# On Constructor Rewrite Systems and the Lambda-Calculus

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#### Abstract

We prove that orthogonal constructor term rewrite systems and lambda-calculus with weak (i.e., no reduction is allowed under the scope of a lambda-abstraction) call-by-value reduction can simulate each other with a linear overhead. In particular, weak call-by-value beta-reduction can be simulated by an orthogonal constructor term rewrite system in the same number of reduction steps. Conversely, each reduction in an orthogonal term rewrite system can be simulated by a constant number of weak call-by-value beta-reduction steps. This is relevant to implicit computational complexity, because the number of beta steps to normal form is polynomially related to the actual cost (that is, as performed on a Turing machine) of normalization, under weak call-by-value reduction. Orthogonal constructor term rewrite systems and lambda-calculus are thus both polynomially related to Turing machines, taking as notion of cost their natural parameters.

### 1 Motivations

Implicit computational complexity is a young research area, whose main aim is the description of complexity phenomena based on language restrictions, and not on external measure conditions or on explicit machine models. It borrows techniques and results from mathematical logic (model theory, recursion theory, and proof theory) and in doing so it has allowed the incorporation of aspects of computational complexity into areas such as formal methods in software development and programming language design. The most developed area of implicit computational complexity is probably the model theoretic one – finite model theory being a very successful way to describe complexity problems. In the design of programming language tools (e.g., type systems), however, syntactical techniques prove more useful. In the last years we have seen much work restricting recursive schemata and developing general proof theoretical techniques to enforce resource bounds on programs. Important achievements have been the characterizations of several complexity classes by means of limitations of recursive definitions (e.g., [3, 9]) and, more recently, by using the "light" fragments of linear logic [6]. Moreover, rewriting techniques such as recursive path orderings and the interpretation method have recently been proved useful in the field [10]. By borrowing the terminology from software design technology, we may dub this area as implicit computational complexity in the large, aiming at a broad, global view on complexity classes. We may have also an implicit computational complexity in the small — using logic to study single machine-free models of computation. Indeed, many models of computations do not come with a natural cost model — a definition of cost which is both intrinsically rooted in the model of computation, and, at the same time, it is polynomially related to the cost of implementing that model of computation on a standard Turing machine. The main example is the  $\lambda$ -calculus: The most natural intrinsic parameter of a computation is its number of beta-reductions, but this very parameter bears no relation, in general, with the actual cost of performing that computation, since

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a beta-reduction may involve the duplication of arbitrarily big subterms<sup>1</sup>. What we call implicit computational complexity in the small, therefore, gives complexity significance to notions and results for computation models where such natural cost measures do not exist, or are not obvious. In particular, it looks for cost-explicit simulations between such computational models.

The present paper applies this viewpoint to the relation between  $\lambda$ -calculus and orthogonal (constructor) term rewrite systems. We will prove that these two machine models simulate each other with a linear overhead. That each constructor term rewrite system could be simulated by  $\lambda$ -terms and beta-reduction is well known, in view of the availability, in  $\lambda$ -calculus, of fixed-point operators, which may be used to solve the mutual recursion expressed by first-order rewrite rules. Here (Section 4) we make explicit the complexity content of this simulation, by showing that any first-order rewriting of *n* steps can be simulated by *kn* beta steps, where *k* depends on the specific rewrite system but *not* on the size of the involved terms. Crucial to this result is the encoding of constructor terms using Scott's schema for numerals [17]. Indeed, Parigot [11] (see also [12]) shows that in the pure  $\lambda$ -calculus Church numerals do not admit a predecessor working in a constant number of beta steps. Moreover, Splawski and Urzyczyn [15] show that it is unlikely that our encoding could work in the typed context of System F.

Section 3 studies the converse – the simulation of (weak)  $\lambda$ -calculus reduction by means of orthogonal constructor term rewrite systems. We give an encoding of  $\lambda$ -terms into a (first-order) constructor term rewrite system. We write  $[\cdot]_{\Phi}$  for the map returning a first-order term, given a  $\lambda$ -term;  $[M]_{\Phi}$  is, in a sense, a complete defunctionalization of the  $\lambda$ -term M, where any  $\lambda$ -abstraction is represented by an atomic constructor. This is similar, although not technically the same, to the use of supercombinators (e.g., [8]). We show that  $\lambda$ -reduction is simulated step by step by first-order rewriting (Theorem 1).

As a consequence, taking the number of beta steps as a cost model for weak  $\lambda$ -calculus is equivalent (up to a linear function) to taking the number of rewritings in orthogonal constructor term rewrite systems. This is relevant to implicit computational complexity "in the small", because the number of beta steps to normal form is polynomially related to the actual cost (that is, as performed on a Turing machine) of normalization, under weak call-by-value reduction. This has been established by Sands, Gustavsson, and Moran [14], by a fine analysis of a  $\lambda$ -calculus implementation based on a stack machine. Constructor term rewrite systems and  $\lambda$ -calculus are thus both *reasonable* machines (see the "invariance thesis" in [16]), taking as notion of cost their natural, instrinsic parameters.

As a byproduct, in Section 5 we sketch a different proof of the cited result in [14]. Instead of using a stack machine, we show how we could encode constructor term rewriting in term graph rewriting. In term graph rewriting we avoid the explicit duplication and substitution inherent to rewriting (and thus also to beta-reduction) and, moreover, we exploit the possible sharing of subterms.

## 2 Preliminaries

The language we study is the pure untyped  $\lambda$ -calculus endowed with weak (that is, we never reduce under an abstraction) call-by-value reduction.

**Definition 1** The following definitions are standard:

• Terms are defined as follows:

$$M ::= x \mid \lambda x.M \mid MM,$$

where x ranges a denumerable set  $\Upsilon$ .  $\Lambda$  denotes the set of all  $\lambda$ -terms. We assume the existence of a fixed, total, order on  $\Upsilon$ ; this way FV(M) will be a sequence (without repetitions) of variables, not a set. A term M is said to be closed if  $FV(M) = \varepsilon$ , where  $\varepsilon$  is the empty sequence.

• Values are defined as follows:

$$V ::= x \mid \lambda x.M.$$

 $<sup>^{1}</sup>$  In full beta-reduction, the size of the duplicated term is indeed arbitrary and does not depend on the size of the original term the reduction started from. The situation is much different with weak reduction, as we will see.

 Weak call-by-value reduction is denoted by →<sub>v</sub> and is obtained by closing call-by-value reduction under any applicative context:

$$\frac{M \to_v N}{(\lambda x.M)V \to_v M\{V/x\}} \qquad \qquad \frac{M \to_v N}{ML \to_v NL} \qquad \qquad \frac{M \to_v N}{LM \to_v LN}$$

Here M ranges over terms, while V ranges over values.

• The length |M| of M is defined as follows, by induction on M: |x| = 1,  $|\lambda x.M| = |M| + 1$  and |MN| = |M| + |N| + 1.

Weak call-by-value reduction enjoys many nice properties. In particular, the one-step diamond property holds and, as a consequence, the number of beta steps to normal form (if any) is invariant on the reduction order [4] (this justifies the way we defined reduction, which is slightly more general than Plotkin's one [13]). It is then meaningful to define  $Time_v(M)$  as the number of beta steps to normal form (or  $\omega$  if such a normal form does not exist). This cost model will be referred to as the unitary cost model, since each beta (weak call-by-value) reduction step counts for 1 in the global cost of normalization. Moreover, notice that  $\alpha$ -conversion is not needed during reduction of closed terms: if  $M \rightarrow_v N$  and M is closed, then the reduced redex will be in the form  $(\lambda x.L)V$ , where V is a closed value. As a consequence, arguments are always closed and open variables cannot be captured.

The following lemma gives us a generalization of the fixed-point (call-by-value) combinator (but observe the explicit limit k on the reduction length):

**Lemma 1** For every natural number n, there are terms  $H_1, \ldots, H_n$  and a natural number m such that for any sequence of values  $V_1, \ldots, V_n$  and for any  $1 \le i \le n$ :

$$H_iV_1\ldots V_n \to_v^k V_i(\lambda x.H_1V_1\ldots V_n x)\ldots (\lambda x.H_nV_1\ldots V_n x),$$

where  $k \leq m$ .

**Proof.** The terms we are looking for are simply the following:

$$H_i \equiv M_i M_1 \dots M_n$$

where, for every  $1 \leq j \leq n$ ,

$$M_j \equiv \lambda x_1 \dots \lambda x_n \lambda y_1 \dots y_n y_j (\lambda z \cdot x_1 x_1 \dots x_n y_1 \dots y_n z) \dots (\lambda z \cdot x_n x_1 \dots x_n y_1 \dots y_n z).$$

The natural number m is simply 2n.

We will consider in this paper orthogonal constructor (term) rewrite systems (CRS, see [2]). A constructor (term) rewrite system is a pair  $\Xi = (\Sigma_{\Xi}, \mathcal{R}_{\Xi})$  where:

- Symbols in the signature  $\Sigma_{\Xi}$  can be either constructors or function symbols.
  - terms in  $\mathcal{C}(\Xi)$  are those built from constructors and are called *constructor terms*;
  - terms in  $\mathcal{P}(\Xi, \Upsilon)$  are those built from constructors and variables and are called *patterns*;
  - terms in  $\mathcal{T}(\Xi)$  are those built from constructor and function symbols and are called *closed* terms.
  - terms in V(Ξ, Υ) are those built from constructors, functions symbols and variables in Υ and are dubbed *terms*.
- Rules in  $\mathcal{R}_{\Xi}$  are in the form  $\mathbf{f}(\mathbf{p}_1, \dots, \mathbf{p}_n) \to_{\Xi} t$  where  $\mathbf{f}$  is a function symbol,  $\mathbf{p}_1, \dots, \mathbf{p}_n \in \mathcal{P}(\Xi, \Upsilon)$  and  $t \in \mathcal{V}(\Xi, \Upsilon)$ . We here consider orthogonal rewrite systems only, i.e. we assume that no distinct two rules in  $\mathcal{R}_{\Xi}$  are overlapping and that every variable appears at most once in the lhs of rules in  $\mathcal{R}_{\Xi}$ . Moreover, we assume that reduction is call-by-value, i.e. the substitution triggering any reduction must assign constructor terms to variables. This restriction is anyway natural in constructor rewriting.

For any term t in a CRS, |t| denotes the number of symbol occurrences, while  $|t|_{\mathbf{f}}$  denotes the number of occurrences of the symbol  $\mathbf{f}$  in t.

