# The communication complexity of non-signaling distributions 

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#### Abstract

We study a model of communication complexity that encompasses many well-studied problems, including classical and quantum communication complexity, the complexity of simulating distributions arising from bipartite measurements of shared quantum states, and XOR games. In this model, Alice gets an input $x$, Bob gets an input $y$, and their goal is to each produce an output $a, b$ distributed according to some pre-specified joint distribution $p(a, b \mid x, y)$. Our results apply to any non-signaling distribution, that is, those where Alice's marginal distribution does not depend on Bob's input, and vice versa.

By taking a geometric view of the non-signaling distributions, we introduce a simple new technique based on affine combinations of lower-complexity distributions, and we give the first general technique to apply to all these settings, with elementary proofs and very intuitive interpretations. Specifically, we introduce two complexity measures, one which gives lower bounds on classical communication, and one for quantum communication. These measures can be expressed as convex optimization problems. We show that the dual formulations have a striking interpretation, since they coincide with maximum violations of Bell and Tsirelson inequalities. The dual expressions are closely related to the winning probability of XOR games. Despite their apparent simplicity, these lower bounds subsume many known communication complexity lower bound methods, most notably the recent lower bounds of Linial and Shraibman for the special case of Boolean functions.

We show that as in the case of Boolean functions, the gap between the quantum and classical lower bounds is at most linear in the size of the support of the distribution, and does not depend on the size of the inputs. This translates into a bound on the gap between maximal Bell and Tsirelson inequality violations, which was previously known only for the case of distributions with Boolean outcomes and uniform marginals. It also allows us to show that for some distributions, information theoretic methods are necessary to prove strong lower bounds.

Finally, we give an exponential upper bound on quantum and classical communication complexity in the simultaneous messages model, for any non-signaling distribution. One consequence of this is a simple proof that any quantum distribution can be approximated with a constant number of bits of communication.


## 1 Introduction

Communication complexity of Boolean functions has a long and rich past, stemming from the paper of Yao in 1979 [Yao79], whose motivation was to study the area of VLSI circuits. In the years that followed, tremendous progress has been made in developing a rich array of lower bound techniques for various models of communication complexity (see e.g. [KN97]).

From the physics side, the question of studying how much communication is needed to simulate distributions arising from physical phenomena, such as measuring bipartite quantum states, was posed in 1992 by Maudlin, a philosopher of science, who wanted to quantify the non-locality inherent to these systems [Mau92]. Maudlin, and the authors who followed [BCT99, Ste00, TB03, CGMP05, DLR07] (some independently of his work, and of each other) progressively improved upper bounds on simulating correlations of the 2 qubit singlet state. In a recent breakthrough, Regev and Toner [RT10] proved that two bits of communication suffice to simulate the correlations arising from two-outcome measurements of arbitrary-dimension bipartite quantum states. In the more general case of non-binary

[^0]outcomes, Shi and Zhu gave a protocol to approximate quantum distributions within constant error, using constant communication [SZ08]. No non-trivial lower bounds are known for this problem.

In this paper, we consider the more general framework of simulating non-signaling distributions. These are distributions of the form $p(a, b \mid x, y)$, where Alice gets input $x$ and produces an output $a$, and Bob gets input $y$ and outputs $b$. The non-signaling condition is a fundamental property of bipartite physical systems, which states that the players gain no information on the other player's input. In particular, distributions arising from quantum measurements on shared bipartite states are non-signaling, and Boolean functions may be reduced to extremal non-signaling distributions with Boolean outcomes and uniform marginals.

Outside of the realm of Boolean functions, a very limited number of tools are available to analyze the communication complexity of distributed tasks, especially for quantum distributions with non-uniform marginals. In such cases, the distributions live in a larger-dimensional space and cannot be cast as communication matrices, so standard techniques do not apply. The structure of non-signaling distributions has been the object of much study in the quantum information community, yet outside the case of distributions with Boolean inputs or outcomes [JM05, BP05], or with uniform marginal distributions, much remains to be understood.

We introduce a new method to study all non-signaling distributions, including the case of non-Boolean outcomes and non-uniform marginals. Our starting point is the observation that non-signaling distributions coincide with affine (instead of convex) combinations of distributions that do not require any communication, called local distributions. With this elegant geometric formulation in mind, we show how to relate communication to non-locality, where we measure non-locality by how far, in terms of its "best" affine representation, a distribution is from the convex set of local distributions. Although they are formulated, and proven, in quite a different way, our lower bounds turn out to subsume Linial and Shraibman's nuclear and factorization norm lower bounds [LS09], in the restricted case of Boolean functions. Similarly, our upper bounds extend the upper bounds of Shi and Zhu for approximating quantum distributions [SZ08] to all non-signaling distributions (in particular distributions obtained by protocols using entanglement and quantum communication).

Our complexity measures can be expressed as convex optimization problems. We may consider dual expressions, and these turn out to correspond precisely to maximal Bell inequality violations in the case of classical communication, and Tsirelson inequality violations for quantum communication. This confirms the long-held physics intuition that large Bell inequality violations should lead to large lower bounds on communication complexity.

We also show that there cannot be a large gap between the classical and quantum expressions. This was previously known only in the case of distributions with Boolean outcomes and uniform marginals, and followed by Tsirelson's theorem and Grothendieck's inequality, neither of which are known to extend beyond this special case. This also shows that our method, as was already the case for Linial and Shraibman's bounds, cannot hope to prove large gaps between classical and quantum communication complexity. While this is a negative result, it also sheds some light on the relationship between the Linial and Shraibman family of lower bound techniques, and the information theoretic methods, such as the recent subdistribution bound [JKN08], one of the few lower bound techniques not known to follow from Linial and Shraibman. We give an example of a problem [BCT99] for which rectangle size gives an exponentially better lower bound than our method.
Summary of results The paper is organized as follows. In Section 2 we give the required definitions and models of communication complexity and characterizations of the classes of distributions we consider.

In Section 3, we prove our lower bound on classical and quantum communication (Theorem 3), and show that it coincides with Linial and Shraibman's method in the special case of Boolean functions (Theorems 4 and 5). Our lower bounds are convex optimization programs (linear programs in the classical case), and in Section 4 we show that the dual programs have a natural interpretation in quantum information, as they coincide with Bell (or Tsirelson) inequality violations (Theorem (6). We give a dual expression which also has a natural interpretation, as the maximum winning probability of an associated XOR game (Corollary 3). The primal form turns out to be the multiplicative inverse of the maximum winning probability of the associated XOR game, where all inputs have the same winning probability.

In Section 5e compare the two methods and show that the quantum and classical lower bound expressions can differ by at most a factor that is linear in the number of outcomes (Theorem 7). When viewed as maximum Bell inequality violations, our results imply that if Alice and Bob each have $k$ possible outcomes, then the largest Bell inequality violation for quantum distributions is at most $O\left(k^{2}\right)$.

Finally, in Section 6, we give upper bounds on simultaneous messages complexity in terms of our lower bound expression (Theorem (8). We use fingerprinting methods [BCWdW01, Yao03, SZ08, GKd06] to give very simple proofs that classical communication with shared randomness, or quantum communication with shared entanglement, can be simulated in the simultaneous messages model, with exponential blowup in communication, and in particular that any quantum distribution can be approximated with constant communication.

Related work The use of affine combinations for non-signaling distributions has roots in the quantum logic community, where quantum non-locality has been studied within the setting of more general probability theories [FR81, RF81, KRF87, Wil92]. Until recently, this line of work was largely unknown in the quantum information theory community [Bar07, BBLW07].

The structure of the non-signaling polytope has been the object of much study. A complete characterization of the vertices has been obtained in some, but not all cases: for two players, the case of binary inputs [BLM ${ }^{+}$05], and the case of binary outputs [BP05, JM05] are known, and for $n$ players, the case of Boolean inputs and outputs is known [BP05].

The work on simulating quantum distributions has focused mainly on providing upper bounds, and most results apply to simulating the correlations only. In particular, Toner and Bacon show that projective measurements on a maximally entangled qubit pair may be simulated using one bit of communication [TB03], and Regev and Toner extend this result by showing that the correlations arising from binary measurements on any entangled state may be simulated using two bits of communication only [RT10]. A few results address the simulation of quantum distributions with non-uniform marginals. Bacon and Toner give an upper bound of 2 bits for non-maximally entangled qubit pairs [TB03]. Shi and Zhu [SZ08] show a constant upper bound for approximating any quantum distribution (including the marginals) to within a constant.

Pironio gives a general lower bound technique based on Bell-like inequalities [Pir03]. There are a few ad hoc lower bounds on simulating quantum distributions, including a linear lower bound for a distribution based on Deutsch-Jozsa's problem [BCT99], and a recent lower bound of Gavinsky [Gav09].

The $\gamma_{2}$ method was first introduced as a measure of the complexity of matrices [LMSS07]. It was shown to be a lower bound on communication complexity [LS09], and to generalize many previously known methods. Lee et al. use it to establish direct product theorems and relate the dual norm of $\gamma_{2}$ to the value of XOR games [LSŠ08]. Lee and Shraibman [LS08] use a multidimensional generalization of a related quantity $\mu$ (where the norm-1 ball consists of cylinder intersections) to prove a lower bound in the multiparty number-on-the-forehead-model, for the disjointness function.

Since the first publication of this work, several extensions and improvements have been made to the upper bounds on Bell inequality violations of Section 5, and related lower bounds on the possible violations have been proved [JPPG ${ }^{+} 10 \mathrm{~b}, \mathrm{JPPG}^{+}$10a, JP10, BRSdW10].

## 2 Preliminaries

In this paper, we extend the framework of communication complexity to non-signaling distributions. This framework encompasses the standard models of communication complexity of Boolean functions but also total and partial nonBoolean functions and relations, as well as distributions arising from the measurements of bipartite quantum states. Most results we present also extend to the multipartite setting.

### 2.1 Definitions of the distribution classes

Throughout this article, we consider bipartite conditional distributions $p(a, b \mid x, y)$ where $x \in \mathcal{X}, y \in \mathcal{Y}$ are the inputs of the players, and they are required to each produce an outcome $a \in \mathcal{A}, b \in \mathcal{B}$, distributed according to $p(a, b \mid x, y)$. We will focus on so-called non-signaling distributions, where the marginal distribution of a given player's outcome does not depend on the other player's input. These include as a special case different classes of distributions, which we define in the following subsections.

### 2.1.1 Local distributions

In the quantum information literature, the distributions that can be simulated with shared randomness and no communication (also called a local hidden variable model) are called local distributions.

Definition 1. Local deterministic distributions are of the form $p(a, b \mid x, y)=\delta_{a=\lambda_{A}(x)} \cdot \delta_{b=\lambda_{B}(y)}$ where $\lambda_{A}: \mathcal{X} \rightarrow \mathcal{A}$ and $\lambda_{B}: \mathcal{Y} \rightarrow \mathcal{B}$, and $\delta$ is the Kronecker delta. A distribution is local if it can be written as a convex combination of local deterministic distributions.

We index by $\Lambda$ the set of local deterministic distributions $\left\{\mathbf{p}^{\lambda}\right\}_{\lambda \in \Lambda}$ and denote by $\mathcal{L}$ the set of local distributions.

### 2.1.2 Quantum distributions

Of particular interest in the study of quantum non-locality are the distributions arising from measuring bipartite quantum states. We will use the following definition:

Definition 2. A distribution $\mathbf{p}$ is quantum if there exists a bipartite quantum state $|\psi\rangle$ in a Hilbert space $\mathcal{H}=\mathcal{H}_{A} \otimes \mathcal{H}_{B}$ and measurement operators $\left\{E_{a}(x): a \in \mathcal{A}, x \in \mathcal{X}\right\}$ acting on $\mathcal{H}_{A}$ and $\left\{E_{b}(y): b \in \mathcal{B}, y \in \mathcal{Y}\right\}$ acting on $\mathcal{H}_{B}$, such that $p(a, b \mid x, y)=\langle\psi| E_{a}(x) \otimes E_{b}(y)|\psi\rangle$, with the measurement operators satisfying

1. $E_{a}(x)^{\dagger}=E_{a}(x)$ and $E_{b}(y)^{\dagger}=E_{b}(y)$,
2. $E_{a}(x) \cdot E_{a^{\prime}}(x)=\delta_{a a^{\prime}} E_{a}(x)$ and $E_{b}(y) \cdot E_{b^{\prime}}(y)=\delta_{b b^{\prime}} E_{b}(y)$,
3. $\sum_{a} E_{a}(x)=\mathbb{1}_{A}$ and $\sum_{b} E_{b}(x)=\mathbb{1}_{B}$, where $\mathbb{1}_{A}$ and $\mathbb{1}_{B}$ are the identity operators on $\mathcal{H}_{A}$ and $\mathcal{H}_{B}$, respectively.

We denote by $\mathcal{Q}$ the set of all quantum distributions.

### 2.1.3 Non-signaling distributions

Non-signaling, a fundamental postulate of physics, states that no observation on part of a system can instantaneously affect a remote part of the system, or similarly, that no signal can travel instantaneously. For a bipartite probability distribution $p(a, b \mid x, y)$ describing observations on two distant physical systems, this means that no choice of measurement $y$ on Bob's side can affect the marginal distribution of the observed outcome $a$ on Alice's side, and vice versa. Mathematically, non-signaling (also called causality) is defined as follows.
Definition 3 (Non-signaling distributions). A bipartite, conditional distribution $\mathbf{p}$ is non-signaling if

$$
\begin{array}{lr}
\forall a, x, y, y^{\prime}, & \sum_{b} p(a, b \mid x, y)=\sum_{b} p\left(a, b \mid x, y^{\prime}\right) \\
\forall b, x, x^{\prime}, y, & \sum_{a} p(a, b \mid x, y)=\sum_{a} p\left(a, b \mid x^{\prime}, y\right)
\end{array}
$$

For any non-signaling distribution, the marginal distribution on Alice's output $p(a \mid x, y)=\sum_{b} p(a, b \mid x, y)$ does not depend on $y$, so we write $p(a \mid x)$, and similarly $p(b \mid y)$ for the marginal distribution on Bob's output. We denote by $\mathcal{C}$ the set of all non-signaling distributions.

In the case of binary outcomes, that is, $\mathcal{A}=\mathcal{B}=\{ \pm 1\}$, it is known that a non-signaling distribution is uniquely determined by the (expected) correlations, defined as $C(x, y)=E(a \cdot b \mid x, y)$, and the (expected) marginals, defined as $M_{A}(x)=E(a \mid x), M_{B}(y)=E(b \mid y)$.

Proposition 1. For any functions $C: \mathcal{X} \times \mathcal{Y} \rightarrow[-1,1], M_{A}: \mathcal{X} \rightarrow[-1,1], M_{B}: \mathcal{Y} \rightarrow[-1,1]$, satisfying $1+a \cdot b C(x, y)+a M_{A}(x)+b M_{B}(y) \geq 0 \forall(x, y) \in \mathcal{X} \times \mathcal{Y}$ and $a, b \in\{ \pm 1\}$, there is a unique non-signaling distribution $\mathbf{p}$ such that $\forall x, y, E(a \cdot b \mid x, y)=C(x, y)$ and $E(a \mid x)=M_{A}(x)$ and $E(b \mid y)=M_{B}(y)$, where $a, b$ are distributed according to $\mathbf{p}$.

