-functions concentrated in points $\beta = \beta_1, \beta_2$. The Born's rule for complex amplitudes (4) can be rewritten in the following form: $p^b_C(\beta) = |\langle \psi_C, e^b_\beta \rangle|^2$, where the scalar product in the space $\Phi(X_b, C)$ is defined by the standard formula: $\langle \phi, \psi \rangle = \sum_{\beta \in X_b} \phi(\beta) \bar{\psi}(\beta)$. The system of functions $\{e^b_\beta\}_{\beta \in X_b}$ is an orthonormal basis in the Hilbert space $H_{ab} = (\Phi, \langle \cdot, \cdot \rangle)$.

Let $X_b \subset \mathbf{R}$. By using the Hilbert space representation of the Born's rule we obtain the Hilbert space representation of the expectation of the observable b: $E(b|C) = \sum_{\beta \in X_b} \beta |\psi_C(\beta)|^2 = \sum_{\beta \in X_b} \beta \langle \psi_C, e_\beta^b \rangle \overline{\langle \psi_C, e_\beta^b \rangle} = \langle \hat{b}\psi_C, \psi_C \rangle$, where the (self-adjoint) operator $\hat{b}: H_{ab} \to H_{ab}$ is determined by its eigenvectors: $\hat{b}e_\beta^b = \beta e_\beta^b, \beta \in X_b$. This is the multiplication operator in the space of complex functions $\Phi(X_b, \mathbf{C}): \hat{b}\psi(\beta) = \beta\psi(\beta)$. It is natural to represent the b-observable (in the Hilbert space model) by the operator \hat{b} .

We would like to have Born's rule not only for the b-variable, but also for the a-variable: $p_C^a(\alpha) = |\langle \varphi, e_\alpha^a \rangle|^2$, $\alpha \in X_a$.

How can we define the basis $\{e_{\alpha}^{a}\}$ corresponding to the *a*-observable? Such a basis can be found starting with interference of probabilities. We set $u_{j}^{a} = \sqrt{p_{C}^{a}(\alpha_{j})}$, $p_{ij} = p(\beta_{j}|\alpha_{i})$, $u_{ij} = \sqrt{p_{ij}}$, $\theta_{j} = \theta_{C}(\beta_{j})$. We have:

$$\varphi = u_1^a e_{\alpha_1}^a + u_2^a e_{\alpha_2}^a, \tag{7}$$

where

$$e_{\alpha_1}^a = (u_{11}, u_{12}), e_{\alpha_2}^a = (e^{i\theta_1}u_{21}, e^{i\theta_2}u_{22})$$
 (8)

The condition R1) implies that the system $\{e^a_{\alpha_i}\}$ is an orthonormal basis iff the probabilistic phases satisfy the constraint:

$$\theta_2 - \theta_1 = \pi \mod 2\pi$$

but, as we have seen [31], we can always choose such phases (under the condition R1).

In this case the a-observable is represented by the operator \hat{a} which is diagonal with eigenvalues α_1, α_2 in the basis $\{e^a_{\alpha_i}\}$. The conditional average of the observable a coincides with the quantum Hilbert space average: $E(a|C) = \sum_{\alpha \in X_a} \alpha p^a_C(\alpha) = \langle \hat{a}\psi_C, \psi_C \rangle$.

If condition R3) is violated, then we obtain nonconventional QL-representations of probabilistic data, for example, in the hyperbolic analogue of the complex Hilbert space [31].

It is important to remark that map (6) is not one-to one!!! Different contexts can be mapped into the same complex probability amplitude, we can also say that the same wave function may represent a few different contexts, cf. section 10.2.

Finally, we mention one recent result of Karl Svozil [49] which seems to be coupled to the number of reference observables – two – producing QL-representation in our approach.

6 Flyes in a packet

We consider a metal box. At different points inside this box there is food which is attractive for flyes. Its distribution is not uniformly weighted, in some points there is more food than in others, there are domains without food. An external observer (who is staying outside this box) has no idea about the real distribution of food in the box, but a "Einstein demon" living inside this box knows well this distribution. We put a population of flyes, say Ω , inside this box. After while they will be distributed in space inside the box by coupling to sites with food. Our Einstein's demon can find the probability distribution $\mathbf{P}(x,y,z)$ to observe a fly at the point with coordinates (x,y,z). It is assumed to be stationary (at least for a while). In principle, some flyes can move between attractive points, but statistically the number of flyes at each site with food is stable.

