

# Dynamic vs Oblivious Routing in Network Design

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**Abstract.** Consider the robust network design problem of finding a minimum cost network with enough capacity to route all traffic demand matrices in a given polytope. We investigate the impact of different routing models in this robust setting: in particular, we compare *oblivious* routing, where the routing between each terminal pair must be fixed in advance, to *dynamic* routing, where routings may depend arbitrarily on the current demand. Our main result is a construction that shows that the optimal cost of such a network based on oblivious routing (fractional or integral) may be a factor of  $\Omega(\log n)$  more than the cost required when using dynamic routing. This is true even in the important special case of the asymmetric hose model. This answers a question in [6], and is tight up to constant factors. Our proof technique builds on a connection between expander graphs and robust design for single-sink traffic patterns [7].

## 1 Introduction

One of the most widely studied applications of robustness in discrete optimization has been in the context of network design. This is partly motivated by the fact that traffic demands in modern data networks are often hard to determine and/or are rapidly changing. In one general model (cf. [5]), the input consists of a graph (network topology) where each edge comes with a cost to reserve capacity. In addition, a universe of possible demand matrices is specified as a polyhedron  $\mathcal{P}$  (or more generally, as a convex body). In this paper our focus is on undirected demands and so for a demand matrix  $D$ , the entries  $D_{ij}$  and  $D_{ji}$  normally represent the same demand, and are hence equal. The problem is to design a minimum cost network such that each demand matrix in the polytope can be routed (according to routing models we describe shortly) in the resulting capacitated network. Typically we seek to install edge capacities so that the sum of costs is minimized, but other cost measures such as minimizing the maximum

congestion are also considered in the literature. We refer to the recent survey by Chekuri [6] for a discussion of these models and previous work.

Since demands are potentially changing, there are two prime natural routing models that are considered. The first is *dynamic routing*: for any given demand  $D \in \mathcal{P}$ , we may use a traffic routing tailored to this demand. We consider only the case where the routing may be an arbitrary multicommodity flow, i.e. traffic flows may be fractional. We also refer to this routing model as FR. Dynamic routing, out of all possible routing models, clearly leads to the cheapest possible solution. However, this model is typically considered impractical.

On the other extreme, *oblivious routing* models, inspired by routing in packet networks, ask for a *routing template* that defines ahead of time how any future demands will be routed. For each node pair  $i, j$ , the template  $f$  specifies a unit network flow  $f_{ij}$  between  $i$  and  $j$ . The interpretation is that if there is a future demand of  $D_{ij}$  between nodes  $i, j$ , then along each  $ij$  path  $P$ , we should route  $D_{ij}f_{ij}(P)$  flow on this path. This scheme is obviously much simpler than dynamic routing, and has the advantage that routings are stable, which can be important in maintaining Quality of Service guarantees.

Flow templates may be either fractional, in which case they are called *multi-path routings* (MPR), or integral, in which case they are called *single-path routings* (SPR). We also discuss a special case of SPR templates called *tree* templates where the support of  $f$  induces a tree in the network; we refer to this model as TR. We can now formally define the robust network design problem (cf. [7]):

**Definition 1.** *Given a graph  $G = (V, E)$  with  $|V| = n$ , edge costs  $c : E \rightarrow \mathbb{R}^+$ , a polytope  $\mathcal{P}$  of demand matrices, and a routing model (FR, SPR, MPR, TR), the robust network design problem is defined as follows. Find a minimum cost capacity installation of edge capacities  $u : E \rightarrow \mathbb{R}^+$  so that all demand matrices in  $\mathcal{P}$  can be routed in the given routing model. The cost of capacity installation  $u$  is given by  $\sum_{e \in E} u(e)c(e)$ .*

For a given instance of robust network design  $(G, c, \mathcal{P})$ , we use  $\text{OPT}_{FR}(G, c, \mathcal{P})$ ,  $\text{OPT}_{MPR}(G, c, \mathcal{P})$ ,  $\text{OPT}_{SPR}(G, c, \mathcal{P})$  and  $\text{OPT}_{TR}(G, c, \mathcal{P})$  to denote the corresponding cost of an optimally designed robust network for the four routing models. If the context is clear, we may simply write, for instance,  $\text{OPT}_{FR}$ .

Obviously we have

$$\text{OPT}_{FR} \leq \text{OPT}_{MPR} \leq \text{OPT}_{SPR} \leq \text{OPT}_{TR}. \quad (1)$$

It was already known that the gap between  $\text{OPT}_{FR}$  and  $\text{OPT}_{SPR}$  is  $O(\log n)$  (credited to A. Gupta, cf. [6]). This follows by an application of the approximation of arbitrary metrics by tree metrics [9]. One can further show, by similar arguments but now using a theorem of [1] instead, that the gap between  $\text{OPT}_{FR}$  and  $\text{OPT}_{TR}$  is at most  $\tilde{O}(\log n)$ , where  $\tilde{O}$  hides an  $O(\text{poly } \log \log n)$  factor.

*Our Results.* In this paper, we seek to understand to what extent these gaps are realizable; in other words, for any pair of routing methods, what is the maximum possible gap between the costs of their optimal solution?