Proof. Fix $x, y . C, M_{A}, M_{B}$ are obtained from $\mathbf{p}$ by the following full rank system of equations.

$$
\left(\begin{array}{rrrr}
1 & -1 & -1 & 1 \\
1 & 1 & -1 & -1 \\
1 & -1 & 1 & -1 \\
1 & 1 & 1 & 1
\end{array}\right)\left(\begin{array}{c}
p(+1,+1 \mid x, y) \\
p(+1,-1 \mid x, y) \\
p(-1,+1 \mid x, y) \\
p(-1,-1 \mid x, y)
\end{array}\right)=\left(\begin{array}{c}
C(x, y) \\
M_{A}(x) \\
M_{B}(y) \\
1
\end{array}\right)
$$

Computing the inverse yields $p(a, b \mid x, y)=\frac{1}{4}\left(1+a \cdot b C(x, y)+a M_{A}(x)+b M_{B}(y)\right)$.
We will write $\mathbf{p}=\left(C, M_{A}, M_{B}\right)$ and use both notations interchangeably when considering distributions over binary outcomes. We also denote by $\mathcal{C}_{0}$ the set of non-signaling distributions with uniform marginals, that is, $\mathbf{p}=$ ( $C, 0,0$ ), and write $C \in \mathcal{C}_{0}$, omitting the marginals when there is no ambiguity.

Since local and quantum distributions are non-signaling, we use similar notation for local and quantum distributions where binary outcomes are concerned. In the case of local distributions, since the vertices of the polytope are deterministic strategies, correlations and marginals can be written using $\pm 1$ vectors. Let conv $(A)$ denote the convex hull of $A$.

Proposition 2. $\mathcal{L}=\operatorname{conv}\left(\left\{\left(u^{T} v, u, v\right): u \in\{ \pm 1\}^{\mathcal{X}}, v \in\{ \pm 1\}^{\mathcal{Y}}\right\}\right)$.
We also denote by $\mathcal{L}_{0}$ the set of local correlations over binary outcomes with uniform marginals and we let $\mathcal{Q}_{0}$ be the set of all quantum correlations.

### 2.1.4 Boolean functions

There is a natural way to map a Boolean function $f: \mathcal{X} \times \mathcal{Y} \rightarrow\{ \pm 1\}$ to a non-signaling distribution $p_{f}(a, b \mid x, y)$ over binary outcomes $a, b \in\{ \pm 1\}$, as follows:

Definition 4. For a function $f: \mathcal{X} \times \mathcal{Y} \rightarrow\{-1,1\}$, denote $\mathbf{p}_{f}$ the distribution defined by $p_{f}(a, b \mid x, y)=\frac{1}{2}$ if $f(x, y)=a \cdot b$ and 0 otherwise. Equivalently, $\mathbf{p}_{f}=\left(C_{f}, 0,0\right)$ where $C_{f}(x, y)=f(x, y)$.

By stipulating that the product of the players' outputs equals the value of the function, we see that the distribution has the same communication complexity as the function (up to an additional bit of communication, for Bob to output $f(x, y)$ ). As we shall see in Section 2.2.1 it so happens that the distributions associated to Boolean functions are extremal points of the non-signaling polytope.

In the case of randomized communication complexity, a protocol that simulates a Boolean function with error probability $\epsilon$ corresponds to simulating correlations $C^{\prime}$ scaled down by a factor at most $1-2 \epsilon$, that is, $\forall x, y, \operatorname{sgn}\left(C^{\prime}(x, y)\right)=$ $C_{f}(x, y)$ and $\left|C^{\prime}(x, y)\right| \geq 1-2 \epsilon$. While we will not consider these cases in full detail, non-Boolean functions, partial functions and some classes of relations may be handled in a similar fashion, hence our techniques can be used to show lower bounds in these settings as well.

### 2.2 Characterizations and relations among the distribution classes

### 2.2.1 Non-signaling distributions

The quantum information literature reveals a great deal of insight into the structure of the classical, quantum, and non-signaling distributions. It is well known that $\mathcal{L}$ and $\mathcal{C}$ are polytopes. While the extremal points of $\mathcal{L}$ are simply the local deterministic distributions, the non-signaling polytope $\mathcal{C}$ has a more complex structure [JM05, [BP05]. In the case of $\mathcal{C}_{0}$, it is the convex hull of the distributions obtained from Boolean functions.

Proposition 3. $\mathcal{C}_{0}=\operatorname{conv}\left(\left\{\left(C_{f}, 0,0\right): C_{f} \in\{ \pm 1\}^{\mathcal{X}} \times \mathcal{Y}\right\}\right)$.
We show that $\mathcal{C}$ is the affine hull of the local polytope (restricted to the positive orthant since all probabilities $p(a, b \mid x, y)$ must be positive). We give a simple proof for the case of binary outcomes but this carries over to the general case. This was shown independently of us, on a few occasions in different communities [RF81, FR81, KRF87, Wil92, Bar07].

Theorem 1. $\mathcal{C}=\operatorname{aff}^{+}(\mathcal{L})$, where $\operatorname{aff}^{+}(\mathcal{L})$ is the restriction to the positive orthant of the affine hull of $\mathcal{L}$, and $\operatorname{dim} \mathcal{C}=$ $\operatorname{dim} \mathcal{L}=|\mathcal{X}| \times|\mathcal{Y}|+|\mathcal{X}|+|\mathcal{Y}|$.

Proof. We show that $\operatorname{aff}(\mathcal{C})=\operatorname{aff}(\mathcal{L})$. The theorem then follows by restricting to the positive orthant, and using the fact that $\mathcal{C}=\operatorname{aff}^{+}(\mathcal{C})$.
$[\operatorname{aff}(\mathcal{L}) \subseteq \operatorname{aff}(\mathcal{C})]$ Since any local distribution satisfies the (linear) non-signaling constraints in Def. 1 , this is also true for any affine combination of local distributions.
$[\operatorname{aff}(\mathcal{C}) \subseteq \operatorname{aff}(\mathcal{L})]$ For any $(\sigma, \pi) \in \mathcal{X} \times \mathcal{Y}$, we define the distribution $\mathbf{p}_{\sigma \pi}=\left(C_{\sigma \pi}, u_{\sigma \pi}, v_{\sigma \pi}\right)$ with correlations $C_{\sigma \pi}(x, y)=\delta_{x=\sigma} \delta_{y=\pi}$ and marginals $u_{\sigma \pi}(x)=0, v_{\sigma \pi}(y)=0$. Similarly, we define for any $\sigma \in \mathcal{X}$ the distribution $\mathbf{p}_{\sigma \cdot}=\left(C_{\sigma \cdot}, u_{\sigma \cdot}, v_{\sigma .}\right)$ with $C_{\sigma \cdot}(x, y)=0, u_{\sigma \cdot}(x)=\delta_{x=\sigma}, v_{\sigma \cdot}(y)=0$, and for any $\pi \in \mathcal{Y}$ the distribution $\mathbf{p}_{\cdot \pi}=$ $\left(C_{\cdot \pi}, u \cdot \pi, v_{\cdot \pi}\right)$ with $C_{\cdot \pi}(x, y)=0, u \cdot \pi(x)=0, v_{\cdot \pi}(y)=\delta_{y=\pi}$. It is straightforward to check that these $|\mathcal{X}| \times|\mathcal{Y}|+$ $|\mathcal{X}|+|\mathcal{Y}|$ distributions are local, and that they constitute a basis for the vector space embedding aff $(\mathcal{C})$, which consists of vectors of the form $(C, u, v)$.

This implies that while local distributions are convex combinations of local deterministic distributions $\mathbf{p}^{\lambda} \in \Lambda$, non-signaling distributions are affine combinations of these distributions.

Corollary 1 (Affine model). A distribution $\mathbf{p} \in \mathcal{C}$ if and only if $\exists q_{\lambda} \in \mathbb{R}$ with $\mathbf{p}=\sum_{\lambda \in \Lambda} q_{\lambda} \mathbf{p}^{\lambda}$.
Note that since $\mathbf{p}$ is a distribution, this implies $\sum_{\lambda \in \Lambda} q_{\lambda}=1$. Since weights in an affine combination may be negative, but still sum up to one, this may be interpreted as a quasi-mixture of local distributions, some distributions being used with possibly "negative probability". Surprisingly this is not a new notion; see for example Groenewold [Gro85] who gave an affine model for quantum distributions; or a discussion of "negative probability" by Feynman [Fey86].

### 2.2.2 Quantum distributions

The following fundamental theorem of Tsirelson relates measurements on quantum states to the inner product of vectors.

Theorem 2 ([Tsi85]). Let $\mathbb{S}_{n}$ be the set of unit vectors in $\mathbb{R}^{n}$, and $\mathcal{H}^{d}$ be a d-dimensional Hilbert space.

1. If $\left(C, M_{A}, M_{B}\right) \in \mathcal{Q}$ is a probability distribution obtained by performing binary measurements on a quantum state $|\psi\rangle \in \mathcal{H}^{d} \otimes \mathcal{H}^{d}$, then there exists vectors $\vec{a}(x), \vec{b}(y) \in \mathbb{S}_{2 d^{2}}$ such that $C(x, y)=\vec{a}(x) \cdot \vec{b}(y)$.
2. If $\vec{a}(x), \vec{b}(y)$ are unit vectors in $\mathbb{S}_{n}$, then there exists a probability distribution $(C, 0,0) \in \mathcal{Q}$ obtained by performing binary measurements on a maximally entangled state $|\psi\rangle \in \mathcal{H}^{\lfloor\lfloor n / 2\rfloor} \otimes \mathcal{H}^{2^{\lfloor n / 2\rfloor}}$ such that $C(x, y)=$ $\vec{a}(x) \cdot \vec{b}(y)$.
Corollary 2. $\mathcal{Q}_{0}=\{C: C(x, y)=\vec{a}(x) \cdot \vec{b}(y),\|\vec{a}(x)\|=\|\vec{b}(y)\|=1 \forall x, y\}$.
Clearly, $\mathcal{L} \subseteq \mathcal{Q} \subseteq \mathcal{C}$. As first noted by Tsirelson, Grothendieck's inequality [Gro53] implies the following statement.

Proposition 4 ([Tsi85]). $\mathcal{L}_{0} \subseteq \mathcal{Q}_{0} \subseteq K_{G} \mathcal{L}_{0}$, where $K_{G}$ is Grothendieck's constant.

### 2.3 Models of communication complexity

We consider the following model of communication complexity of non-signaling distributions $\mathbf{p}$. Alice gets input $x$, Bob gets input $y$, and after exchanging bits or qubits, Alice has to output $a$ and Bob $b$ so that the joint distribution is $p(a, b \mid x, y) . R_{0}(\mathbf{p})$ denotes the communication complexity of simulating $\mathbf{p}$ exactly, using private randomness and classical communication. $Q_{0}(\mathbf{p})$ denotes the communication complexity of simulating $\mathbf{p}$ exactly, using quantum communication. We use superscripts "pub" and "ent" in the case where the players share random bits or quantum entanglement. For $R_{\epsilon}(\mathbf{p})$, we are only required to simulate some distribution $\mathbf{p}^{\prime}$ such that $\delta\left(\mathbf{p}, \mathbf{p}^{\prime}\right) \leq \epsilon$, where $\delta\left(\mathbf{p}, \mathbf{p}^{\prime}\right)=\max \left\{\left|p(\mathcal{E} \mid x, y)-p^{\prime}(\mathcal{E} \mid x, y)\right|: x, y \in \mathcal{X} \times \mathcal{Y}, \mathcal{E} \subseteq \mathcal{A} \times \mathcal{B}\right\}$ is the total variation distance (or statistical distance) between two distributions.

For distributions with binary outcomes, we write $R_{\epsilon}\left(C, M_{A}, M_{B}\right)$ and $Q_{\epsilon}\left(C, M_{A}, M_{B}\right)$. In the case of Boolean functions, $R_{\epsilon}(C)=R_{\epsilon}(C, 0,0)$ corresponds to the usual notion of computing $f$ with probability at least $1-\epsilon$, where $C$ is the $\pm 1$ communication matrix of $f$. From the point of view of communication, distributions with uniform marginals are the easiest to simulate. Suppose we have a protocol that simulates correlations $C$ with arbitrary marginals. By using just an additional shared random bit, both players can flip their outcome whenever the shared random bit is 1 . Since each players' marginal outcome is now an even coin flip, this protocol simulates the distribution $(C, 0,0)$.

Proposition 5. For any Boolean non-signaling distribution $\left(C, M_{A}, M_{B}\right)$, we have $R_{\epsilon}^{\mathrm{pub}}(C, 0,0) \leq R_{\epsilon}^{\mathrm{pub}}\left(C, M_{A}, M_{B}\right)$ and $Q_{\epsilon}^{\mathrm{ent}}(C, 0,0) \leq Q_{\epsilon}^{\mathrm{ent}}\left(C, M_{A}, M_{B}\right)$.

## 3 Lower bounds for non-signaling distributions

In this section we prove our main theorem, a lower bound on quantum and classical communication complexity for non-signaling distributions, based on their affine representations.

Let us define the following quantities, which as we will see may be considered as extensions of the $\nu$ and $\gamma_{2}$ quantities of [LS09] (defined in Section 3.3) to distributions.

Definition 5. - $\tilde{\nu}(\mathbf{p})=\min \left\{\sum_{i}\left|q_{i}\right|: \exists \mathbf{p}_{i} \in \mathcal{L}, q_{i} \in \mathbb{R}, \mathbf{p}=\sum_{i} q_{i} \mathbf{p}_{i}\right\}$,

- $\tilde{\gamma}_{2}(\mathbf{p})=\min \left\{\sum_{i}\left|q_{i}\right|: \exists \mathbf{p}_{i} \in \mathcal{Q}, q_{i} \in \mathbb{R}, \mathbf{p}=\sum_{i} q_{i} \mathbf{p}_{i}\right\}$,
- $\tilde{\nu}^{\epsilon}(\mathbf{p})=\min \left\{\tilde{\nu}\left(\mathbf{p}^{\prime}\right): \delta\left(\mathbf{p}, \mathbf{p}^{\prime}\right) \leq \epsilon\right\}$,
- $\tilde{\gamma}_{2}^{\epsilon}(\mathbf{p})=\min \left\{\tilde{\gamma}_{2}\left(\mathbf{p}^{\prime}\right): \delta\left(\mathbf{p}, \mathbf{p}^{\prime}\right) \leq \epsilon\right\}$.

Notice that $\sum_{i} q_{i} \mathbf{p}_{i}=\mathbf{p}$ implies in particular $\sum_{i} q_{i}=1$. The quantities $\tilde{\nu}(\mathbf{p})$ and $\tilde{\gamma}_{2}(\mathbf{p})$ show how well $\mathbf{p}$ may be represented as an affine combination of local or quantum distributions, a good affine combination being one where the sum of absolute values of coefficients $q_{i}$ is as low as possible. Figure 1 represents the decomposition of a distribution into an affine combination of local distributions. For a local distribution, we may take positive coefficients $q_{i}$, and therefore obtain the minimum possible value $\tilde{\nu}(\mathbf{p})=1$, and similarly for quantum distributions, so that

Lemma 1. $\mathbf{p} \in \mathcal{L} \Longleftrightarrow \tilde{\nu}(\mathbf{p})=1$, and $\mathbf{p} \in \mathcal{Q} \Longleftrightarrow \tilde{\gamma}_{2}(\mathbf{p})=1$.
In other words, the set of local distributions $\mathcal{L}$ form the unit sphere of $\tilde{\nu}$, and similarly the set of quantum distributions $\mathcal{Q}$ form the unit sphere of $\tilde{\gamma}_{2}$. In the binary case, observe that by Proposition5, we have $\tilde{\gamma}_{2}(C) \leq \tilde{\gamma}_{2}(C, u, v)$ and $\tilde{\nu}(C) \leq \tilde{\nu}(C, u, v)$. By Proposition $4 \tilde{\gamma}_{2}(C) \leq \tilde{\nu}(C) \leq K_{G} \tilde{\gamma}_{2}(C)$. Similar properties hold for the approximate versions $\tilde{\nu}^{\epsilon}(C)$ and $\tilde{\gamma}_{2}^{\epsilon}(C)$.

