New bounds on the average distance from the Fermat-Weber center of a planar convex body

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Abstract

The Fermat-Weber center of a planar body Q is a point in the plane from which the average distance to the points in Q is minimal. We first show that for any convex body Q in the plane, the average distance from the Fermat-Weber center of Q to the points of Q is larger than $\frac{1}{6} \cdot \Delta(Q)$, where $\Delta(Q)$ is the diameter of Q. This proves a conjecture of Carmi, Har-Peled and Katz. From the other direction, we prove that the same average distance is at most $\frac{2(4-\sqrt{3})}{13} \cdot \Delta(Q) < 0.3490 \cdot \Delta(Q)$. The new bound substantially improves the previous bound of $\frac{2}{3\sqrt{3}} \cdot \Delta(Q) \approx 0.3849 \cdot \Delta(Q)$ due to Abu-Affash and Katz, and brings us closer to the conjectured value of $\frac{1}{3} \cdot \Delta(Q)$. We also confirm the upper bound conjecture for centrally symmetric planar convex bodies.

1 Introduction

The Fermat-Weber center of a measurable planar set Q with positive area is a point in the plane that minimizes the average distance to the points in Q. Such a point is the ideal location for a base station (e.g., fire station or a supply station) serving the region Q, assuming the region has uniform density. Given a measurable set Q with positive area and a point p in the plane, let $\mu_Q(p)$ be the average distance between p and the points in Q, namely,

$$\mu_Q(p) = \frac{\int_{q\in Q} \mathrm{dist}(p,q)\ \mathrm{d}q}{\mathrm{area}(Q)},$$

where $\mathrm{dist}(p,q) = |pq|$ is the Euclidean distance between p and q. Let FW_Q be the Fermat-Weber center of Q, and write $\mu_Q^* = \min\{\mu_Q(p) : p \in \mathbb{R}^2\} = \mu_Q(FW_Q)$.

Carmi, Har-Peled and Katz [3] showed that there exists a constant $c > 0$ such that $\mu_Q^* \geq c\cdot\Delta(Q)$ holds for any convex body Q, where $\Delta(Q)$ denotes the diameter of Q. The convexity is necessary, since it is easy to construct nonconvex regions where the average distance from the Fermat-Weber center is arbitrarily small compared to the diameter. Of course the opposite inequality $\mu_Q^* \leq c'\cdot\Delta(Q)$ holds for any body Q (convexity is not required), since we can trivially take $c' = 1$.

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Let c_1 denote the infimum, and c_2 denote the supremum of $\mu_Q^*/\Delta(Q)$ over all convex bodies Q in the plane. Carmi, Har-Peled and Katz [3] conjectured that $c_1 = \frac{1}{6}$ and $c_2 = \frac{1}{3}$. Moreover, they conjectured that the supremum c_2 is attained for a circular disk D, where $\mu_D^* = \frac{1}{3} \cdot \Delta(D)$. They also proved that $\frac{1}{7} \leq c_1 \leq \frac{1}{6}$. The inequality $c_1 \leq \frac{1}{6}$ is given by an infinite sequence of rhombi, P_ε, where one diagonal has some fixed length, say 2, and the other diagonal tends to zero; see Fig. 1. By symmetry, the Fermat-Weber center of a rhombus is its center of symmetry, and one can verify that $\mu_{P_\varepsilon}^*/\Delta(P_\varepsilon)$ tends to $\frac{1}{6}$. The lower bound for c_1 has been recently further improved by Abu-Affash and Katz from $\frac{1}{7}$ to $\frac{4}{25}$ [1]. Here we establish that $c_1 = \frac{1}{6}$ and thereby confirm the first of the two conjectures of Carmi, Har-Peled and Katz.

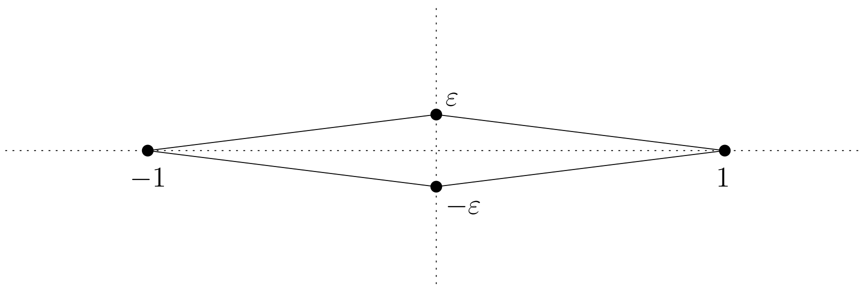

Figure 1: A flat rhombus P_ε, with $\lim_{\varepsilon \to 0} \mu_{P_\varepsilon}^*/\Delta(P_\varepsilon) = \frac{1}{6}$.

Regarding the second conjecture, recently Abu-Affash and Katz proved that $c_2 \leq \frac{2}{3\sqrt{3}} = 0.3849\ldots$. Here we further improve this bound and bring it closer to the conjectured value of $\frac{1}{3}$. Finally, we also confirm the upper bound conjecture for centrally symmetric convex bodies Q.

Our main results are summarized in the following two theorems:

Theorem 1 *For any convex body Q in the plane, we have $\mu_Q^* > \frac{1}{6} \cdot \Delta(Q)$.*

Theorem 2 *For any convex body Q in the plane, we have*

$$\mu_Q^* \leq \frac{2(4-\sqrt{3})}{13} \cdot \Delta(Q) < 0.3490 \cdot \Delta(Q).$$

Moreover, if Q is centrally symmetric, then $\mu_Q^* \leq \frac{1}{3} \cdot \Delta(Q)$.

Remarks. 1. The average distance from a point p in the plane can be defined analogously for finite point sets and for rectifiable curves. Observe that for a line segment I (a one-dimensional convex set), we would have $\mu_I^*/\Delta(I) = \frac{1}{4}$. It might be interesting to note that while the thin rhombi mentioned above tend in the limit to a line segment, the value of the limit $\mu_{P_\varepsilon}^*/\Delta(P_\varepsilon)$ equals $\frac{1}{6}$, not $\frac{1}{4}$.