In short, the answer is that except for the pair  $\{\text{OPT}_{MPR}, \text{OPT}_{SPR}\}$ , the gap between any pair in (1) can be as large as  $\Omega(\log n)$ ; this is essentially tight. The exception, the gap between  $\text{OPT}_{MPR}$  and  $\text{OPT}_{SPR}$ , is at least polylogarithmically large ( $\Omega(\log^{1/4-\epsilon}(n))$  for any  $\epsilon > 0$ ). This follows indirectly from an approximation-preserving reduction [16] from the uniform buy-at-bulk problem to the general robust network design ( $\text{OPT}_{SPR}$ ). Andrews [2,3] showed that under a plausible complexity theoretic assumption ( $NP \not\subseteq ZPTIME(n^{\text{polylog } n})$ ), there is no polytime algorithm for uniform buy-at-bulk with approximation guarantee within  $O(\log^{1/4-\epsilon}(n))$ , for any  $\epsilon > 0$ . These two results imply that the gap between  $\text{OPT}_{MPR}, \text{OPT}_{SPR}$  must be similarly large, since  $\text{OPT}_{MPR}$  is polytime computable, and could otherwise be used to approximate (the decision form of) uniform buy-at-bulk. We will demonstrate all the other gaps in this paper; most of the work is on the most interesting case, between  $\text{OPT}_{FR}$  and  $\text{OPT}_{MPR}$ .

*Discussion.* In the robustness paradigm, the question of how large these gaps can be is asked for specific classes of demand polyhedra. A class that has received much attention consists of the so-called “hose models” which come in symmetric and asymmetric flavours. In the symmetric hose model, each terminal  $v$  has an associated *marginal*  $b_v$ , which represents an upper bound on the *total* amount of traffic that can terminate at  $v$ . The demand polytope consists of all symmetric demands which do not violate these “hose” constraints; i.e.  $\sum_j D_{ij} \leq b_i$  for each terminal  $i$ . The asymmetric hose problem is similar, but the terminals are divided into sources and sinks; all demand is between source and sink nodes, and again, total demand to or from a terminal cannot exceed its marginal. Classes such as the hose model arise naturally in switch design, but they were also motivated by applications in data networks [10,11]; one of these is referred to as the *virtual private network* (VPN) problem.

It is implicit in Fingerhut et al. [10] and explicit in Gupta et al. [11] that in the *symmetric* hose model,  $\text{OPT}_{MPR} \leq \text{OPT}_{SPR} \leq 2\text{OPT}_{FR}$ . However, the gap instance between  $\text{OPT}_{MPR}$  and  $\text{OPT}_{FR}$  that we demonstrate in this paper is in fact an instance of the asymmetric hose problem, and hence there is a logarithmic gap for this latter model.<sup>4</sup> We describe a class of graphs  $G$ , cost function  $c$ , and a demand polytope  $\mathcal{P}$ , such that  $\text{OPT}_{FR}(G, c, \mathcal{P}) = O(n)$  but  $\text{OPT}_{SPR}(G, c, \mathcal{P}) = \Omega(n \log n)$  and  $\text{OPT}_{MPR}(G, c, \mathcal{P}) = \Omega(n \log n)$ . The polytope  $\mathcal{P}$  has the property that all demands share a common “sink” node.

It turns out that the problem of designing an SPR routing template for our gap instance corresponds to the well-studied rent-or-buy network flow problem, which is a generalization of the Steiner tree problem. In this problem there is only one demand matrix instead of a polytope of demands, but the cost function is concave. We sketch the lower bound argument for  $\text{OPT}_{SPR}$  separately in Section 2.3 since it is much simpler; it proceeds by showing that the optimal SPR templates may be assumed to be tree templates for our gap instance.

The lower bound for  $\text{OPT}_{MPR}$  is more involved. We show that the cost of an MPR template for our gap instance can be characterized by a network design

<sup>4</sup> This rectifies an earlier assertion (cf. Theorem 4.6 in [6]).

problem that we call *buy-and-rent*. Again there is only one demand to be satisfied, but the cost function is more complex. The buy-and-rent cost function seems to be new and natural: briefly, instead of asking that each edge be either rented or bought, it allows that capacity may be partially bought and the rest rented. This new cost function is more amenable to analysis, and leads to our lower bound for  $\text{OPT}_{MPR}$ .

*Relation to congestion lower bounds.* We remark that our lower bounds for the total cost model also imply lower bounds for minimizing the maximum congestion, essentially because if every edge had congestion at most  $\alpha$ , the total cost would also be bounded by a factor  $\alpha$ . Since the polytope  $\mathcal{P}$  we use is a subset of the single-sink demands routable in  $G$ , this also implies a result in [13] which gives an  $\Omega(\log n)$  bound for congestion via oblivious routing of single sink demands (although their analysis also extends to the case of lower bounding performance of a general online algorithm). Congestion minimization problems can be seen as equivalent to a robust optimization where one uses maximum edge congestion as a cost function; simply take the polytope consisting of *all* single-sink demands which are routable in  $G$  (this is a superset of our choice  $\mathcal{P}$ ). The construction in [13] uses meshes (grids), building on work of [4,14]. This construction does not seem to extend to the total cost model however, and we use instead a construction based on expanders, extending and simplifying a connection shown in earlier work [7].

*Gaps for tree routing.* In Section 3 we give a family of instances (using a different demand polyhedron) showing that the gap between  $\text{OPT}_{SPR}$  and  $\text{OPT}_{TR}$  can be  $\Omega(\log n)$ . This immediately implies that the gaps between  $\text{OPT}_{FR}$  and  $\text{OPT}_{TR}$  and between  $\text{OPT}_{MPR}$  and  $\text{OPT}_{TR}$  is  $\Omega(\log n)$  for this family of instances.