As in the example "firefly in the box", one can divide this box in two ways: a) by the vertical wall -a, see Figure 2; b) by horizontal wall -b, see Figure 3. Here $a(\omega) = \alpha_1$ if Einstein's demon finds a fly ω in the left-hand part and $a(\omega) = \alpha_2$ if he finds a fly ω in the right-hand part (e.g. $\alpha = \pm 1$). We define b in a similar way: $b(\omega) = \beta_1$ if Einstein's demon finds a fly ω in the top part and $b(\omega) = \beta_2$ if he finds a fly ω in the bottom part (e.g. $\beta = \pm 1$). Einstein's demon can consider populations of flyes:

$$\Omega_{\alpha} = \{ \omega \in \Omega : a(\omega) = \alpha \}, \quad \Omega_{\beta} = \{ \omega \in \Omega : a(\omega) = \beta \}.$$

By assuming that $\mathbf{P}(\Omega_{\alpha})$, $\mathbf{P}(\Omega_{\beta}) > 0$, he can define transition probabilities:

$$p^{b|a}(\beta|\alpha) = \mathbf{P}(\Omega_{\beta}|\Omega_{\alpha}) \equiv \frac{\mathbf{P}(\Omega_{\beta} \cap \Omega_{\alpha})}{\mathbf{P}(A_{\alpha})}$$

and in the same way probabilities $p^{a|b}(\alpha|\beta)$.

Let C be some domain inside the box. We shall consider it as a geometric-context. Einstein's demon can be find (by using the Bayes' formula) conditional probability distribution:

$$\mathbf{P}_{C}(U) = \frac{\mathbf{P}(\Omega_{U} \cap \Omega_{C})}{\mathbf{P}(\Omega_{C})},\tag{9}$$

for any subset U of box. Here $\Omega_C = \{\omega \in \Omega : (x_\omega, y_\omega, z_\omega) \in C\}$ is the populattion of flyes which are concentrated inside the configuration C. The population Ω_U is defined in the same way.

This probability distribution \mathbf{P}_C provides the probabilistic representation of the domain C. Einstein's demon encoded geometry by probability. Of course, probability provides only rough images of geometric structures, since the map:

$$C \to \mathbf{P}_C$$

is not one-to-one. Denote now by \mathcal{F} some σ -algebra of subsets of the box such that the probability \mathbf{P} – flyes' distribution – can be defined on it. Denote also the set of all probability measures on the \mathcal{F} by the symbol \mathcal{P} . Then we have the map:

$$J: \mathcal{F} \to \mathcal{P}.$$
 (10)

This is the classical probabilistic representation of geometry (of distribution of food). It is avalaible for any internal observer (Einstein's demon) who lives inside this box. In this mapping a lot of geometric information is neglected. However, the whole probabilistic information is taken into account. This is the end of the classical story!

Remark 6. 1. (Food and flyes version of fields and particles) This representation has one interesting feature. Geometry of food distribution is represented by ensembles of flyes. We can make the following analogy: electromagnetic field can be represented by photons. One can compare the food distribution with a kind of a "food-field" and flyes with particles representing this field. If we put another type of insects into the box, they may be not interested in this sort of food. They would not reproduce the distribution $\mathbf{P}(x,y,z)$. Thus we may speak about various food-fields which are represented by corresponding types of insects-particles. In some sense this picture reminds Bohmian mechanics [50].

Now we modify the previous framework. We have the same box with the same ditribution of fly-attractive food. But flyes are put not directly in the box, but in a plastic packet, say C. The geometric configuration is unknown for us – external observers. Moreover, we are not able to find its configuration directly (even by making a hole in the box), because packet's surface is covered by a "B2-bomber type" material. Thus we look inside the box, but we see nothing.¹¹ Nevertheless, we (external observers) would like to get at least partial information about this packet configuration by using flyes distribution. The problem is complified by the assumption that any attempt to open the metal box will induce destruction of the packet which in its turn induces redistribution of flyes in the space. Such a hard problem...