Our main theorem gives a lower bound on communication complexity in terms of the quantities $\tilde{\nu}$ and $\tilde{\gamma}_{2}$.
Theorem 3. For any non-signaling distribution $\mathbf{p}$ and correlation matrix $C$,

1. $R_{0}^{\text {pub }}(\mathbf{p}) \geq \log (\tilde{\nu}(\mathbf{p}))-1$, and $R_{\epsilon}^{\text {pub }}(\mathbf{p}) \geq \log \left(\tilde{\nu}^{\epsilon}(\mathbf{p})\right)-1$.
2. $Q_{0}^{\text {ent }}(\mathbf{p}) \geq \frac{1}{2} \log \left(\tilde{\gamma}_{2}(\mathbf{p})\right)-1$, and $Q_{\epsilon}^{\text {ent }}(\mathbf{p}) \geq \frac{1}{2} \log \left(\tilde{\gamma}_{2}^{\epsilon}(\mathbf{p})\right)-1$.
3. $Q_{0}^{\mathrm{ent}}(C) \geq \log \left(\tilde{\gamma}_{2}(C)\right)$, and $Q_{\epsilon}^{\mathrm{ent}}(C) \geq \log \left(\tilde{\gamma}_{2}^{\epsilon}(C)\right)$.

The proof, minus the details, goes as follows. Assume that there is a $t$ bit protocol for $\mathbf{p}$. We derive a noisy, local distribution from $\mathbf{p}$ as follows (Lemma2). Simulate the protocol, but instead of communicating, guess a transcript. If both players agree that this was the correct transcript, then they output according to $\mathbf{p}$. This occurs with probability $2^{-t}$. Otherwise, output something random. The resulting distribution is $p^{\prime}=2^{-t} \mathbf{p}+\left(1-2^{-t}\right) \mathbf{q}$ where $\mathbf{q}$ is some random noise. But $\mathbf{p}^{\prime}$ and $\mathbf{q}$ are local, so this gives an affine representation of $\mathbf{p}=2^{t} \mathbf{p}^{\prime}-2^{t}\left(1-2^{-t}\right) \mathbf{q}$, showing that $\tilde{\nu}(\mathbf{p}) \leq 2^{t+1}-1$. The rest of this section is devoted to the details. The only complication arises from handling arbitrary marginal distributions and setting up the distribution they should output from when they disagree with the random transcript. However, the proof is straightforward, as above, when the marginals are uniform, which is the case for Boolean functions.


Figure 1: $\mathbf{p}$ is an affine combination of $\mathbf{p}^{+}$and $\mathbf{p}^{-}$

### 3.1 Producing a noisy local distribution from a communication protocol

We first show that if a distribution $\mathbf{p}$ may be simulated with $t$ bits of communication (or $q$ qubits of quantum communication), then there is a noisy version of this distribution that is local (or quantum).

Lemma 2. Let $\mathbf{p}$ be a non-signaling distribution over $\mathcal{A} \times \mathcal{B}$ with input set $\mathcal{X} \times \mathcal{Y}$.

1. Assume that $R_{0}^{\mathrm{pub}}(\mathbf{p}) \leq t$, then there exist two marginal distributions $p_{A}(a \mid x)$ and $p_{B}(b \mid y)$ such that the distribution $p_{l}(a, b \mid x, y)=\frac{1}{2^{t}} p(a, b \mid x, y)+\left(1-\frac{1}{2^{t}}\right) p_{A}(a \mid x) p_{B}(b \mid y)$ is local.
2. Assume that $Q_{0}^{\mathrm{ent}}(\mathbf{p}) \leq q$, then there exist two marginal distributions $p_{A}(a \mid x)$ and $p_{B}(b \mid y)$ such that the distribution $p_{l}(a, b \mid x, y)=\frac{1}{2^{2 q}} p(a, b \mid x, y)+\left(1-\frac{1}{2^{2 q}}\right) p_{A}(a \mid x) p_{B}(b \mid y)$ is quantum.
3. Assume that $\mathbf{p}=(C, 0,0)$ and $Q_{0}^{\mathrm{ent}}(C) \leq q$, then $C / 2^{q} \in \mathcal{Q}_{0}$.

Proof. We assume that the length of the transcript is exactly $t$ bits for each execution of the protocol, adding dummy bits if necessary. We now fix some notations. In the original protocol, the players pick a random string $\lambda$ and exchange some communication whose transcript is denoted $T(x, y, \lambda)$. Alice then outputs some value $a$ according to a probability distribution $p_{P}(a \mid x, \lambda, T)$. Similarly, Bob outputs some value $b$ according to a probability distribution $p_{P}(b \mid y, \lambda, T)$.

From Alice's point of view, on input $x$ and shared randomness $\lambda$, only a subset of the set of all $t$-bit transcripts can be produced: the transcripts $S \in\{0,1\}^{t}$ for which there exists a $y$ such that $S=T(x, y, \lambda)$. We will call these transcripts the set of valid transcripts for $(x, \lambda)$. The set of valid transcripts for Bob is defined similarly. We denote these sets respectively $U_{x, \lambda}$ and $V_{y, \lambda}$.

We now define a local protocol for the distribution $p_{l}(a, b \mid x, y)$ :

- As in the original protocol, Alice and Bob initially share some random string $\lambda$.
- Using additional shared randomness, Alice and Bob choose a transcript $T$ uniformly at random in $\{0,1\}^{t}$.
- If $T$ is a valid transcript for $(x, \lambda)$, she outputs $a$ according to the distribution $p_{P}(a \mid x, \lambda, T)$. If it is not, Alice outputs $a$ according to a distribution $p_{A}(a \mid x)$ which we will define later.
- Bob does the same. We will also define the distribution $p_{B}(b \mid y)$ later.

Let $\mu$ be the distribution over the randomness and the $t$-bit strings in the local protocol. By definition, the distribution produced by this protocol is

$$
\begin{aligned}
p_{l}(a, b \mid x, y) & =\sum_{\lambda} \mu(\lambda)\left[\sum_{T \in U_{x, \lambda} \cap V_{y, \lambda}} \mu(T) p_{P}(a \mid x, \lambda, T) p_{P}(b \mid y, \lambda, T)+p_{B}(b \mid y) \sum_{T \in U_{x, \lambda} \cap \bar{V}_{y, \lambda}} \mu(T) p_{P}(a \mid x, \lambda, T)\right. \\
& \left.+p_{A}(a \mid x) \sum_{T \in \bar{U}_{x, \lambda} \cap V_{y, \lambda}} \mu(T) p_{P}(b \mid y, \lambda, T)+p_{B}(b \mid y) p_{A}(a \mid x) \sum_{T \in \bar{U}_{x, \lambda} \cap \bar{V}_{y, \lambda}} \mu(T)\right]
\end{aligned}
$$

We now analyze each term separately. For fixed inputs $x, y$ and shared randomness $\lambda$, there is only one transcript which is valid for both Alice and Bob, and when they use this transcript for each $\lambda$, they output according to the distribution $\mathbf{p}$. Therefore, we have

$$
\sum_{\lambda} \mu(\lambda) \sum_{T \in U_{x, \lambda} \cap V_{y, \lambda}} \mu(T) p_{P}(a \mid x, \lambda, T) p_{P}(b \mid y, \lambda, T)=\frac{1}{2^{t}} p(a, b \mid x, y) .
$$

Let $A_{x}$ be the event that Alice's transcript is valid for $x$ (over random $\lambda, T$ ), and $\bar{A}_{x}$ its negation (similarly $B_{y}$ and $\bar{B}_{y}$ for Bob$)$. We denote

$$
p_{P}\left(a \mid x, A_{x} \cap \bar{B}_{y}\right)=\frac{\sum_{\lambda} \mu(\lambda) \sum_{T \in U_{x, \lambda} \cap \bar{V}_{y, \lambda}} \mu(T) p_{P}(a \mid x, \lambda, T)}{\mu\left(A_{x} \cap \bar{B}_{y}\right)}
$$

where, by definition, we have $\mu\left(A_{x} \cap \bar{B}_{y}\right)=\sum_{\lambda} \mu(\lambda) \sum_{T \in U_{x, \lambda} \cap \bar{V}_{y, \lambda}} \mu(T)$. We will show that this distribution is independent of $y$ and that the corresponding distribution $p_{P}\left(b \mid y, \bar{A}_{x} \cap B_{y}\right)$ for Bob is independent of $x$. Using these distributions, we may write $p_{l}(a, b \mid x, y)$ as

$$
\begin{aligned}
p_{l}(a, b \mid x, y) & =\frac{1}{2^{t}} p(a, b \mid x, y)+\mu\left(A_{x} \cap \bar{B}_{y}\right) p_{B}(b \mid y) p_{P}\left(a \mid x, A_{x} \cap \bar{B}_{y}\right) \\
& +\mu\left(\bar{A}_{x} \cap B_{y}\right) p_{A}(a \mid x) p_{P}\left(b \mid x, \bar{A}_{x} \cap B_{y}\right)+\mu\left(\bar{A}_{x} \cap \bar{B}_{y}\right) p_{B}(b \mid y) p_{A}(a \mid x)
\end{aligned}
$$

Summing over $b$, and using the fact that $\mathbf{p}_{l}$ and $\mathbf{p}$ are non-signaling, we have

$$
\begin{aligned}
p_{l}(a \mid x) & =\frac{1}{2^{t}} p(a \mid x)+\mu\left(A_{x} \cap \bar{B}_{y}\right) p_{P}\left(a \mid x, A_{x} \cap \bar{B}_{y}\right) \\
& +\mu\left(\bar{A}_{x} \cap B_{y}\right) p_{A}(a \mid x)+\mu\left(\bar{A}_{x} \cap \bar{B}_{y}\right) p_{A}(a \mid x) \\
& =\frac{1}{2^{t}} p(a \mid x)+\mu\left(A_{x} \cap \bar{B}_{y}\right) p_{P}\left(a \mid x, A_{x} \cap \bar{B}_{y}\right)+\mu\left(\bar{A}_{x}\right) p_{A}(a \mid x)
\end{aligned}
$$

Note that by definition, $\mu\left(A_{x}\right)=\sum_{\lambda} \mu(\lambda) \sum_{T \in U_{x, \lambda}} \mu(T)$ is independent of $y$, therefore so is $\mu\left(A_{x} \cap \bar{B}_{y}\right)=\mu\left(A_{x}\right)-$ $\mu\left(A_{x} \cap B_{y}\right)=\mu\left(A_{x}\right)-\frac{1}{2^{2}}$. From the expression for $p_{l}(a \mid x)$, we can conclude that $p_{P}\left(a \mid x, A_{x} \cap \bar{B}_{y}\right)$ is independent of $y$ and can be evaluated by Alice (and similarly for the analogue distribution for Bob). We now set

$$
\begin{aligned}
p_{A}(a \mid x) & =p_{P}\left(a \mid x, A_{x} \cap \bar{B}_{y}\right) \\
p_{B}(b \mid y) & =p_{P}\left(b \mid y, \bar{A}_{x} \cap B_{y}\right) .
\end{aligned}
$$

Therefore, the final distribution obtained from the local protocol may be written as

$$
\begin{aligned}
p_{l}(a, b \mid x, y) & =\frac{1}{2^{t}} p(a, b \mid x, y)+\mu\left(A_{x} \cap \bar{B}_{y}\right) p_{A}(a \mid x) p_{B}(b \mid y) \\
& +\mu\left(\bar{A}_{x} \cap B_{y}\right) p_{A}(a \mid x) p_{B}(b \mid y)+\mu\left(\bar{A}_{x} \cap \bar{B}_{y}\right) p_{A}(a \mid x) p_{B}(b \mid y) \\
& =\frac{1}{2^{t}} p(a b \mid x y)+\left(1-\frac{1}{2^{t}}\right) p_{A}(a \mid x) p_{B}(b \mid y) .
\end{aligned}
$$

For quantum protocols, we first simulate quantum communication using shared entanglement and teleportation, which uses 2 bits of classical communication for each qubit. Starting with this protocol using $2 q$ bits of classical communication, we may use the same idea as in the classical case, that is choosing a random $2 q$-bit string interpreted as the transcript, and replacing the players' respective outputs by independent random outputs chosen according to $p_{A}$ and $p_{B}$ if the random transcript does not match the bits they would have sent in the original protocol.

In the case of binary outputs with uniform marginals, that is, $\mathbf{p}=(C, 0,0)$, we may improve the exponent of the scaling-down coefficient $2^{2 q}$ by a factor of 2 using a more involved analysis and a variation of a result by [Kre95, Yao93, LS09] (the proof is given in Appendix Afor completeness).

Lemma 3 ([Kre95, Yao93, LS09]). Let $\left(C, M_{A}, M_{B}\right)$ be a distribution simulated by a quantum protocol with shared entanglement using $q_{A}$ qubits of communication from Alice to Bob and $q_{B}$ qubitsfrom Bob to Alice. There exist vectors $\vec{a}(x), \vec{b}(y)$ with $\|\vec{a}(x)\| \leq 2^{q_{B}}$ and $\|\vec{b}(y)\| \leq 2^{q_{A}}$ such that $C(x, y)=\vec{a}(x) \cdot \vec{b}(y)$.

The fact that $C / 2^{q} \in \mathcal{Q}_{0}$ then follows from Theorem 2 part 2.

### 3.2 Deriving an affine model and the lower bound from the noisy distribution

In this section we show that using Lemma 2a an explicit affine model can be derived from a (classical or quantum) communication protocol for $\mathbf{p}$, which gives us a lower bound technique for communication complexity in terms of how "good" the affine model is. We now are ready to complete the proof of Theorem 3

Proof of Theorem 3 We give a proof for the classical case, the quantum case follows the same lines. Let $c$ be the number of bits exchanged. From Lemma 2, we know that there exists marginal distributions $p_{A}(a \mid x)$ and $p_{B}(b \mid y)$ such that $p_{l}(a, b \mid x, y)=\frac{1}{2^{t}} p(a, b \mid x, y)+\left(1-\frac{1}{2^{t}}\right) p_{A}(a \mid x) p_{B}(b \mid y)$ is local. This gives an affine model for $p(a, b \mid x, y)$, as the following combination of two local distributions:

$$
p(a, b \mid x, y)=2^{t} p_{l}(a, b \mid x, y)+\left(1-2^{t}\right) p_{A}(a \mid x) p_{B}(b \mid y)
$$

Then $\tilde{\nu}(\mathbf{p}) \leq 2^{t+1}-1$.
In the case of binary outputs with uniform marginals, $\mathbf{p}_{l}=\left(C / 2^{t}, 0,0\right)$, and Lemma 2 implies that $C / 2^{t} \in \mathcal{L}_{0}$. By following the local protocol for $C / 2^{t}$ and letting Alice flip her output, we also get a local protocol for $-C / 2^{t}$, so $-C / 2^{t} \in \mathcal{L}_{0}$ as well. Notice that we may build an affine model for $C$ as a combination of $C / 2^{t}$ and $-C / 2^{t}$ :

$$
C=\frac{1}{2}\left(2^{t}+1\right) \frac{C}{2^{t}}-\frac{1}{2}\left(2^{t}-1\right) \frac{C}{2^{t}}
$$

Then, $\tilde{\nu}(C) \leq 2^{t}$.

### 3.3 Factorization norm and related measures

In the special case of distributions over binary variables with uniform marginals, the quantities $\tilde{\nu}$ and $\tilde{\gamma}_{2}$ become equivalent to the original quantities defined in [LMSS07, LS09] (at least for the interesting case of non-local correlations, that is correlations with non-zero communication complexity). When the marginals are uniform we omit them and write $\tilde{\nu}(C)$ and $\tilde{\gamma}_{2}(C)$. The following are reformulations as Minkowski functionals of the definitions appearing in LMSS07, LS09.