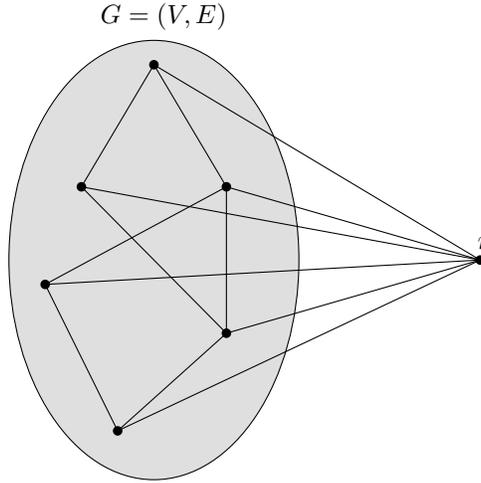
## 2 A gap example

### 2.1 A robust network design instance

Let  $G = (V, E)$  be a graph on  $n$  nodes with constant degree  $d \geq 3$  and edge-expansion at least 1; i.e. we have that  $|\delta_G(S)| \geq |S|$  for all  $S \subseteq V$  with  $|S| \leq n/2$ . Here  $\delta_G(S)$  denotes the set of edges in  $E$  with one end-point in  $S$  and the other outside  $S$ . It is well-known that such edge-expanders with the above parameters exist. Now add a special sink node  $r$  to  $V$  to obtain our instance  $\bar{G} = (\bar{V}, \bar{E}) = (V \cup \{r\}, E \cup \{vr : v \in V\})$ ; see Figure 1.

We look at a single-sink hose model (cf. [7]), where our demands come from a polytope  $\mathcal{P}$  defined as follows. We have a specified marginal capacity  $b_v$  at each node:  $b_r = \beta n$  (where  $0 < \beta < 1$ ), and  $b_v = 1$  for all  $v \in V$ . Each demand matrix  $D_{ij} \in \mathcal{P}$  has the property that  $\sum_j D_{ij} \leq b_i$  for each node  $i \in \bar{V}$ , and  $D_{ij} > 0$  only if  $r \in \{i, j\}$ . Although we often think of nodes routing flow towards the sink, the demands and flows are undirected in this paper.

Thus each demand matrix we must support, identifies a single-sink network flow problem. It is a simple exercise to see that:



**Fig. 1.** The gap instance.  $G$  is a  $d$ -regular expander

**Lemma 1.** *If  $b_r$  is an integer, then our network is robust for  $\mathcal{P}$  and a given routing model if and only if for each subset  $X$  of  $b_r$  nodes in  $G$ , there is enough capacity to route one unit from each node in  $X$  to  $r$ , using the prescribed routing model.*

We use this fact below. Finally, we also assign costs to the edges: each edge of  $G$  has cost 1, and each edge in  $\delta_{\bar{G}}(r)$  has cost  $1/\beta$ .

Our main result is the following theorem:

**Theorem 1.** *For  $\beta = 1/\log n$ , there is a dynamic routing for the single-sink hose model instance (defined above) of cost  $O(n)$ , but every MPR solution (and hence every SPR solution) has cost  $\Omega(n \log n)$ .*

The first assertion is proved in the next section. In Section 2.3, we see that determining  $\text{OPT}_{SPR}$  for single-sink hose models is equivalent to the well-studied *single-sink rent-or-buy* problem, and the rent-or-buy problem always has a tree solution. This can be used to show that  $\text{OPT}_{SPR} = \Omega(n \log n)$  for our instance with  $\beta = 1/\log n$ . We give a sketch of a proof of this since it is considerably simpler than (but implied by) the proof of the corresponding bound for MPR. This MPR lower bound is demonstrated in Section 2.4.

We assume throughout the paper that  $b_r = \beta n$  is an integer.

## 2.2 A solution for the dynamic routing model

Put capacity  $\beta$  on each edge of  $\delta_{\bar{G}}(r)$ , and capacity 1 on each edge of  $G$ . Clearly, the cost of this reservation is  $O(n)$  independent of  $\beta$ . We show that this is a valid FR capacity reservation. Using Lemma 1 it suffices to show that for any subset of  $\beta n$  nodes  $X$  in  $G$ , all nodes in  $X$  can simultaneously route a unit flow to  $r$ . To

this end, we add a new node  $t$  to  $\tilde{G}$  and edges  $vt$  for  $v \in X$  with unit capacity to form graph  $G'$ . We show that  $G'$  supports a  $t$ - $r$  flow of size  $|X| = \beta n$ . By the max-flow min-cut theorem it suffices to show that all  $r$ - $t$  cuts in  $G'$  have size at least  $\beta n$ , i.e. that for each  $S \subseteq V$  we have  $|\delta_{G'}(S \cup \{t\})| \geq \beta n$ .

We have

$$|\delta_{G'}(S \cup \{t\})| = \beta|S| + |X \setminus S| + |\delta_G(S)|.$$

Now, if  $|S| \leq n/2$  then using the fact that for  $G$  we have  $|\delta_G(S)| \geq |S|$  we get

$$\begin{aligned} |\delta_{G'}(S \cup \{t\})| &\geq \beta|S| + |X \setminus S| + |S| \\ &\geq \beta|S| + |X| \\ &\geq |X|. \end{aligned}$$

And if  $|S| > n/2$  then using the fact that for  $G$  we have  $|\delta_G(S)| \geq n - |S|$  we get

$$\begin{aligned} |\delta_{G'}(S \cup \{t\})| &\geq \beta|S| + |X \setminus S| + n - |S| \\ &\geq \beta|S| + |X \setminus S| + \beta(n - |S|) \\ &= \beta n + |X \setminus S| \\ &\geq \beta n. \end{aligned}$$

Hence the above capacity reservation can support the FR routing model and costs  $O(n)$ .

### 2.3 Rent-or-buy: an $\Omega(\log n)$ gap between FR and SPR

Note that the optimal cost oblivious SPR network can be cast as a minimum cost (unsplittable) flow problem as follows. Each node  $v \in V$  must route one unit of flow on a path  $P_v$  to  $r$  and the overall (truncated) cost of path choices is:  $\sum_e c(e) \min\{N(e), b_r\}$ , where  $N(e)$  is the number of nodes  $v$  whose path to  $r$  used the edge  $e$ . Clearly, if the capacity of each edge is  $\min\{N(e), b_r\}$ , then we have sufficient capacity to route any demand matrix in  $\mathcal{P}$  using as a template the paths  $P_v$ . The converse is in fact also true and easy; any template gives rise to a corresponding integer flow whose truncated cost is the same.