We do the following. As in the firefly-example we introduce fuzzy coordinates a and b. We measure them in the following way. We assume that we can put very quickly either vertical or horyzontal wall into the box. Such a moving wall divides (pactically instanteneously, at least in comparation with fly's velocity) the box into two sub-boxes, but at the same time it destroys (of course) the plastic packet. It is assumed that after this act we can open each sub-box and find numbers of flyes in each part of the box.

At the moment, cf. section 8, we consider *nondisturbing measure-ments:* walls do not change food distributions in corresponding parts of the box (those walls are negligibly thin and destruction of the packet does not change the distribution of food). However, opening of any box induces a strong disturbing effect, flyes are essentially redistributed.

Thus first we do the a-measuring by using the vertical wall. It divides the box into two parts, say C_{α_1} and C_{α_2} . In this way we get probabilities $p_C^a(\alpha)$ that a fly was located in the α -side of the box. Since the vertical wall moves quicky relatively to fly's velocity, the number of flyes which were able to change the left-hand part of the box to the right-hand part

¹¹The "Einstein demon" also gets a problem, but he can still investigate packet's geometry just by moving over its surface. Of course, if the packet is disconnected, so it has a few components, a few "Einstein demons" should be employed.

or vice versa is statistically negligible. In principle, we might try to use the classical formula:

$$p_C^a(\alpha) = \mathbf{P}_C(\Omega_\alpha) \equiv \frac{\mathbf{P}(\Omega_\alpha \cap \Omega_C)}{\mathbf{P}(\Omega_C)},$$

However, it is totally unuseful for us, because we do not know the configuration C and hence \mathbf{P}_C .

We point out that if we do not open sub-boxes C_{α} and if after while the corresponding "Einstein demons" measure the *b*-coordinate of flyes in each part C_{α} of the box they will obtain the original transition probabilities $p^{b|a}(\beta|\alpha)$, since flyes will again redistribute in the domain C_{α} according to the food-field.¹² However, the original distribution of flyes in the domain $C \cap C_{\alpha}$ has been lost for ever even for the "Einstein demons." We (external observers) are not able to find transition probabilities in this way, since opening of a box produces redistribution of flyes in it.

We also remark that trivially $a(\omega) = \alpha$ on the α -part of the box.

Remark 6.2. (Reaction of "food-field" to space reconfiguration) At the moment we proceed under the assumption that the "food-field" is not sensitive to the disturbing effect of the moving wall (separating the box into two sub-boxes). Moreover, the "food-field" is not sensitive to changes of the geometry of space ("boundary conditions"). In principle, we can imagine the following situation, see section 7. The appearance of a separating wall does not induce a disturbing effect which could move food in space. However, the wall by itself can have some physical properties influencing the food distribution. For example, food is placed in charged capsulas and walls of the box (including walls used in separation experiments) also carry electric charges. Thus even "mechanically peaceful appearance" of a separating wall will induce (after a while) redistribudtion of food in the sub-box.

Remark 6.3. (Fair sampling) At the moment we proceed under the assumption of "fair sampling," cf. [43], [44], [45]. Moving walls do not kill statistically non-negligible populations of flyes.

To construct the QL-representation of the context C by a complex probability amplitude, we need also probabilities:

$$p_C^b(\beta) = \mathbf{P}_C(\Omega_\beta) \equiv \frac{\mathbf{P}(\Omega_\beta \cap \Omega_C)}{\mathbf{P}(\Omega_C)},$$

However, since we do not know the configuration C, we are not able to apply Bayes' formula directly. We should repeat previous considerations, but by using now the horizontal wall which separates quickly the box into top and bottom parts, C_{β_1} and C_{β_2} . Then by opening these sub-boxes and counting flyes in each of them we find the probabilities $p_C^b(\beta)$.

Of course, we should have two boxes with the same configuration C, because each falling wall destroys this configuration. Thus we should be able to make such a preparation a few times. Moreover, if one wants to exclude effects of interaction between flyes (as one does in QM), there

¹²Two "Einstein demons" should be involved – one for each sub-box.

should be created an ensemble of boxes, each box containing just one fly. It is assumed that flyes would reproduce the food distribution.

