Succinct Greedy Geometric Routing in the Euclidean Plane*

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Abstract

In greedy geometric routing, messages are passed in a network embedded in a metric space according to the greedy strategy of always forwarding messages to nodes that are closer to the destination. We show that greedy geometric routing schemes exist for the Euclidean metric in $\mathbf{R}^2$, for 3-connected planar graphs, with coordinates that can be represented *succinctly*, that is, with $O(\log n)$ bits, where n is the number of vertices in the graph. Moreover, our embedding strategy introduces a coordinate system for $\mathbf{R}^2$ that supports distance comparisons using our succinct coordinates. Thus, our scheme can be used to significantly reduce bandwidth, space, and header size over other recently discovered greedy geometric routing implementations for $\mathbf{R}^2$.

1 Introduction

In an intriguing confluence of computational geometry and networking, *geometric routing* has shown how simple geometric rules can replace cumbersome routing tables to facilitate effective message passing in a network (e.g., see [5, 13, 24, 25, 29, 30, 31]). Geometric routing algorithms perform message passing using geometric information stored at the nodes and edges of a network. For example, geometric information could come from the latitude and longitude GPS coordinates of the nodes in a wireless sensor network or this information could come from an embedded doubly-connected edges list representation of a planar subgraph of such a network. Indeed, in one of the early works on the subject, Bose *et al.* [5] show how to do geometric routing in an embedded planar subgraph of a wireless sensor network by using a geometric subdivision traversal algorithm of Kranakis *et al.* [28], which was first introduced in the computational geometry literature.

1.1 Greedy Geometric Routing

Perhaps the simplest routing rule is the *greedy* one:

- If a node v receives a message M intended for a destination $w \neq v$, then v should forward M to a neighbor that is closer to w than v is.

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This rule can be applied in any metric space, of course, but simple and natural metric spaces are preferred over cumbersome or artificial ones.

The greedy routing rule traces its roots back to the original "degrees-of-separation" small-world experiment of Milgram [34], where he asked randomly chosen individuals to forward 296 letters, initiating in Omaha, Nebraska and Wichita, Kansas, all intended for a lawyer in Boston, using the rule that requires each letter to be forwarded to an acquaintance that is closer to the destination.

In the modern context, researchers are interested in solutions that use a paradigm introduced by Rao *et al.* [39] of doing greedy geometric routing in geometric graphs that assigns virtual coordinates in a metric space to each node in the network, rather than relying on physical coordinates. For example, GPS coordinates may be unavailable for some sensors or the physical coordinates of network nodes may be known only to a limited degree of certainty. Thus, we are interested in greedy routing schemes that assign network nodes to virtual coordinates in a natural metric space.

Interestingly, the feasibility of the greedy routing rule depends heavily on the geometry of the underlying metric space used to define the notion of "closer to the destination." For example, it is easy to see that star graphs (consisting of a central vertex adjacent to every node in an arbitrarily large independent set) cannot support greedy geometric routing in any fixed-dimensional Euclidean space. By a simple packing argument, there has to be two members of the large independent set, in such a graph, that will be closer to each other than the central vertex. Likewise, even for bi-connected or tri-connected planar graphs embedded in $\mathbf{R}^2$, a network may have "holes" where greedy routing algorithms could get "stuck" in a local metric minimum (e.g., see Funke [16] for related work on hole detection in sensor networks). Alternatively, several researchers (e.g., see [13, 25, 35]) have shown that greedy geometric routing is possible, for any connected graph, in fixed-dimensional hyperbolic spaces. Our interest in this paper, however, is on greedy geometric routing in $\mathbf{R}^2$ under the Euclidean metric, since this space more closely matches the geometry of wireless sensor networks.

Interest in greedy geometric routing in fixed-dimensional Euclidean spaces has expanded greatly since the work by Papadimitriou and Ratajczak [37], who showed that any 3-connected planar graph can be embedded in $\mathbf{R}^3$ so as to support greedy geometric routing. Indeed, their conjecture that such embeddings are possible in $\mathbf{R}^2$ spawned a host of additional papers (e.g., see [1, 10, 11, 13, 33, 35]). Leighton and Moitra [32] settled this conjecture by giving an algorithm to produce a greedy embedding of any 3-connected planar graph in $\mathbf{R}^2$, and a similar result was independently found by Angelini *et al.* [1]. Greedy embeddings in $\mathbf{R}^2$ were previously known only for graphs containing power diagrams [10], graphs containing Delaunay triangulations [33], and existentially (but not algorithmically) for triangulations [11].