This truncated routing cost problem is simply a so-called single-sink *rent-or-buy* (SSROB) problem (see e.g. [8,12]). We are given a network  $G$  with edge costs  $c(e)$ , and a special sink node  $t$ . A parameter  $B \geq 1$  is also given (this will equal  $b_r$  in the instance corresponding to SPR). We also have a list of sources  $s_i$  for  $i = 1, 2, \dots, p$ ; each source needs to route to the sink  $t$ . For each edge in the network, we may either purchase it at a cost of  $Bc(e)$ , in which case it is deemed to have infinite capacity, or otherwise we may rent it. In that case, we must pay  $c(e)$  per unit of capacity that we use on the edge. The goal is to find which edges to buy and which to rent in order to support a flow from each node to  $t$ , at the smallest possible cost. In other words, we seek a fractional flow  $\mathbf{f}$  of the demands that minimizes  $\sum_{e \in E} c(e) \min\{f(e), B\}$  (we will see next that the optimal solution will always be integral, ensuring that we do in fact have a

correspondence with SPR). In general, we may also consider such *single-sink flow problems with concave costs*  $\sum_e g_e(f(e))$  where each  $g_e$  is a concave function.

The following result is immediate from the concavity of the cost function:

**Proposition 1.** *Any single-sink flow problem with nondecreasing concave costs has an optimal solution whose support is a tree. In particular, such an optimal solution always exists for the SSROB problem.*

*Proof.* Let  $\mathbf{f}$  be a flow giving an optimal solution to the flow problem, chosen so that  $\text{supp}(\mathbf{f})$  is setwise minimal. We show that then  $\text{supp}(\mathbf{f})$  must form a tree.

Let us consider  $\mathbf{f}$  as a directed flow, where each terminal sends flow to the sink. If there is any directed cycle in the support of  $\mathbf{f}$ , then we may simply reduce flow on this cycle until some arc becomes zero; this does not increase the cost since our cost function is nondecreasing. So we may assume our support is acyclic in the directed sense. Suppose now that there is some undirected cycle  $C$  in the support which by assumption corresponds to some forward (traversing  $C$  in order) arcs  $F$  and some reverse arcs  $R$ . Let  $\epsilon = \min\{f(a) : a \in R \cup F\}$ . Define two solutions  $\mathbf{f}^+, \mathbf{f}^-$  by  $f^\pm(a) = f(a) \pm \epsilon$  for  $a \in F$ , and  $f^\pm(a) = f(a) \mp \epsilon$  for  $a \in R$ . By concavity,  $C(\mathbf{f}) \geq (1/2)[C(\mathbf{f}^+) + C(\mathbf{f}^-)]$ . Then since  $\mathbf{f}$  was an optimal solution,  $C(\mathbf{f}^+) = C(\mathbf{f}^-) = C(\mathbf{f})$ . Hence both  $\mathbf{f}^+$  and  $\mathbf{f}^-$  are optimal, and one of them must have smaller support than  $\mathbf{f}$ , a contradiction.  $\square$

Note that the preceding result shows that in the case of single-sink hose models,  $\text{OPT}_{SPR} = \text{OPT}_{TR}$ . It is not the case that  $\text{OPT}_{MPR} = \text{OPT}_{TR}$  in this setting however. If that were the case, SSROB would be polynomially solvable, but the case where  $b_r = 1$  already captures the Steiner Tree problem. Because of this tree structure, arguing why the gap holds in the case of SPR is considerably simpler. The argument contains some intuition as to why the gap also holds for MPR, so we describe the approach now.

By Proposition 1, we may represent the optimal SPR solution with a tree  $T$ , which we think of as being rooted at  $r$ . First let us suppose that the solution uses only one edge  $rv$  from  $\delta(r)$ , so that all terminals must route via  $v$ . Since  $G$  was bounded degree this means that many nodes (a constant fraction) must use long paths, of length  $\log_d(n)$ . If these all had to pay one unit along their whole path, then this already costs  $\Omega(n \log n)$ . But it is not as easy as that; if we have a subtree  $T_w$  rooted at node  $w$  that contains at least  $b_r = \beta n$  nodes, then in fact we only need to pay for  $b_r$  units on the edge out of  $w$ .

Imagine removing the edges of  $T$  which are used by more than  $\beta n$  terminals, leaving a number of subtrees, each containing at most  $\beta n$  terminals. If  $T$  is fairly balanced, there are around  $\Theta(n/(\beta n)) = \Theta(1/\beta)$  such subtrees. (If  $T$  is very unbalanced, there could be many more—consider a caterpillar. For the full proof, one must use the larger distances of leaves to the root to get the required bound.) Roughly speaking, in each such subtree, a good fraction of the leaves are a distance roughly  $\log \beta n$  from the root of this subtree. Since there is no cost sharing within this subtree, these nodes really do pay  $\beta n \log(\beta n)$ . Thus the subtrees combined pay

$$\Omega(1/\beta \cdot \beta n \log(\beta n)) = \Omega(n \log(\beta n)).$$

If we set  $\beta = \frac{1}{\log n}$ , this yields a cost of  $\Omega(n \log n)$ .

To make the above argument precise, we must balance the use of multiple edges into  $r$ . Label the children of  $r$  in  $T$ , 1 through  $m$ , and let  $k_i$  be the size of subtree  $i$ . Let  $L$  be the set of *heavy* children of  $r$  in  $T$ , i.e.,  $k_i > \beta n$ ; let  $R$  be the set of *light* children of  $r$ .