In particular, for $C=C_{\alpha}$, i.e., the configuration C which coinsides with the α -part of the box we get: $\Omega_{C_{\alpha}}=\Omega_{\alpha}$ and

$$p_{C_{\alpha}}^{b}(\beta) = p^{b|a}(\beta|\alpha).$$

However, we do not know from the very beginning that a hidden geometric configuration is the half-box C_{α} . Therefore this is not an experimental way to find transition probabilities.

To find transition probabilities, we assume that each half-box C_{α} can be devided by the horyzontal wall (as in the original *b*-measurement in the whole box) in two parts, say $C_{\beta|\alpha}$, $\beta = \beta_1, \beta_2$. By counting flyes in each of these boxes we find the transition probabilities. At the moment we proceed under the same assumptions as before: by puting the horyzontal walls in the box C_{α} we do not change the distribution of food in it.

Now everything is prepared for application of QLRA. A nessesary condition is given by R2), since in QM matrices of transition probabilities are symmetrically conditioned. Thus from the very beginning one should assume that the distribution of attracting sites in the box induces this condition. This happens iff $\mathbf{P}(\Omega_{\alpha}) = \mathbf{P}(\Omega_{\beta}) = 1/2$.

The next condition is that variables are statistically conjugate, i.e., $\mathbf{P}(\Omega_{\alpha} \cap \Omega_{\beta}) \neq 0$ for all α and β .

Finally, the context C should be "large enough" with respect to both variables: $\mathbf{P}(\Omega_C \cap \Omega_\beta)$, $\mathbf{P}(\Omega_C \cap \Omega_\alpha) > 0$. Statistically small configurations could not be represented in the QL-way (they are simply neglected in the incomplete QL-representation of information).

We also know that, becides a complex probability amplitude, some contexts can be represented by hyperbolic amplitudes, thus to guarantee real QM-like representation we should have $|\lambda| \leq 1$ for the coefficient of interference.

Thus we represent all "trigonometric configurations" C by complex vectors and the observables a and b by self-adjoint operators. The map:

$$J^{b|a}: \mathcal{C}^{\mathrm{tr}} \to H$$

is a QL-analogue of the classical map J given by (10). Of course, the map (10) is "better" than the QL-map. However, we are not able to use it in the situation with invisible configuration C.

As was remarked, for some contexts, hyperbolic ones, $|\lambda| > 1$. They are mapped into hyperbolic amplitudes:

$$J^{b|a}: \mathcal{C}^{\mathrm{hyp}} \to H_{\mathrm{hyp}},$$

where $H_{\rm hyp}$ is the hyperbolic analogue of Hilbert space [31]. Appearance of such amplitudes is not surprising from the viewpoint of general contextual probability theory. Why should the coefficient λ be always bounded by one? It is surprising that we do not have them in conventional QM. In some way it happens that all physical quantum contexts are trigonometric (or that physical hyperbolic contexts have not yet been observed?).

Remark 6.4. (Complementarity or supplementarity?) We point out that, although the "reference observables" a and b corresponding to two

different separations of the box are represented by noncommutative operators, they can be considered as simutaneously existing: each fly has the definite position in the box and hence its location in each part of the box is well defined. This is the typical situation for the Kolmogorov approach: the values of both random variables $a(\omega)$ and $b(\omega)$ are well defined for each $\omega \in \Omega$. Thus "properties" a and b of a fly are not mutually exclusive, in spite of noncommutativity: $[\hat{a}, \hat{b}] \neq 0$. Since Nils Bohr reserved the term complementarity for mutually exclusive properties, it might be better to call a and b supplementary observables, see [51]. It is clear that a result of measurement of b produces supplementary information with respect to the result of preceding measurement of a and vice versa.

7 How far can one proceed with the quantumlike representation of the Kolmogorov model?

In spite of the presence of the underlyning Kolmogorov space, we constructed the QL-representation of probabilistic data for macroscopic configurations (essentially incomplete representation) which has all distinguishing features of the conventional quantum representation of probabilistic data for a pair of incompatible observables: intereference formula for probabilities, Born's rule, representation of these observables by selfadjoint operators. As was mentioned, the map $J^{b|a}$ is not injective. We no ask: Is it surjective? Can one get any quantum state ψ and any pair of quantum observables \hat{a} and \hat{b} in such a way? The answer is no. This is a consequence of Bell's type inequality for transition probabilities, see [38] and appendix.