### 3 From Lambda-Calculus to Constructor Term Rewriting

**Definition 2 (The CRS**  $\Phi$ ) The constructor rewrite system  $\Phi$  is defined as a set of rules  $\mathcal{R}_{\Phi}$  over an infinite signature  $\Sigma_{\Phi}$ . In particular:

• The signature  $\Sigma_{\Phi}$  includes the binary function symbol **app** and constructor symbols  $\mathbf{c}_{x,M}$  for every  $M \in \Lambda$  and every  $x \in \Upsilon$ . The arity of  $\mathbf{c}_{x,M}$  is the length of  $\mathsf{FV}(\lambda x.M)$ . To every term  $M \in \Lambda$  we can associate a term  $[M]_{\Phi} \in \mathcal{V}(\Phi, \Upsilon)$  as follows:

$$\begin{split} & [x]_{\Phi} &= x; \\ & [\lambda x.M]_{\Phi} &= \mathbf{c}_{x,M}(x_1,\ldots,x_n), \text{ where } \mathsf{FV}(\lambda x.M) = x_1,\ldots,x_n; \\ & [MN]_{\Phi} &= \mathbf{app}([M]_{\Phi},[N]_{\Phi}). \end{split}$$

Observe that if M is closed, then  $[M]_{\Phi} \in \mathcal{T}(\Phi)$ .

• The rewrite rules in  $\mathcal{R}_{\Phi}$  are all the rules in the following form:

$$\mathbf{app}(\mathbf{c}_{x,M}(x_1,\ldots,x_n),x)\to [M]_{\Phi},$$

where  $FV(\lambda x.M) = x_1, \ldots, x_n$ .

• A term  $t \in \mathcal{T}(\Phi)$  is canonical if either  $t \in \mathcal{C}(\Phi)$  or  $t = \mathbf{app}(u, v)$  where u and v are themselves canonical.

Notice that the signature  $\Sigma_{\Phi}$  contains an infinite amount of constructors.

**Example 1** Consider the  $\lambda$ -term  $M = (\lambda x.xx)(\lambda y.yy)$ .  $[M]_{\Phi}$  is  $t \equiv \operatorname{app}(\mathbf{c}_{x,xx}, \mathbf{c}_{y,yy})$ . Moreover,  $t \to \operatorname{app}(\mathbf{c}_{y,yy}, \mathbf{c}_{y,yy}) \equiv u$ , as expected. Finally, we have  $u \to u$ .

To any term in  $\mathcal{V}(\Phi, \Upsilon)$  corresponds a  $\lambda$ -term in  $\Lambda$ :

**Definition 3** To every term  $t \in \mathcal{V}(\Phi, \Upsilon)$  we can associate a term  $\langle t \rangle_{\Lambda} \in \Lambda$  as follows:

$$\begin{aligned} \langle x \rangle_{\Lambda} &= x \\ \langle \mathbf{app}(u, v) \rangle_{\Lambda} &= \langle u \rangle_{\Lambda} \langle v \rangle_{\Lambda} \\ \langle \mathbf{c}_{x,M}(t_1, \dots t_n) \rangle_{\Lambda} &= (\lambda x.M) \{ \langle t_1 \rangle_{\Lambda} / x_1, \dots, \langle t_n \rangle_{\Lambda} / x_n \} \end{aligned}$$

where  $FV(\lambda x.M) = x_1, \ldots, x_n$ .

Canonicity holds for terms in  $\Phi$  obtained as images of (closed)  $\lambda$ -terms via  $[\cdot]_{\Phi}$ . Moreover, canonicity is preserved by reduction in  $\Phi$ :

**Lemma 2** For every closed  $M \in \Lambda$ ,  $[M]_{\Phi}$  is canonical. Moreover, if t is canonical and  $t \to u$ , then u is canonical.

**Proof.**  $[M]_{\Phi}$  is canonical for any  $M \in \Lambda$  by induction on the structure of M (which, by hypothesis, is either an abstraction or an application NL where both N and L are closed). We can further prove that  $v = [M]_{\Phi}\{t_1/x_1, \ldots, t_n/x_n\}$  is canonical whenever  $t_1, \ldots, t_n \in \mathcal{C}(\Phi)$  and  $x_1, \ldots, x_n$  includes all the variables in FV(M):

- If  $M = x_i$ , then  $v = t_i$ , which is clearly canonical.
- If M = NL, then

$$v = [NL]_{\Phi} \{ t_1/x_1, \dots t_n/x_n \}$$
  
= **app** ([N]\_{\Phi} \{ t\_1/x\_1, \dots t\_n/x\_n \}, [L]\_{\Phi} \{ t\_1/x\_1, \dots t\_n/x\_n \})

which is canonical, by IH.

• If  $M = \lambda y . N$ , then

$$v = [\lambda y.N]_{\Phi} \{ t_1/x_1, \dots t_n/x_n \}$$
  
=  $\mathbf{c}_{y,N}(x_{i_1}, \dots, x_{i_m}) \{ t_1/x_1, \dots t_n/x_n \}$   
=  $\mathbf{c}_{y,N}(t_{i_1}, \dots, t_{i_m})$ 

which is canonical, because each  $t_i$  is in  $\mathcal{C}(\Phi)$ .

This implies the rhs of any instance of a rule in  $\mathcal{R}_{\Phi}$  is canonical. As a consequence, u is canonical whenever  $t \to u$  and t is canonical. This concludes the proof.

For canonical terms, being a normal form is equivalent of being mapped to a normal form via  $\langle \cdot \rangle_{\Lambda}$ . This is not true, in general: take as a counterexample  $\mathbf{c}_{x,y}(\mathbf{app}(\mathbf{c}_{z,z},\mathbf{c}_{z,z}))$ , which corresponds to  $\lambda x.(\lambda z.z)(\lambda z.z)$  via  $\langle \cdot \rangle_{\Lambda}$ .

**Lemma 3** A canonical term t is a normal form iff  $\langle t \rangle_{\Lambda}$  is a normal form.

**Proof.** If a canonical t is a normal form, then t does not contain the function symbol **app** and, as a consequence,  $\langle t \rangle_{\Lambda}$  is an abstraction, which is always a normal form. Conversely, if  $\langle t \rangle_{\Lambda}$  is a normal form, then t is not in the form **app**(u, v), because otherwise  $\langle t \rangle_{\Lambda}$  will be a (closed) application, which cannot be a normal form. But since t is canonical,  $t \in C(\Phi)$ , which only contains terms in normal form.

**Lemma 4** For every term  $t \in \mathcal{V}(\Phi, \Upsilon)$  and every  $t_1, \ldots, t_n \in \mathcal{C}(\Phi)$ ,

$$\langle t\{t_1/x_1,\ldots,t_n/x_n\}\rangle_{\Lambda} = \langle t\rangle_{\Lambda}\{\langle t_1\rangle_{\Lambda}/x_1,\ldots,\langle t_n\rangle_{\Lambda}/x_n\}$$

whenever  $x_1, \ldots, x_n$  includes all the variables in t.

**Proof.** By induction on *t*:

• If  $t = x_i$ , then

$$\begin{aligned} \langle t\{t_1/x_1, \dots, t_n/x_n\} \rangle_{\Lambda} &= \langle x_i\{t_1/x_1, \dots, t_n/x_n\} \rangle_{\Lambda} \\ &= \langle t_i \rangle_{\Lambda} \\ &= x_i\{\langle t_1 \rangle_{\Lambda}/x_1, \dots, \langle t_n \rangle_{\Lambda}/x_n\} \\ &= t\{\langle t_1 \rangle_{\Lambda}/x_1, \dots, \langle t_n \rangle_{\Lambda}/x_n\}. \end{aligned}$$

• If  $t = \mathbf{app}(u, v)$ , then

$$\langle t\{t_1/x_1, \dots, t_n/x_n\} \rangle_{\Lambda} = \langle \mathbf{app}(u, v)\{t_1/x_1, \dots, t_n/x_n\} \rangle_{\Lambda} = \langle \mathbf{app}(u\{t_1/x_1, \dots, t_n/x_n\}, v\{t_1/x_1, \dots, t_n/x_n\}) \rangle_{\Lambda} = \langle u\{t_1/x_1, \dots, t_n/x_n\} \rangle_{\Lambda} \langle v\{t_1/x_1, \dots, t_n/x_n\} \rangle_{\Lambda} = \langle u\rangle_{\Lambda} \{\langle t_1 \rangle_{\Lambda}/x_1, \dots, \langle t_n \rangle_{\Lambda}/x_n\} \langle v\rangle_{\Lambda} \{\langle t_1 \rangle_{\Lambda}/x_1, \dots, \langle t_n \rangle_{\Lambda}/x_n\} = \langle u\rangle_{\Lambda} \langle v\rangle_{\Lambda} \{\langle t_1 \rangle_{\Lambda}/x_1, \dots, \langle t_n \rangle_{\Lambda}/x_n\} = \langle \mathbf{app}(u, v) \rangle_{\Lambda} \{\langle t_1 \rangle_{\Lambda}/x_1, \dots, \langle t_n \rangle_{\Lambda}/x_n\} = \langle t\rangle_{\Lambda} \{\langle t_1 \rangle_{\Lambda}/x_1, \dots, \langle t_n \rangle_{\Lambda}/x_n\}.$$

• If  $t = \mathbf{c}_{y,N}(u_1, \ldots, u_m)$ , then

$$\begin{split} \langle t\{t_1/x_1,\ldots,t_n/x_n\}\rangle_{\Lambda} &= \langle \mathbf{c}_{y,N}(u_1,\ldots,u_m)\{t_1/x_1,\ldots,t_n/x_n\}\rangle_{\Lambda} \\ &= \langle \mathbf{c}_{y,N}(u_1\{t_1/x_1,\ldots,t_n/x_n\},\ldots,u_m\{t_1/x_1,\ldots,t_n/x_n\})\rangle_{\Lambda} \\ &= (\lambda y.N)\{\langle u_1\{t_1/x_1,\ldots,t_n/x_n\}\rangle_{\Lambda}/x_{i_1} \\ &\quad ,\ldots, \\ \langle u_m\{t_1/x_1,\ldots,t_n/x_n\}\rangle_{\Lambda}/x_{i_m}\} \\ &= (\lambda y.N)\{\langle u_1\rangle_{\Lambda}\{\langle t_1\rangle_{\Lambda}/x_1,\ldots,\langle t_n\rangle_{\Lambda}/x_n\}/x_{i_1} \\ &\quad ,\ldots, \\ \langle u_m\rangle_{\Lambda}\{\langle t_1\rangle_{\Lambda}/x_1,\ldots,\langle t_n\rangle_{\Lambda}/x_n\}/x_{i_m}\} \\ &= ((\lambda y.N)\{\langle u_1\rangle_{\Lambda}/x_1,\ldots,u_m/x_{i_1}\})\{\langle t_1\rangle_{\Lambda}/x_1,\ldots,\langle t_n\rangle_{\Lambda}/x_n\} \\ &= \langle \mathbf{c}_{y,N}(u_1,\ldots,u_m)\rangle_{\Lambda}\{\langle t_1\rangle_{\Lambda}/x_1,\ldots,\langle t_n\rangle_{\Lambda}/x_n\} \\ &= \langle t\rangle_{\Lambda}\{\langle t_1\rangle_{\Lambda}/x_1,\ldots,\langle t_n\rangle_{\Lambda}/x_n\} \end{split}$$

This concludes the proof.