Suppose  $i$  is a heavy child. The subtree  $T_i$  rooted at  $i$  has some set of heavy edges  $H$ , i.e., edges with flow more than  $b_r = \beta n$ ; let  $p = |H|$ . Now consider the tree  $T'_i$  obtained from  $T_i$  by contracting the edges in  $H$ . The root of  $T'_i$  has degree at most  $pd$ ; all other nodes have maximum degree  $d$ . The maximum number of nodes that are a distance less than  $j$  from the root is

$$\sum_{i=1}^j pd^i = pd(d^j - 1)/(d - 1).$$

Taking  $j = \log_d(k_i/10p)$ , the above is a constant fraction of the nodes, and the rest must be further away. So a constant fraction of the nodes are a distance  $\Omega(\log(k_i/p))$  away from the root of  $T'_i$ . Since the edges in  $T'_i$  are not heavy, these nodes contribute

$$\Omega(k_i \log(k_i/p)) = \Omega(k_i \log(\beta n/p))$$

to the cost of tree  $T_i$ . Adding the cost of the heavy edges, we get a total cost of

$$\Omega(p \cdot \beta n + k_i \log(k_i/p)).$$

We verify that this is at least  $\Omega(k_i \ln(\beta n))$ . It suffices to show that  $p \cdot \beta n + k_i \ln(k_i/p) \geq k_i \ln(\beta n)$ . But this is equivalent to

$$\frac{p\beta n}{k_i} \geq \ln\left(\frac{p\beta n}{k_i}\right),$$

which is clearly true since  $x \geq \ln x$  for all  $x > 0$ .

For  $i \in R$ , a lower bound can be obtained by considering only the edge  $ir$ , which contributes  $k_i/\beta$  (remembering that edges adjacent to  $r$  have cost  $1/\beta$ ). So we have the following lower bound on the cost of the tree solution:

$$\sum_{i \in R} k_i/\beta + \sum_{i \in L} k_i \log(\beta n)/C,$$

where  $C$  is some constant. This is at least

$$n \cdot \min(1/\beta, \log(\beta n)/C).$$

Setting  $\beta = 1/\log n$  gives the result.

## 2.4 Buy-and-rent: an $\Omega(\log n)$ gap between FR and MPR

The main difficulty in analyzing the MPR model is that we can no longer restrict to tree like routings as we could for SPR; there is no equivalent of Proposition 1

for MPR. In particular, the MPR problem for our instance is not captured by a SSROB problem. Instead, we get a new kind of routing cost model as explained below.

Let us first examine more closely the cost on edges induced by an MPR routing template for a single-sink hose design problem. As in Lemma 1, it is sufficient to consider the cases where we wish the network to support the routing of any  $\beta n$  of the nodes in  $V$  to the sink  $r$  simultaneously. Suppose that  $f_i(e)$  represents the flow that node  $i$  sends on edge  $e$  in a template, then for the single sink hose design problem, the formula for the capacity needed by  $e$  is:

$$\max_{D \in \mathcal{P}} \sum_{i \in V} D_{ir} f_i(e) = \max_{W \subseteq V: |W| = \beta n} \sum_{i \in W} f_i(e), \quad (2)$$

where  $\mathcal{P}$  is the set of single-sink hose matrices. In other words, the capacity needed on edge  $e$  is just the sum of the  $\beta n$  largest values of  $f_i(e)$ .

We introduce a new routing cost model which we call (single-sink) *buy-and-rent* (BAR). This exactly models the MPR cost model defined above, but is more manageable in terms of analysis. In the buy-and-rent problem, there are costs on the edges, and unit demands from some subset  $W$  of nodes called *terminals*. Each terminal wishes to (fractionally) route one unit of demand to the sink  $r$ . Apart from the costs  $c(e)$  on the edges, we also have a parameter  $k$ . The difference from rent-or-buy is that we may now purchase some capacity amount  $\gamma(e) \in [0, 1]$  (in rent-or-buy we would buy an infinite capacity link) and the interpretation is that every terminal is allowed to use up to  $\gamma(e)$  units of capacity on the edge. If it chooses to route any more on that edge, then it must pay for the additional rental cost. The cost of purchasing capacity on an edge  $e$  is  $k\gamma(e)c(e)$ .

Buy-and-rent can be considered as an LP relaxation of (single-sink) rent-or-buy; this formulation is in fact very similar to the LP relaxation used by Swamy and Kumar [17] to give constant factor approximation algorithms for connected facility location and single-sink rent-or-buy. Their formulation is stronger however (in that the optimum for their LP lies between the BAR and SPR optima), and so does not exactly model the MPR problem. In particular, in buy-and-rent, solutions may conceivably use flow paths that alternate several times between rented capacity and purchased capacity. In contrast, a solution to the LP of Swamy and Kumar [17] always has a connected ‘‘core’’ of purchased edges containing the sink node and terminals use rented capacity to route to that core.