1.2 Succinct Geometric Routing

In spite of their theoretical elegance, these results settling the Papadimitriou-Ratajczak conjecture have an unfortunate drawback, in that the virtual coordinates of nodes in these solutions require $\Omega(n \log n)$ bits each in the worst case. These space inefficiencies reduce the applicability of these results for greedy geometric routing, since one could alternatively keep routing tables of size $O(n \log n)$ bits at each network node to support message passing. Indeed, such routing tables would allow for network nodes to be identified using labels of only $O(\log n)$ bits each, which would significantly cut down on the space, bandwidth, and packet header size needed to communicate the destination for each packet being routed. Thus, for a solution to be effectively solving the routing problem using a greedy geometric routing scheme, we desire that it be *succinct*, that is, it should use $O(\log n)$ bits per virtual coordinate. Succinct greedy geometric routing schemes are known for fixed-dimensional hyperbolic spaces [13, 35], but we are unaware of any prior work on succinct greedy geometric routing in fixed-dimensional Euclidean spaces. We are therefore interested in this paper in a method for succinct greedy geometric routing in $\mathbf{R}^2$, with distance comparisons being consistent with the standard Euclidean L_2 metric.

1.3 Additional Related Prior Work

In addition to the greedy geometric routing schemes referenced above, there is a hybrid scheme, for example, as outlined by Karp and Kung [24], which combines a greedy routing strategy with face routing [5]. Similar hybrid schemes were subsequently studied by several other researchers (e.g., see [15, 29, 30, 31]). An alternative hybrid augmented greedy scheme is introduced by Carlsson and Eager [9]. In addition, Gao *et al.* [17] show how to maintain a geometric spanner in a mobile network so as to support hybrid routing schemes. Although such schemes are *local*, in that routing decisions can be made at a node v simply using information about v's neighbors, we are interested in this paper in routing methods that are purely greedy.

As mentioned above, Rao *et al.* [39] introduce the idea of doing greedy geometric routing using virtual coordinates, although they make no theoretical guarantees, and Papadimitriou and Ratajczak [37] are the first to prove such a method exists in $\mathbf{R}^3$, albeit with a non-standard metric. In addition, we also mentioned above how Leighton and Moitra [32] and Angelini *et al.* [1] have settled the Papadimitriou-Ratajczak conjecture, albeit with solutions that are not succinct. Moreover, the only known succinct greedy geometric routing schemes are for fixed-dimensional hyperbolic spaces [13, 35]. Thus, there does not appear to be any prior work on succinct greedy geometric routing in $\mathbf{R}^2$ using the standard Euclidean L_2 metric.

The problem of constructing succinct greedy geometric routing schemes in $\mathbf{R}^2$ is related to the general area of compressing geometric and topological data for networking purposes. Examples of such work includes the compression schemes of Suri *et al.* [44] for two-dimensional routing tables, and the coordinate and mesh compression work of Isenburg *et al.* [23]. We should stress, therefore, that we are not primarily interested in this paper in compression schemes for greedy geometric routing; we are interested primarily in coordinate systems for greedy routing, since they have a better applicability in distributed settings. In particular, we are not interested in a compression scheme where the computation of the coordinates in $\mathbf{R}^2$ of a network node v depends on anything other than a succinct label for v. That is, we want a succinct *coordinate system*, not simply an efficient compression scheme that supports greedy routing. Indeed, we show that succinct compression schemes are trivial, given known Euclidean greedy geometric routing methods [1, 32].

Another area of related work is on methods for routing in geometric graphs, such as road networks (e.g., see [2, 19, 22, 26, 40, 41, 45]). For example, Sedgewick and Vitter [41] and Goldberg and Harrelson [19] study methods based on applying AI search algorithms, and Bast *et al.* [2] explore routing methods based on the use of transit nodes. In this related work, the coordinates of the network nodes are fixed geometric points, whereas, in the greedy geometric routing problems we study in this paper, vertices are assigned virtual coordinates so as to support greedy routing.

1.4 Our Results

We provide a succinct greedy geometric routing scheme for 3-connected planar graphs in $\mathbf{R}^2$. At the heart of our scheme is a new greedy embedding for 3-connected planar graphs in $\mathbf{R}^2$ which exploits the tree-like topology of a spanning (Christmas cactus) subgraph. Our embedding allows us to form a coordinate system which uses $O(\log n)$ bits per vertex, and allows distance comparisons to be done just using our coordinate representations consistently with the Euclidean metric. Although we are primarily interested in such a coordinate system for greedy geometric routing, we also give a simple global compression scheme for greedy geometric routing, based on the approach of Leighton and Moitra [32] and Angelini *et al.* [1], which achieves $O(\log n)$ bits per vertex, which is asymptotically optimal.