**Lemma 5** For every  $\lambda$ -term  $M \in \Lambda$ ,  $\langle [M]_{\Phi} \rangle_{\Lambda} = M$ .

**Proof.** By induction on *M*:

• If M = x, then

$$\langle [M]_{\Phi} \rangle_{\Lambda} = \langle [x]_{\Phi} \rangle_{\Lambda} = \langle x \rangle_{\Lambda} = x$$

• If M = NL, then

$$\langle [M]_{\Phi} \rangle_{\Lambda} = \langle \mathbf{app}([N]_{\Phi}, [L]_{\Phi}) \rangle_{\Lambda} = \langle [N]_{\Phi} \rangle_{\Lambda} \langle [L]_{\Phi} \rangle_{\Lambda} = NL.$$

• If  $M = \lambda y . N$ , then

$$\langle [M]_{\Phi} \rangle_{\Lambda} = \langle \mathbf{c}_{y,N}(x_1, \dots, x_n) \rangle_{\Lambda} = (\lambda y.N) \{ x_1/x_1, \dots, x_n/x_n \} = \lambda y.N = M.$$
  
les the proof.

This concludes the proof.

The previous two lemmas implies that if  $M \in \Lambda$ ,  $t_1, \ldots, t_n \in \mathcal{C}(\Phi)$  and  $x_1, \ldots, x_n$  includes all the variables in FV(M), then:

$$\langle [M]_{\Phi}\{t_1/x_1, \dots, t_n/x_n\} \rangle_{\Lambda} = M\{\langle t_1 \rangle_{\Lambda}/x_1, \dots, \langle t_n \rangle_{\Lambda}/x_n\}.$$
(1)

Reduction in  $\Phi$  can be simulated by reduction in the  $\lambda\text{-calculus},$  provided the starting term is canonical.

**Lemma 6** If t is canonical and  $t \to u$ , then  $\langle t \rangle_{\Lambda} \to_v \langle u \rangle_{\Lambda}$ .

**Proof.** Consider the (instance of the) rewriting rule which turns t into u. Let it be

$$\mathbf{app}(\mathbf{c}_{y,M}(t_1,\ldots,t_n),v)\to [M]_{\Phi}\{t_1/x_1,\ldots,t_n/x_n,v/y\}.$$

Clearly,

$$\langle \operatorname{\mathbf{app}}(\mathbf{c}_{y,M}(t_1,\ldots,t_n),v) \rangle_{\Lambda} = ((\lambda y.M)\{t_1/x_1,\ldots,t_n/x_n\}) \langle v \rangle_{\Lambda}$$

while, by (1):

$$\langle [M]_{\Phi}\{t_1/x_1,\ldots,t_n/x_n,v/y\}\rangle_{\Lambda} = M\{\langle t_1\rangle_{\Lambda}/x_1,\ldots,\langle t_n\rangle_{\Lambda}/x_n,\langle v\rangle_{\Lambda}/y\}$$

which implies the thesis.

Conversely, call-by-value reduction in the  $\lambda$ -calculus can be simulated in  $\Phi$ :

**Lemma 7** If  $M \to_v N$ , t is canonical and  $\langle t \rangle_{\Lambda} = M$ , then  $t \to u$ , where  $\langle u \rangle_{\Lambda} = N$ .

**Proof.** Let  $(\lambda x.L)V$  be the redex fired in M when rewriting it to N. There must be a corresponding subterm v of t such that  $\langle v \rangle_{\Lambda} = (\lambda x.L)V$ . Then

$$v = \mathbf{app}(\mathbf{c}_{x,P}(t_1,\ldots,t_n),w),$$

where  $\langle \mathbf{c}_{x,P}(t_1,\ldots,t_n) \rangle_{\Lambda} = \lambda x.L.$  and  $\langle w \rangle_{\Lambda} = V.$  Observe that, by definition,

$$\langle \mathbf{c}_{x,P}(t_1,\ldots,t_n) \rangle_{\Lambda} = (\lambda x.P) \{ \langle t_1 \rangle_{\Lambda} / x_1,\ldots, \langle t_n \rangle_{\Lambda} / x_n \}$$

where  $FV(P) = x_1, \ldots, x_n$ . Since t is canonical,  $t_1, \ldots, t_n \in C(\Phi)$ . Moreover, since V is a value, w itself is in  $C(\Phi)$ . This implies

$$\mathbf{app}(\mathbf{c}_{x,P}(t_1,\ldots,t_n),w)\to [P]_{\Phi}\{t_1/x_1,\ldots,t_n/x_n,w/x\}$$

By (1):

$$\langle [P]_{\Phi} \{ t_1/x_1, \dots, t_n/x_n, w/x \} \rangle_{\Lambda} = P\{ \langle t_1 \rangle_{\Lambda}/x_1, \dots, \langle t_n \rangle_{\Lambda}/x_n, \langle w \rangle_{\Lambda}/x \}$$
  
=  $(P\{ \langle t_1 \rangle_{\Lambda}/x_1, \dots, \langle t_n \rangle_{\Lambda}/x_n \}) \{ \langle w \rangle_{\Lambda}/x \}$   
=  $(\lambda x.L) \{ V/x \}$ 

This concludes the proof.

The previous lemmas altogether imply the following theorem, by which  $\lambda$ -calculus normalization can be mimicked (step-by-step) by reduction in  $\Phi$ :

**Theorem 1 (Term Reducibility)** Let  $M \in \Lambda$  be a closed term. The following two conditions are equivalent:

- 1.  $M \rightarrow_v^n N$  where N is in normal form;
- 2.  $[M]_{\Phi} \rightarrow^{n} t$  where  $\langle t \rangle_{\Lambda} = N$  and t is in normal form.

**Proof.** Suppose  $M \to_v^n N$ , where N is in normal form. Then, by applying Lemma 7, we obtain a term t such that  $[M]_{\Phi} \to^n t$  and  $\langle t \rangle_{\Lambda} = N$ . By Lemma 2, t is canonical and, by Lemma 3, it is in normal form. Now, suppose  $[M]_{\Phi} \to^n t$  where  $\langle t \rangle_{\Lambda} = N$  and t is in normal form. By applying n times Lemma 6, we obtain  $\langle [M]_{\Phi} \rangle_{\Lambda} \to_v^n \langle t \rangle_{\Lambda} = N$ . But  $\langle [M]_{\Phi} \rangle_{\Lambda} = M$  by Lemma 5 and N is a normal form by Lemma 3, since  $[M]_{\Phi}$  and t are canonical by Lemma 2.

There is another nice property of  $\Phi$ , that will be crucial in proving the main result of this paper:

**Proposition 1** For every  $M \in \Lambda$ , for every t with  $[M]_{\Phi} \to^* t$  and for every occurrence of a constructor  $\mathbf{c}_{x,N}$  in t, N is a subterm of M.

**Proof.** Assume  $[M]_{\Phi} \to^n t$  and proceed by induction on n.

**Example 2** Let us consider the  $\lambda$ -term  $M = (\lambda x.(\lambda y.x)x)(\lambda z.z)$ . Notice that

$$M \to_v (\lambda y.(\lambda z.z))(\lambda z.z) \to_v \lambda z.z.$$

Clearly  $[M]_{\Phi} = \operatorname{app}(\mathbf{c}_{x,(\lambda y,x)x}, \mathbf{c}_{z,z})$ . Moreover:

$$\operatorname{app}(\mathbf{c}_{x,(\lambda y.x)x},\mathbf{c}_{z,z}) o \operatorname{app}(\mathbf{c}_{y,x}(\mathbf{c}_{z,z}),\mathbf{c}_{z,z}) o \mathbf{c}_{z,z}.$$

For every constructor  $\mathbf{c}_{w,N}$  occurring in any term in the previous reduction sequence, N is a subterm of M.

A remark on  $\Phi$  is now in order.  $\Phi$  is an infinite CRS, since  $\Sigma_{\Phi}$  contains an infinite amount of constructor symbols and, moreover, there are infinitely many rules in  $\mathcal{R}_{\Phi}$ . As a consequence, what we have presented here is an embedding of the (weak, call-by-value)  $\lambda$ -calculus into an infinite (orthogonal) CRS. Consider, now, the following scenario: suppose the  $\lambda$ -calculus is used to write a *program* M, and suppose that inputs to M form an infinite set of  $\lambda$ -terms  $\Theta$  which can anyway be represented by a finite set of constructors in  $\Phi$ . In this scenario, Proposition 1 allows to conclude the existence of finite subsets of  $\Sigma_{\Phi}$  and  $\mathcal{R}_{\Phi}$  such that every MN (where  $N \in \Theta$ ) can be reduced via  $\Phi$  by using only constructors and rules in those finite subsets. As a consequence, we can see the above schema as one that puts any program M in correspondence to a finite CRS. Finally, observe that assuming data to be representable by a finite number of constructors in  $\Phi$  is reasonable. Scott's scheme [17], for example, allows to represent any term in a given free algebra in a finitary way, e.g. the natural number 0 becomes  $[0] \equiv \mathbf{c}_{y,\lambda z. x}$  while n + 1 becomes  $[n + 1] \equiv \mathbf{c}_{y,\lambda z. yx}([n])$ . Church's scheme, on the other hand, does not have this property.

### 4 From Constructor Term Rewriting to Lambda-Calculus

In this Section, we will show that any rewriting step of a constructor rewrite system can be simulated by a fixed number of weak call-by-value beta-reductions.

Let  $\Xi$  be an orthogonal constructor rewrite system over a finite signature  $\Sigma_{\Xi}$ . Let  $\mathbf{c}_1, \ldots, \mathbf{c}_g$  be the constructors of  $\Xi$  and let  $\mathbf{f}_1, \ldots, \mathbf{f}_h$  be the function symbols of  $\Xi$ . The following constructions work independently of  $\Xi$ .