**Proposition 2.** *The buy-and-rent problem with parameter  $k = \beta n$ , and the single-sink hose design problem in the MPR routing model have the same optimal solution.*

*Proof.* Suppose that  $(\mathbf{f}_i)$  is an MPR routing template for the robust hose design problem. Consider the BAR solution for parameter  $k = \beta n$  obtained as follows. For each edge  $e$ , order the terminals so that  $f_{\pi(1)}(e) \geq f_{\pi(2)}(e) \geq \dots \geq f_{\pi(n)}(e)$ . Then we purchase  $\gamma(e) = f_{\pi(k)}(e)$  units of capacity on edge  $e$ , and we use the same routing  $\mathbf{f}_i$  as the MPR solution. This guarantees that for any edge, none of the terminals  $\pi(j)$  with  $j > k$ , pays to route on edge  $e$  since we purchased enough

capacity for them to travel for free. For each terminal  $\pi(j)$  with  $j \leq k$ , it must pay the rental cost to route  $f_{\pi(j)}(e) - f_{\pi(k)}(e) \geq 0$ . This costs  $c(e)$  times the amount  $\sum_{j \leq k} (f_{\pi(j)}(e) - f_{\pi(k)}(e)) = \sum_{j \leq k} f_{\pi(j)}(e) - k f_{\pi(k)}(e)$ . Since the purchased capacity cost  $k f_{\pi(k)}(e) c(e)$ , the total buy-and-rent cost is  $c(e) \sum_{j \leq k} f_{\pi(j)}(e)$  which is the cost of edge  $e$  in the MPR template using (2).

Conversely, suppose that we have a minimum cost solution for BAR and consider the robust design cost for using the same routing as a template. Without loss of generality  $\gamma(e) = f_{\pi(k)}(e)$  since if  $\gamma(e)$  was larger than this, then by reducing the capacity bought by sufficiently small  $\epsilon > 0$ , the rental costs are unaffected for terminals  $\pi(j)$  for  $j \geq k$ . And for terminals  $\pi(j)$  with  $j < k$ , their rental cost increases by at most  $\epsilon c(e)$ . Hence the total rental cost increases by  $(k-1)\epsilon c(e)$ , and the total cost of bought capacity reduces by  $k\epsilon c(e)$ , thus decreasing the overall cost.

Similarly, if  $\gamma(e) < f_{\pi(k)}(e)$ , then increasing the bought capacity  $\gamma$  by some small  $\epsilon > 0$ , has cost of  $k\epsilon c(e)$ . But the reduction in rental costs is at least the reduction in rental cost of the first  $k$  terminals which is  $k\epsilon c(e)$ , and thus the overall cost does not increase as a result of increasing  $\gamma$ . Hence the cost of edge  $e$  is just the purchase cost  $c(e) \cdot k f_{\pi(k)}(e)$  plus the rental cost  $c(e) \sum_{j \leq k} (f_{\pi(j)}(e) - f_{\pi(k)}(e))$  and this is identical to the robust design cost when using the same template.  $\square$

We again take  $\beta = 1/\log n$  (so  $k = n/\log n$ ). We now prove that any solution to the BAR problem on our expander instance is expensive; this together with the preceding proposition implies our main result, Theorem 1.

**Theorem 2.** *Any solution to the BAR problem with  $\beta = 1/\log n$  on the expander instance (defined in Sec. 2.1) has cost  $\Omega(n \log n)$ .*

*Proof.* Consider an arbitrary BAR solution, determined by bought capacity  $\gamma_e$  on each edge, and a flow template ( $f_i$  : for each terminal  $i$ ).

For a set  $A$  of edges, let  $\gamma(A) := \sum_{e \in A} \gamma_e$ . Thus  $\gamma(\delta(r)) := \sum_{v \in V} \gamma_{vr}$  is the total bought capacity on the *port edges* (these are the edges connecting  $r$  to the nodes in  $V$ ), and  $\gamma(E) := \sum_{e \in E} \gamma_e$  is the capacity bought in the expander. The cost of buying capacity in the expander is then  $k \cdot \gamma(E)$ , so we may assume that  $\gamma(E) < \log^2 n$ , or else the solution already costs  $\Omega(n \log n)$ . A similar argument for port edges (but recalling that these edges cost  $\log n$ ) allows us to assume that  $\gamma(\delta(r)) < \log n$ .

For a terminal  $v$ , let  $B_i(v)$  be the set of nodes (or sometimes, their induced graph) in the expander that are a distance at most  $i$  from  $v$ . We are particularly interested in balls of radius  $R := \lfloor \log_d \sqrt{n} \rfloor - 1 = \lfloor \log n / (2 \log d) \rfloor - 1$ ; we use  $B(v)$  as shorthand for  $B_R(v)$ . Note that since  $G$  is  $d$ -regular,

$$|B(v)| \leq \sum_{i=0}^R d^i \leq d^{R+1} \leq n^{1/2}.$$

Let  $\gamma^E(v) := \sum_{e \in E: e \subset B(v)} \gamma(e)$  and  $\gamma^P(v) := \sum_{w \in B(v)} \gamma(wr)$ . A single  $\gamma(e)$  for an edge  $e = u_1 u_2$  contributes to many  $\gamma^E(v)$ 's, but not too many:

$$|\{v : e \subset B(v)\}| \leq |\{v : u_1 \in B(v)\}| = |B(u_1)| \leq n^{1/2}.$$

So we must have that

$$\sum_{v \in V} \gamma^E(v) \leq n^{1/2} \gamma(E) \leq n^{1/2} \log^2 n. \quad (3)$$

Similarly,

$$\sum_{v \in V} \gamma^P(v) \leq n^{1/2} \log n. \quad (4)$$

Consider an arbitrary terminal  $v$ . The unit of flow from  $v$  can be divided up into three types depending on how the flow enters  $r$ :

- A fraction  $\mu_v^r$  of flow that rents on the port edge it uses.
- A fraction  $\mu_v^b$  of flow that uses bought port capacity, on a port within a distance  $R$  from  $v$ .
- A fraction  $\mu_v^t$ , representing all remaining flow; this flow must “travel” and use port edges that are further than  $R$  from  $v$ .

Clearly  $\mu_v^r + \mu_v^b + \mu_v^t = 1$ .