Definition 6. - $\nu(C)=\min \left\{\Lambda>0: \frac{1}{\Lambda} C \in \mathcal{L}_{0}\right\}$,

- $\gamma_{2}(C)=\min \left\{\Lambda>0: \frac{1}{\Lambda} C \in \mathcal{Q}_{0}\right\}$,
- $\nu^{\alpha}(C)=\min \left\{\nu\left(C^{\prime}\right): 1 \leq C(x, y) C^{\prime}(x, y) \leq \alpha, \forall x, y \in \mathcal{X} \times \mathcal{Y}\right\}$,
- $\gamma_{2}^{\alpha}(C)=\min \left\{\gamma_{2}\left(C^{\prime}\right): 1 \leq C(x, y) C^{\prime}(x, y) \leq \alpha, \forall x, y \in \mathcal{X} \times \mathcal{Y}\right\}$.

Theorem 4. For any correlation matrix $C: \mathcal{X} \times \mathcal{Y} \rightarrow[-1,1]$,

1. $\tilde{\nu}(C)=1$ iff $\nu(C) \leq 1$, and $\tilde{\gamma}_{2}(C)=1$ iff $\gamma_{2}(C) \leq 1$,
2. $\tilde{\nu}(C)>1 \Longrightarrow \nu(C)=\tilde{\nu}(C)$,
3. $\tilde{\gamma}_{2}(C)>1 \Longrightarrow \gamma_{2}(C)=\tilde{\gamma}_{2}(C)$.

Proof. The first item follows by definition of $\nu$ and $\gamma_{2}$. For the next items, we give the proof for $\nu$, and the proof for $\gamma_{2}$ is similar. The key to the proof is that if $C \in \mathcal{L}_{0}$, then $-C \in \mathcal{L}_{0}$ (it suffices for one of the players to flip his output).
$[\tilde{\nu}(C) \leq \nu(C)]$ If $\tilde{\nu}(C)>1$, then $\Lambda=\nu(C)>1$. Let $C^{+}=\frac{C}{\Lambda}$ and $C^{-}=-\frac{C}{\Lambda}$. By definition of $\nu(C)$, both $C^{+}$ and $C^{-}$are in $\mathcal{L}_{0}$. Furthermore, let $q_{+}=\frac{1+\Lambda}{2} \geq 0$ and $q_{-}=\frac{1-\Lambda}{2} \leq 0$. Since $C=q_{+} C^{+}+q_{-} C^{-}$, this determines an affine model for $C$ with $\left|q_{+}\right|+\left|q_{-}\right|=\Lambda$.
[ $\tilde{\nu}(C) \geq \nu(C)]$ Let $\Lambda=\tilde{\nu}(C)$. By definition of $\tilde{\nu}(C)$, there exists $C_{i}$ and $q_{i}$ such that $C=\sum_{i} q_{i} C_{i}$ and $\Lambda=\sum_{i}\left|q_{i}\right|$. Let $\tilde{C}_{i}=\operatorname{sgn}\left(q_{i}\right) C_{i}$ and $p_{i}=\frac{\left|q_{i}\right|}{\Lambda}$. Then, $\frac{C}{\Lambda}=\sum_{i} p_{i} \tilde{C}_{i}$ and therefore $\frac{1}{\Lambda} C \in \mathcal{L}_{0}$ since $\tilde{C}_{i} \in \mathcal{L}_{0}$.

In the special case of sign matrices (corresponding to Boolean functions, as shown above), we also have the following correspondence between $\tilde{\nu}^{\epsilon}, \tilde{\gamma}_{2}^{\epsilon}$, and $\nu^{\alpha}, \gamma_{2}^{\alpha}$.

Theorem 5. Let $0 \leq \epsilon<1 / 2$ and $\alpha=\frac{1}{1-2 \epsilon}$. For any sign matrix $C: \mathcal{X} \times \mathcal{Y} \rightarrow\{-1,1\}$,

1. $\tilde{\nu}^{\epsilon}(C)>1 \Longrightarrow \nu^{\alpha}(C)=\frac{\tilde{\nu}^{\epsilon}(C)}{1-2 \epsilon}$,
2. $\tilde{\gamma}_{2}^{\epsilon}(C)>1 \Longrightarrow \gamma_{2}^{\alpha}(C)=\frac{\tilde{\gamma}_{2}^{\epsilon}(C)}{1-2 \epsilon}$.

Proof. We give the proof for $\nu^{\alpha}$, the proof for $\gamma_{2}^{\alpha}$ is similar.
$\left[\nu^{\alpha}(C) \leq \frac{\tilde{\nu}^{\epsilon}(C)}{1-2 \epsilon}\right]$ By definition of $\tilde{\nu}^{\epsilon}(C)$, there exists a correlation matrix $C^{\prime}$ such that $\tilde{\nu}\left(C^{\prime}\right)=\tilde{\nu}^{\epsilon}(C)$ and $\left|C(x, y)-C^{\prime}(x, y)\right|^{\leq} \leq 2 \epsilon$ for all $x, y \in \mathcal{X} \times \mathcal{Y}$. Since $C$ is a sign matrix, and $C^{\prime}$ is a correlation matrix, $\operatorname{sgn}\left(C^{\prime}(x, y)\right)=$ $C(x, y)$ and $1-2 \epsilon \leq\left|C^{\prime}(x, y)\right| \leq 1$. Hence $1 \leq C(x, y) \frac{C^{\prime}(x, y)}{1-2 \epsilon} \leq \frac{1}{1-2 \epsilon}=\alpha$. This implies that $\nu^{\alpha}(C) \leq \nu\left(\frac{C^{\prime}}{1-2 \epsilon}\right)=$ $\frac{\nu\left(C^{\prime}\right)}{1-2 \epsilon}=\frac{\tilde{\nu}\left(C^{\prime}\right)}{1-2 \epsilon}$, where we used the fact that $\nu\left(C^{\prime}\right)=\tilde{\nu}\left(C^{\prime}\right)$ since $\tilde{\nu}\left(C^{\prime}\right)>1$.
$\left[\nu^{\alpha}(C) \geq \frac{\tilde{\nu}^{\epsilon}(C)}{1-2 \epsilon}\right]$ By definition of $\nu^{\alpha}(C)$, there exists a (not necessarily correlation) matrix $C^{\prime}$ such that $\nu\left(C^{\prime}\right)=$ $\nu^{\alpha}(C)$ and $1 \leq C(x, y) C^{\prime}(x, y) \leq \alpha$ for all $x, y$. Since $C$ is a sign matrix, this implies $\operatorname{sgn}\left(C^{\prime}(x, y)\right)=C(x, y)$ and $1-2 \epsilon \leq\left|\frac{C^{\prime}(\overline{x, y)}}{\alpha}\right| \leq 1$. Therefore, $\left|C(x, y)-\frac{C^{\prime}(x, y)}{\alpha}\right| \leq 2 \epsilon$ for all $x, y$. This implies that $\tilde{\nu}^{\epsilon}(C) \leq \tilde{\nu}\left(\frac{C^{\prime}}{\alpha}\right)=\nu\left(\frac{C^{\prime}}{\alpha}\right)=$ $(1-2 \epsilon) \nu\left(C^{\prime}\right)$, where we have used the fact that $\tilde{\nu}\left(\frac{C^{\prime}}{\alpha}\right)=\nu\left(\frac{C^{\prime}}{\alpha}\right)$ since $\tilde{\nu}\left(\frac{C^{\prime}}{\alpha}\right) \geq \tilde{\nu}^{\epsilon}(C)>1$.
Discussion. Just as the special case $\nu(C), \tilde{\nu}(\mathbf{p})$ may be expressed as a linear program. However, while $\gamma_{2}(C)$ could be expressed as a semidefinite program, this may not be true in general for $\tilde{\gamma}_{2}(\mathbf{p})$ (even though it can still be studied by SDP relaxation, as shown in [NPA08, DLTW08]).

Lemmas 4 and 5 establish that Corollary 3 is a generalization of Linial and Shraibman's factorization norm lower bound technique. Note that Linial and Shraibman use $\gamma_{2}^{\alpha}$ to derive a lower bound not only on the quantum communication complexity $Q_{\epsilon}^{\text {ent }}$, but also on the classical complexity $R_{\epsilon}^{\text {pub }}$. In the case of binary outcomes with uniform marginals (which includes Boolean functions, studied by Linial and Shraibman, as a special case), we obtain a similar result by combining our bound for $Q_{\epsilon}^{\mathrm{ent}}(C)$ with the fact that $Q_{\epsilon}^{\mathrm{ent}}(C) \leq\left\lceil\frac{1}{2} R_{\epsilon}^{\text {pub }}(C)\right\rceil$, which follows from superdense coding. This implies $R_{\epsilon}^{\text {pub }}(C) \geq 2 \log \left(\gamma_{2}^{\epsilon}(C)\right)-1$. In the general case, however, we can only prove that $R_{\epsilon}^{\text {pub }}(\mathbf{p}) \geq \log \left(\gamma_{2}^{\epsilon}(\mathbf{p})\right)-1$. This may be due to the fact that the result holds in the much more general setting of non-signaling distributions with arbitrary outcomes and marginals.

Because of Proposition 4 we know that $\nu(C) \leq K_{G} \gamma_{2}(C)$ for correlations. Note also that although $\gamma_{2}$ and $\nu$ are matrix norms, this fails to be the case for $\tilde{\gamma}_{2}$ and $\tilde{\nu}$, even in the case of correlations. Nevertheless, it is still possible to formulate dual quantities, which turn out to have sufficient structure, as we show in the next section.

## 4 Duality, Bell inequalities, and XOR games

In their primal formulation, the $\tilde{\gamma}_{2}$ and $\tilde{\nu}$ methods are difficult to apply since they are formulated as a minimization problem. Transposing to the dual space not only turns the method into a maximization problem; we show it also has a very natural, well-understood interpretation since it coincides with maximal violations of Bell and Tsirelson inequalities. This is particularly relevant to physics, since it formalizes in very precise terms the intuition that distributions with large Bell inequality violations should require more communication to simulate.

Recall that for any norm $\|\cdot\|$ on a vector space $V$, the dual norm is $\|B\|^{*}=\max _{v \in V:\|v\| \leq 1} B(v)$, where $B$ is a linear functional on $V$.

### 4.1 Bell and Tsirelson inequalities

Bell inequalities were first introduced by Bell [Bel64], as bounds on the correlations that could be achieved by any local physical theory. He showed that quantum correlations could violate these inequalities and therefore exhibited non-locality. Tsirelson later proved that quantum correlations should also respect some bound (known as the Tsirelson bound), giving a first example of a "Tsirelson-like" inequality for quantum distributions Tsi80].

Since the set of non-signaling distributions $\mathcal{C}$ lies in an affine space aff $(\mathcal{C})$, we may consider the isomorphic dual space of linear functionals over this space. The dual quantity $\tilde{\nu}^{*}$ (technically not a dual norm since $\tilde{\nu}$ itself is not a norm in the general case) is the maximum value of a linear functional in the dual space on local distributions, and $\tilde{\gamma}_{2}^{*}$
is the maximum value of a linear functional on quantum distributions. These are exactly what is captured by the Bell and Tsirelson inequalities.

Definition 7 (Bell and Tsirelson inequalities). Let $B: \operatorname{aff}(\mathcal{C}) \mapsto \mathbb{R}$ be a linear functional on the (affine hull of the) set of non-signaling distributions, $B(\mathbf{p})=\sum_{a, b, x, y} B_{a b x y} p(a, b \mid x, y)$. Define $\tilde{\nu}^{*}(B)=\max _{\mathbf{p} \in \mathcal{L}}|B(\mathbf{p})|$ and $\tilde{\gamma}_{2}^{*}(B)=\max _{\mathbf{p} \in \mathcal{Q}}|B(\mathbf{p})|$. A Bell inequality is a linear inequality satisfied by any local distribution:

$$
B(\mathbf{p}) \leq \tilde{\nu}^{*}(B)(\forall \mathbf{p} \in \mathcal{L})
$$

and a Tsirelson inequality is a linear inequality satisfied by any quantum distribution:

$$
B(\mathbf{p}) \leq \tilde{\gamma}_{2}^{*}(B)(\forall \mathbf{p} \in \mathcal{Q})
$$

By linearity (Proposition (1) Bell inequalities are often expressed as linear functionals over the correlations in the case of binary outputs and uniform marginals.

Finally, $\tilde{\gamma}_{2}$ and $\tilde{\nu}$ amount to finding a maximum violation of a (normalized) Bell or Tsirelson inequality.
Theorem 6. For any distribution $\mathbf{p} \in \mathcal{C}$,

1. $\tilde{\nu}(\mathbf{p})=\max \left\{B(\mathbf{p}): \forall \mathbf{p}^{\prime} \in \mathcal{L},\left|B\left(\mathbf{p}^{\prime}\right)\right| \leq 1\right\}$, and
2. $\tilde{\gamma}_{2}(\mathbf{p})=\max \left\{B(\mathbf{p}): \forall \mathbf{p}^{\prime} \in \mathcal{Q},\left|B\left(\mathbf{p}^{\prime}\right)\right| \leq 1\right\}$,
where the maximization is over linear functionals $B: \operatorname{aff}(\mathcal{C}) \mapsto \mathbb{R}$.
Proof. The proof of item 1 follows by LP duality from the definition of $\tilde{\nu}$. Nevertheless, we give an alternative proof that can be easily adapted to prove item 2 (it suffices to replace $\tilde{\nu}$ by $\tilde{\gamma}_{2}$ and $\mathcal{L}$ by $\mathcal{Q}$ ). The key idea of the proof is to use the convex conjugate of $\tilde{\nu}$ (written $\tilde{\nu}^{\star}$ ) which is closely related to the dual expression (written $\tilde{\nu}^{*}$ ), and apply it twice.

We first recall basic facts about convex conjugate functions (See [BV04] for full details). For a function $f: \mathbb{R}^{n} \rightarrow$ $\mathbb{R}$, the convex conjugate function $f^{\star}: \mathbb{R}^{n} \rightarrow \mathbb{R}$ is defined as:

$$
f^{\star}(y)=\sup _{x \in \operatorname{dom}(f)}\left(y^{T} x-f(x)\right)
$$

where $\operatorname{dom}(f)$ denotes the domain of $f$. It is known that $f^{\star \star}=f$ provided that $f$ is convex and closed i.e., its epigraph is closed.

By grouping negative and positive terms together, it is easy to see that $\tilde{\nu}(\mathbf{p})=\min \left\{k^{+}+k^{-}: k^{+}, k^{-} \in\right.$ $\left.\mathbb{R}^{+}, \exists \mathbf{p}^{+}, \mathbf{p}^{-} \in \mathcal{L}, \mathbf{p}=k^{+} \mathbf{p}^{+}-k^{-} \mathbf{p}^{-}\right\}$. We consider $\tilde{\nu}$ as a function over aff $(\mathcal{L})$. Then, it is straightforward to verify that $\tilde{\nu}$ is convex, and since its domain $\operatorname{aff}(\mathcal{L})$ is closed, $\tilde{\nu}$ is also a closed function.