We will first concentrate on constructor terms, encoding them as  $\lambda$ -terms using Scott's schema [17]. Constructor terms can be easily put in correspondence with  $\lambda$ -terms by way of a map  $\langle\!\langle \cdot \rangle\!\rangle_{\Lambda}$  defined by induction as follows:

$$\langle\!\langle \mathbf{c}_i(t_1\ldots,t_n)\rangle\!\rangle_{\Lambda} \equiv \lambda x_1\ldots\lambda x_q.\lambda y.x_i\langle\!\langle t_1\rangle\!\rangle_{\Lambda}\ldots\langle\!\langle t_n\rangle\!\rangle_{\Lambda}.$$

This way constructors become functions:

$$\langle\!\langle \mathbf{c}_i \rangle\!\rangle_{\Lambda} \equiv \lambda x_1 \dots \lambda x_{ar(\mathbf{c}_i)} \cdot \lambda y_1 \dots \lambda y_g \cdot \lambda z \cdot y_i x_1 \dots x_{ar(\mathbf{c}_i)}$$

Trivially,  $\langle\!\langle \mathbf{c}_i \rangle\!\rangle_{\Lambda} \langle\!\langle t_1 \rangle\!\rangle_{\Lambda} \dots \langle\!\langle t_n \rangle\!\rangle_{\Lambda}$  rewrites to  $\langle\!\langle \mathbf{c}_i(t_1 \dots t_n) \rangle\!\rangle_{\Lambda}$  in  $ar(\mathbf{c}_i)$  steps. The  $\lambda$ -term  $\lambda x_1 \dots \lambda x_g \cdot \lambda y \cdot y$  corresponds to an error value, and is denoted as  $\perp$ . A  $\lambda$ -term built in this way, i.e. a  $\lambda$ -term which is either  $\perp$  or in the form  $\langle\!\langle t \rangle\!\rangle_{\Lambda}$  is denoted with metavariables like X or Y.

The map  $\langle\!\langle \cdot \rangle\!\rangle_{\Lambda}$  defines encodings of constructor terms. But what about terms containing function symbols? The goal is defining another map  $[\cdot]_{\Lambda}$  returning a  $\lambda$ -term given any term t in  $\mathcal{T}(\Xi)$ , in such a way that  $t \to^* u$  and  $u \in \mathcal{C}(\Xi)$  implies  $[t]_{\Lambda} \to^*_v \langle\!\langle u \rangle\!\rangle_{\Lambda}$ . Moreover,  $[t]_{\Lambda}$  should rewrite to  $\bot$  whenever the rewriting of t causes an error (i.e. whenever t has a normal form containing a function symbol). First of all, we can define the  $\lambda$ -term  $[\mathbf{c}_i]_{\Lambda}$  corresponding to any constructor  $\mathbf{c}_i$ . To do that, define a  $\lambda$ -term  $M^i_{x_1,\ldots,x_m}$  for every  $1 \le i \le g$ , for every  $0 \le m \le ar(\mathbf{c}_i)$  and for every variables  $x_1,\ldots,x_m$  by induction on  $ar(\mathbf{c}_i) - m$ :

$$\begin{aligned} M_{x_1,\dots,x_{ar(\mathbf{c}_i)}}^i &\equiv \lambda y_1,\dots,\lambda y_g, y_i x_1\dots, x_{ar(\mathbf{c}_i)};\\ \forall 0 \leq m < ar(\mathbf{c}_i).M_{x_1,\dots,x_m}^i &\equiv \lambda y.y N_{1,i}^m \dots N_{g,i}^m L_i^m; \end{aligned}$$

where:

$$N_{j,i}^m \equiv \lambda z_1 \dots \lambda z_{ar(\mathbf{c}_j)} (\lambda x_{m+1} . M_{x_1,\dots,x_{m+1}}^i) M_{z_1,\dots,z_j}^{ar(\mathbf{c}_j)};$$
  

$$L_i^m \equiv \lambda z_{m+2} \dots \lambda z_{ar(\mathbf{c}_i)} . \bot.$$

**Lemma 8** There is a constant  $n \in \mathbb{N}$  such that for every *i* and for every *m*:

$$M^{i}_{x_{1},\ldots,x_{m}}\{\langle t_{1}\rangle_{\Lambda}/x_{1},\ldots,\langle t_{m}\rangle_{\Lambda}/x_{m}\}\langle t_{m+1}\rangle_{\Lambda}\ldots\langle t_{ar(\mathbf{c}_{i})}\rangle_{\Lambda}\rightarrow^{k}\langle\!\langle \mathbf{c}_{i}(t_{1}\ldots t_{ar(\mathbf{c}_{i})})\rangle\!\rangle_{\Lambda}$$

(where  $k \leq n$ ) and

$$M^{i}_{x_{1},\ldots,x_{m}}\{\langle\langle t_{1}\rangle\rangle_{\Lambda}/x_{1},\ldots,\langle\langle t_{m}\rangle\rangle_{\Lambda}/x_{m}\}X_{m+1}\ldots X_{ar(\mathbf{c}_{i})}\to^{l}\bot$$

(where  $l \leq n$ ) whenever  $X_j$  is either  $\langle \langle t_j \rangle \rangle_{\Lambda}$  or  $\perp$  but at least one among  $X_{m+1} \dots X_{ar(\mathbf{c}_i)}$  is  $\perp$ .

**Proof.** We proceed by induction on  $ar(\mathbf{c}_i) - m$ : • If  $m = ar(\mathbf{c}_i)$ , then

$$\begin{split} &M_{x_1,\ldots,x_{ar(\mathbf{c}_i)}}^i \{ \langle \langle t_1 \rangle \rangle_{\Lambda} / x_1,\ldots, \langle \langle t_{ar(\mathbf{c}_i)} \rangle \rangle_{\Lambda} / x_{ar(\mathbf{c}_i)} \} \\ &\equiv \ (\lambda y_1,\ldots,\lambda y_g y_i x_1\ldots x_{ar(\mathbf{c}_i)}) \{ \langle \langle t_1 \rangle \rangle_{\Lambda} / x_1,\ldots, \langle \langle t_{ar(\mathbf{c}_i)} \rangle \rangle_{\Lambda} / x_{ar(\mathbf{c}_i)} \} \\ &\equiv \ \lambda y_1,\ldots,\lambda y_g. y_i \langle \langle t_1 \rangle \rangle_{\Lambda} \ldots \langle \langle t_{ar(\mathbf{c}_i)} \rangle \rangle_{\Lambda} \\ &\equiv \ \langle \langle \mathbf{c}_i(t_1,\ldots,t_{ar(\mathbf{c}_i)}) \rangle \rangle_{\Lambda}. \end{split}$$

• If  $m < ar(\mathbf{c}_i)$ , we use the following abbreviations:

$$P_{j,i}^m \equiv X_{j,i}^m \{ \langle \langle t_1 \rangle \rangle_{\Lambda} / x_1, \dots, \langle \langle t_m \rangle \rangle_{\Lambda} / x_m \}; Q_j^m \equiv L_j^m \{ \langle \langle t_1 \rangle \rangle_{\Lambda} / x_1, \dots, \langle \langle t_m \rangle \rangle_{\Lambda} / x_m \}.$$

Let's distinguish two cases:

• If  $X_{m+1} \equiv \bot$ , then:

$$M_{x_1,\ldots,x_m}^i\{\langle t_1 \rangle\rangle_{\Lambda}/x_1,\ldots,\langle t_m \rangle\rangle_{\Lambda}/x_m\}X_{m+1}\ldots X_{ar(\mathbf{c}_i)}$$
  

$$\rightarrow_v \quad (\perp P_{1,i}^m \ldots P_{g,i}^m Q_i^m)X_{m+2}\ldots X_{ar(\mathbf{c}_i)}$$
  

$$\rightarrow_v^* \quad Q_i^m X_{m+2}\ldots X_{ar(\mathbf{c}_i)}$$
  

$$\rightarrow_v^* \quad \perp$$

• Let  $X_{m+1}$  be  $\langle\!\langle t_{m+1} \rangle\!\rangle_{\Lambda}$ , where  $t_{m+1} \equiv \mathbf{c}_j(u_1, \ldots, u_{ar(\mathbf{c}_j)})$ . Then:

$$\begin{split} & M_{x_{1},\ldots,x_{m}}^{i}\{\langle\langle t_{1}\rangle\rangle_{\Lambda}/x_{1},\ldots,\langle\langle t_{m}\rangle\rangle_{\Lambda}/x_{m}\}X_{m+1}\ldots X_{ar(\mathbf{c}_{i})} \\ & \rightarrow_{v} \quad (\langle\langle \mathbf{c}_{j}(u_{1},\ldots,u_{ar(\mathbf{c}_{j})})\rangle\rangle_{\Lambda}P_{1,i}^{m}\ldots P_{g,i}^{m}Q_{i}^{m})X_{m+2}\ldots X_{ar(\mathbf{c}_{i})} \\ & \rightarrow_{v}^{*} \quad P_{j,i}^{m}\langle\langle u_{1}\rangle\rangle_{\Lambda}\ldots,\langle\langle u_{ar(\mathbf{c}_{j})}\rangle\rangle_{\Lambda}X_{m+2}\ldots X_{ar(\mathbf{c}_{i})} \\ & \rightarrow_{v}^{*} \quad (\lambda x_{m+1}.M_{x_{1},\ldots,x_{m+1}}^{i}\{\langle\langle t_{1}\rangle\rangle_{\Lambda}/x_{1},\ldots,\langle\langle t_{ar(\mathbf{c}_{j})}\rangle\rangle_{\Lambda}/x_{m}\}) \\ & \quad (M_{z_{1},\ldots,z_{ar(\mathbf{c}_{j})}}^{j}\{\langle\langle u_{1}\rangle\rangle_{\Lambda}/y_{1},\ldots,\langle\langle t_{ar(\mathbf{c}_{j})}\rangle\rangle_{\Lambda}/y_{ar(\mathbf{c}_{j})}\})X_{m+2}\ldots X_{ar(\mathbf{c}_{i})} \\ & \rightarrow_{v}^{*} \quad (\lambda x_{m+1}.M_{x_{1},\ldots,x_{m+1}}^{i}\{\langle\langle t_{1}\rangle\rangle_{\Lambda}/x_{1},\ldots,\langle\langle t_{m}\rangle\rangle_{\Lambda}/x_{m}\}) \\ & \quad (\langle\langle \mathbf{c}_{j}(u_{1},\ldots,u_{ar(\mathbf{c}_{j})})\rangle\rangle_{\Lambda})X_{m+2}\ldots X_{ar(\mathbf{c}_{i})} \\ & \rightarrow_{v}^{*} \quad M_{x_{1},\ldots,x_{m+1}}^{i}\{\langle\langle t_{1}\rangle\rangle_{\Lambda}/x_{1},\ldots,\langle\langle t_{m+1}\rangle\rangle_{\Lambda}/x_{m+1}\}X_{m+2}\ldots X_{ar(\mathbf{c}_{i})} \end{split}$$

and, by the inductive hypothesis, the last term in the reduction sequence reduces to the correct normal form. The existence of a natural number n with the prescribed properties can be proved by observing that none of the reductions above have a length which depends on the parameters  $\langle\!\langle t_1 \rangle\!\rangle_{\Lambda}, \ldots, \langle\!\langle t_m \rangle\!\rangle_{\Lambda}$  and  $X_{m+1} \ldots X_{ar(\mathbf{c}_i)}$ . 