2. In some applications, the cost of serving a location q from a facility at point p is $\text{dist}^\kappa(p,q)$ for some exponent $\kappa \geq 1$, rather than $\text{dist}(p,q)$. We can define $\mu_Q^\kappa(p) = \left(\int_{q \in Q} \text{dist}^\kappa(p,q)\, \mathrm{d}q\right)/\text{area}(Q)$ and $\mu_Q^{\kappa *} = \inf\{\mu_Q^\kappa(p) : p \in \mathbb{R}^2\}$, which is invariant under congruence. The ratio $\mu_Q^{\kappa *}/\Delta^\kappa(Q)$ is also invariant under similarity. The proof of Theorem 1 carries over for this variant and shows that $\mu_Q^{\kappa *}/\Delta^\kappa(Q) > \frac{1}{(\kappa+2)2^\kappa}$ for any convex body Q, and $\lim_{\varepsilon \to 0} \mu_{P_\varepsilon}^{\kappa *}/2^\kappa = \frac{1}{(\kappa+2)2^\kappa}$. For the upper bound, the picture is not so clear: $\mu_Q^*/\Delta(Q)$ is conjectured to be maximal for the circular disk, however, there is a $\kappa \geq 1$ such that $\mu_Q^{\kappa *}/\Delta^\kappa(Q)$ cannot be maximal for the disk. In particular, if D is a disk of diameter 2 and R is a convex body of diameter 2 whose smallest enclosing disk has diameter more than 2 (e.g., a regular or a Reuleaux triangle of diameter 2), then $\mu_D^{\kappa *} < \mu_R^{\kappa *}$, for a sufficiently large $\kappa > 1$. Let o be an arbitrary point in the plane, and let D be centered at o. Then $\int_{q \in D} \mathrm{dist}^\kappa(o,q)\ \mathrm{d}q = \int_0^{2\pi} \int_0^1 r^\kappa \cdot r\ \mathrm{d}r\ \mathrm{d}\theta = \frac{2\pi}{\kappa+2}$, and so $\lim_{\kappa\to\infty} \mu_D^{\kappa *} \leq \lim_{\kappa\to\infty} \frac{2}{\kappa+2} = 0$. On the other hand, for any region R' lying outside of D and for any $\kappa \geq 1$, we have $\int_{q\in R'} \mathrm{dist}^\kappa(o,q)\ \mathrm{d}q \geq \mathrm{area}(R') > 0$. If $R' = R \setminus D$ is the part of R lying outside D, then $\lim_{\kappa\to\infty} \mu_R^{\kappa *} \geq \mathrm{area}(R')/\pi > 0$.

Related work. Fekete, Mitchell, and Weinbrecht [8] studied a continuous version of the problem for polygons with holes, where the distance between two points is measured by the L_1 geodesic distance. A related question on Fermat-Weber centers in a discrete setting deals with stars and Steiner stars [5, 7]. The reader can find more information on other variants of the Fermat-Weber problem in [4, 11].

2 Lower bound: proof of Theorem 1

In a nutshell the proof goes as follows. Given a convex body Q, we take its Steiner symmetrization with respect to a supporting line of a diameter segment cd, followed by another Steiner symmetrization with respect to the perpendicular bisector of cd. The two Steiner symmetrizations preserve the area and the diameter, and do not increase the average distance from the corresponding Fermat-Weber centers. In the final step, we prove that the inequality holds for a convex body with two orthogonal symmetry axes.

Steiner symmetrization with respect to an axis. Steiner symmetrization of a convex figure Q with respect to an axis (line) ℓ consists in replacing Q by a new figure $S(Q,\ell)$ with symmetry axis ℓ by means of the following construction: Each chord of Q orthogonal to ℓ is displaced along its line to a new position where it is symmetric with respect to ℓ, see [12, pp. 64]. The resulting figure $S(Q,\ell)$ is also convex, and obviously has the same area as Q.

A body Q is x-monotone if the intersection of Q with every vertical line is either empty or is connected (that is, a point or a line segment). Every x-monotone body Q is bounded by the graphs of some functions $f : [a,b] \to \mathbb{R}$ and $g : [a,b] \to \mathbb{R}$ such that $g(x) \leq f(x)$ for all $x \in [a,b]$. The Steiner symmetrization with respect to the x-axis ℓ_x transforms Q into an x-monotone body $S(Q,\ell_x)$ bounded by the functions $\frac{1}{2}(f(x)-g(x))$ and $\frac{1}{2}(g(x)-f(x))$ for $x \in [a,b]$. As noted earlier, $\mathrm{area}(S(Q,\ell_x)) = \mathrm{area}(Q)$. The next two lemmas do not require the convexity of Q.

Lemma 1 *Let Q be an x-monotone body in the plane with a diameter parallel or orthogonal to the x-axis, then $\Delta(Q) = \Delta(S(Q,\ell_x))$.*

Proof. Let $Q' = S(Q,\ell_x)$. If Q has a diameter parallel to the x-axis, then the diameter is $[(a,c),(b,c)]$, with a value $c \in \mathbb{R}$, $g(a) = c = f(a)$ and $g(b) = c = f(b)$. That is, $\Delta(Q) = b - a$. In this case, the diameter of Q' is at least $b-a$, since both points $(a,0)$ and $(b,0)$ are in Q'. If Q has a diameter orthogonal to the x-axis, then the diameter is $[(x_0, f(x_0)), (x_0, g(x_0))]$ for some $x_0 \in [a,b]$, and $\Delta(Q) = f(x_0) - g(x_0)$. In this case, the diameter of Q' is at least $f(x_0) - g(x_0)$, since both points $(x_0, \frac{1}{2}(f(x_0) - g(x_0)))$ and $(x_0, \frac{1}{2}(g(x_0) - f(x_0)))$ are in Q'. Therefore, we have $\Delta(Q') \geq \Delta(Q)$.