We then have by definition

$$
\begin{aligned}
\tilde{\nu}^{\star}(B) & =\max _{\mathbf{p} \in \operatorname{aff}(\mathcal{L})}(B(\mathbf{p})-\tilde{\nu}(\mathbf{p})), \\
& =\max _{\mathbf{p}_{1}, \mathbf{p}_{2} \in \mathcal{L}, k_{1}-k_{2}=1}\left(B\left(k_{1} \mathbf{p}_{1}-k_{2} \mathbf{p}_{2}\right)-\left(k_{1}+k_{2}\right)\right) \\
& =\max _{\mathbf{p}_{1}, \mathbf{p}_{2} \in \mathcal{L}, k_{1}-k_{2}=1}\left(k_{1}\left(B\left(\mathbf{p}_{1}\right)-1\right)-k_{2}\left(B\left(\mathbf{p}_{2}\right)+1\right)\right) .
\end{aligned}
$$

Therefore,

$$
\tilde{\nu}^{\star}(B)= \begin{cases}\max _{\mathbf{p} \in \mathcal{L}}|B(\mathbf{p})|-1 & \text { if } \max _{\mathbf{p} \in \mathcal{L}}|B(\mathbf{p})| \leq 1 \\ +\infty & \text { otherwise }\end{cases}
$$

Taking the convex conjugate a second time, we obtain

$$
\tilde{\nu}^{\star \star}(\mathbf{p})=\max _{B}\left(B(\mathbf{p})-\tilde{\nu}^{\star}(B)\right)
$$

From the expression for $\tilde{\nu}^{\star}(B)$ above, it is clear that the maximum is achieved for a linear functional $B$ such that $\max _{\mathbf{p} \in \mathcal{L}}|B(\mathbf{p})| \leq 1$. Let the maximum be achieved by a linear functional $\bar{B}$, and let us consider $\bar{B}_{\max }=$ $\max _{\mathbf{p} \in \mathcal{L}} \bar{B}(\mathbf{p})$ and $\bar{B}_{\text {min }}=\min _{\mathbf{p} \in \mathcal{L}} \bar{B}(\mathbf{p})$. We show that we can assume without loss of generality that $\left|\bar{B}_{\min }\right| \leq$ $\bar{B}_{\text {max }}=1$. Indeed, we must have $\left|\bar{B}_{\text {min }}\right| \leq \bar{B}_{\text {max }}$, otherwise $\bar{B}$ could not achieve the maximum since $-\bar{B}$ would yield a larger value. This implies that $\tilde{\nu}^{\star}(\bar{B})=\bar{B}_{\text {max }}-1$ and $\tilde{\nu}^{\star \star}(\mathbf{p})=\bar{B}(\mathbf{p})-\bar{B}_{\text {max }}+1$. Then, the maximum is also achieved by the linear functional $\bar{B}^{\prime}(\mathbf{p})=\bar{B}(\mathbf{p})-\bar{B}_{\text {max }}+1$, which satisfies $\bar{B}_{\max }^{\prime}=1$ and therefore $\tilde{\nu}^{\star}\left(\bar{B}^{\prime}\right)=0$. From the expression for $\tilde{\nu}^{\star \star}$, we therefore obtain $\tilde{\nu}^{\star \star}(\mathbf{p})=\tilde{\nu}(\mathbf{p})=\max \left\{B(\mathbf{p}): \forall \mathbf{p}^{\prime} \in \mathcal{L},\left|B\left(\mathbf{p}^{\prime}\right)\right| \leq 1\right\}$.

### 4.2 XOR games and Bell inequalities for correlations

In the special case of XOR games, there is a close connection between winning probability and Bell inequalities, which we make explicit in this section.

In an XOR game, Alice is given some input $x$ and Bob is given an input $y$, and they should output $a= \pm 1$ and $b= \pm 1$. They win if $a \cdot b$ equals some $\pm 1$ function $G(x, y)$. Since they are not allowed to communicate, their strategy may be represented as a local correlation matrix $S \in \mathcal{L}_{0}$. We consider the distributional version of this game, where $\mu$ is a distribution on the inputs. The winning bias given some strategy $S$ with respect to $\mu$ is $\epsilon_{\mu}(G \| S)=\sum_{x, y} \mu(x, y) G(x, y) S(x, y)$, and $\epsilon_{\mu}^{\mathrm{pub}}(G)=\max _{S \in \mathcal{L}_{0}} \epsilon_{\mu}(G \| S)$ is the maximum winning bias of any local (classical) strategy (for convenience, we consider the bias instead of game value $\left.\omega_{\mu}^{\mathrm{pub}}(G)=\left(1+\epsilon_{\mu}^{\mathrm{pub}}(G)\right) / 2\right)$. We define $\epsilon_{\mu}^{\text {ent }}(G)$ similarly for quantum strategies. When the input distribution is not fixed, we define the game biases as $\epsilon^{\mathrm{pub}}(G)=\min _{\mu} \epsilon_{\mu}^{\mathrm{pub}}(G)$ and $\epsilon^{\mathrm{ent}}(G)=\min _{\mu} \epsilon_{\mu}^{\mathrm{ent}}(G)$.
Lemma 4. There is a bijection between XOR games $(G, \mu)$ and normalized correlation Bell inequalities.
Proof. For a given XOR game $G$, and a local strategy $C$, its winning probability, or more simply its bias, can be written as a linear equation, which we write $G \circ \mu(C)=\epsilon_{\mu}(G \| C)$ where $\circ$ is the Hadamard (entrywise) product. This can be seen as a linear functional over the space of strategies. By Definition $7 \nu^{*}(G \circ \mu)=\epsilon_{\mu}^{\text {pub }}(G)$, and $\epsilon_{\mu}(G \| C) \leq \epsilon_{\mu}^{\mathrm{pub}}(G)$ is a Bell inequality satisfied by any local correlation matrix $C$. Similarly, when the players are allowed to use entanglement, we get a Tsirelson inequality on quantum correlations, $\epsilon_{\mu}(G \| C) \leq \epsilon_{\mu}^{\text {ent }}(G)$ (the quantum bias is also equivalent to a dual norm $\left.\epsilon_{\mu}^{\text {ent }}(G)=\gamma_{2}^{*}(G \circ \mu)\right)$.

Conversely, consider a general linear functional $B(C)=\sum_{x, y} B_{x y} C(x, y)$ on aff $\left(\mathcal{C}_{0}\right)$, defining a correlation Bell inequality $B(C) \leq \nu^{*}(B) \forall C \in \mathcal{L}_{0}$. Dividing this Bell inequality by $N=\sum_{x, y}\left|B_{x y}\right|$, we see that it determines an XOR game specified by a sign matrix $G(x, y)=\operatorname{sgn}\left(B_{x y}\right)$ and an input distribution $\mu_{x y}=\frac{\left|B_{x y}\right|}{N}$, and having a game bias $\epsilon_{\mu}^{\mathrm{pub}}(G)=\frac{\nu^{*}(B)}{N}$.

By Theorem6and the previous bijection (see also Lee et al. LSŠ08]):
Corollary 3. 1. $\nu(C)=\max _{\mu, G} \frac{\epsilon_{\mu}(G \| C)}{\epsilon_{\mu}^{\text {pub }}(G)}$,
2. $\nu(C) \geq \frac{1}{\epsilon^{\mathrm{pub}}(C)}$.

The second part follows by letting $G=C$. Even though playing correlations $C$ for a game $G=C$ allows us to win with probability one, there are cases where some other game $G \neq C$ yields a larger ratio. In these cases, we have $\nu(C)>\frac{1}{\epsilon^{\mathrm{pub}}(C)}$ so that $\nu$ gives a stronger lower bound for communication complexity than the game value (which has been shown to be equivalent to the discrepancy method [LSŠ08]. Similar properties hold for the quantum values, in particular, we have $\gamma_{2}(C) \geq \frac{1}{\epsilon^{\text {ent }}(C)}$.

We can characterize when the inequality is tight. Let $\epsilon_{=}^{\operatorname{pub}}(C)=\max _{S \in \mathcal{L}_{0}}\{\beta: \forall x, y, C(x, y) S(x, y)=\beta\}$, that is, we only consider strategies that win the game with equal bias with respect to all distributions. For the sake of comparison, the game bias may also be expressed as vN28]:

$$
\epsilon^{\mathrm{pub}}(C)=\max _{S \in \mathcal{L}_{0}}\{\beta: \forall x, y, C(x, y) S(x, y) \geq \beta\}=\max _{S \in \mathcal{L}_{0}} \min _{x, y} C(x, y) S(x, y)
$$

Lemma 5. $\nu(C)=\frac{1}{\epsilon_{\underline{\underline{p u b}}}^{\text {put }}(C)}$.
We can also relate the game value to $\nu^{\alpha}(C)$, as it was shown in [SSŠ08] that for $\alpha \rightarrow \infty, \nu^{\infty}(C)$ is exactly the inverse of the game bias $\frac{1}{\epsilon^{\mathrm{pub}}(C)}$. We show that this holds as soon as $\alpha=\frac{1}{1-2 \epsilon}$ is large enough for $C$ to be local up to an error $\epsilon$, completing the picture given in Lemma 5
Lemma 6. Let $0 \leq \epsilon<1 / 2$ and $\alpha=\frac{1}{1-2 \epsilon}$. For any sign matrix $C: \mathcal{X} \times \mathcal{Y} \rightarrow\{-1,1\}$,

1. $\tilde{\nu}^{\epsilon}(C)=1 \Longleftrightarrow \epsilon \geq 1-\omega^{\mathrm{pub}}(C) \Longleftrightarrow \alpha \geq \frac{1}{\epsilon^{\operatorname{pub}}(C)} \Longleftrightarrow \nu^{\alpha}(C)=\nu^{\infty}(C)=\frac{1}{\epsilon^{\operatorname{pub}}(C)}$,
2. $\tilde{\gamma}_{2}^{\epsilon}(C)=1 \Longleftrightarrow \epsilon \geq 1-\omega^{\mathrm{ent}}(C) \Longleftrightarrow \alpha \geq \frac{1}{\epsilon^{\mathrm{ent}}(C)} \Longleftrightarrow \gamma_{2}^{\alpha}(C)=\gamma_{2}^{\infty}(C)=\frac{1}{\epsilon^{\mathrm{ent}}(C)}$.

Proof. By von Neumann's minmax principle [vN28],

$$
\begin{aligned}
\epsilon^{\mathrm{pub}}(C) & =\max _{S \in \mathcal{L}_{0}} \min _{x, y} C(x, y) S(x, y) \\
& =\max _{S \in \mathcal{L}_{0}} \min _{x, y} 1-|C(x, y)-S(x, y)|
\end{aligned}
$$

where we used the fact that $C$ is a sign matrix. This implies that $\tilde{\nu}^{\epsilon}(C)=1 \Leftrightarrow \epsilon \geq \frac{1-\epsilon^{\mathrm{pub}}(C)}{2} \Leftrightarrow \alpha \geq \frac{1}{\epsilon^{\mathrm{pub}}(C)}$.
By Lemma 5, this in turn implies that $\nu^{\alpha}(C)=\frac{\tilde{\nu}^{\epsilon}(C)}{1-2 \epsilon}$ for all $\epsilon<\frac{1-\epsilon^{\mathrm{pub}}(C)}{2}$. By continuity, taking the limit $\epsilon \rightarrow \frac{1-\epsilon^{\mathrm{pub}}(C)}{2}$ yields $\nu^{\alpha}(C)=\frac{1}{\epsilon^{\mathrm{pub}}(C)}$ for $\alpha=\frac{1}{\epsilon^{\mathrm{pub}}(C)}$. From LSŠ08], $\nu^{\infty}(C)=\frac{1}{\epsilon^{\mathrm{pub}}(C)}$, and the lemma follows by the monotonicity of $\nu^{\alpha}(C)$ as a function of $\alpha$.

## 5 Bounding the violation of Bell inequalities

In this section, we give bounds on the maximal violations of Bell inequalities. By Theorem 6, this is equivalent to bounding the ratio between $\tilde{\gamma}_{2}$ and $\tilde{\nu}$. In the case of distributions over binary outcomes with uniform marginals (correlations), the theorems of Tsirelson (Theorem 2) and Grothendieck (Proposition 4) imply that $\gamma_{2}$ and $\nu$ differ by at most a constant. This is bad news for anyone trying to find a Boolean function with high randomized communication complexity and considerably smaller quantum communication complexity, since it means that any randomized lower bound obtained by using $\nu$ will yield a similar quantum lower bound. Although neither of these theorems are known to hold beyond the Boolean setting with uniform marginals, we show in this section that this surprisingly also extends to non-signaling distributions. This is also bad news for anyone looking for large Bell inequality violations by quantum distributions, since in this case, $\tilde{\gamma}_{2}(\mathbf{p})=1$, and the maximum Bell inequality we can hope for will be bounded above by the expressions below.

Theorem 7. For any distribution $\mathbf{p} \in \mathcal{C}$, with inputs in $\mathcal{X} \times \mathcal{Y}$ and outcomes in $\mathcal{A} \times \mathcal{B}$ with $A=|\mathcal{A}|, B=|\mathcal{B}|$,

1. $\tilde{\nu}(\mathbf{p}) \leq\left(2 K_{G}+1\right) \tilde{\gamma}_{2}(\mathbf{p})$ when $A=B=2$,
2. $\tilde{\nu}(\mathbf{p}) \leq\left[2 A B\left(K_{G}+1\right)-1\right] \tilde{\gamma}_{2}(\mathbf{p})$ for any $A, B$.

Therefore, one cannot hope to prove separations between classical and quantum communication using this method, except in the case where the number of outcomes is large. For binary outcomes at least, this says that arguments based on analyzing the distance to the quantum set only, without taking into account the particular structure of the distribution, will not suffice to prove large separations; and other techniques, such as information theoretic arguments, may be necessary.

For example, Brassard et al. [BCT99] give a (promise) distribution based on the Deutsch-Jozsa problem, which can be obtained exactly with entanglement and no communication, but which requires linear communication to simulate exactly. The lower bound is proven using a corruption bound [BCW98], which is closely related to the information theoretic subdistribution bound [JKN08]. For this problem, $\mathcal{X}=\mathcal{Y}=\{0,1\}^{n}$ and $\mathcal{A}=\mathcal{B}=[n]$, therefore our method
can only prove a lower bound logarithmic in $n$. This is the first example of a problem for which the corruption bound gives an exponentially better lower bound than the Linial and Shraibman family of methods.

On the positive side, this is very interesting for quantum information, since (by Theorem6), it tells us that the set of quantum distributions cannot be much larger than the local polytope, for any number of inputs and outcomes. For binary correlations, this follows from the theorems of Tsirelson (Theorem 2) and Grothendieck (Proposition 4), but no extensions are known for these results in the more general setting.

The proof of Theorem 7 proceeds by showing that an arbitrary quantum distribution may be written as an affine combination of quantum distributions over binary outcomes with uniform marginals. We can then conclude using Grothendieck's inequality. For the details of the proof, we will need two rather straightforward lemmas. The first is a subadditivity-type property for $\tilde{\nu}$, and the second allows us to extend the support of a distribution without affecting the value of $\tilde{\nu}$.

Lemma 7. If $\mathbf{p}=\sum_{i \in[I]} q_{i} \mathbf{p}_{i}$, where $\mathbf{p}_{i} \in \mathcal{C}$ and $q_{i} \in \mathbb{R}$ for all $i \in[I]$, then $\tilde{\nu}(\mathbf{p}) \leq \sum_{i \in[I]}\left|q_{i}\right| \tilde{\nu}\left(\mathbf{p}_{i}\right)$.
Proof. By definition, for each $\mathbf{p}_{i}$, there exists $\mathbf{p}_{i}^{+}, \mathbf{p}_{i}^{-} \in \mathcal{L}$ and $q_{i}^{+}, q_{i}^{-} \geq 0$ such that $\mathbf{p}_{i}=q_{i}^{+} \mathbf{p}_{i}^{+}-q_{i}^{-} \mathbf{p}_{i}^{-}$, and $q_{i}^{+}+q_{i}^{-}=\tilde{\nu}\left(\mathbf{p}_{i}\right)$. Therefore, $\mathbf{p}=\sum_{i \in[I]} q_{i}\left(q_{i}^{+} \mathbf{p}_{i}^{+}-q_{i}^{-} \mathbf{p}_{i}^{-}\right)$and $\sum_{i \in[I]}\left(\left|q_{i} q_{i}^{+}\right|+\left|q_{i} q_{i}^{-}\right|\right)=\sum_{i}\left|q_{i}\right|\left(q_{i}^{+}+q_{i}^{-}\right)=$ $\sum_{i}\left|q_{i}\right| \tilde{\nu}\left(\mathbf{p}_{i}\right)$.