This concludes the proof.

So, the required lambda term  $[\mathbf{c}_i]_{\Lambda}$  is simply  $M_{\epsilon}^i$ . Interpreting function symbols is more difficult: we have to "embed" reduction rules into the  $\lambda$ -term interpreting the function symbol. To do that, we need a preliminary result:

**Lemma 9 (Pattern matching)** Let  $\alpha_1, \ldots, \alpha_n$  be non-overlapping sequences of patterns of the same length m. Then there are a term  $M^m_{\alpha_1,\ldots,\alpha_n}$  and an integer l such that for every sequence of values  $V_1, \ldots, V_n$ , if  $\alpha_i = \mathbf{p}_1, \ldots, \mathbf{p}_m$  then

$$M^{m}_{\alpha_{1},\ldots,\alpha_{n}} \langle\!\langle \mathbf{p}_{1}(t_{1}^{1},\ldots,t_{1}^{k_{1}}) \rangle\!\rangle_{\Lambda} \ldots \langle\!\langle \mathbf{p}_{m}(t_{m}^{1},\ldots,t_{m}^{k_{m}}) \rangle\!\rangle_{\Lambda} V_{1} \ldots V_{n} \rightarrow^{k}_{v} V_{i} \langle\!\langle t_{1}^{1} \rangle\!\rangle_{\Lambda} \ldots \langle\!\langle t_{1}^{k_{1}} \rangle\!\rangle_{\Lambda} \ldots \langle\!\langle t_{m}^{k_{m}} \rangle\!\rangle_{\Lambda} \ldots \langle\!\langle t_{m}^{k_{m}} \rangle\!\rangle_{\Lambda},$$

where  $k \leq l$ , whenever the  $t_i^j$  are constructor terms. Moreover,

 $M^m_{\alpha_1,\ldots,\alpha_n} X_1,\ldots,X_m V_1\ldots V_n \to^k_v \bot$ 

where  $k \leq l$ , whenever  $X_1, \ldots, X_m$  do not unify with any of the sequences  $\alpha_1, \ldots, \alpha_n$  or any of the  $X_1, \ldots, X_m$  is itself  $\perp$ .

**Proof.** We go by induction on  $p = \sum_{i=1}^{n} ||\alpha_i||$ , where  $||\alpha_i||$  is the number of constructors occurrences in patterns inside  $\alpha_i$ :

• If p = 0 and n = 0, then we should always return  $\perp$ :

$$M^m_{\varepsilon} \equiv \lambda x_1 \dots \lambda x_m \dots$$

- If p = 0 and n = 1 and  $\alpha_1$  is simply a sequence of variables  $x_1, \ldots, x_m$  (because the  $\alpha_i$  are assuming to be non-overlapping). Then  $M_{x_1,...,x_m}^m$  is a term defined by induction on m which returns  $\perp$  only if one of its first m arguments is  $\perp$  and otherwise returns its m + 1-th argument applied to its first m arguments.
- If  $p \ge 1$ , then there must be integers i and j with  $1 \le i \le m$  and  $1 \le j \le n$  such that

$$\alpha_j = \mathbf{p}_1, \dots, \mathbf{p}_{i-1}, \mathbf{c}_k(\mathbf{r}_1, \dots, \mathbf{r}_{ar(\mathbf{c}_k)}), \mathbf{p}_{i+1}, \dots, \mathbf{p}_m$$

for a constructor  $\mathbf{c}_k$  and for some patterns  $\mathbf{p}_p$  and some  $\mathbf{r}_q$ . Now, for every  $1 \leq p \leq n$  and for every  $1 \leq j \leq g$  we define sequences of patterns  $\beta_p^j$  and values  $W_p^j$  as follows:

$$\alpha_p = \mathbf{p}_1, \dots, \mathbf{p}_{i-1}, \mathbf{c}_j(\mathbf{r}_1, \dots, \mathbf{r}_{ar(\mathbf{c}_i)}), \mathbf{p}_{i+1} \dots \mathbf{p}_m$$

then  $\beta_p^j$  is defined to be the sequence

$$\mathbf{p}_1,\ldots,\mathbf{p}_{i-1},\mathbf{r}_1,\ldots,\mathbf{r}_{ar(\mathbf{c}_k)},\mathbf{p}_{i+1},\ldots,\mathbf{p}_m.$$

Moreover,  $W_p$  is simply the indentity  $\lambda x.x$ .

$$\alpha_p = \mathbf{p}_1, \dots, \mathbf{p}_{i-1}, \mathbf{c}_s(\mathbf{r}_1, \dots, \mathbf{r}_{ar(\mathbf{c}_s)}), \mathbf{p}_{i+1} \dots \mathbf{p}_m$$

where  $s \neq j$  then  $\beta_p^j$  and  $W_p^j$  are both undefined.

• Finally, if

$$\alpha_p = \mathbf{p}_1, \dots, \mathbf{p}_{i-1}, x, \mathbf{p}_{i+1} \dots \mathbf{p}_m$$

then  $\beta_p^j$  is defined to be the sequence

$$\mathbf{p}_1,\ldots,\mathbf{p}_{i-1},x_1,\ldots,x_{ar(\mathbf{c}_i)},\mathbf{p}_{i+1},\ldots,\mathbf{p}_m$$

and  $W_p^j$  is the following  $\lambda$ -term

$$\lambda x.\lambda y_1.\ldots\lambda y_t.x_1.\ldots\lambda x_{ar(\mathbf{c}_k)}.\lambda z_1.\ldots\lambda z_u.xy_1\ldots y_t(\langle\!\langle \mathbf{c}_j \rangle\!\rangle_{\Lambda} x_1\ldots x_{ar(\mathbf{c}_j)})z_1\ldots z_u$$

where t is the number of variables in  $\mathbf{p}_1, \ldots, \mathbf{p}_{i-1}$  and u is the number of variables in  $\mathbf{p}_{i+1}, \ldots, \mathbf{p}_m$ .

As a consequence, for every  $1 \leq j \leq g$ , we can find a natural number  $t_j$  and a sequence of pairwise distinct natural numbers  $i_1, \ldots, i_{t_j}$  such that  $\beta_{i_1}^j, \ldots, \beta_{i_{t_j}}^j$  are exactly the sequences which can be defined by the above construction. We are now able to formally define  $M_{\alpha_1,\ldots,\alpha_n}^m$ ; it is the term

$$\lambda x_1 \dots \lambda x_m \lambda y_1 \dots \lambda y_n ((x_i V_1 \dots V_g V_\perp) x_1 \dots x_{i-1} x_{i+1} \dots x_m) y_1 \dots y_n$$

where

$$\begin{aligned} \forall 1 \leq j \leq g. V_j &\equiv \lambda z_1 \dots \lambda z_{ar(\mathbf{c}_j)} \lambda x_1 \dots \lambda x_{i-1} \lambda x_{i+1} \dots \lambda x_m \lambda y_1 \dots \lambda y_n. \\ & M_{\beta_{i_1}^j, \dots, \beta_{i_j}^j}^{m-1+ar(\mathbf{c}_j)} x_1 \dots x_{i-1} z_1 \dots z_{ar(\mathbf{c}_j)} x_{i+1} \dots x_m (W_{i_1}^j y_{i_1}) \dots (W_{i_{t_j}}^j y_{i_{t_j}}) \\ & V_{\perp} &\equiv \lambda x_1 \dots \lambda x_{i-1} \lambda x_{i+1} \dots \lambda x_m \lambda y_1 \dots \lambda y_n. \end{aligned}$$

Notice that, for every j,  $p > \sum_{v=1}^{t_j} ||\beta_v^j||$ . Moreover, for every j any  $\beta_v^j$  has the same length  $m-1+ar(\mathbf{c}_j)$ . This justifies the application of the induction hypothesis above. This concludes the proof.

For every function symbol  $\mathbf{f}_i$ , let

$$\mathbf{f}_i(\alpha_i^1) \to t_i^1, \dots, \mathbf{f}_i(\alpha_i^{n_i}) \to t_i^{n_i}$$

be the rules for  $\mathbf{f}_i$ . Moreover, suppose that the variables appearing in the patterns in  $\alpha_i^j$  are  $z_i^{j,1}, \ldots, z_i^{j,m_{i,j}}$ . Recall that we have a signature with  $\mathbf{f}_1, \ldots, \mathbf{f}_h$  function symbols. For any  $1 \le i \le h$  the lambda term interpreting  $\mathbf{f}_i$  is defined to be:

$$[\mathbf{f}_i]_{\Lambda} \equiv H_i V_1 \dots V_h$$

where

$$V_i \equiv \lambda x_1 \dots \lambda x_h \lambda y_1 \dots \lambda y_{ar(\mathbf{f}_i)} M_{\alpha_i^1, \dots, \alpha_i^n} y_1 \dots y_{ar(\mathbf{f}_i)} W_i^1 \dots W_i^{n_i}$$
$$W_i^j \equiv \lambda z_1 \dots \lambda z_{m_{i,j}} \langle t_i^j \rangle_{\Lambda}$$

• If

whenever  $1 \le i \le h$  and  $1 \le j \le n_i$ . Moreover  $\langle \cdot \rangle_{\Lambda}$  is defined by induction as follows:

$$\langle x \rangle_{\Lambda} = x \langle \mathbf{c}_{i}(t_{1}, \dots, t_{ar(\mathbf{c}_{i})}) \rangle_{\Lambda} = [\mathbf{c}_{i}]_{\Lambda} \langle t_{1} \rangle_{\Lambda} \dots \langle t_{ar(\mathbf{c})} \rangle_{\Lambda} \langle \mathbf{f}_{i}(t_{1}, \dots, t_{ar(\mathbf{f}_{i})}) \rangle_{\Lambda} = x_{i} \langle t_{1} \rangle_{\Lambda} \dots \langle t_{ar(\mathbf{c})} \rangle_{\Lambda}$$

Now, we have all the necessary ingredients to extend the mapping  $[\cdot]_{\Lambda}$  to every term in  $\mathcal{T}(\Xi)$ :

$$[\mathbf{c}(t_1, \dots, t_{ar(\mathbf{c}_i)})]_{\Lambda} = [\mathbf{c}_i]_{\Lambda}[t_1]_{\Lambda} \dots [t_{ar(\mathbf{c})}]_{\Lambda}$$
  
$$[\mathbf{f}_i(t_1, \dots, t_{ar(\mathbf{f}_i)})]_{\Lambda} = [\mathbf{f}_i]_{\Lambda}[t_1]_{\Lambda} \dots [t_{ar(\mathbf{c})}]_{\Lambda}$$

**Theorem 2** There is a natural number k such that for every function symbol **f** and for every  $t_1, \ldots, t_{ar(\mathbf{f})} \in \mathcal{C}(\Xi)$ , the following three implications hold (where u stands for  $\mathbf{f}(t_1, \ldots, t_{ar(\mathbf{f})})$ ) and M stands for  $[\mathbf{f}]_{\Lambda}\langle\langle t_1 \rangle\rangle_{\Lambda} \ldots \langle\langle t_{ar(\mathbf{f})} \rangle\rangle_{\Lambda}$ ):

- If u rewrites to  $v \in \mathcal{C}(\Xi)$  in *n* steps, then *M* rewrites to  $\langle\!\langle v \rangle\!\rangle_{\Lambda}$  in at most kn steps.
- If u rewrites to a normal form  $v \notin C(\Xi)$ , then M rewrites to  $\bot$ .
- If u diverges, then M diverges.