We now aim to find a lower bound on the total rental cost paid by the terminals. Flow that rents the port edge must pay  $\log n$  just for this edge, giving a cost of  $\mu_v^r \log n$ . Now consider the  $\mu_v^t$  fraction of flow that travels outside the ball  $B(v)$  in the expander before using a port edge. This flow must cross each of the cuts  $C_i := \delta(B_i(v))$ , for  $0 \leq i \leq R$ .

The maximum amount of flow that can travel across cut  $C_i$  for free (using the bought capacity) is  $\gamma(C_i)$ , and so there is a rental cost of at least  $\mu_v^t - \gamma(C_i)$  in crossing cut  $C_i$ . Summing over all the cuts, we find that the rental cost associated with this travelling flow is at least

$$\sum_{i=0}^{R-1} (\mu_v^t - \gamma(C_i)) \geq R\mu_v^t - \gamma^E(v).$$

Thus the rental cost associated with terminal  $v$  is at least

$$\log n \cdot \mu_v^r + R\mu_v^t - \gamma^E(v).$$

Summing this over all terminals  $v$ , we obtain a total rental cost of at least

$$\begin{aligned} C(\text{rent}) &\geq \sum_{v \in V} (\log n \cdot \mu_v^r + R \cdot \mu_v^t) - \sum_{v \in V} \gamma^E(v) \\ &\geq R \sum_{v \in V} (\mu_v^r + \mu_v^t) - \sum_{v \in V} \gamma^E(v) && \text{since } R \leq \log n \\ &\geq R \sum_{v \in V} (\mu_v^r + \mu_v^t) - n^{1/2} \log^2 n && \text{by (3)}. \end{aligned}$$

Finally, note that

$$\begin{aligned} \sum_{v \in V} (\mu_v^r + \mu_v^t) &= \sum_{v \in V} (1 - \mu_v^b) \geq \sum_{v \in V} (1 - \gamma^P(v)) \\ &\geq n - n^{1/2} \log n \end{aligned} \quad \text{by (4).}$$

Thus

$$\begin{aligned} C(\text{rent}) &\geq R \cdot (n - n^{1/2} \log n) - n^{1/2} \log^2 n \\ &= \Omega(n \log n), \end{aligned}$$

since  $R = \Theta(\log n)$ . □

### 3 Single path routing vs. tree routing

As discussed in the introduction, for any robust network design problem we have  $\text{OPT}_{TR} = \tilde{O}(\log n) \text{OPT}_{FR}$ . We now show that this is essentially best possible by exhibiting a problem instance such that  $\text{OPT}_{TR} = \Omega(\log n) \text{OPT}_{SPR}$ , and so also  $\text{OPT}_{TR} = \Omega(\log n) \text{OPT}_{FR}$ .

Consider a graph on  $n$  vertices with girth (length of the shortest cycle in the graph)  $\Omega(\log n)$ , and with  $cn$  edges, where  $c$  is a constant strictly larger than 1. The requirement  $c > 1$  makes the existence of such graphs nontrivial, but they do exist. For example, Lemma 15.3.2 in [15] states that there exist graphs with girth  $\ell$  and  $\frac{1}{9}n^{1+1/(\ell-1)}$  edges. Taking  $\ell := (\log n)/100$  gives a graph satisfying our requirements.

This graph defines the network topology for our problem instance: all the nodes are terminals, and all edges have unit cost. The demand polytope is given by a single demand: there is a unit demand between terminals connected by an edge.

Clearly, a good SPR template is the network itself, and its cost is  $cn$ , the number of edges. Now, if we take any tree template, then edges of the network that are not included in the tree have to be routed on a path of length  $\Omega(\log n)$  because of the girth property. There are at least  $cn - (n - 1) = (c - 1)n + 1$  such edges, and so the total cost of the tree template is  $\Omega(n \log n)$ .

### 4 Conclusions

We have shown that oblivious routing (even splittable) can perform quite poorly compared to dynamic routing in some situations. However, fully dynamic routing is problematic to implement. Is it possible that some tradeoff between the two extremes of dynamic and oblivious routing could produce significantly better results while remaining practical?

Another very natural question concerns the gap between MPR and SPR for the single-sink robust network design problem with arbitrary demand polytopes. We

are not aware of any single-sink instances for which the gap is superconstant. The single-sink robust design problem (computing  $\text{OPT}_{SPR}$ ) could still conceivably have a constant factor approximation algorithm for well-described polytopes. This would be of interest since it generalizes a host of well-known problems such as Steiner tree, single-sink rent-or-buy, and single-sink buy-at-bulk (the last follows from a transformation given in [16]).

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## A Upper bounds on the gaps

For the sake of completeness, in this appendix we give a proof of Gupta’s observation that the gap between  $\text{OPT}_{FR}$  and  $\text{OPT}_{SPR}$  is  $O(\log n)$ . We also show that the gap between  $\text{OPT}_{FR}$  and  $\text{OPT}_{TR}$  is  $\tilde{O}(\log n)$ , via a similar proof. A sketch proof of Gupta’s observation appears in Chekuri [6].

We use basic notions about finite metric spaces; an excellent exposition of this topic is found in Matousek’s book [15]. We begin with some notation and state a theorem that we need. We are given an instance  $(G, c, \mathcal{P})$  of the robust network design problem on  $n$  nodes. The cost function  $c$  induces a metric  $d_G(\cdot, \cdot)$  on nodes of  $G$  in the usual way: the distance  $d_G(x, y)$  between nodes  $x$  and  $y$  is given by the length of the shortest  $x$ - $y$  path in  $G$ , with edge  $e$  having length  $c(e)$ . We also define the complete graph  $C_G$  on  $V$  where edge  $\{x, y\}$  has length  $d_G(x, y)$ .