To apply conditional Bell's inequality to our macroscoipic situation, it is better to consider a ball bounded by the metal sphere, instead of the box. We now can divide this ball into parts with the aid of central planes. To simplify considerations, we can consider a boundle of planes which are enumerated by the angle ϕ . Then we shall obtain a familty of observables a_{ϕ} , say taking values \pm . Parts of the ball obtained by the ϕ -separation are $C_{\phi,+} = \{\theta : \phi \leq \theta < \phi + \pi\}$ and $C_{\phi,-} = \{\theta : \phi + \pi \leq \theta < \phi\}$, respectively.

For each pair of them we find transition probabilities $p^{\phi_1|\phi_2}(\epsilon_1|\epsilon_2)$. For each context C (a plastic packet with flyes inside it; this paket is placed inside the metal ball; any attempt to open the ball would destroy this paket) and any ϕ -section, we find probabilities $p_C^{\phi}(\epsilon)$, $\epsilon=\pm$. If we choose a context C such that $p_C^{\phi}(+1)=p_C^{\phi}(-1)=1/2$ for all ϕ , then we can apply arguments of appendix and we see that some types of transition probabilities could not be obtained from a single Kolmogorov model.

One Kolmogorov space is too small to generate all quantum (or better to say quantum-like) states and observables.

8 Disturbing measurements

However, we can easily modify our example to destroy the (hidden) Kolmogorov structure of the model. Suppose now that everything is as it was before with only one difference: destruction of the packet by a wall

(encoded by some ϕ -plane) induces not only the possibility for flyes to move outside the packet, but also induces a redistribution of food sites, cf. Remark 2. The latter is determined by the wall. Thus after e.g. the ϕ -plane separation of the ball the distribution of sites with food in its parts $C_{\phi,+}$ and $C_{\phi,-}$ is not such as it was before this separation. Therefore, for any successive ϕ' -separation of the sectors $C_{\phi,\epsilon}$ (which were produced by the previous ϕ -separation), the transition probabilities $p^{\phi'|\phi}(\epsilon'|\epsilon)$ obtained by an external observer do not coincide with the transition probabilities which would be obtained by Einstein's demon on the basis of the original ensemble. Hence Bell's type inequality for transition probabilities, see appendix, cannot be applied.

In fact, by using random generators we can simulate probabilities for any complex probability amplitude and any pair of self-adjoint operators in the two dimensional Hilbert space.

For example, suppose that we would like to simulate the transition probabilities for successive measurements of spin projections as well as the uniform probability distribution for the a_{ϕ} measurements for the original context C (state ψ_C). To provide the latter condition, we start with the uniform distribution of food. It would induce probabilities $p_C^{\phi}(+1) = p_C^{\phi}(-1) = 1/2$.

Now to simplify considerations, we consider not three dimensional configurations, but just two dimensional, in particular, we consider a circle, instead of a ball, and sections by central lines, instead of planes.

We assume that disturbance induced by the a_{ϕ_0} -measurement, $0 \le \phi_0 < \pi$, induces redustribution of food in the sectors $C_{\phi_0,+}$ and $C_{\phi_0,-}$ and, finally, generates e.g. in the sector $C_{\phi_0,+}$ the density of flyes:

$$\rho_{\phi_0}^+(r,\theta) = \sin(\theta - \phi_0). \tag{11}$$

(We assume that the circle has unit radius). Then we separate the sector $C_{\phi_0,+}$ by the ϕ -plane, say $\phi > \phi_0$. Then the probability

$$p^{\phi|\phi_0}(+|+) = \int_0^1 r dr \int_{\phi}^{\phi_0 + \pi} \sin(\theta - \phi_0) d\theta = \cos^2 \frac{\phi - \phi_0}{2},$$

$$p^{\phi|\phi_0}(-|+) = \int_0^1 r dr \int_{\phi_0}^{\phi} \sin(\theta - \phi_0) d\theta = \sin^2 \frac{\phi - \phi_0}{2}.$$