**Proof.** By an easy combinatorial argument following from the definition of  $[\cdot]_{\Lambda}$ .

Clearly, the constant k in Theorem 2 depends on  $\Xi$ , but is independent on the particular term u.

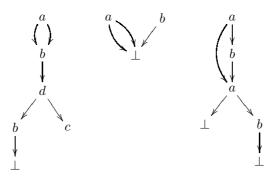
### 5 Graph Representation

In this Section, we introduce term graph rewriting, following [1] but adapting the framework to call-by-value constructor rewriting. Contrarily to what we did in Section 3, we will stay abstract here: our attention will not be restricted to the particular graph rewrite system that is needed to implement reduction in the  $\lambda$ -calculus.

**Definition 4 (Labelled Graph)** Given a signature  $\Sigma$ , a labelled graph over  $\Sigma$  consists of a directed acyclic graph together with an ordering on the outgoing edges of each node and a (partial) labelling of nodes with symbols from  $\Sigma$  such that the out-degree of each node matches the arity of the corresponding symbols (and is 0 if the labelling is undefined). Formally, a labelled graph is a triple  $G = (V, \alpha, \delta)$  where:

- V is a set of vertices.
- $\alpha: V \to V^*$  is a (total) ordering function.
- $\delta: V \to V$  is a (partial) labelling function such that the length of  $\alpha(v)$  is the arity of  $\delta(v)$  if  $\delta(v)$  is defined and is 0 otherwise.
- A labelled graph  $(V, \alpha, \delta)$  is closed iff  $\delta$  is a total function.

Consider the signature  $\Sigma = \{a, b, c, d\}$ , where arities of a, b, c, d are 2, 1, 0, 2 respectively, and b, c, d are constructors. Examples of labelled graphs over the signature  $\Sigma$  are the following ones:

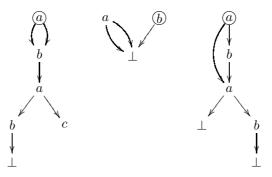


The symbol  $\perp$  denotes vertices where the underlying labelling function is undefined (and, as a consequence, no edge departs from such vertices). Their role is similar to the one of variables in terms.

If one of the vertices of a labelled graph is selected as the *root*, we obtain a term graph:

**Definition 5 (Term Graphs)** A term graph, is a quadruple  $G = (V, \alpha, \delta, r)$ , where  $(V, \alpha, \delta)$  is a labelled graph and  $r \in V$  is the root of the term graph.

The following are graphic representations of some term graphs.



The root is the only vertex drawn inside a circle.

There are some classes of paths which are particularly relevant for our purposes

**Definition 6 (Paths)** A path  $v_1, \ldots, v_n$  in a labelled graph  $G = (V, \alpha, \delta)$  is said to be:

- A constructor path iff for every  $1 \le i \le n$ , the symbol  $\delta(v_i)$  is a constructor;
- A pattern path iff for every  $1 \le i \le n$ ,  $\delta(v_i)$  is either a constructor symbol or is undefined;
- A left path iff  $n \ge 1$ , the symbol  $\delta(v_1)$  is a function symbol and  $v_2, \ldots, v_n$  is a pattern path.

**Definition 7 (Homomorphisms)** An homomorphism between two labelled graphs  $G = (V_G, \alpha_G, \delta_G)$ and  $H = (V_H, \alpha_H, \delta_H)$  over the same signature  $\Sigma$  is a function  $\varphi$  from  $V_G$  to  $V_H$  preserving the term graph structure. In particular

$$\delta_H(\varphi(v)) = \delta_G(v)$$
  

$$\alpha_H(\varphi(v)) = \varphi^*(\alpha_G(v))$$

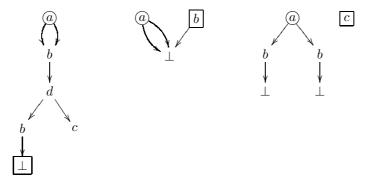
for any  $v \in dom(\delta)$ , where  $\varphi^*$  is the obvious generalization of  $\varphi$  to sequences of vertices. An homomorphism between two term graphs  $G = (V_G, \alpha_G, \delta_G, r_G)$  and  $H = (V_H, \alpha_H, \delta_H, r_H)$  is an homomorphism between  $(V_G, \alpha_G, \delta_G)$  and  $(V_H, \alpha_H, \delta_H)$  such that  $\varphi(r_G) = r_H$ . Two labelled graphs G and H are isomorphic iff there is a bijective homomorphism from G to H; in this case, we write  $G \cong H$ . Similarly for term graphs.

In the following, we will consider term graphs modulo isomorphism, i.e., G = H iff  $G \cong H$ . Observe that two isomorphic term graphs have the same graphical representation.

**Definition 8 (Graph Rewrite Rules)** A graph rewrite rule over a signature  $\Sigma$  is a triple  $\rho = (G, r, s)$  such that:

- G is a labelled graph;
- r, s are vertices of G, called the left root and the right root of  $\rho$ , respectively.
- Any path starting in r is a left path.

The following are examples of graph rewriting rules, assuming a to be a function symbol and b, c, d to be constructors:



**Definition 9 (Subgraphs)** Given a labelled graph  $G = (V_G, \alpha_G, \delta_G)$  and any vertex  $v \in V_G$ , the subgraph of G rooted at v, denoted  $G \downarrow v$ , is the term graph  $(V_{G \downarrow v}, \alpha_{G \downarrow v}, \delta_{G \downarrow v}, r_{G \downarrow v})$  where

- $V_{G\downarrow v}$  is the subset of  $V_G$  whose elements are vertices which are reachable from v in G.
- $\alpha_{G\downarrow v}$  and  $\delta_{G\downarrow v}$  are the appropriate restrictions of  $\alpha_G$  and  $\delta_G$  to  $V_{G\downarrow v}$ .
- $r_{G\downarrow v}$  is v.

**Definition 10 (Redexes)** Given a labelled graph G, a redex for G is a pair  $(\rho, \varphi)$ , where  $\rho$  is a rewrite rule (H, r, s) and  $\varphi$  is an homomorphism between  $H \downarrow r$  and G such that for any vertex  $v \in V_{H\downarrow r}$  with  $v \notin dom(\delta_{H\downarrow r})$ , any path starting in  $\varphi(v)$  is a constructor path.

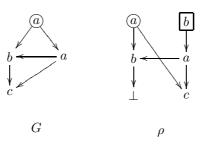
The last condition in the definition of a redex is needed to capture the call-by-value nature of the rewriting process.

Given a term graph G and a redex  $((H, r, s), \varphi)$ , the result of firing the redex is another term graph obtained by successively applying the following three steps to G:

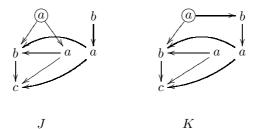
- 1. The build phase: create an isomorphic copy of the portion of  $H \downarrow s$  not contained in  $H \downarrow r$ , and add it to G, obtaining J. The underlying ordering and labelling functions are defined in the natural way.
- 2. The redirection phase: all edges in J pointing to  $\varphi(r)$  are replaced by edges pointing to the copy of s. If  $\varphi(r)$  is the root of G, then the root of the newly created graph will be the newly created copy of s. The graph K is obtained.
- 3. The garbage collection phase: all vertices which are not accessible from the root of K are removed. The graph I is obtained.

We will write  $G \xrightarrow{(H,r,s)} I$  (or simply  $G \to I$ , if this does not cause ambiguity) in this case.

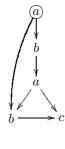
As an example, consider the term graph G and the rewriting rule  $\rho = (H, r, s)$ :



There is an homomorphism  $\varphi$  from  $H \downarrow r$  to G. In particular,  $\varphi$  maps r to the rightmost vertex in G. Applying the build phase and the redirection phase we get J and K as follows:



Finally, applying the garbage collection phase, we get the result of firing the redex  $(\rho, \varphi)$ :



Ι

**Definition 11** A constructor graph rewrite system (CGRS) over a signature  $\Sigma$  consists of a set of graph rewrite rules  $\mathcal{G}$  on  $\Sigma$ .

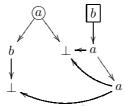
#### 5.1 From Term Rewriting to Graph Rewriting

Any term t over the signature  $\Sigma$  can be turned into a graph G in the obvious way: G will be a tree and vertices in G will be in one-to-one correspondence with symbol occurrences in t. Conversely, any term graph G over  $\Sigma$  can be turned into a term t over  $\Sigma$  (remember: we only consider acyclic graphs here). Similarly, any term rewrite rule  $t \to u$  over the signature  $\Sigma$  can be translated into a graph rewrite rule (G, r, s) as follows:

- Take the graph representing t and u. They are trees, in fact.
- From the union of these two trees, share those nodes representing the same variable in t and u. This is G.
- Take r to be the root of t in G and s to be the root of u in G. As an example, consider the rewriting rule

$$a(b(x), y) \rightarrow b(a(y, a(y, x))).$$

Its translation as a graph rewrite rule is the following:



An arbitrary constructor rewriting system can be turned into a constructor graph rewriting system:

**Definition 12** Given a constructor rewriting system  $\mathcal{R}$  over  $\Sigma$ , the corresponding constructor graph rewriting system  $\mathcal{G}$  is defined as the class of graph rewrite rules corresponding to those in  $\mathcal{R}$ . Given a term t,  $[t]_{\mathcal{G}}$  will be the corresponding graph, while the term graph  $\mathcal{G}$  corresponds to the term  $\langle \mathcal{G} \rangle_{\mathcal{R}}$ .