Lemma 8. Let $\mathbf{p}, \mathbf{p}^{\prime} \in \mathcal{C}$ be non-signaling distributions with inputs in $\mathcal{X} \times \mathcal{Y}$ for both distributions, outcomes in $\mathcal{A} \times \mathcal{B}$ for $\mathbf{p}$, and outcomes in $\mathcal{A}^{\prime} \times \mathcal{B}^{\prime}$ for $\mathbf{p}^{\prime}$, such that $\mathcal{A} \subseteq \mathcal{A}^{\prime}$ and $\mathcal{B} \subseteq \mathcal{B}^{\prime}$. If, for any $(a, b) \in \mathcal{A} \times \mathcal{B}$ $p^{\prime}(a, b \mid x, y)=p(a, b \mid x, y)$, then $\tilde{\nu}\left(\mathbf{p}^{\prime}\right)=\tilde{\nu}(\mathbf{p})$.

Proof. Let $\mathcal{E}=\left(\mathcal{A}^{\prime} \times \mathcal{B}^{\prime}\right) \backslash(\mathcal{A} \times \mathcal{B})$. First, note that since $p^{\prime}(a, b \mid x, y)=p(a, b \mid x, y)$ for any $(a, b) \in \mathcal{A} \times \mathcal{B}$, we have, by normalization of $\mathbf{p}, p^{\prime}(a, b \mid x, y)=0$ for any $(a, b) \in \mathcal{E}$.
$\left[\tilde{\nu}\left(\mathbf{p}^{\prime}\right) \leq \tilde{\nu}(\mathbf{p})\right]$ Let $\mathbf{p}=q_{+} \mathbf{p}^{+}-q_{-} \mathbf{p}^{-}$be an affine model for $\mathbf{p}$. Obviously, this implies an affine model for $\mathbf{p}^{\prime}$ by extending the local distributions $\mathbf{p}^{+}, \mathbf{p}^{-}$from $\mathcal{A} \times \mathcal{B}$ to $\mathcal{A}^{\prime} \times \mathcal{B}^{\prime}$, by setting $p^{+}(a, b \mid x, y)=p^{-}(a, b \mid x, y)=0$ for any $(a, b) \in \mathcal{E}$, so $\tilde{\nu}\left(\mathbf{p}^{\prime}\right) \leq \tilde{\nu}(\mathbf{p})$.
$\left[\tilde{\nu}\left(\mathbf{p}^{\prime}\right) \geq \tilde{\nu}(\mathbf{p})\right]$ Let $\mathbf{p}^{\prime}=q_{+} \mathbf{p}^{\prime+}-q_{-} \mathbf{p}^{\prime-}$ be an affine model for $\mathbf{p}^{\prime}$. We may not immediately derive an affine model for $\mathbf{p}$ since it could be the case that $p^{\prime+}(a, b \mid x, y)$ or $p^{\prime-}(a, b \mid x, y)$ is non zero for some $(a, b) \in \mathcal{E}$. However, we have $q_{+} p^{\prime+}(a, b \mid x, y)-q_{-} p^{\prime-}(a, b \mid x, y)=p^{\prime}(a, b \mid x, y)=0$ for any $(a, b) \in \mathcal{E}$, so we may define an affine model $\mathbf{p}=q_{+} \mathbf{p}^{+}-q_{-} \mathbf{p}^{-}$, where $\mathbf{p}^{+}$and $\mathbf{p}^{-}$are distributions on $\mathcal{A} \times \mathcal{B}$ such that

$$
p^{+}(a, b \mid x, y)=p^{\prime+}(a, b \mid x, y)+\frac{1}{A} \sum_{a^{\prime} \notin \mathcal{A}} p^{\prime+}\left(a^{\prime}, b \mid x, y\right)+\frac{1}{B} \sum_{b^{\prime} \notin \mathcal{B}} p^{\prime+}\left(a, b^{\prime} \mid x, y\right)+\frac{1}{A B} \sum_{a^{\prime} \notin \mathcal{A}, b^{\prime} \notin \mathcal{B}} p^{\prime+}\left(a^{\prime}, b^{\prime} \mid x, y\right),
$$

and similarly for $\mathbf{p}^{-}$. These are local since it suffices for Alice and Bob to use the local protocol for $\mathbf{p}^{\prime+}$ or $\mathbf{p}^{\prime-}$ and for Alice to replace any output $a \notin \mathcal{A}$ by a uniformly random output $a^{\prime} \in \mathcal{A}$ (similarly for Bob). Therefore, we also have $\tilde{\nu}\left(\mathbf{p}^{\prime}\right) \geq \tilde{\nu}(\mathbf{p})$.

Before proving Theorem 7 , we first consider the special case of quantum distributions, for which $\tilde{\gamma}_{2}(\mathbf{p})=1$. As we shall see in Section 6 this special case implies the constant upper bound of Shi and Zhu on approximating any quantum distribution [SZ08], which they prove using diamond norms. This also immediately gives an upper bound on maximum Bell inequality violations for quantum distributions, by Theorem6 which may be of independent interest in quantum information theory.

Proposition 6. For any quantum distribution $\mathbf{p} \in \mathcal{Q}$, with inputs in $\mathcal{X} \times \mathcal{Y}$ and outcomes in $\mathcal{A} \times \mathcal{B}$ with $A=|\mathcal{A}|, B=$ $|\mathcal{B}|$,

1. $\tilde{\nu}(\mathbf{p}) \leq 2 K_{G}+1$ when $A=B=2$,
2. $\tilde{\nu}(\mathbf{p}) \leq 2 A B\left(K_{G}+1\right)-1$ for any $A, B$.

Proof. 1. Since $A=B=2$, we may write the distribution as correlations and marginals, $\mathbf{p}=\left(C, M_{A}, M_{B}\right)$. Since $\left(C, M_{A}, M_{B}\right) \in \mathcal{Q}$, we also have $(C, 0,0) \in \mathcal{Q}$, and by Tsirelson's theorem, $\left(C / K_{G}, 0,0\right) \in \mathcal{L}$. Moreover, it is immediate that $\left(M_{A} M_{B}, M_{A}, M_{B}\right),\left(M_{A} M_{B}, 0,0\right)$ and $(0,0,0)$ are local distributions as well, so that we have the following affine model for $\left(C, M_{A}, M_{B}\right)$

$$
\left(C, M_{A}, M_{B}\right)=K_{G}\left(C / K_{G}, 0,0\right)+\left(M_{A} M_{B}, M_{A}, M_{B}\right)-\left(M_{A} M_{B}, 0,0\right)-\left(K_{G}-1\right)(0,0,0)
$$

This implies that $\tilde{\nu}\left(C, M_{A}, M_{B}\right) \leq 2 K_{G}+1$.
2. For the general case, we will reduce to the binary case. Let us introduce an additional output $\varnothing$, and set $\mathcal{A}^{\prime}=\mathcal{A} \cup\{\varnothing\}$ and $\mathcal{B}^{\prime}=\mathcal{B} \cup\{\varnothing\}$. We first extend the distribution $\mathbf{p}$ to a distribution $\mathbf{p}^{\prime}$ on $\mathcal{A}^{\prime} \times \mathcal{B}^{\prime}$ by setting $p^{\prime}(a, b \mid x, y)=p(a, b \mid x, y)$ for any $(a, b) \in \mathcal{A} \times \mathcal{B}$, and $p^{\prime}(a, b \mid x, y)=0$ otherwise. By Lemma 8 , we have $\tilde{\nu}(\mathbf{p})=\tilde{\nu}\left(\mathbf{p}^{\prime}\right)$.
For each $(\alpha, \beta) \in \mathcal{A} \times \mathcal{B}$, we also define a probability distribution $\mathbf{p}_{\alpha \beta}$ on $\mathcal{A}^{\prime} \times \mathcal{B}^{\prime}$ :

$$
p_{\alpha \beta}(a, b \mid x, y)= \begin{cases}p(\alpha, \beta \mid x, y) & \text { if }(a, b)=(\alpha, \beta) \\ p(\alpha \mid x)-p(\alpha, \beta \mid x, y) & \text { if }(a, b)=(\alpha, \varnothing) \\ p(\beta \mid y)-p(\alpha, \beta \mid x, y) & \text { if }(a, b)=(\varnothing, \beta) \\ 1-p(\alpha \mid x)-p(\beta \mid y)+p(\alpha, \beta \mid x, y) & \text { if }(a, b)=(\varnothing, \varnothing) \\ 0 & \text { otherwise }\end{cases}
$$

Notice that $p_{\alpha \beta} \in \mathcal{Q}$, since a protocol for $p_{\alpha \beta}$ can be obtained from a protocol for $p$ : Alice outputs $\varnothing$ whenever her outcome is not $\alpha$, similarly for Bob. Let $\mathcal{A}_{\alpha}=\{\alpha, \varnothing\}$ and $\mathcal{B}_{\beta}=\{\beta, \varnothing\}$. Since $p_{\alpha \beta}(a, b \mid x, y)=0$ when $(a, b) \notin \mathcal{A}_{\alpha} \times \mathcal{B}_{\beta}$, we may define distributions $\mathbf{p}_{\alpha \beta}^{\prime}$ on $\mathcal{A}_{\alpha} \times \mathcal{B}_{\beta}$ such that $p_{\alpha \beta}^{\prime}(a, b \mid x, y)=p_{\alpha \beta}(a, b \mid x, y)$ for all $(a, b) \in \mathcal{A}_{\alpha} \times \mathcal{B}_{\beta}$. By Lemma 8 , these are such that $\tilde{\nu}\left(\mathbf{p}_{\alpha \beta}^{\prime}\right)=\tilde{\nu}\left(\mathbf{p}_{\alpha \beta}\right)$, and since these are binary distributions, $\tilde{\nu}\left(\mathbf{p}_{\alpha \beta}^{\prime}\right) \leq 2 K_{G}+1$. Let us define three distributions $\mathbf{p}_{\mathbf{A}}, \mathbf{p}_{\mathbf{B}}, \mathbf{p}_{\varnothing}$ on $\mathcal{A}^{\prime} \times \mathcal{B}^{\prime}$ as follows. We let $\mathbf{p}_{\mathbf{A}}(a, \varnothing \mid x, y)=$ $p(a \mid x), \mathbf{p}_{\mathbf{B}}(\varnothing, b \mid x, y)=p(b \mid y)$, and 0 everywhere else; and $p_{\varnothing}(a, b \mid x, y)=1$ if $(a, b)=(\varnothing, \varnothing)$, and 0 otherwise. These are product distributions, so $\mathbf{p}_{\mathbf{A}}, \mathbf{p}_{\mathbf{B}}, \mathbf{p}_{\varnothing} \in \mathcal{L}$ and $\tilde{\nu}=1$ for all three distributions.
We may now build the following affine model for $\mathbf{p}^{\prime}$

$$
\mathbf{p}^{\prime}=\sum_{(\alpha, \beta) \in \mathcal{A} \times \mathcal{B}} \mathbf{p}_{\alpha \beta}^{\prime}-(B-1) \mathbf{p}_{\mathbf{A}}-(A-1) \mathbf{p}_{\mathbf{B}}-(A B-A-B+1) \mathbf{p}_{\varnothing} .
$$

From Lemma7 we conclude that $\tilde{\nu}\left(\mathbf{p}^{\prime}\right) \leq A B\left(2 K_{G}+2\right)-1$.

The proof of Theorem 7 immediately follows.
Proof of Theorem 7 By definition of $\tilde{\gamma}_{2}(\mathbf{p})$, there exists $\mathbf{p}^{+}, \mathbf{p}^{-} \in \mathcal{Q}$ and $q_{+}, q_{-} \geq 0$ such that $\mathbf{p}=q_{+} \mathbf{p}^{+}-q_{-} \mathbf{p}^{-}$ and $q_{+}+q_{-}=\tilde{\gamma}_{2}(\mathbf{p})$. From Lemma $7 \tilde{\nu}(\mathbf{p}) \leq q_{+} \tilde{\nu}\left(\mathbf{p}^{+}\right)+q_{-} \tilde{\nu}\left(\mathbf{p}^{-}\right)$, and Proposition6immediately concludes the proof.

## 6 Upper bounds for non-signaling distributions

We have seen that if a distribution can be simulated using $t$ bits of communication, then it may be represented by an affine model with coefficients exponential in $t$ (Lemma 2]. In this section, we consider the converse: how much communication is sufficient to simulate a distribution, given an affine model? This approach allows us to show that any (shared randomness or entanglement-assisted) communication protocol can be simulated with simultaneous messages, with an exponential cost to the simulation, which was previously known only in the case of Boolean functions Yao03, SZ08, GKd06]. Our results imply for example that for any quantum distribution $\mathbf{p} \in \mathcal{Q}, Q_{\varepsilon}^{\|}(\mathbf{p})=O(\log (n))$, where $n$ is the input size. This in effect replaces arbitrary entanglement in the state being measured, with logarithmic quantum
communication (using no additional resources such as shared randomness). We use the superscript || to indicate the simultaneous messages model, where Alice and Bob each send a message to the referee, who without knowing the inputs, outputs the value of the function, or more generally, outputs $a, b$ with the correct probability distribution conditioned on the inputs $x, y$.
Theorem 8. For any distribution $\mathbf{p} \in \mathcal{C}$ with inputs in $\mathcal{X} \times \mathcal{Y}$ with $|\mathcal{X} \times \mathcal{Y}| \leq 2^{n}$, and outcomes in $\mathcal{A} \times \mathcal{B}$ with $A=|\mathcal{A}|, B=|\mathcal{B}|$, and any $\epsilon, \delta<1 / 2$,

1. $R_{\epsilon+\delta}^{\|, \text {pub }}(\mathbf{p}) \leq 16\left[\frac{A B \tilde{\nu}^{\epsilon}(\mathbf{p})}{\delta}\right]^{2} \ln \left[\frac{4 A B}{\delta}\right] \log (A B)$,
2. $Q_{\epsilon+\delta}^{\|}(\mathbf{p}) \leq O\left((A B)^{5}\left[\frac{\tilde{\nu}^{\epsilon}(\mathbf{p})}{\delta}\right]^{4} \ln \left[\frac{A B}{\delta}\right] \log (n)\right)$.

The general idea of the proof is to build a communication protocol for $\mathbf{p}$ based on an affine combination $\mathbf{p}=$ $q_{+} \mathbf{p}^{+}-q_{-} \mathbf{p}^{-}$, where $\mathbf{p}^{+}$and $\mathbf{p}^{-}$are local (or quantum) distributions. By sending sufficiently many samples of $\mathbf{p}^{+}$ and $\mathbf{p}^{-}$to the referee (which does not require any communication between Alice and Bob), the referee can estimate these distributions and therefore simulate their affine combination $\mathbf{p}$. To quantify the number of samples that are necessary to achieve some precision, we use Hoeffding's inequality [McD91].

Proposition 7 (Hoeffding's inequality). Let $X$ be a random variable with values in $[a, b]$. Let $X_{t}$ be the $t$-th of $T$ independent trials of $X$, and $S=\frac{1}{T} \sum_{t=1}^{T} X_{t}$.

Then, $\operatorname{Pr}[S-E(X) \geq \beta] \leq e^{-\frac{2 T \beta^{2}}{(b-a)^{2}}}$, and $\operatorname{Pr}[E(X)-S \geq \beta] \leq e^{-\frac{2 T \beta^{2}}{(b-a)^{2}}}$, for any $\beta \geq 0$.
We will also use the following lemma.
Lemma 9. Let $\mathbf{p}$ be a probability distribution on $\mathcal{V}$ with $V=|\mathcal{V}|$, and $e: \mathbb{R}^{+} \rightarrow \mathbb{R}^{+}$. For each $v \in \mathcal{V}$, let $Q_{v}$ be a random variable such that $\forall \beta \geq 0, \operatorname{Pr}\left[Q_{v} \geq p(v)+\beta\right] \leq e(\beta)$ and $\operatorname{Pr}\left[Q_{v} \leq p(v)-\beta\right] \leq e(\beta)$.