Our coordinate scheme for greedy geometric routing in a graph G is based on a three-phase approach. In the first phase, we find a spanning subgraph, C, of G, called a Christmas cactus graph [32]. In the second phase, we find a graph-theoretic dual to C, which is a tree, T, and we form a heavy path decomposition on T. Finally, in the third phase, we show how to use T and C to embed G in $\mathbf{R}^2$ to support greedy routing with coordinates that can be represented using $O(\log^2 n)$ bits, and then we show how this can be further reduced to $O(\log n)$ bits per node.

2 Finite-Length Coordinate Systems

Let us begin by formally defining what we mean by a coordinate system, and how that differs, for instance, from a simple compression scheme. Let Σ be an alphabet, and let Σ^* a set of finite-length strings over Σ. We define a *coordinate system* f for a space S:

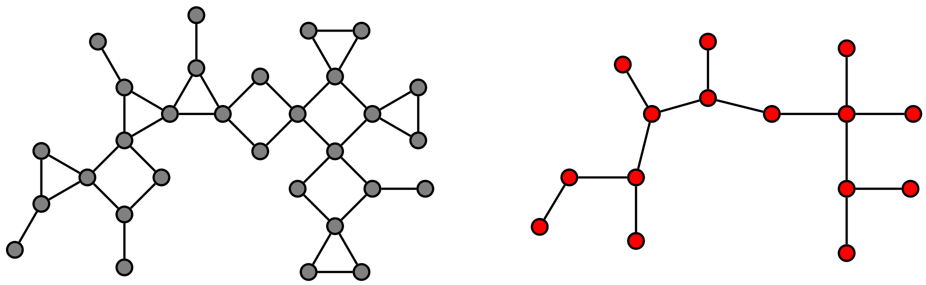
1. f is a map, $f : \Sigma^* \to S$, which assigns character strings to points of S.
2. f may be *parameterized*: the assignment of strings to points may depend on a fixed set of parameters.
3. f is *oblivious*: the value of f on any given $x \in \Sigma^*$ must depend only on f's parameters and x itself. It cannot rely on any other character strings in Σ^*, points in S, or other values of f.

Clearly, this is a computationally-motivated definition of a coordinate system, since real-world computations performed on actual points must use finite representations of those points. This is an issue and theme present, for instance, in computational geometry (e.g., see [3, 4, 6, 7, 8, 12, 14, 20, 21, 23, 36, 38, 42, 44]). Note also that our definition can be used to define finite versions of all the usual coordinate systems, since it allows for the use of symbols like "π", "/," and k-th root symbols. Thus, it supports finite coordinates using rational and algebraic numbers, for example. In addition, note that it supports points in non-Cartesian coordinate systems, such as a finite-length polar coordinate system, in that we can allow strings of the form "(x, y)" where x is a string representing a value $r \in \mathbf{R}^+$ and y is a string representing a value $\theta \in [0, 2\pi)$, which may even use "π". It also allows for non-unique representations, like the homogeneous coordinate system for $\mathbf{R}^2$, which uses triples of strings, with each triple representing a point in the Euclidean plane, albeit in a non-unique way. If f is lacking property 3, we prefer to think of f as a *compression scheme*. Examples of compression schemes are mappings that use look-up tables, which are built incrementally based on sequences of previous point assignments [23]. Given a compression scheme $f : \Sigma_f^* \to S$, note that it is possible to construct a coordinate system $f' : \Sigma_{f'}^* \to S$ by augmenting strings in Σ_f^* with the data required to evaluate f (such as the assignments of other points in a set of interest).

3 Greedy Routing in Christmas Cactus Graphs

Our method is a non-trivial adaptation of the Leighton and Moitra scheme [32], so we begin by reviewing some of the ideas from their work.

A graph G is said to be a *Christmas cactus graph* if: (1) each edge of G is in at most one cycle, (2) G is connected, and (3) removing any vertex disconnects G into at most two components. For ease of discussion, we consider any edge in a Christmas cactus graph that is not in a simple cycle to be a simple cycle itself (a 2-cycle); hence, every edge in is in exactly one simple cycle. The *dual tree* of a Christmas cactus graph G is a tree containing a vertex for each simple cycle in G with an edge between two vertices if their corresponding cycles in G share a vertex. Rooting the dual tree at an arbitrary vertex creates what we call a *depth tree.*(See Fig. 1.)