Let us now consider graph rewrite rules corresponding to rewrite rules in  $\mathcal{R}$ . It is easy to realize that the following invariant is preserved while performing rewriting in  $[\mathcal{R}]_{\mathcal{G}}$ : whenever any vertex v can be reached by two distinct paths starting at the root (i.e., v is *shared*), any path starting at v is a constructor path. A term graph satisfying this invariant is said to be *constructor-shared*.

Constructor-sharedness holds for term graphs coming from terms and is preserved by graph rewriting:

**Lemma 10** For every closed term t,  $[t]_{\mathcal{G}}$  is constructor-shared. Moreover, if G is closed and constructor-shared and  $G \to I$ , then I is constructor-shared.

**Proof.** The fact  $[t]_{\mathcal{G}}$  is constructor-shared for every t follows from the way the  $[\cdot]_{\mathcal{G}}$  map is defined: it does not introduce any sharing. Now, suppose G is constructor-shared and

$$G \xrightarrow{(H,r,s)} I$$

where (H, r, s) corresponds to a term rewrite rule  $t \to u$ . The term graph J obtained from G by the build phase is itself constructor-shared: it is obtained from G by adding some new nodes, namely an isomorphic copy of the portion of  $H \downarrow s$  not contained in  $H \downarrow r$ . Notice that J is constructor-shared in a stronger sense: any vertex which can be reached from the newly created copy of s by two distinct paths must be a constructor path. This is a consequence of (H, r, s) being a graph rewrite rule corresponding to a term rewrite rule  $t \to u$ , where the only shared vertices are those where the labelling function is undefined. The redirection phase preserves itself constructor-sharedness, because only one pointer is redirected (the vertex is labelled by a function symbol) and the destination of this redirection is a vertex (the newly created copy of s) which had no edge incident to it. Clearly, the garbage collection phase preserve constructor-sharedness.  $\Box$ 

**Lemma 11** A closed term graph G in  $\mathcal{G}$  is a normal form iff  $\langle G \rangle_{\mathcal{R}}$  is a normal form.

**Proof.** Clearly, if a closed term graph G is in normal form, then  $\langle G \rangle_{\mathcal{R}}$  is a term in normal form, because each redex in G translates to a redex in  $\langle G \rangle_{\mathcal{R}}$ . On the other hand, if  $\langle G \rangle_{\mathcal{R}}$  is in normal form, then G is in normal form: each redex in  $\langle G \rangle_{\mathcal{R}}$  translates back to a redex in G.  $\Box$ 

Reduction at the level of graphs correctly simulates reduction at the level of terms, but only if the underlying graphs are constructor shared:

**Lemma 12** If G is closed and constructor-shared and  $G \to I$ , then  $\langle G \rangle_{\mathcal{R}} \to \langle I \rangle_{\mathcal{R}}$ .

**Proof.** The fact each reduction step starting in G can be mimicked by n reduction steps in  $\langle G \rangle_{\mathcal{R}}$  is known from the literature. If G is constructor-shared, then n = 1, because any redex in a constructor-shared term graph cannot be shared.

A counterexample, when G in not constructor-shared can be easily built: consider the term rewrite rule  $a(c, c) \rightarrow c$  and the following term graph, which is not constructor-shared and correspond to a(a(c, c), a(c, c)):



The term graph rewrites in *one* step to the following one



while the term a(a(c,c), a(c,c)) rewrites to a(c,c) in two steps.

As can be expected, graph reduction is even complete with respect to term reduction, with the only *proviso* that term graphs must be constructor-shared:

**Lemma 13** If  $t \to u$ , G is constructor-shared and  $\langle G \rangle_{\mathcal{R}} = t$ , then  $G \to I$ , where  $\langle I \rangle_{\mathcal{R}} = u$ .

**Theorem 3 (Graph Reducibility)** For every constructor rewrite system  $\mathcal{R}$  over  $\Sigma$  and for every term t over  $\Sigma$ , the following two conditions are equivalent:

- 1.  $t \rightarrow^n u$ , where u is in normal form;
- 2.  $[t]_{\mathcal{G}} \to^n G$ , where G is in normal form and  $\langle G \rangle_{\mathcal{R}} = u$ .

**Proof.** Suppose  $t \to^n u$ , where u is in normal form. Then, by applying Lemma 13, we obtain a term graph G such that  $[t]_{\mathcal{G}} \to^n G$  and  $\langle G \rangle_{\mathcal{R}} = u$ . By Lemma 10, G is canonical and, by Lemma 11, it is in normal form. Now, suppose  $[t]_{\mathcal{G}} \to^n G$  where  $\langle G \rangle_{\mathcal{R}} = u$  and G is in normal form. By applying n times Lemma 12, we obtain that  $\langle [t]_{\mathcal{G}} \rangle_{\mathcal{R}} \to^n \langle G \rangle_{\mathcal{R}} = u$ . But  $\langle [t]_{\mathcal{G}} \rangle_{\mathcal{R}} = t$  and u is a normal form by Lemma 11, since  $[t]_{\mathcal{G}}$  and G are constructor shared due to Lemma 10.  $\Box$ 

There are *term* rewrite systems which are not graph reducible, i.e. for which the two conditions of Theorem 3 are not equivalent (see [1]). However, any *constructor* rewrite system is graph reducible, due to the strict constraints on the shape of rewrite rules. This result can be considered as a by-product of our analysis, for which graph rewriting is only instrumental.

#### 5.2 Lambda-Terms Can Be Efficiently Reduced by Graph Rewriting

As a corollary of Theorem 3 and Theorem 1, we obtain the possibility of reducing  $\lambda$ -terms by term graphs over  $\Sigma_{\Phi}$ . To this purpose, we can use the CGRS  $\Theta$  corresponding to  $\Phi$ :

**Corollary 1** Let  $M \in \Lambda$  be a closed term. The following two conditions are equivalent:

- 1.  $M \rightarrow_v^n N$  where N is in normal form;
- 2.  $[[M]_{\Phi}]_{\Theta} \to^{n} G$  where  $\langle\!\langle \langle G \rangle_{\Phi} \rangle\!\rangle_{\Lambda} = N$  and G is in normal form.

However, there are some missing tales. Let us analyze more closely the combinatorics of graph rewriting in  $\Theta$ :

- Consider a closed  $\lambda$ -term M and a term graph G such that  $[[M]_{\Phi}]_{\Theta} \to^* G$ . By Proposition 1 and Lemma 12, for every constructor  $\mathbf{c}_{x,N}$  appearing as a label of a vertex in G, N is a subterm of M.
- As a consequence, if  $[[M]_{\Phi}]_{\Theta} \to^* G \to H$ , then the difference |H| |G| cannot be too big: at most |M|. As a consequence, if  $[[M]_{\Phi}]_{\Theta} \to^n G$  then  $|G| \leq (n+1)|M|$ . Here, we exploit in an essential way the possibility of sharing constructors.
- Whenever  $[[M]_{\Phi}]_{\Theta} \to^{n} G$ , computing a graph H such that  $G \to H$  takes polynomial time in |G|, which is itself polynomially bounded by n and |M|.

Hence:

**Theorem 4** There is a polynomial  $p : \mathbb{N}^2 \to \mathbb{N}$  such that for every  $\lambda$ -term M, the normal form of  $[[M]_{\Phi}]_{\Theta}$  can be computed in time at most  $p(|M|, Time_v(M))$ .

As we mentioned in the introduction, this cannot be achieved when using explicit representations of  $\lambda$ -terms. Moreover, reading back a  $\lambda$ -term from a term graph can take exponential time, as we mentioned in the introduction.

We can complement Theorem 4 with a completeness statement — any universal computational model with an invariant cost model can be embedded in the  $\lambda$ -calculus with a polynomial overhead. We can exploit for this the analogous result we proved in [4] (Theorem 1) — the unitary cost model is easily proved to be more parsimonious than the difference cost model considered in [4].

**Theorem 5** Let  $f: \Sigma^* \to \Sigma^*$  be computed by a Turing machine  $\mathcal{M}$  in time g. Then, there are a  $\lambda$ -term  $N_{\mathcal{M}}$  and a suitable encoding  $\lceil \cdot \rceil : \Sigma^* \to \Lambda$  such that  $N_{\mathcal{M}} \lceil v \rceil$  normalizes to  $\lceil f(v) \rceil$  in O(g(|v|)) beta steps.

### 6 Variations: Call-by-Name Reduction

Our purpose in this last section is showing that similar techniques can be applied to call-by-name evaluation of  $\lambda$ -terms.

In the previous sections,  $\lambda$ -calculus was endowed with weak call-by-value reduction. The same technique, however, can be applied to weak call-by-name reduction, as we will sketch in this section.  $\Lambda$  is now endowed with a relation  $\rightarrow_h$  defined as follows:

$$\frac{M \to_h N}{(\lambda x.M)N \to_h M\{N/x\}} \qquad \qquad \frac{M \to_h N}{ML \to_h NL}$$

Similarly to the call-by-value case,  $Time_h(M)$  stands for the number of reduction steps to the normal form of M (if any). Since the relation  $\rightarrow_h$  is deterministic (i.e., functional),  $Time_h(M)$  is well-defined.