Then, given samples $\left\{Q_{v}: v \in \mathcal{V}\right\}$, and without knowing $\mathbf{p}$, we may simulate a probability distribution $\mathbf{p}^{\prime}$ such that $\delta\left(\mathbf{p}^{\prime}, \mathbf{p}\right) \leq 2 V[\beta+e(\beta)]$.
Proof. In order to use the variables $Q_{v}$ as estimations for $p(v)$, we must first make them positive, and then renormalize them so that they sum up to 1 . Let $R_{v}=\max \left\{0, Q_{v}\right\}$. Then we may easily verify that

$$
\begin{aligned}
\operatorname{Pr}\left[R_{v} \geq p(v)+\beta\right] & \leq e(\beta) \\
\operatorname{Pr}\left[R_{v} \leq p(v)-\beta\right] & \leq e(\beta)
\end{aligned}
$$

For any subset $\mathcal{E} \subseteq \mathcal{V}$ of size $E=|\mathcal{E}|$, we also define the estimates $R_{\mathcal{E}}=\sum_{v \in \mathcal{E}} R_{v}$ for $p(\mathcal{E})$. For any $v$, we have $R_{v}-p(v) \geq \beta$ with probability at least $1-e(\beta)$. Therefore, with probability at least $1-E e(\beta)$, we have $R_{v}-p(v) \geq \beta$ simultaneously for all $v \in \mathcal{E}$, and therefore by summation also $R_{\mathcal{E}}-p(\mathcal{E}) \geq E \beta$. Similarly, with probability at least $1-E e(\beta)$, we have $p(v)-R_{v} \geq \beta$ simultaneously for all $v \in \mathcal{E}$, and therefore also $p(\mathcal{E})-R_{\mathcal{E}} \geq E \beta$. Hence, we have the following bounds for $R_{\mathcal{E}}$ For any subset $\mathcal{E} \subseteq \mathcal{V}$ of size $E=|\mathcal{E}|$, we also define the estimates $R_{\mathcal{E}}=\sum_{v \in \mathcal{E}} R_{v}$ for $p(\mathcal{E})$. By summing,

$$
\begin{aligned}
\operatorname{Pr}\left[R_{\mathcal{E}} \geq p(\mathcal{E})+E \beta\right] & \leq E e(\beta) \\
\operatorname{Pr}\left[R_{\mathcal{E}} \leq p(\mathcal{E})-E \beta\right] & \leq E e(\beta)
\end{aligned}
$$

In order to renormalize the estimated probabilities, let $R_{\mathcal{V}}=\sum_{v \in \mathcal{V}} R_{v}$. If $R_{\mathcal{V}}>1$, we use as final estimates $S_{v}=R_{v} / R_{\mathcal{V}}$. On the other hand, if $R_{\mathcal{V}} \leq 1$, we keep $S_{v}=R_{v}$ and introduce a dummy output $\varnothing \notin \mathcal{V}$ with estimated probability $S_{\varnothing}=1-R_{\mathcal{V}}$ (we extend the original distribution to $\mathcal{V} \cup\{\varnothing\}$, setting $p(\varnothing)=0$ ). By outputting $v$ with probability $S_{v}$, we then simulate some distribution $p^{\prime}(v)=E\left(S_{v}\right)$, and it suffices to show that $\left|E\left(S_{\mathcal{E}}\right)-p(\mathcal{E})\right| \leq$ $2 V[\beta+e(\beta)]$ for any $\mathcal{E} \subseteq \mathcal{V} \cup\{\varnothing\}$.

We first upper bound $E\left(S_{\mathcal{E}}\right)$ for $\mathcal{E} \in \mathcal{V}$. Since $S_{\mathcal{E}} \leq R_{\mathcal{E}}$, we obtain from the bounds on $R_{\mathcal{E}}$ that $\operatorname{Pr}\left[S_{\mathcal{E}} \geq\right.$ $p(\mathcal{E})+E \beta] \leq E e(\beta)$. Therefore, we have $S_{\mathcal{E}}<p(\mathcal{E})+E \beta$ with probability at least $1-E e(\beta)$, and $S_{\mathcal{E}} \leq 1$ with probability at most $E e(\beta)$. This implies that $E\left(S_{\mathcal{E}}\right) \leq p(\mathcal{E})+E[\beta+e(\beta)]$.

To lower bound $E\left(S_{\mathcal{E}}\right)$, we note that with probability at least $1-E e(\beta)$, we have $R_{\mathcal{E}}>p(\mathcal{E})-E \beta$, and with probability at least $1-V e(\beta)$, we have $R_{\mathcal{V}}<1+V \beta$. Therefore, with probability at least $1-(E+V) e(\beta)$, both these events happen at the same time, so that $S_{\mathcal{E}}=R_{\mathcal{E}} / R_{\mathcal{V}}>(p(\mathcal{E})-E \beta)(1-V \beta) \geq p(\mathcal{E})-(E+V) \beta$. This implies that $E\left(S_{\mathcal{E}}\right) \geq p(\mathcal{E})-(E+V)[\beta+e(\beta)]$. Since $S_{\varnothing}=1-S_{\mathcal{V}}$, this also implies that $E\left(S_{\varnothing}\right) \leq 2 V[\beta+e(\beta)]$.

Proof of Theorem 8 1. Let $\Lambda=\tilde{\nu}(\mathbf{p}), \mathbf{p}=q_{+} \mathbf{p}^{+}-q_{-} \mathbf{p}^{-}$, with $q_{+}, q_{-} \geq 0, q_{+}+q_{-}=\Lambda$ and $\mathbf{p}^{+}, \mathbf{p}^{-} \in \mathcal{L}$. Let $P^{+}, P^{-}$be protocols for $\mathbf{p}^{+}$and $\mathbf{p}^{-}$, respectively. These protocols use shared randomness but no communication.

To simulate $\mathbf{p}$, Alice and Bob make $T$ independent runs of $P^{+}$, where we label the outcome of the $t$-th run $\left(a_{t}^{+}, b_{t}^{+}\right)$. Similarly, let $\left(a_{t}^{-}, b_{t}^{-}\right)$be the outcome of the $t$-th run of $P^{-}$. They send the list of outcomes to the referee.

The idea is for the referee to estimate $p(a, b \mid x, y)$ based on the $2 T$ samples, and output according to the estimated distribution. Let $P_{t, a, b}^{+}$be an indicator variable which equals 1 if $a_{t}^{+}=a$ and $b_{t}^{+}=b$, and 0 otherwise. Define $P_{t, a, b}^{-}$ similarly. Furthermore, let $P_{t, a, b}=q_{+} P_{t, a, b}^{+}-q_{-} P_{t, a, b}^{-}$. Then $E\left(P_{t, a, b}\right)=p(a, b \mid x, y)$ and $P_{t, a, b} \in\left[-q_{-}, q_{+}\right]$.

Let $P_{a, b}=\frac{1}{T} \sum_{t=1}^{T} P_{t, a, b}$ be the referee's estimate for $p(a, b \mid x, y)$. By Hoeffding's inequality,

$$
\begin{aligned}
& \operatorname{Pr}\left[P_{a, b} \geq p(a, b \mid x, y)+\beta\right] \leq e^{-\frac{2 T \beta^{2}}{\Lambda^{2}}} \\
& \operatorname{Pr}\left[P_{a, b} \leq p(a, b \mid x, y)-\beta\right] \leq e^{-\frac{2 T \beta^{2}}{\Lambda^{2}}}
\end{aligned}
$$

Lemma 9 with $\mathcal{V}=\mathcal{A} \times \mathcal{B}, Q_{a, b}=P_{a, b}$ and $e(\beta)=e^{-\frac{2 T \beta^{2}}{\Lambda^{2}}}$ then implies that the referee may simulate a probability distribution $\mathbf{p}^{\prime}$ such that $\delta\left(\mathbf{p}^{\prime}, \mathbf{p}\right) \leq 2 A B\left(\beta+e^{-\frac{2 T \beta^{2}}{\Lambda^{2}}}\right)$. It then suffices to set $\beta=\frac{\delta}{4 A B}$, and $T=8\left[\frac{A B \Lambda}{\delta}\right]^{2} \ln \left[\frac{4 A B}{\delta}\right]$ to conclude the proof, since Alice sends $2 T \log A$ and Bob sends $2 T \log B$ bits to the referee.

For $\tilde{\nu}^{\epsilon}$, apply this proof to the distribution $\mathbf{p}^{\prime \prime}$ with statistical distance $\delta\left(\mathbf{p}, \mathbf{p}^{\prime \prime}\right) \leq \epsilon$ and $\tilde{\nu}\left(\mathbf{p}^{\prime \prime}\right)=\tilde{\nu}^{\epsilon}(\mathbf{p})$.
Note that the same proof gives an upper bound on $R_{\epsilon+\delta}^{\|, \text {ent }}$ in terms of $\tilde{\gamma}_{2}$.
2. If shared randomness is not available but quantum messages are, then we can use quantum fingerprinting [BCWdW01, Yao03] to send the results of the repeated protocol to the referee. Let $\left(a^{+}(r), b^{+}(r)\right)$ be the outcomes of $P^{+}$using $r$ as shared randomness. We use the random variable $A_{a}^{+}(r)$ as an indicator variable for $a^{+}(r)=a$; similarly $B_{b}^{+}$, and $P_{\mathcal{E}}^{+}=\sum_{(a, b) \in \mathcal{E}} A_{a}^{+} B_{b}^{+}$.

We can easily adapt the proof of Newman's Theorem [New91], to show that there exists a set of $L$ random strings $\mathcal{R}=\left\{r_{1}, \ldots r_{L}\right\}$ such that $\forall x, y,\left|E_{r_{i} \in \mathcal{R}}\left(\tilde{P}_{\mathcal{E}}^{+}\left(r_{i}\right)\right)-E\left(P_{\mathcal{E}}^{+}\right)\right| \leq \alpha$ provided $L \geq \frac{4 n}{\alpha^{2}}$, where $n$ is the input length, and $\tilde{P}_{\mathcal{E}}^{+}$is the random variable where randomness is taken from $\mathcal{R}$. In other words, by taking the randomness from $\mathcal{R}$, we may simulate a probability distribution $\tilde{\mathbf{p}}^{+}$such that $\delta\left(\tilde{\mathbf{p}}^{+}, \mathbf{p}^{+}\right) \leq \alpha$.

For each $a, b \in \mathcal{A} \times \mathcal{B}$, Alice and Bob send $T$ copies of the states $\left|\phi_{a}^{+}\right\rangle=\frac{1}{\sqrt{L}} \sum_{1 \leq i \leq L}\left|A_{a}^{+}\left(r_{i}\right)\right\rangle|1\rangle|i\rangle$ and $\left|\phi_{b}^{+}\right\rangle=\frac{1}{\sqrt{L}} \sum_{1 \leq i \leq L}|1\rangle\left|B_{a}^{+}\left(r_{i}\right)\right\rangle|i\rangle$ to the referee. The inner product is

$$
\left\langle\phi_{a}^{+} \mid \phi_{b}^{+}\right\rangle=\frac{1}{L} \sum_{1 \leq i \leq L}\left\langle A_{a}^{+}\left(r_{i}\right) \mid 1\right\rangle\left\langle 1 \mid B_{b}^{+}\left(r_{i}\right)\right\rangle=\tilde{p}^{+}(a, b \mid x, y)
$$

where the expectation is taken over the random choices $r_{1}, \ldots r_{L}$.
The referee then uses inner product estimation [BCWdW01]: for each copy, he performs a measurement on $\left|\phi_{a}^{+}\right\rangle \otimes$ $\left|\phi_{b}^{+}\right\rangle$to obtain a random variable $Z_{t, a, b}^{+} \in\{0,1\}$ such that $\operatorname{Pr}\left[Z_{t, a, b}^{+}=1\right]=\frac{1-\left|\left\langle\phi_{b}^{+} \mid \phi_{a}^{+}\right\rangle\right|^{2}}{2}$, then he sets $Z_{a, b}^{+}=$ $\frac{1}{T} \sum_{t=1}^{T} Z_{t, a, b}^{+}$. Let $Q_{a, b}^{+}=\sqrt{1-2 Z_{a, b}^{+}}$if $Z_{a, b}^{+} \leq 1 / 2$ and $Q_{a, b}^{+}=0$ otherwise. This serves as an approximation for $\tilde{p}^{+}(a, b \mid x, y)=\left|\left\langle\phi_{b}^{+} \mid \phi_{a}^{+}\right\rangle\right|$, and Hoeffding's inequality then yields

$$
\begin{aligned}
& \operatorname{Pr}\left[Q_{a, b}^{+} \geq \tilde{p}^{+}(a, b \mid x, y)+\beta\right] \leq e^{-\frac{T \beta^{4}}{2}} \\
& \operatorname{Pr}\left[Q_{a, b}^{+} \leq \tilde{p}^{+}(a, b \mid x, y)-\beta\right] \leq e^{-\frac{T \beta^{4}}{2}}
\end{aligned}
$$

Let $Q_{a, b}^{-}$be an estimate for $\tilde{p}^{-}(a, b \mid x, y)$ obtained using the same method. The referee then obtains an estimate for
$\tilde{p}(a, b \mid x, y)=q_{+} \tilde{p}^{+}(a, b \mid x, y)-q_{-} \tilde{p}^{-}(a, b \mid x, y)$, by setting $Q_{a, b}=q_{+} Q_{a, b}^{+}+q_{-} Q_{a, b}^{-}$, such that

$$
\begin{aligned}
& \operatorname{Pr}\left[Q_{a, b} \geq \tilde{p}(a, b \mid x, y)+\beta\right] \leq 2 e^{-\frac{T \beta^{4}}{2 \Lambda^{4}}}, \\
& \operatorname{Pr}\left[Q_{a, b} \leq \tilde{p}(a, b \mid x, y)-\beta\right] \leq 2 e^{-\frac{T \beta^{4}}{2 \Lambda^{4}}} .
\end{aligned}
$$

Lemma 0 with $e(\beta)=2 e^{-\frac{T \beta^{4}}{2 \Lambda^{4}}}$ then implies that the referee may simulate a probability distribution $\mathbf{p}^{s}$ such that $\delta\left(\mathbf{p}^{s}, \tilde{\mathbf{p}}\right) \leq 2 A B\left(\beta+2 e^{-\frac{T \beta^{4}}{2 \Lambda^{4}}}\right)$. Since $\delta(\tilde{\mathbf{p}}, \mathbf{p}) \leq \Lambda \alpha$, we need to pick $T, L=\frac{4 n}{\alpha}$ large enough so that $\Lambda \alpha+$ $2 A B\left[\beta+2 e^{-T \beta^{4} / 2 \Lambda^{4}}\right] \leq \delta$. Setting $\alpha=\frac{\delta}{2 \Lambda}, \beta=\frac{\delta}{8 A B}, T=2 \frac{\Lambda^{4}}{\beta^{4}} \ln \left(\frac{16 A B}{\delta}\right)=2^{13}\left[\frac{A B \Lambda}{\delta}\right]^{4} \ln \left(\frac{16 A B}{\delta}\right)$ and $L=$ $\frac{4 n}{\alpha^{2}}=\frac{16 n \Lambda^{2}}{\delta^{2}}$, the total complexity of the protocol is $4 A B T(\log (L)+2)=O\left((A B)^{5}\left[\frac{\Lambda}{\delta}\right]^{4} \ln \left[\frac{A B}{\delta}\right] \log (n)\right.$ ), (we may assume that $\frac{\Lambda}{\delta} \leq n^{1 / 4}$, otherwise this protocol performs worse than the trivial protocol).