Let A_1 and A_2 be two points on the boundary of Q' such that $\Delta(Q') = \mathrm{dist}(A_1, A_2)$. Since Q' is symmetric to the x-axis, points A_1 and A_2 cannot both be on the upper (resp., lower) boundary of Q'. Assume w.l.o.g. that $A_1 = (x_1, \frac{1}{2}(f(x_1) - g(x_1)))$ and $A_2 = (x_2, \frac{1}{2}(g(x_2) - f(x_2)))$ for some $a \leq x_1, x_2 \leq b$.

$$\Delta(Q') = \mathrm{dist}(A_1, A_2) = \sqrt{(x_2 - x_1)^2 + \left(\frac{f(x_1) + f(x_2) - g(x_1) - g(x_2)}{2}\right)^2}.$$

Now consider the following two point pairs in Q. The distance between $B_1 = (x_1, f(x_1))$ and $B_2 = (x_2, g(x_2))$ is $\mathrm{dist}(B_1, B_2) = \sqrt{(x_2 - x_1)^2 + (f(x_1) - g(x_2))^2}$. Similarly, the distance between $C_1 = (x_1, g(x_1))$ and $C_2 = (x_2, f(x_2))$ is $\mathrm{dist}(C_1, C_2) = \sqrt{(x_2 - x_1)^2 + (g(x_1) - f(x_2))^2}$. Using the inequality between the arithmetic and quadratic means, we have

$$\left(\frac{f(x_1) + f(x_2) - g(x_1) - g(x_2)}{2}\right)^2 \leq \frac{(f(x_1) - g(x_2))^2 + (g(x_1) - f(x_2))^2}{2}.$$

This implies that $\mathrm{dist}(A_1, A_2) \leq \max(\mathrm{dist}(B_1, B_2), \mathrm{dist}(C_1, C_2))$, and so $\Delta(Q') \leq \Delta(Q)$. We conclude that $\Delta(Q) = \Delta(S(Q, \ell_x))$. □

Lemma 2 *If Q is an x-monotone body in the plane, then $\mu^*_Q \geq \mu^*_{S(Q,\ell_x)}$.*

Proof. If (x_0, y_0) is the Fermat-Weber center of Q, then

$$\mu^*_Q = \frac{\int_a^b \int_{g(x)}^{f(x)} \sqrt{(x - x_0)^2 + (y - y_0)^2}\, \mathrm{d}y\, \mathrm{d}x}{\mathrm{area}(Q)}.$$

Observe that $\int_{g(x)}^{f(x)} \sqrt{(x - x_0)^2 + (y - y_0)^2}\, \mathrm{d}y$ is the integral of the distances of the points in a line segment of length $f(x) - g(x)$ from a point at distance $|x - x_0|$ from the supporting line of the segment. This integral is minimal if the point is on the orthogonal bisector of the segment. That is, we have

$$\begin{aligned}
\int_{g(x)}^{f(x)} \sqrt{(x - x_0)^2 + (y - y_0)^2}\, \mathrm{d}y \quad &\geq \quad \int_{g(x)}^{f(x)} \sqrt{(x - x_0)^2 + \left(y - \frac{f(x) - g(x)}{2}\right)^2}\, dy \\
&= \quad \int_{\frac{1}{2}(g(x) - f(x))}^{\frac{1}{2}(f(x) - g(x))} \sqrt{(x - x_0)^2 + y^2}\, \mathrm{d}y.
\end{aligned}$$

Therefore, we conclude that

$$\begin{aligned}
\mu^*_Q \quad &= \quad \frac{\int_a^b \int_{g(x)}^{f(x)} \sqrt{(x - x_0)^2 + (y - y_0)^2}\, \mathrm{d}y\, \mathrm{d}x}{\mathrm{area}(Q)} \\
&\geq \quad \frac{\int_a^b \int_{\frac{1}{2}(g(x) - f(x))}^{\frac{1}{2}(f(x) - g(x))} \sqrt{(x - x_0)^2 + y^2}\, \mathrm{d}y\, \mathrm{d}x}{\mathrm{area}(S(Q, x))} = \mu_{S(Q,\ell_x)}((x_0, 0)) \geq \mu^*_{S(Q,\ell_x)}.
\end{aligned}$$

□

Triangles. We next consider right triangles of a special kind, lying in the first quadrant, and show that the average distance from the origin to their points is larger than $\frac{1}{3}$.

Lemma 3 *Let T a right triangle in the first quadrant based on the x-axis, with vertices $(a,0)$, (a,b), and $(1,0)$, where $0 \le a < 1$, and $b > 0$. Then $\mu_T(o) > \frac{1}{3}$.*

Proof. We use the simple fact that the x-coordinate of a point is a lower bound to the distance from the origin.

$$\begin{aligned}
\mu_T(o) &= \frac{\int_a^1 (\int_0^{b(1-x)/(1-a)} \sqrt{x^2+y^2}\, \mathrm{d}y)\, \mathrm{d}x}{b(1-a)/2} > \frac{\int_a^1 (\int_0^{b(1-x)/(1-a)} x\, \mathrm{d}y)\, \mathrm{d}x}{b(1-a)/2} \\
&= \frac{\frac{b}{1-a}\int_a^1 x(1-x)\, \mathrm{d}x}{b(1-a)/2} = \frac{2}{(1-a)^2}\left(\frac{x^2}{2} - \frac{x^3}{3}\right)\Big|_a^1 \\
&= \frac{2}{(1-a)^2} \cdot \frac{(2a^3 - 3a^2 + 1)}{6} = \frac{2}{(1-a)^2} \cdot \frac{(1-a)(1+a-2a^2)}{6} \\
&= \frac{1}{(1-a)} \cdot \frac{(1+a-2a^2)}{3} \ge \frac{1}{3}.
\end{aligned}$$

The last inequality in the chain follows from $0 \le a < 1$. The inequality in the lemma is strict, since $\sqrt{x^2+y^2} > x$ for all points above the x-axis. □

Corollary 1 *Let P be any rhombus. Then $\mu_P^* > \frac{1}{6} \cdot \Delta(P)$.*