(a) (b)

Figure 1: (a) A Christmas cactus graph and (b) its dual tree.

Having a depth tree allows us to apply the rooted tree terminology to cycles in G. In particular: *root*, *depth*, *parent*, *child*, *ancestor*, and *descendant* all retain their familiar definitions. We define the *depth* of a node v to be the minimum depth of any cycle containing v. The unique node that a cycle C shares with its parent is called the *primary node* of C. Node v is a *descendant* of a cycle C if v is in a cycle that is a descendant of C and v is not the primary node of C. Node v is a *descendant* of node u if removing neighbors of u with depth less than or equal to u leaves u and v in the same component.

3.1 Greedy Routing with a Christmas Cactus Graph Embedding

Leighton and Moitra [32] show that every 3-connected planar graph contains a spanning Christmas cactus subgraph and that every Christmas cactus graph has a greedy embedding in $\mathbf{R}^2$, which together imply that 3-connected planar graphs have greedy embeddings in $\mathbf{R}^2$. Working level by level in a depth tree, Leighton and Moitra [32] embed the cycles of a Christmas cactus graph on semi-circles of increasing radii, centered at the origin. Within the embedding we say that vertex u is above vertex v if u is embedded farther from the origin than v, and we say that u is to the left of v if u is embedded in the positive angular direction relative to v. We can define below and right similarly. These comparisons naturally give rise to directions of movement between adjacent vertices in the embedding: up, down, left, and right.

Routing from start vertex s to a terminal vertex t in a Christmas cactus graph embedding can be broken down into two cases: (1) t is a descendant of s, and (2) t is not a descendant of s.

1. As shown in Fig. 2(a), if t is a descendant of s, then we can route to t by a simple path of up and right hops, up and left hops, or a combination of the two.

2. As shown in Fig. 2(b), if t is not a descendant of s, then we route to the least common (cycle) ancestor of s and t. Suppose, without loss of generality, that t is to the left of s, then we can reach this cycle by a sequence of down and left hops. Once on the cycle, we can move left until we reach an ancestor of t. Now we are back in case 1.

3.2 A Succinct Compression Scheme

Using the Christmas cactus graph embedding discussed above, we can assign succinct integer values to each vertex, allowing us perform greedy routing according to the Euclidean L_2 metric. Our embedding $f : V(G) \rightarrow \mathbf{Z}_n^3$ produces a triple of the following integers: radialOrder(v): the number of vertices to the right of v; level(v): the number of semi-circles between the vertex and the origin, excluding the semi-circle that v is embedded on; and boundary(v): the smallest radialOrder value of all vertices that are descendants of v. The Leighton-Moitra embedding has the property that all descendants of v fall between v and the vertex embedded immediately to the right of v on the same level as v. Since each element of the triple can take on values in the range $[0, n]$, the triple can be stored using $O(\log n)$ bits.

We can implement each step of the routing scheme using only the triples of s, the neighbors of s, and t. Queries of the form u *is left/right of* v, involve a straightforward comparison of the radialOrder element of the triple. Likewise for u *is above/below* v, using level. The same comparisons can be used to determine which neighbors of s are a left, right, down, or up move away. Finally, queries of the form u *is a descendant of* v are true if and only if boundary(v) $\leq$ radialOrder(u) $\leq$ radialOrder(v) and level(v) $\leq$ level(u).

To extend this routing scheme to graphs that have a spanning Christmas cactus subgraph, we need to ensure that the routing scheme does not fail by following edges that are not in the Christmas cactus subgraph. Since the Christmas cactus graph has bounded degree 4, for a node v, we can store the triples of neighbors of v in the Christmas cactus graph, in addition to storing the triple for v, and only allow our greedy routing scheme to choose vertices that are neighbors in the Christmas cactus subgraph. Storing these extra triples in the coordinate does not increase its

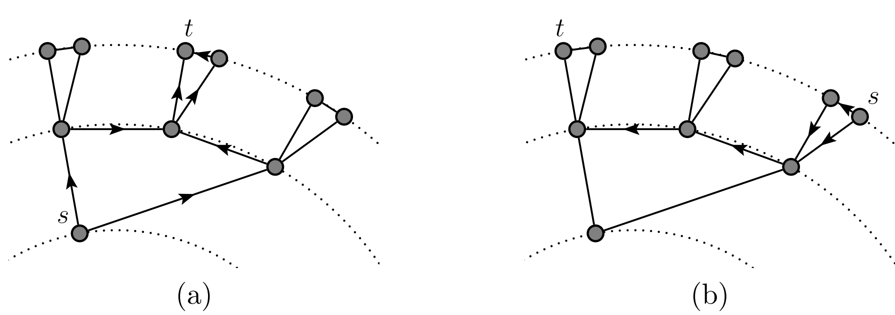


Figure 2: Arrows indicate valid greedy hops. (a) Descendants of s can be reached by a simple path of up and right hops, up and left hops, or a combination of the two. (b) If t is not a descendant of s, then we route down and (left or right) in the direction of t until we reach an ancestor of t.

asymptotic bit-complexity.