For the sector $C_{\phi_0,-}$, we choose the probability distribution

$$\rho_{\phi_0}^-(r,\theta) = -\sin(\theta - \phi_0). \tag{12}$$

Here transition probabilities are given by

$$p^{\phi|\phi_0}(+|-) = -\int_0^1 r dr \int_{\phi_0 + \pi}^{\phi + \pi} \sin(\theta - \phi_0) d\theta = \sin^2 \frac{\phi - \phi_0}{2},$$

$$p^{\phi|\phi_0}(-|-) = -\int_0^1 r dr \int_{\phi+\pi}^{\phi_0+2\pi} \sin(\theta - \phi_0) d\theta = \cos^2 \frac{\phi - \phi_0}{2}.$$

Remark 8.1. (Complementarity or supplementarity?) Since we consider disturbing measurements, we (external observers) are not able to

measure two observables, a_{ϕ_1} and a_{ϕ_2} , simultaneously. Thus these are *incompatible observables*. However, such measurement incompatibility does not exclude that an element of reality can be assigned to each fly – the pair $a_{\phi_1}(\omega)$, $a_{\phi_2}(\omega)$. We recall that we consider such separations that they do not induce redistribution of flyes between sectors: the ϕ -plane moves so quickly that flyes are not able to change sectors (or at least only statistically negligible number of flyes could make such changes). Moreover, only negligible number of flyes can be killed by a moving-separating plane. Thus the values of $a_{\phi_1}(\omega)$ and $a_{\phi_2}(\omega)$ which would be obtained by an external observer coincide with the values which have been known by Einstein's demon before measurements. Therefore complementarity (in the sense of mutual exclusivity) is only external observer's complementarity. Einstein's demon still has supplementarity, in the sense of additional information (of course, fuzzy) about fly's location.

Remark 9.1. (Counterfactual arguments) We point out that already in the previous remark we have applied counterfactual arguments – by using "would be obtained by an external observer." In fact, one cannot escape them, because an external observer is not able to assign both values $a_{\phi_1}(\omega), a_{\phi_2}(\omega)$ to the same fly ω .

9 Can the classical probabilistic structure be violated without disturbance effects?

In section 7 we pointed out that by Bell's inequality for transition probabilities it is impossible to find a single underlying classical probabilistic space which would reproduce all possible wave functions and pairs of self-adjoint noncommutative operators in the contextual probabilistic framework. One can not find such a Kolmogorov probability space that by choosing different pairs of reference observables a, b and corresponding families of trigonometric contexts $C^{\rm tr}(a,b)$ (represented by sets from the σ -algebra of the Kolmogorov space) he would (by alphyling QLRA) cover the whole unit sphere of Hilbert state space as well as obtain all pairs of noncommutative self-adjoint operators. In section 8 we showed that by considering disturbing measurements we can reproduce all quantum structures. Can one approach the same result without disturbance? In principle, yes!

9.1 Unfair sampling

One of possibilities is to proceed under unfair sampling assumption, see Remark 6.3, cf. [43], [44], [45]. We can assume that moving planes separating the metal ball do not produce food redistribution, thus the "food-field" is not changed. However, these planes kill subensembles of flyes¹³,

¹³We remark that we really consider unfair samplig and not "the detectors efficiency." We operate with macroscopic systems – flyes which are are detected with probability one.

depending on ϕ . Then we can easily violate the Bell's inequality for conditional probabilities.

9.2 Ensemble fluctuations

Another important point is that in section 7 we proceeded by using counterfactual arguments, cf. remark 9.1. To be on really realistic ground, we should consider at least three different balls and perform on them conditional measurents for pairs of observables a_{ϕ_i}, a_{ϕ_j} . In principle, we cannot guarantee that we would be able to reproduce statistically identical distributions of food in balls and idential hidden configurations. As was emphasized in section 3, the map, see (6), from the collection of trigonometric contexts into complex probability amplitudes is not injection, various contexts can be mapped in the same complex probability amplitude. Even if we are sure that we have the same QL-state given by the same complex probability amplitude, ψ , we could never be sure that contexts in different balls are the same. It may be that $\psi = \psi_{C_1} = \psi_{C_2} = \dots = \psi_{C_N}$ and moreover it may be that $N \to \infty$. Therefore we should work in multi-Kolmogorovian framework and the Bell's inequality for conditional probabilities can also be violated without any disturbance.