Proof. Without loss of generality, we may assume that P is symmetric with respect to both the x-axis and the y-axis. Let us denote the vertices of P by $(-1,0)$, $(1,0)$, $(0,-b)$, and $(0,b)$, where $b \le 1$. We have $\Delta(P) = 2$. By symmetry, μ_P^* equals the average distance between the origin $(0,0)$ and the points in one of the four congruent right triangles forming P. Consider the triangle T in the first quadrant. By Lemma 3 (with $a = 0$), we have $\mu_P^* = \mu_T(o) > \frac{1}{3}$. Since $\Delta(P) = 2$, we have $\mu_P^* > \frac{1}{6} \cdot \Delta(P)$, as desired. □

Lemma 4 *Let T be a triangle in the first quadrant with a vertical side on the line $x = a$, where $0 \le a < 1$, and a third vertex at $(1,0)$. Then $\mu_T(o) > \frac{1}{3}$.*

Proof. Refer to Fig. 2(ii). Let U be a right triangle obtained from T by translating each vertical chord of T down until its lower endpoint is on the x-axis. Note that $\text{area}(T) = \text{area}(U)$. Observe also that the average distance from the origin decreases in this transformation, namely $\mu_T(o) \ge \mu_U(o)$. By Lemma 3, we have $\mu_U(o) > \frac{1}{3}$, and so $\mu_T(o) > \frac{1}{3}$, as desired. □

We now have all necessary ingredients to prove Theorem 1.

Proof of Theorem 1. Refer to Fig. 2. Let Q be a convex body in the plane, and let $c, d \in Q$ be two points at $\Delta(Q)$ distance apart. We may assume that $c = (-1,0)$ and $d = (1,0)$, by a similarity transformation if necessary, so that $\Delta(Q) = 2$ (the ratio $\mu_Q^*/\Delta(Q)$ is invariant under similarities). Apply a Steiner symmetrization with respect to the x-axis, and then a second Steiner symmetrization with respect to the y-axis. The resulting body $Q' = S(S(Q,\ell_x),\ell_y)$ is convex, and it is symmetric with respect to both coordinate axes. We have $\Delta(Q') = \Delta(Q) = 2$ by Lemma 1, and in fact $c, d \in Q'$. We also have $\mu_{Q'}^* \le \mu_Q^*$ by Lemma 2.

Let Q_1 be the part of Q' lying in the first quadrant: $Q_1 = \{(x, y) \in Q' : x, y \geq 0\}$. By symmetry, $FW_{Q'} = o$ and we have $\mu^*_{Q'} = \mu_{Q'}(o) = \mu_{Q_1}(o)$. Let γ be the portion of the boundary of Q' lying in the first quadrant, between points $b = (0, h)$, with $0 < h \leq 1$, and $d = (1, 0)$. For any two points $p, q \in \gamma$ along γ, denote by $\gamma(p, q)$ the portion of γ between p and q. Let r be the intersection point of γ and the vertical line $x = \frac{1}{3}$.

For a positive integer n, subdivide Q_1 into at most $2n + 2$ pieces as follows. Choose $n + 1$ points $b = q_1, q_2 \ldots, q_{n+1} = r$ along $\gamma(b, r)$ such that q_i is the intersection of γ and the vertical line $x = (i - 1)/3n$. Connect each of the $n + 1$ points to d by a straight line segment. These segments subdivide Q_1 into $n + 2$ pieces: the right triangle $T_0 = \Delta bod$; a convex body Q_0 bounded by rd and $\gamma(r, d)$; and n curvilinear triangles $\Delta q_i d q_{i+1}$ for $i = 1, 2, \ldots, n$. For simplicity, we assume that neither Q_0, nor any of the curvilinear triangles are degenerate; otherwise they can be safely ignored (they do not contribute to the value of $\mu^*_{Q'}$). Subdivide each curvilinear triangle $\Delta q_i d q_{i+1}$ along the vertical line through q_{i+1} into a small curvilinear triangle S_i on the left and a triangle T_i incident to point d on the right. The resulting subdivision has $2n + 2$ pieces, under the nondegeneracy assumption.

By Lemma 3, we have $\mu_{T_0}(o) > \frac{1}{3}$. Observe that the difference $\mu_{T_0}(o) - \frac{1}{3}$ does not depend on n, and let $\delta = \mu_{T_0}(o) - \frac{1}{3}$. By Lemma 4, we also have $\mu_{T_i}(o) > \frac{1}{3}$, for each $i = 1, 2, \ldots, n$. Since every point in Q_0 is at distance at least $\frac{1}{3}$ from the origin, we also have $\mu_{Q_0}(o) \geq \frac{1}{3}$.

For the n curvilinear triangles S_i, $i = 1, 2, \ldots, n$, we use the trivial lower bound $\mu_{S_i}(o) \geq 0$. We now show that their total area $s_n = \sum_{i=1}^{n} \text{area}(S_i)$ tends to 0 if n goes to infinity. Recall that the y-coordinates of the points q_i are at most 1, and their x-coordinates are at most $\frac{1}{3}$. This implies that the slope of every line $q_i d$, $i = 1, 2, \ldots, n + 1$, is in the interval $[-3/2, 0]$. Therefore, S_i is contained in a right triangle bounded by a horizontal line through q_i, a vertical line through q_{i+1}, and the line $q_i d$. The area of this triangle is at most $\frac{1}{2}(\frac{1}{3n} \cdot (\frac{3}{2} \cdot \frac{1}{3n})) = 1/(12n^2)$. That is, $s_n = \sum_{i=1}^{n} \text{area}(S_i) \leq 1/(12n)$. In particular, $s_n \leq \delta \cdot \text{area}(T_0)$ for a sufficiently large n. Then we can write

$$\begin{aligned}
\mu_{Q_1}(o) &= \frac{\int_{p \in Q_1} \text{dist}(o, p)\, \text{d}p}{\text{area}(Q_1)} \geq \frac{\mu_{Q_0}(o) \cdot \text{area}(Q_0) + \sum_{i=0}^{n} \mu_{T_i}(o) \cdot \text{area}(T_i)}{\text{area}(Q_1)} \\
&\geq \frac{\frac{1}{3}(\text{area}(Q_1) - s_n) + \delta \cdot \text{area}(T_0)}{\text{area}(Q_1)} \geq \frac{1}{3} + \frac{2\delta \cdot \text{area}(T_0)}{3 \cdot \text{area}(Q_1)} > \frac{1}{3}.
\end{aligned}$$