Now using the result and notation of [9], a metric  $d$  can be approximated by distribution over dominating tree metrics in the following sense. A metric  $(V, d')$  is said to *dominate* a metric  $(V, d)$  if for all  $x, y \in V$  we have  $d'(x, y) \geq d(x, y)$ . Given a probability distribution  $\mathcal{D}$  over a family of tree metrics  $\mathcal{S}$  on  $V$ , we say that  $(\mathcal{S}, \mathcal{D})$   $\alpha$ -probabilistically approximates a metric  $(V, d)$  if every metric in  $\mathcal{S}$  dominates  $d$ , and for all  $x, y \in V$  we have  $\mathbb{E}_{d' \in (\mathcal{S}, \mathcal{D})}[d'(x, y)] \leq \alpha \cdot d(x, y)$ .

Building upon previous work, Fakcharoenphol et al. [9] proved that every finite metric on  $n$  nodes can be  $O(\log n)$ -probabilistically approximated by a distribution over tree metrics.

Now getting back to our robust network design instance, we find a distribution  $(\mathcal{S}, \mathcal{D})$  over tree metrics which  $O(\log n)$ -probabilistically approximates  $d_G$ . Trees in  $\mathcal{S}$  can be taken to be spanning trees of  $C_G$  (they need not be subtrees of  $G$  though). For a capacity reservation  $u$  on the edges of  $G$ , its cost is  $\text{cost}_G(u) := \sum_{e \in E} c(e)u(e)$ . We can also define the cost of this reservation on a tree metric  $T$  by  $\text{cost}_T(u) := \sum_{e \in E} d_T(e)u(e)$ . Let  $u^*$  be the optimum capacity reservation for the FR routing model, so  $\text{cost}_G(u^*) = \text{OPT}_{FR}$ . By the theorem of [9] we have by linearity of expectation applied to  $\text{cost}_T(u^*)$ :

$$\text{cost}_G(u^*) \leq \mathbb{E}_{d' \in (\mathcal{S}, \mathcal{D})} \text{cost}_T(u^*) \leq O(\log n) \text{cost}_G(u^*).$$

So there exists a tree  $T \in \mathcal{S}$  such that

$$\text{cost}_G(u^*) \leq \text{cost}_T(u^*) \leq O(\log n) \text{cost}_G(u^*).$$

Let  $u_T$  denote the optimal capacity vector for the robust network design problem on the graph  $T$ <sup>5</sup>; N.B. all routing models are equivalent on a tree  $T$ . We also have  $\text{cost}_T(u_T) \leq \text{cost}_T(u^*)$ . This is because the dynamic solution gives an oblivious solution for  $T$  with cost at most  $\text{cost}_T(u^*)$  as follows. For any edge  $e$  in  $G$ , add  $u^*(e)$  units of capacity on the path in  $T$  between the endpoints of  $e$ . The overall capacity  $u'$  installed then costs  $\text{cost}_T(u^*)$ . For any valid demand, the dynamic solution satisfied the demand by assigning flows  $f(P)$  to paths in  $G$ . Moreover,  $\sum_{P:e \in P} f(P) \leq u^*(e)$  and thus by definition,  $u'$  has enough capacity to support routing all such flow paths  $P$ , between  $u, v$  say, on the unique  $u - v$  path in  $T$ . Since  $\text{cost}_T(u_T) \leq \text{cost}(u')$  we are done. Hence

$$\text{cost}_G(u^*) \leq \text{cost}_T(u_T) \leq O(\log n) \text{cost}_G(u^*).$$

Let  $f_T$  be the routing template on  $T$  that determines  $u_T$ . This can be transferred to an SPR routing template in  $G$  with a capacity reservation  $u_{G(T)}$  on  $G$  with the same cost as follows: Each edge in  $T$  corresponds to a path in  $G$ . For edge  $xy$  in  $T$  we reserve  $u_T(xy)$  capacity on each edge on the path in  $G$  corresponding to edge  $xy$ . If an edge in  $G$  lies on several such paths then the capacity reserved on it is the sum of the  $u_T$ -values for all of these paths. Clearly the cost of the resulting capacity reservation  $u_{G(T)}$  on  $G$  is the same as the cost of  $u_T$ . Also, the routing template  $f_T$  for  $u_T$  can be simulated on  $G$  in the natural fashion in the SPR routing model. Thus, we have a reservation  $u_{G(T)}$  supporting a SPR routing on  $G$  and with  $\text{cost}_G(u_{G(T)}) = \text{cost}_T(u_T) \leq O(\log n) \text{cost}_G(u^*)$ . Noting that  $\text{OPT}_{SPR} \leq \text{cost}_G(u_{G(T)})$  and  $\text{OPT}_{FR} = \text{cost}_G(u^*)$  completes the proof that  $\text{OPT}_{SPR} = O(\log n) \text{OPT}_{FR}$ .

Note that the above proof does not give us that  $\text{OPT}_{TR} = O(\log n) \text{OPT}_{FR}$  because the support of  $u_{G(T)}$  need not be a tree. We can prove a slightly weaker result, namely  $\text{OPT}_{TR} = \tilde{O}(\log n) \text{OPT}_{FR}$  by invoking a theorem of [1]: For any metric  $d_G$  induced by a graph  $G$  on  $n$  nodes there is a distribution on the *spanning trees* of  $G$  which  $\tilde{O}(\log n)$ -probabilistically approximates  $d_G$ . The remaining details are essentially the same as in the above, except that this time the support of  $u_{G(T)}$  is indeed a tree.

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<sup>5</sup> Computing the capacity of an edge  $e \in T$  amounts to solving a linear program over  $\mathcal{P}$  with objective  $\sum_{i \in A, j \in B} D_{ij}$  where  $A, B$  are the two components of  $T - e$ .