This argument (but for composite systems) was presented at the first time by De Baere [46], then by the author [40] and recently by Hess and Philipp [48]. Moreover, they pointed out in [48] to the old paper of Soviet mathematician Vorobjev [52] who studied the problem of the possibility to realize a number of observables on a single Kolmogorv space. This problem is equivalent to the problem of violation of Bell's inequality for transition probabilities.

9.3 Communication

In principle, we may also produce redistribution of flyes without redistribution of the "food-field" if we assume that flyes can communicate. For example, each separation measurement starts communication between flyes. As the result, they can come to the agreement to concentrate in each sector $C_{\phi_0,\pm}$ in such a way that e.g. sin-type dsitribution of section 8 would be produced. If they communicate by using signals which we, external observers, are not able to detect, then this communication would be hidden from us.

10 ERP-Bohm type experiments with flyes

We have considered in very detail measurements (in fact, position-type measurements) for ensembles of single flyes. In principle, we could consider the real EPR-Bohm type experiment for pairs of "entangled flyes" which we put into different metal balls. One of technological problems is to produce such pairs of flyes. However, this is not the main point. The main point is that in the macroscopic framework such experiments would not give so much more than experiments with single flyes. In contrast to

photons or electrons, we have no doubts that flyes have objective properties, in particular, the position. Therefore the only consequence of the EPR-Bohm type experiment with flyes would be that disturbing effects should be excluded.¹⁴

Thus as well as in the case of a single system we have tree choices: a) unfair sampling; b) ensemble fluctuations; c) nonrelativistic communications between flyes.

The last condition cannot be completely rejected even for human beings, but the EPR-type experiment could not be used to provide the crucial argument in its favor.

Conclusion. We shown that macroscopic configurations can be naturally represented in the QL-way – by complex probability amplitudes – with the aid of pairs of "supplementary observables" which in turn are represented by noncommutative self-adjoint operators. Classical probabilistic structure can be violated. In particular, Bell's type inequality can be violated. Such violations have nothing to do with "death of reality." They could be induced either by disturbing effects of measurements, or unfair sampling, or ensemble fluctuations, or nonlocal communication between macroscopic systems. The latter assumption is not so much reasonable for macroscopic biological systems (however, it could not be completely excluded).

11 Appendix: Bell's inequality for transition probabilities

Theorem. Let $a, b, c = \pm 1$ be dichotomous uniformly distributed random variables on a single Kolmogorov space. Then the following inequality holds true:

$$\mathbf{P}(a = +1|b = +1) + \mathbf{P}(c = +1|b = -1) \ge \mathbf{P}(a = +1|c = +1) \tag{13}$$

Proof. We have

$$P(b = +1) = P(b = -1) = P(a = +1) = P(a = -1) = P(c = +1) = P(c = -1) = 1/2.$$

Thus

$$\mathbf{P}(a=+1|b=+1) + \mathbf{P}(c=+1|b=-1) = 2\mathbf{P}(a=+1,b=+1) + 2\mathbf{P}(c=+1,b=-1)$$

and

$$P(a = +1|c = +1) = 2P(a = +1, c = +1).$$

Hence by the well know Wiegner inequality [55] we get (13).

¹⁴We remind that we consider not only mechanical disturbance by moving planes, but also the field type disturbance. To exclude the latter type of disturbance, one should be sure that the effect of the "food-field" (e.g. smell) from one ball would be not able to propagate to another ball. If balls have small windows (or produced not of metal, but of some less isolating material), then smell can propagate from one ball to another. We recall that insects can find smell-traces on huge distances. Thus to exclude completely disturbing effects, we should either isolate balls completely or to make measurements on balls with a time-window such that a signal from one ball would not be able to approach another during this time window, cf. [53]

We underline again that the main distinguishing feature of (13) is the presence of only transition probabilities. Transition probabilities can always be calculated by using quantum formalism for noncomposite systems. In fact, we need not consider pairs of particles.

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