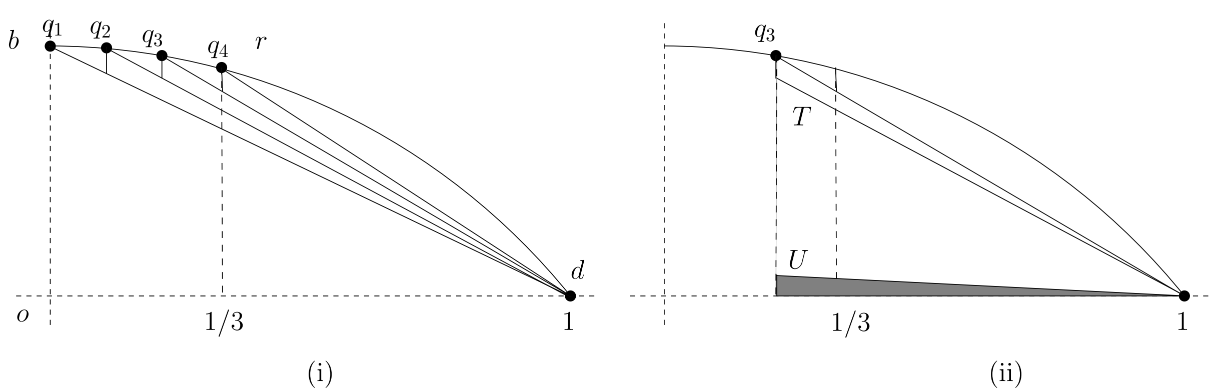

Figure 2: (i) The subdivision of Q_1 for $n = 3$. Here $o = (0, 0)$, $q_1 = b = (0, h)$, $q_4 = r$, $d = (1, 0)$. (ii) Transformation in the proof of Lemma 4.

This concludes the proof of Theorem 1. □

Remark. A finite triangulation, followed by taking the limit suffices to prove the slightly weaker, non-strict inequality: $\mu_Q^* \geq \frac{1}{6} \cdot \Delta(Q)$.

3 Upper bounds: proof of Theorem 2

Let Q be a planar convex body and let $D = \Delta(Q)$. Let ∂Q denote the boundary of Q, and let $\mathrm{int}(Q)$ denote the interior of Q. Let Ω be the smallest disk enclosing Q, and let o and R be the center and respectively the radius of Ω. Write $a = \frac{2(4-\sqrt{3})}{13}$. By the convexity of Q, $o \in Q$, as observed in [1]. Moreover, Abu-Affash and Katz [1] have shown that the average distance from o to the points in Q satisfies

$$\mu_Q(o) \leq \frac{2}{3\sqrt{3}} \cdot \Delta(Q) < 0.3850 \cdot \Delta(Q).$$

Here we further refine their analysis and derive a better upper bound on the average distance from o to the points in Q:

$$\mu_Q(o) \leq \frac{2(4-\sqrt{3})}{13} \cdot \Delta(Q) < 0.3490 \cdot \Delta(Q).$$

Since the average distance from the Fermat-Weber center of Q is not larger than that from o, we immediately get the same upper bound on c_2. We need the next simple lemma established in [1]. Its proof follows from the definition of average distance.

Lemma 5 [1]. *Let Q_1, Q_2 be two (not necessarily convex) disjoint bodies in the plane, and p be a point in the plane. Then $\mu_{(Q_1 \cup Q_2)}(p) \leq \max(\mu_{Q_1}(p), \mu_{Q_2}(p))$.*

By induction, Lemma 5 yields:

Lemma 6 *Let $Q_1, Q_2, \ldots, Q_n$ be n (not necessarily convex) pairwise disjoint bodies in the plane, and p be a point in the plane. Then*

$$\mu_{(Q_1 \cup \ldots \cup Q_n)}(p) \leq \max(\mu_{Q_1}(p), \ldots \mu_{Q_n}(p)).$$

We also need the following classical result of Jung [10]; see also [9].

Theorem 3 (Jung [10]). *Let S be a set of diameter $\Delta(S)$ in the plane. Then S is contained in a circle of radius $\frac{1}{\sqrt{3}} \cdot \Delta(S)$.*

By Theorem 3 we have

$$\frac{1}{2} D \leq R \leq \frac{1}{\sqrt{3}} D. \tag{1}$$

Observe that the average distance from the center of a circular sector of radius r and center angle α to the points in the sector is

$$\frac{\int_0^r \alpha x^2 \, \mathrm{d}x}{\int_0^r \alpha x \, \mathrm{d}x} = \frac{\alpha r^3/3}{\alpha r^2/2} = \frac{2r}{3}. \tag{2}$$

Proof of Theorem 2. If $o \in \partial Q$ then Q is contained in a halfdisk Θ of Ω, of the same diameter D, with o as the midpoint of this diameter. Then by (2), it follows that $\mu_Q(o) \leq \frac{1}{3} \cdot D$, as required.