We need another CRS, called  $\Psi$ , which is similar to  $\Phi$  but designed to simulate weak call-byname reduction:

• The signature  $\Sigma_{\Psi}$  includes the binary function symbol **app** and constructor symbols  $\mathbf{c}_{x,M}$  for every  $M \in \Lambda$  and every  $x \in \Upsilon$ , exactly as  $\Sigma_{\Phi}$ . Moreover, there is another binary constructor symbol **capp**. To every term  $M \in \Lambda$  we can associate terms  $\{M\}_{\Psi}, [M]_{\Psi} \in \mathcal{V}(\Psi, \Upsilon)$  as follows:

$$\{x\}_{\Psi} = x \{\lambda x.M\}_{\Psi} = \mathbf{c}_{x,M}(x_1,\ldots,x_n), \text{ where } \mathsf{FV}(\lambda x.M) = x_1,\ldots,x_n \{MN\}_{\Psi} = \mathbf{capp}(\{M\}_{\Psi},\{N\}_{\Psi}) [x]_{\Psi} = x [\lambda x.M]_{\Psi} = \mathbf{c}_{x,M}(x_1,\ldots,x_n), \text{ where } \mathsf{FV}(\lambda x.M) = x_1,\ldots,x_n [MN]_{\Psi} = \mathbf{app}([M]_{\Psi},\{N\}_{\Psi})$$

• The rewrite rules in  $\mathcal{R}_{\Psi}$  are all the rules in the following form:

$$\begin{aligned} \mathbf{app}(\mathbf{c}_{z,z},\mathbf{capp}(w,f)) &\to \mathbf{app}(w,f) \\ \mathbf{app}(\mathbf{c}_{z,z},\mathbf{c}_{x,M}(x_1,\ldots,x_n)) &\to \mathbf{c}_{x,M}(x_1,\ldots,x_n) \\ \mathbf{app}(\mathbf{c}_{z,w}(\mathbf{capp}(f,g)),h) &\to \mathbf{app}(f,g) \\ \mathbf{app}(\mathbf{c}_{z,w}(\mathbf{c}_{x,M}(x_1,\ldots,x_n)),h) &\to \mathbf{c}_{x,M}(x_1,\ldots,x_n) \\ \mathbf{app}(\mathbf{c}_{y,N}(y_1,\ldots,y_m),y) &\to [N]_{\Psi} \end{aligned}$$

where M ranges over  $\lambda$ -terms, N ranges over abstractions and applications,  $FV(\lambda x.M) = x_1, \ldots, x_n$  and  $FV(\lambda y.N) = y_1, \ldots, y_m$ . These rewrite rules are said to be *ordinary rules*. We also need the following *administrative* rule:

$$app(capp(x, y), z) = app(app(x, y), z)$$

The CTRS  $\Psi$  is slightly more complicated than  $\Phi$ : some additional overhead is needed to force reduction to happen only in head position. As usual, to every term  $t \in \mathcal{V}(\Psi, \Upsilon)$  we can associate a term  $\langle t \rangle_{\Lambda}$ :

$$\langle x \rangle_{\Lambda} = x \langle \mathbf{app}(u,v) \rangle_{\Lambda} = \langle \mathbf{capp}(u,v) \rangle_{\Lambda} = \langle u \rangle_{\Lambda} \langle v \rangle_{\Lambda} \langle \mathbf{c}_{x,M}(t_1,\ldots,t_n) \rangle_{\Lambda} = (\lambda x.M) \{ \langle t_1 \rangle_{\Lambda} / x_1,\ldots, \langle t_n \rangle_{\Lambda} / x_n \}$$

where  $FV(\lambda x.M) = x_1, \ldots, x_n$ . A term  $t \in \mathcal{T}(\Psi)$  is canonical if either  $t = \mathbf{c}_{x,M}(t_1, \ldots, t_n) \in \mathcal{C}(\Psi)$ or  $t = \mathbf{app}(u, v)$  where u is canonical and  $v \in \mathcal{C}(\Psi)$ .

**Lemma 14** For every closed  $M \in \Lambda$ ,  $[M]_{\Psi}$  is canonical.

**Proof.** By a straightforward induction on M.

The obvious variation on Equation 1 holds here:

$$\langle [M]_{\Psi}\{t_1/x_1, \dots, t_n/x_n\} \rangle_{\Lambda} = M\{\langle t_1 \rangle_{\Lambda}/x_1, \dots, \langle t_n \rangle_{\Lambda}/x_n\}.$$
(2)

 $\Psi$  mimics call-by-name reduction in much the same way  $\Phi$  mimics call-by-value reduction. However, one reduction step in the  $\lambda$ -calculus corresponds to  $n \geq 1$  steps in  $\Psi$ , although n is kept under control:

**Lemma 15** Suppose that  $t \in \mathcal{T}(\Psi)$  is canonical and that  $t \to u$ . Then there is a natural number n such that:

- 1.  $\langle t \rangle_{\Lambda} \to_h \langle u \rangle_{\Lambda};$
- 2. There is a canonical term  $v \in \mathcal{T}(\Psi)$  such that  $u \to^n v$ ;
- 3.  $|w|_{app} = |u|_{app} + m$  whenever  $u \to^m w$  and  $m \le n$ ;
- 4.  $\langle w \rangle_{\Lambda} = \langle u \rangle_{\Lambda}$  whenever  $u \to^m w$  and  $m \leq n$ .

**Proof.** A term t is said to be *semi-canonical* iff  $t = \mathbf{app}(u, v)$ , where  $v \in \mathcal{C}(\Psi)$  and u is either semi-canonical or is itself an element of  $\mathcal{C}(\Psi)$ . We now prove that if t is semi-canonical, there there are a natural number n and a canonical term u such that:

- $t \rightarrow^n u;$
- $|v|_{app} = |t|_{app} + m$  whenever  $t \to^m v$  and  $m \le n$ ;
- $\langle v \rangle_{\Lambda} = \langle t \rangle_{\Lambda}$  whenever  $t \to^m v$  and  $m \leq n$ .

We can proceed by induction on |t|. By definition t is always in the form app(w, d). We distinguish three cases:

- w is semi-canonical. Then, we get what we want by induction hypothesis.
- w is in  $\mathcal{C}(\Psi)$  and has the form  $\mathbf{c}_{x,M}(t_1,\ldots,t_m)$ . Then, n=0 and t is itself canonical.
- w is in  $\mathcal{C}(\Psi)$  and has the form  $\mathbf{capp}(e, f)$ . Then

$$t = \operatorname{app}(\operatorname{capp}(e, f), d) \to \operatorname{app}(\operatorname{app}(e, f), d)$$

We can apply the induction hypothesis to app(e, f) (since its length is strictly smaller than |t|).

We can now proceed as in Lemma 6, since whenever t rewrites to u by one of the ordinary rules, u is semi-canonical.

**Lemma 16** A canonical term  $t \in \mathcal{T}(\Psi)$  is in normal form iff  $\langle t \rangle_{\Lambda}$  is in normal form.

**Proof.** We first prove that any canonical normal form t can be written as  $\mathbf{c}_{x,M}(t_1,\ldots,t_n)$ , where  $t_1,\ldots,t_n \in \mathcal{C}(\Psi)$ . We proceed by induction on t:

- If  $t = \mathbf{c}_{x,M}(t_1, \ldots, t_n)$ , then the thesis holds.
- If  $t = \mathbf{app}(u, v)$ , then u is canonical and in normal form, hence in the form  $\mathbf{c}_{x,M}(t_1, \ldots, t_n)$  by induction hypothesis. As a consequence, t is not a normal form, which is a contraddiction.

We can now prove the statement of the lemma, by distinguishing two cases:

- If  $t = \mathbf{c}_{x,M}(t_1, \ldots, t_n)$ , where  $t_1, \ldots, t_n \in \mathcal{C}(\Psi)$ , then t is in normal form and  $\langle t \rangle_{\Lambda}$  is an abstraction, hence a normal form.
- If  $t = \mathbf{app}(u, v)$ , then t cannot be a normal form, since u is canonical and in normal form and, as a consequence, it can be written as  $\mathbf{c}_{x,M}(t_1, \ldots, t_n)$ .

This concludes the proof.

Observe that this property holds only if t is canonical: a non-canonical term can reduce to another one (canonical or not) even if the underlying  $\lambda$ -term is a normal form.

**Lemma 17** If  $M \to_h N$ , t is canonical and  $\langle t \rangle_{\Lambda} = M$ , then  $t \to_h u$ , where  $\langle u \rangle_{\Lambda} = N$  and  $|u|_{app} + 1 \ge |t|_{app}$ .

**Proof.** Similar to the one of Lemma 17.

The slight mismatch between call-by-name reduction in  $\Lambda$  and reduction in  $\Psi$  is anyway harmless globally: the total number of reduction step in  $\Psi$  is at most two times as large as the total number of call-by-name reduction steps in  $\Lambda$ .

**Theorem 6 (Term Reducibility)** Let  $M \in \Lambda$  be a closed term. The following two conditions are equivalent:

1.  $M \rightarrow_h^n N$  where N is in normal form;

2.  $[M]_{\Psi} \to^m t$  where  $\langle\!\langle t \rangle\!\rangle_{\Lambda} = N$  and t is in normal form. Moreover  $n \leq m \leq 2n$ .

**Proof.** Suppose  $M \to_h^n N$ , where N is in normal form. M is closed and, by Lemma 14,  $[M]_{\Psi}$  is canonical. By iterating over Lemma 15 and Lemma 17, we obtain the existence of a term t such that  $\langle t \rangle_{\Lambda} = u, t$  is in normal form and  $[M]_{\Psi} \to^m t$ , where  $m \ge n$  and

$$|t|_{\mathbf{app}} - |[M]_{\Psi}|_{\mathbf{app}} \ge (m-n) - n.$$

Since  $|t|_{app}$  (since t is in normal form),  $m \leq 2n$ . If  $[M]_{\Psi} \to^m t$  where  $\langle\!\langle t \rangle\!\rangle_{\Lambda} = N$  and t is in normal form, then by iterating over Lemma 15 we obtain that  $M \to_h^n N$  where  $n \leq m \leq 2n$  and N is in normal form.

 $\Xi$  is the graph rewrite system corresponding to  $\Psi$ , in the sense of Section 5. Exactly as for the call-by-value case, computing the normal form of (the graph representation of) any term takes time polynomial in the number of reduction steps to normal form:

**Theorem 7** There is a polynomial  $p : \mathbb{N}^2 \to \mathbb{N}$  such that for every  $\lambda$ -term M, the normal form of  $[[M]_{\Psi}]_{\Xi}$  can be computed in time at most  $p(|M|, Time_h(M))$ .

On the other hand, we cannot hope to *directly* reuse the results in Section 4 when proving the existence of an embedding of CRSs into weak call-by-name  $\lambda$ -calculus: the same  $\lambda$ -term can have distinct normal forms in the two cases. It is widely known, however, that a continuation-passing translation can be used to simulate call-by-value reduction by call-by-name reduction [13]. The only missing tale is about the relative performances: do terms obtained via the CPS translation reduce (in call-by-name) to their normal forms in a number of steps which is *comparable* to the number of (call-by-value) steps to normal form for the original terms? We conjecture the answer is "yes", but we still have to complete a formal proof.

## 7 Conclusions

We have shown that the most naïve cost models for weak call-by-value  $\lambda$ -calculus (each betareduction step has unitary cost) and orthogonal constructor term rewriting (each rule application has unitary cost) are linearly related. Since, in turn, this cost model for  $\lambda$ -calculus is polynomially related to the actual cost of reducing a  $\lambda$ -term on a Turing machine, the two machine models we considered are both *reasonable* machines, when endowed with their natural, intrinsic cost models (see also Gurevich's opus on Abstract State Machine simulation "at the same level of abstraction", e.g. [7]). This strong (the embeddings we consider are compositional), complexity-preserving equivalence between a first-order and a higher-order model is the most important technical result of the paper.

Ongoing and future work includes the investigation of how much of this simulation could be recovered either in a typed setting (see [15] for some of the difficulties), or in the case of  $\lambda$ calculus with strong reduction, where we reduce under an abstraction. Novel techniques have to be developed, since the analysis we performed in the present paper cannot be easily extended to these cases.

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