In the case of Boolean functions, corresponding to correlations $C_{f}(x, y) \in\{ \pm 1\}$ (see Def. [4], the referee's job is made easier by the fact that he only needs to determine the sign of the correlation with probability $1-\delta$. This allows us to get some improvements in the upper bounds. Similar improvements can be obtained for other types of promises on the distribution.

Theorem 9. Let $f:\{0,1\}^{n} \times\{0,1\}^{n} \rightarrow\{0,1\}$, with associated sign matrix $C_{f}$, and $\epsilon, \delta<1 / 2$.

1. $R_{\delta}^{\|, \text {,pub }}(f) \leq 4\left[\frac{\tilde{\nu}^{\epsilon}\left(C_{f}\right)}{1-2 \epsilon}\right]^{2} \ln \left(\frac{1}{\delta}\right)$,
2. $Q_{\delta}^{\|}(f) \leq O\left(\log (n)\left[\frac{\tilde{\nu}^{\epsilon}\left(C_{f}\right)}{1-2 \epsilon}\right]^{4} \ln \left(\frac{1}{\delta}\right)\right)$.

From Lemmas [5]and 6, these bounds may also be expressed in terms of $\gamma_{2}^{\alpha}$, and the best upper bounds are obtained from $\gamma_{2}^{\infty}\left(C_{f}\right)=\frac{1}{\epsilon^{\text {ent }}\left(C_{f}\right)}$. The first item then coincides with the upper bound of [LS09].

Together with the bound between $\tilde{\nu}$ and $\tilde{\gamma}_{2}$ from Section[5, and the lower bounds on communication complexity from Section 3 Theorems 8 and 9 immediately imply the following corollaries.
Corollary 4. Let $f:\{0,1\}^{n} \times\{0,1\}^{n} \rightarrow\{0,1\}$. For any $\epsilon, \delta<1 / 2$, if $Q_{\epsilon}^{\text {ent }}(f) \leq q$, then

1. $R_{\delta}^{\|, \text {pub }}(f) \leq K_{G}^{2} \cdot 2^{2 q+2} \ln \left(\frac{1}{\delta}\right) \frac{1}{\left(1-2 \epsilon \epsilon^{2}\right.}$,
2. $Q_{\delta}^{\|}(f) \leq O\left(\log (n) 2^{4 q} \ln \left(\frac{1}{\delta}\right) \frac{1}{\left(1-2 \epsilon \epsilon^{4}\right.}\right)$.

Let $\mathbf{p} \in \mathcal{C}$ be a distribution with inputs in $\mathcal{X} \times \mathcal{Y}$ with $|\mathcal{X} \times \mathcal{Y}| \leq 2^{n}$, and outcomes in $\mathcal{A} \times \mathcal{B}$ with $A=|\mathcal{A}|, B=|\mathcal{B}|$. For any $\epsilon, \delta<1 / 2$, if $Q_{\epsilon}^{\mathrm{ent}}(\mathbf{p}) \leq q$, then
3. $R_{\epsilon+\delta}^{\|, \mathrm{pub}}(\mathbf{p}) \leq O\left(2^{4 q} \frac{(A B)^{4}}{\delta^{2}} \ln ^{2}\left[\frac{A B}{\delta}\right]\right)$,
4. $Q_{\epsilon+\delta}^{\|}(\mathbf{p}) \leq O\left(2^{8 q} \frac{(A B)^{9}}{\delta^{4}} \ln \left[\frac{A B}{\delta}\right] \log (n)\right)$.

The first two items can be compared to results of Yao, Shi and Zhu, and Gavinsky et al. [Yao03, SZ08, GKd06], who show how to simulate any (logarithmic) communication protocol for Boolean functions in the simultaneous messages model, with an exponential blowup in communication. The last two items extend these results to arbitrary non-signaling distributions.

In particular, Item 3 gives in the special case $q=0$, that is, $\mathbf{p} \in \mathcal{Q}$, a much simpler proof of the constant upper bound on approximating quantum distributions, which Shi and Zhu prove using sophisticated techniques based on diamond norms [SZ08]. Moreover, Item 3 is much more general as it also allows to simulate protocols requiring quantum communication in addition to entanglement. As for Item 4, it also has new interesting consequences. For example, it implies that quantum distributions $(q=0)$ can be approximated with logarithmic quantum communication in the simultaneous messages model, using no additional resources such as shared randomness, and regardless of the amount of entanglement in the bipartite state measured by the two parties.

## 7 Conclusion and open problems

By studying communication complexity in the framework provided by the study of quantum non-locality (and beyond), we have given very natural and intuitive interpretations of the otherwise very abstract lower bounds of Linial and Shraibman. Conversely, bridging this gap has allowed us to port these very strong and mathematically elegant lower bound methods to the much more general problem of simulating non-signaling distributions.

Since many communication problems may be reduced to the task of simulating a non-signaling distribution, we hope to see applications of this lower bound method to concrete problems for which standard techniques do not apply, in particular for cases that are not Boolean functions, such as non-Boolean functions, partial functions or relations. Let us also note that our method can be generalized to multipartite non-signaling distributions, and will hopefully lead to applications in the number-on-the-forehead model, for which quantum lower bounds seem hard to prove.

In the case of binary distributions with uniform marginals (which includes in particular Boolean functions), Tsirelson's theorem (Theorem 2) and the existence of Grothendieck's constant (Proposition (4) imply that there is at most a constant gap between $\nu$ and $\gamma_{2}$. For this reason, it was known that Linial and Shraibman's factorization norm lower bound technique give lower bounds of the same of order for classical and quantum communication (note that this is also true for the related discrepancy method). Despite the fact that Tsirelson's theorem and Grothendieck's inequality are not known to extend beyond the case of Boolean outcomes with uniform marginals, we have shown that in the general case of distributions, there is also a constant gap between $\tilde{\nu}$ and $\tilde{\gamma}_{2}$. While this may be seen as a negative result, this also reveals interesting information about the structure of the sets of local and quantum distributions. In particular, this could have interesting consequences for the study of non-local games.

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## A Proof of Lemma 3

We give the proof of Lemma 3, which relates the outcome of communication protocols to vectors of bounded norm.
Lemma 3 ([Kre95], Yao93, LS09]). Let $\left(C, M_{A}, M_{B}\right)$ be a distribution simulated by a quantum protocol with shared entanglement using $q_{A}$ qubits of communication from Alice to Bob and $q_{B}$ qubits from Bob to Alice. There exist vectors $\vec{a}(x), \vec{b}(y)$ with $\|\vec{a}(x)\| \leq 2^{q_{B}}$ and $\|\vec{b}(y)\| \leq 2^{q_{A}}$ such that $C(x, y)=\vec{a}(x) \cdot \vec{b}(y)$.

The proof relies on the following observation:
Claim 1. Let $\left|\psi_{t}\right\rangle$ be the entangled state shared by Alice and Bob after the first $t=t_{A}+t_{B}$ qubits of communication ( $t_{A}$ bits from Alice to Bob, and $t_{B}$ bits from Bob to Alice). This state may be written as $\left|\psi_{t}\right\rangle=$ $\sum_{i \in I} \mu_{i} \sum_{T \in\{0,1\}^{t}} A_{T}\left|\alpha^{(i)}\right\rangle B_{T}\left|\beta^{(i)}\right\rangle$, where $\sum_{i}\left|\mu_{i}\right|^{2}=1,\left\{\left|\alpha^{(i)}\right\rangle: \forall i \in I\right\}$ and $\left\{\left|\beta^{(i)}: \forall i \in I\right\rangle\right\}$ are orthonormal bases for Alice and Bob's initial registers respectively and $A_{T}, B_{T}$ are linear operators such that:

- $A_{0}, B_{0}$ are the identity operators on Alice and Bob's initial registers, respectively,
- $A_{T}$ are linear operators acting on Alice's initial register and depending on her input only, satisfying

$$
\sum_{T \in\{0,1\}^{t}} \| A_{T}\left|\psi_{A}\right\rangle \|^{2}=2^{t_{B}}
$$

for all (unit) state $\left|\psi_{A}\right\rangle$ of Alice's register.

- $B_{T}$ are linear operators depending on Bob's input only, satisfying $\sum_{T \in\{0,1\}^{t}} \| B_{T}\left|\psi_{B}\right\rangle \|^{2}=2^{t_{A}}$ for all (unit) state $\left|\psi_{B}\right\rangle$ of Bob's register.

Proof of Claim [] We prove this by induction over $t$. This is true for $t=0$, since using Schmidt decomposition, we may write the initial entangled state shared by Alice and Bob, before the quantum communication protocol is initiated, as $\left|\psi_{0}\right\rangle=\sum_{i \in I} \mu_{i}\left|\alpha^{(i)}\right\rangle\left|\beta^{(i)}\right\rangle$, where $\sum_{i}\left|\mu_{i}\right|^{2}=1$ and $\left\{\left|\alpha^{(i)}\right\rangle: \forall i \in I\right\}$ and $\left\{\left|\beta^{(i)}: \forall i \in I\right\rangle\right\}$ are orthonormal bases for Alice and Bob's registers respectively (as is, these are actually just orthonormal, but we can always obtain a basis by setting $\mu_{i}=0$ for the missing basis vectors).

If this is true for $t-1$, then we have $\left|\psi_{t-1}\right\rangle=\sum_{i \in I} \mu_{i} \sum_{T \in\{0,1\}^{t-1}} A_{T}\left|\alpha^{(i)}\right\rangle B_{T}\left|\beta^{(i)}\right\rangle$, where $\sum_{T \in\{0,1\}^{t-1}} \| A_{T}\left|\alpha^{(i)}\right\rangle \|^{2}=2^{t_{B}}$ and $\sum_{T \in\{0,1\}^{t-1}} \| B_{T}\left|\beta^{(i)}\right\rangle \|^{2}=2^{t_{A}-1}$ for all $i \in I$ (we assume without loss of generality that the $t$ 's qubit is sent by Alice to Bob). Alice's operation at turn $t$ will be to apply some unitary operation $U_{t}$ on her register, then send one of the qubits in her register to Bob. By isolating this qubit, we define the linear operators $A_{T 0}$ and $A_{T 1}$ to be such that $U_{t} A_{T}\left|\alpha^{(i)}\right\rangle=A_{T 0}\left|\alpha^{(i)}\right\rangle|0\rangle+A_{T 1}\left|\alpha^{(i)}\right\rangle|1\rangle$ for all $i \in I$. Unitarity then implies that $\| A_{T 0}\left|\alpha^{(i)}\right\rangle\left\|^{2}+\right\| A_{T 1}\left|\alpha^{(i)}\right\rangle\left\|^{2}=\right\| A_{T}\left|\alpha^{(i)}\right\rangle \|^{2}$, and as a consequence $\sum_{T \in\{0,1\} t} \| A_{T}\left|\alpha^{(i)}\right\rangle \|^{2}=2^{t_{B}}$. We then have

$$
\begin{align*}
\left|\psi_{t}\right\rangle & =\sum_{i \in I} \mu_{i} \sum_{T \in\{0,1\}^{t-1}}\left[A_{T 0}\left|\alpha^{(i)}\right\rangle|0\rangle B_{T}\left|\beta^{(i)}\right\rangle+A_{T 1}\left|\alpha^{(i)}\right\rangle|1\rangle B_{T}\left|\beta^{(i)}\right\rangle\right]  \tag{1}\\
& =\sum_{i \in I} \mu_{i} \sum_{T \in\{0,1\}^{t}} A_{T}\left|\alpha^{(i)}\right\rangle B_{T}\left|\beta^{(i)}\right\rangle \tag{2}
\end{align*}
$$

where, for all $T \in\{0,1\}^{t-1}$, we have defined linear operators $B_{T 0}, B_{T 1}$ such that $B_{T 0}\left|\beta^{(i)}\right\rangle=|0\rangle B_{T}\left|\beta^{(i)}\right\rangle$ and $B_{T 1}\left|\beta^{(i)}\right\rangle=|1\rangle B_{T}\left|\beta^{(i)}\right\rangle$ for all $i \in I$, considering that the additional qubit is in Bob's hands at the end of turn $t$. Furthermore, we have $\| B_{T 0}\left|\beta^{(i)}\right\rangle\left\|^{2}+\right\| B_{T 1}\left|\beta^{(i)}\right\rangle\left\|^{2}=2\right\| B_{T}\left|\beta^{(i)}\right\rangle \|^{2}$, and as a consequence $\sum_{T \in\{0,1\}^{t}} \| B_{T}\left|\beta^{(i)}\right\rangle \|^{2}=$ $2^{t_{A}}$, which completes the proof of our claim.

Proof of Lemma 3 At the end of the quantum communication protocol, Alice and Bob share a quantum state $\left|\psi_{q}\right\rangle$ satisfying Claim 1 for $t=q$. Alice and Bob then perform binary $(\{+1,-1\}$-valued) measurements $A$ and $B$ on their respective parts of the state. By orthonormality of the states $\left|\psi_{q}^{(i)}\right\rangle$, we have for the correlation

$$
\begin{align*}
C & =\left\langle\psi_{q}\right| A B\left|\psi_{q}\right\rangle  \tag{3}\\
& =\sum_{i, j \in I} \mu_{i}^{*} \mu_{j} \sum_{T, U \in\{0,1\}^{q}}\left\langle\alpha^{(i)}\right| A_{T}^{\dagger} A A_{U}\left|\alpha^{(j)}\right\rangle\left\langle\beta^{(i)}\right| B_{T}^{\dagger} B B_{U}\left|\beta^{(j)}\right\rangle . \tag{4}
\end{align*}
$$

We may now define the vectors $\vec{a}(x)$ and $\vec{b}(y)$ in a $2^{2 t}|I|^{2}$-dimensional complex vector space, with coordinates

$$
\begin{align*}
a_{T U i j}(x) & =\mu_{i}\left\langle\alpha^{(j)}\right| A_{U}^{\dagger} A A_{T}\left|\alpha^{(i)}\right\rangle  \tag{5}\\
b_{T U i j}(x) & =\mu_{j}\left\langle\beta^{(i)}\right| B_{T}^{\dagger} B B_{U}\left|\beta^{(j)}\right\rangle, \quad \forall T, U \in\{0,1\}^{q}, i, j \in I \tag{6}
\end{align*}
$$

so that $C=\vec{a}(x) \cdot \vec{b}(y)$. Moreover, using the fact that the $\left|\alpha^{(j)}\right\rangle$ 's define an orthonormal basis for Alice's register and the property on the norms of the operators $A_{T}$, we have

$$
\begin{align*}
\|\vec{a}(x)\|^{2} & \left.=\sum_{i, j \in I}\left|\mu_{i}\right|^{2} \sum_{T, U \in\{0,1\}^{q}}\left|\left\langle\alpha^{(j)}\right| A_{U}^{\dagger} A A_{T}\right| \alpha^{(i)}\right\rangle\left.\right|^{2}  \tag{7}\\
& =\sum_{i \in I}\left|\mu_{i}\right|^{2} \sum_{T, U \in\{0,1\}^{q}} \| A_{U}^{\dagger} A A_{T}\left|\alpha^{(i)}\right\rangle \|^{2}  \tag{8}\\
& \leq \sum_{i \in I}\left|\mu_{i}\right|^{2} \sum_{T, U \in\{0,1\}^{q}} \| A_{U}^{\dagger}\left|\phi_{T}^{(i)}\right\rangle\left\|^{2}\right\| A_{T}\left|\alpha^{(i)}\right\rangle \|^{2}=2^{2 q_{B}} \tag{9}
\end{align*}
$$

where $\left|\phi_{T}^{(i)}\right\rangle$ is the renormalized state $A A_{T}\left|\alpha^{(i)}\right\rangle$. So, we have $\|\vec{a}(x)\| \leq 2^{q_{B}}$, and similarly $\|\vec{b}(y)\| \leq 2^{q_{A}}$.


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