We can therefore assume that $o \in \mathrm{int}(Q)$. Let $\varepsilon > 0$ be sufficiently small. For a large positive integer n, subdivide Ω into n congruent circular double sectors (wedges) $W_1, \ldots, W_n$, symmetric about o (the center of Ω), where each sector subtends an angle $\alpha = \pi/n$. Consider a double sector $W_i = U_i \cup V_i$, where U_i and V_i are circular sectors of Ω. Let $X_i \subseteq U_i$, and $Y_i \subseteq V_i$ be two minimal circular sectors centered at o and containing $U_i \cap Q$, and $V_i \cap Q$, respectively: $U_i \cap Q \subseteq X_i$, and $V_i \cap Q \subseteq Y_i$. Let x_i and y_i be the radii of X_i and Y_i, respectively. Let $X_i' \subseteq X_i$, and $Y_i' \subseteq Y_i$ be two circular subsectors of radii $(1-\varepsilon)x_i$ and $(1-\varepsilon)y_i$, respectively. Since $o \in \mathrm{int}(Q)$, we can select $n = n(Q, \varepsilon)$ large enough, so that for each $1 \leq i \leq n$, the subsectors X_i' and Y_i' are nonempty and entirely contained in Q. That is, for every i, we have

$$X_i' \cup Y_i' \subseteq W_i \cap Q \subseteq X_i \cup Y_i. \tag{3}$$

It is enough to show that for any double sector $W = W_i$, we have

$$\lim_{\varepsilon \to 0} \mu_{(W \cap Q)}(o) \leq aD,$$

since then, Lemma 6 (with W_i being the n pairwise disjoint regions) will imply that $\mu_Q(o) \leq aD$, concluding the proof of Theorem 2. For simplicity, write $x = x_i$, and $y = y_i$. Obviously the diameter of $W \cap Q$ is at most D, hence $x + y \leq D$. We can assume w.l.o.g. that $y \leq x$, so by Theorem 3 we also have $x \leq \frac{1}{\sqrt{3}} \cdot D$. Hence so far, our constraints are:

$$0 < y \leq x \leq \frac{1}{\sqrt{3}} \cdot D \qquad \text{and} \qquad x + y \leq D. \tag{4}$$

By the minimality of the disk Ω, the convex body Q either contains three points q_1, q_2, q_3 on the boundary of Ω such that the triangle $q_1q_2q_3$ contains the disk center o in the interior, or contains two points q_1, q_2 on the boundary of Ω such that the segment q_1q_2 goes through the disk center o. In the latter case, the segment q_1q_2 can be viewed as a degenerate triangle $q_1q_2q_3$ with two coinciding vertices q_2 and q_3.

Let r be the radius of the largest disk centered at o that is contained in the convex body Q. Then r is at least the distance from o to the longest side of the triangle $q_1q_2q_3$, say q_1q_2. Since $|q_1q_2| \leq D$, $|oq_1| = |oq_2| = R$, we have

$$r \geq \sqrt{R^2 - D^2/4}.$$

Then the constraints in (4) can be expanded to the following:

$$\sqrt{R^2 - D^2/4} \leq y \leq x \leq R \leq D/\sqrt{3} \quad \text{and} \quad x + y \leq D. \tag{5}$$

By the definition of average distance, we can write

$$\begin{aligned} \mu_{(W \cap Q)}(o) &= \frac{\int_{p \in (W \cap Q)} \mathrm{dist}(o, p)\, \mathrm{d}p}{\mathrm{area}(W \cap Q)} \\ &\leq \frac{\alpha \cdot \frac{x^2}{2} \cdot \frac{2x}{3} + \alpha \cdot \frac{y^2}{2} \cdot \frac{2y}{3}}{\alpha(1-\varepsilon)^2 \cdot \left(\frac{x^2}{2} + \frac{y^2}{2}\right)} = \frac{2}{3} \cdot \frac{x^3 + y^3}{(1-\varepsilon)^2 \cdot (x^2 + y^2)}. \end{aligned} \tag{6}$$

Let

$$f(x, y) = \frac{2}{3} \cdot \frac{x^3 + y^3}{x^2 + y^2}, \quad \text{and} \quad f_1(x, y, \varepsilon) = \frac{2}{3} \cdot \frac{x^3 + y^3}{(1-\varepsilon)^2 \cdot (x^2 + y^2)}. \tag{7}$$

Clearly for any feasible pair (x, y), we have

$$\lim_{\varepsilon \to 0} f_1(x, y, \varepsilon) = f(x, y).$$

It remains to maximize $f(x, y)$ subject to the constraints in (5). We will show that under these constraints,

$$f(x, y) \le \frac{2(4 - \sqrt{3})}{13} \cdot D. \tag{8}$$

Then

$$\lim_{\varepsilon \to 0} \mu_{(W \cap Q)}(o) \le \lim_{\varepsilon \to 0} f_1(x, y, \varepsilon) = f(x, y) \le \frac{2(4 - \sqrt{3})}{13} \cdot D,$$

as required.

We next verify the upper bound in (8). Throughout our analysis, we may assume that D is a fixed constant and x, y, and R are variable parameters. Substituting $z = y/x$ in (7), we have

$$f(x, y) = g(x, z) = \frac{2x}{3} \cdot \frac{1 + z^3}{1 + z^2}.$$

Then, taking the partial derivative of $g(x, z)$ with respect to z, we have

$$\begin{aligned} \frac{\partial}{\partial z} g(x, z) &= \frac{2x}{3} \cdot \left(\frac{3z^2}{1 + z^2} - \frac{1 + z^3}{(1 + z^2)^2} 2z \right) \\ &= \frac{2x}{3} \cdot \frac{3z^2(1 + z^2) - (1 + z^3)2z}{(1 + z^2)^2} = \frac{2x}{3} \cdot \frac{z(z^3 + 3z - 2)}{(1 + z^2)^2}. \end{aligned}$$

The cubic equation $z^3 + 3z - 2 = 0$ has exactly one real root $z_0 = (\sqrt{2} + 1)^{1/3} - (\sqrt{2} - 1)^{1/3} = 0.596\ldots$. Thus for a fixed x, the function $g(x, z)$ is strictly decreasing for $0 \le z \le z_0$ and is strictly increasing for $z_0 \le z \le 1$. Therefore, by the upper bound that $x + y \le D$ and the lower bound that $\sqrt{R^2 - D^2/4} \le r \le y$ in (5), the function $f(x, y)$ is maximized when y takes one of the following two extreme values:

$$y_1 = \sqrt{R^2 - D^2/4} \quad \text{and} \quad y_2 = D - x.$$

By the inequality that $x \le R \le D/\sqrt{3}$ in (5), it follows that $x + y_1 \le R + \sqrt{R^2 - D^2/4} \le D/\sqrt{3} + D/\sqrt{12} < D$. Since $x + y_2 = D$, we have $y_1 < y_2$.

Case 1. We first consider the easy case that $y = y_2$. Then $x + y = D$, and we have

$$f(x, y) = \frac{2}{3} \cdot \frac{x^3 + y^3}{x^2 + y^2} = \frac{2}{3} \cdot \frac{(x + y)^3 - 3(x + y)xy}{(x + y)^2 - 2xy} = \frac{2}{3} \cdot \frac{D^3 - 3Dxy}{D^2 - 2xy}.$$

Substituting $w = xy$, we tranform the function $f(x, y)$ to a function $h_1(w)$:

$$f(x, y) = h_1(w) = \frac{2}{3} \cdot \frac{3Dw - D^3}{2w - D^2}.$$

The function $h_1(w)$ is decreasing in w because

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}w} h_1(w) &= \frac{2}{3} \cdot \left(\frac{3D}{2w - D^2} - \frac{2(3Dw - D^3)}{(2w - D^2)^2} \right) \\ &= \frac{2}{3} \cdot \frac{3D(2w - D^2) - 2(3Dw - D^3)}{(2w - D^2)^2} = \frac{2}{3} \cdot \frac{-D^3}{(2w - D^2)^2} \le 0. \end{aligned}$$

Thus $f(x, y)$ is maximized when xy is minimized. With the sum $x + y$ fixed at D, and under the constraint that $x \le R \le D/\sqrt{3}$ in (5), the product xy is minimized when $x = \frac{1}{\sqrt{3}}D$ and $y = \left(1 - \frac{1}{\sqrt{3}}\right) D$. Thus we have

$$f(x, y) \le \frac{2}{3} \cdot \frac{\left(\frac{1}{\sqrt{3}}\right)^3 + \left(1 - \frac{1}{\sqrt{3}}\right)^3}{\left(\frac{1}{\sqrt{3}}\right)^2 + \left(1 - \frac{1}{\sqrt{3}}\right)^2} D = \frac{2(4 - \sqrt{3})}{13} D = 0.3489 \ldots D. \tag{9}$$

Case 2. We next consider the case[1] that $y = y_1$. With y fixed, the function $f(x, y)$ is maximized when x is as large as possible because

$$\begin{aligned}
\frac{\partial}{\partial x} f(x, y) &= \frac{2}{3} \cdot \left(\frac{3x^2}{x^2 + y^2} - \frac{x^3 + y^3}{(x^2 + y^2)^2} 2x \right) \\
&= \frac{2}{3} \cdot \frac{3x^2(x^2 + y^2) - (x^3 + y^3)2x}{(x^2 + y^2)^2} \\
&= \frac{2}{3} \cdot \frac{x(x^3 + 3xy^2 - 2y^3)}{(x^2 + y^2)^2} \\
&\ge \frac{2}{3} \cdot \frac{x(y^3 + 3y^3 - 2y^3)}{(x^2 + y^2)^2} \ge 0.
\end{aligned}$$

Thus for $y = \sqrt{R^2 - D^2/4}$ and under the constraint that $x \le R$ in (5), the function $f(x, y)$ is maximized when $x = R$ and $y = \sqrt{R^2 - D^2/4} = \sqrt{x^2 - D^2/4}$. It follows that

$$\frac{\mathrm{d}x}{\mathrm{d}R} = 1 \qquad \text{and} \qquad \frac{\mathrm{d}y}{\mathrm{d}R} = \frac{\mathrm{d}\sqrt{x^2 - D^2/4}}{\mathrm{d}R} = \frac{x}{\sqrt{x^2 - D^2/4}} = x/y.$$

Let $h_2(R) = f(R, \sqrt{R^2 - D^2/4})$. We next show that $h_2(R)$ is increasing in R. Taking the derivative, we have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}R} h_2(R) &= \frac{2}{3} \cdot \left(\frac{3x^2 \frac{\mathrm{d}x}{\mathrm{d}R} + 3y^2 \frac{\mathrm{d}y}{\mathrm{d}R}}{x^2 + y^2} - \frac{x^3 + y^3}{(x^2 + y^2)^2} \left(2x \frac{\mathrm{d}x}{\mathrm{d}R} + 2y \frac{\mathrm{d}y}{\mathrm{d}R} \right) \right) \\
&= \frac{2}{3} \cdot \left(\frac{3x^2 + 3y^2(x/y)}{x^2 + y^2} - \frac{x^3 + y^3}{(x^2 + y^2)^2} (2x + 2y(x/y)) \right) \\
&= \frac{2}{3} \cdot \frac{(3x^2 + 3xy)(x^2 + y^2) - (x^3 + y^3)(2x + 2x)}{(x^2 + y^2)^2} \\
&= \frac{2}{3} \cdot \frac{(3x^4 + 3x^2y^2 + 3x^3y + 3xy^3) - (4x^4 + 4xy^3)}{(x^2 + y^2)^2} \\
&= \frac{2}{3} \cdot \frac{(x^4 + 3x^2y^2 + 3x^3y + xy^3) - (2x^4 + 2xy^3)}{(x^2 + y^2)^2} \\
&= \frac{2}{3} \cdot \frac{x^4}{(x^2 + y^2)^2} \cdot \left((1 + y/x)^3 - 2 - 2(y/x)^3 \right).
\end{aligned}$$

[1] This case, when $x + y < D$, has been mistakenly overlooked in the proof given in [6].

Substituting $z = y/x$, we simplify the last factor $(1+y/x)^3 - 2 - 2(y/x)^3$ in the resulting expression above to

$$h_3(z) = (1+z)^3 - 2 - 2z^3.$$

To show that $\frac{\mathrm{d}}{\mathrm{d}R} h_2(R) > 0$, it remains to show that $h_3(z) > 0$. For $0 \le z \le 1$, the function $h_3(z)$ is increasing in z because

$$\frac{\mathrm{d}}{\mathrm{d}z} h_3(z) = 3(1+z)^2 - 6z^2 = -3(1-z)^2 + 6 \ge 6 - 3 > 0.$$

Recall that $x \ge y$. If $R \le \frac{3(4-\sqrt{3})}{13} D$, then we would easily have

$$f(x,y) = \frac{2}{3} \cdot \frac{x^3+y^3}{x^2+y^2} \le \frac{2}{3} \cdot \frac{x^3}{x^2} = \frac{2}{3} x \le \frac{2}{3} R \le \frac{2(4-\sqrt{3})}{13} D,$$

which matches the upper bound in case 1. Now suppose that $R > \frac{3(4-\sqrt{3})}{13} D$. Then

$$D/R < \frac{13}{3(4-\sqrt{3})} \text{ and } z = y/x = \sqrt{1-(D/R)^2/4} > \sqrt{1 - \left(\frac{13}{3(4-\sqrt{3})}\right)^2 \Big/ 4} = 0.2955\ldots.$$

It follows that

$$h_3(z) > h_3\left(\sqrt{1 - \left(\frac{13}{3(4-\sqrt{3})}\right)^2 \Big/ 4}\right) = 0.1226\ldots > 0,$$

hence

$$\frac{\mathrm{d}}{\mathrm{d}R} h_2(R) > 0.$$

We have shown that the function $h_2(R)$ is increasing in R. Then, under the constraint that $R \le D/\sqrt{3}$ in (5), $h_2(R)$ is maximized when $R = \frac{1}{\sqrt{3}} D$. Correspondingly, $f(x,y)$ is maximized when $x = \frac{1}{\sqrt{3}} D$ and $y = \frac{1}{\sqrt{12}} D$. Thus

$$f(x,y) \le \frac{2}{3} \cdot \frac{\left(\frac{1}{\sqrt{3}}\right)^3 + \left(\frac{1}{\sqrt{12}}\right)^3}{\left(\frac{1}{\sqrt{3}}\right)^2 + \left(\frac{1}{\sqrt{12}}\right)^2} D = \frac{\sqrt{3}}{5} D = 0.3464\ldots D, \tag{10}$$

which is (slightly) smaller than the upper bound obtained in case 1. This proves the upper bound in (8).

Centrally symmetric body. Assume now that Q is centrally symmetric with respect to a point q. We repeat the same "double sector" argument. It is enough to observe that: (i) the center of Ω coincides with q, that is, $o = q$; and (ii) $x = y \le \frac{1}{2} \cdot D$ for any double sector W. By (6), the average distance calculation yields now

$$\mu_{(W \cap Q)}(o) \le \frac{2x^3}{3(1-\varepsilon)^2 \cdot x^2} = \frac{2x}{3(1-\varepsilon)^2} \le \frac{D}{3(1-\varepsilon)^2},$$

and by taking the limit when ε tends to zero, we obtain

$$\mu_Q(o) \le \frac{D}{3},$$

as required. The proof of Theorem 2 is now complete. □

4 Applications

1. Carmi, Har-Peled and Katz [3] showed that given a convex polygon Q with n vertices, and a parameter $\varepsilon > 0$, one can compute an ε-approximate Fermat-Weber center $q \in Q$ in $O(n + 1/\varepsilon^4)$ time such that $\mu_Q(q) \le (1+\varepsilon)\mu_Q^*$. Abu-Affash and Katz [1] gave a simple $O(n)$-time algorithm for computing the center q of the smallest disk enclosing Q, and showed that q approximates the Fermat-Weber center of Q, with $\mu_Q(q) \le \frac{25}{6\sqrt{3}}\mu_Q^*$. Our Theorems 1 and 2, combined with their analysis, improves the approximation ratio to about 2.09:

$$\mu_Q(q) \le \frac{12(4-\sqrt{3})}{13}\mu_Q^*.$$

2. The value of the constant c_1 (i.e., the infimum of $\mu_Q^*/\Delta(Q)$ over all convex bodies Q in the plane) plays a key role in the following load balancing problem introduced by Aronov, Carmi and Katz [2]. We are given a convex body D and m points $p_1, p_2, \ldots, p_m$ representing *facilities* in the interior of D. Subdivide D into m convex regions, $R_1, R_2, \ldots, R_m$, of equal area such that $\sum_{i=1}^m \mu_{p_i}(R_i)$ is minimal. Here $\mu_{p_i}(R_i)$ is the *cost* associated with facility p_i, which may be interpreted as the average travel time from the facility to any location in its designated region, each of which has the same area. One of the main results in [2] is a $(8+\sqrt{2\pi})$-factor approximation in the case that D is an $n_1 \times n_2$ rectangle for some integers $n_1, n_2 \in \mathbb{N}$. This basic approximation bound is then used for several other cases, e.g., subdividing a convex fat domain D into m convex regions R_i.

By substituting $c_1 = \frac{1}{6}$ (Theorem 1) into the analysis in [2], the upper bound for the approximation ratio improves from $8 + \sqrt{2\pi} \approx 10.5067$ to $7 + \sqrt{2\pi} \approx 9.5067$. It can be further improved by optimizing another parameter used in their calculation. Let S be a unit square and let $s \in S$ be an arbitrary point in the square. Aronov et al. [2] used the upper bound $\mu_S(s) \le \frac{2}{3}\sqrt{2} \approx 0.9429$. It is clear that $\max_{s\in S}\mu_S(s)$ is attained if s is a vertex of S. The average distance of S from such a vertex, say v, is $\mu_S(v) = \frac{1}{3}\left(\sqrt{2} + \ln(1+\sqrt{2})\right) \approx 0.7652$, and so $\mu_S(s) \le \frac{1}{3}\left(\sqrt{2} + \ln(1+\sqrt{2})\right)$, for any $s \in S$. With these improvements, the upper bound on the approximation ratio becomes $7 + \frac{\sqrt{\pi}}{2}\left(\sqrt{2} + \ln(1+\sqrt{2})\right) \approx 9.0344$.

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