This routing scheme is greedy according to the Euclidean coordinates of the vertices, using the Euclidean L_2 metric. Unfortunately, if we only have access to the integer triples then it is not obvious that there is any metric that we can define that will satisfy the definition for greedy routing using just these integer values. Therefore, we must concede that, while this routing scheme fulfills the spirit of greedy routing, it is not greedy routing in the strictest sense. This is an example of a compression scheme and not a coordinate system.

4 Toward a Succinct Greedy Embedding

Given a 3-connected planar graph, we can find a spanning Christmas cactus subgraph in polynomial time [32]. Therefore, we restrict our attention to Christmas cactus graphs. Our results apply to 3-connected planar graphs with little or no modification. In this section, we construct a novel greedy embedding scheme for any Christmas cactus graph in $\mathbf{R}^2$. We then build a coordinate system from our embedding and show that the coordinates can be represented using $O(\log^2 n)$ bits. In the next section, we show how to achieve an optimal $O(\log n)$-bit representation.

4.1 Heavy Path Decompositions

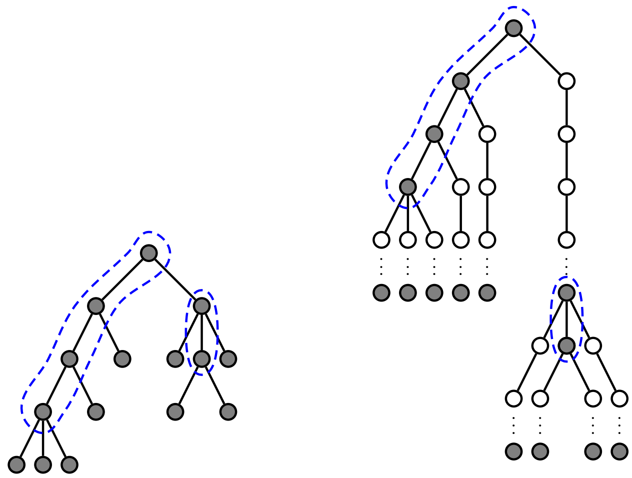
We begin by applying the Sleator and Tarjan [43] *heavy path decomposition* to the depth tree T for G.

Definition 1. *Let T be a rooted tree. For each node v in T, let $n_T(v)$ denote the number of descendants of v in T, including v. For each edge $e = (v, \text{parent}(v))$ in T, label e as a heavy edge if $n_T(v) > n_T(\text{parent}(v))/2$. Otherwise, label e as a light edge. Connected components of heavy edges form paths, called heavy paths. Vertices that are incident only to light edges are considered to be zero-length heavy paths. We call this the* ***heavy path decomposition*** *of T.*

For ease of discussion, we again apply the terminology from nodes in T to cycles in G. A cycle in G is on a heavy path H if its dual node in T is on H. Let H be a heavy path in T. We say that head(H) is the cycle in H that has minimum depth, we define tail(H) similarly. Let C_1 and C_2 be two cycles such that $C_1 = \text{parent}(C_2)$ and let $\{p\} = V(C_1) \cap V(C_2)$. If C_1 and C_2 are on the same heavy path then we call p a *turnpike*. If C_1 and C_2 are on different heavy paths (where $C_1 = \text{tail}(H_1)$ and $C_2 = \text{head}(H_2)$) then we call p an *off-ramp* for H_1 and the vertices $v \in V(C_2) \setminus \{p\}$ *on-ramps* for H_2.

4.2 An Overview of Our Embedding Strategy

Like Leighton and Moitra [32], we lay the cycles from our Christmas cactus graph on concentric semi-circles of radius $1 = R_0 < R_1 < R_2 \ldots$; however, our embedding has the following distinct differences: we have $\Theta(n \log n)$ semi-circles instead of $O(n)$ semi-circles, on-ramps to heavy paths are embedded on special semi-circles which we call *super levels*, turnpikes are placed in a predefined position when cycles are embedded, and the radii of semi-circles can be computed without knowing the topology of the particular Christmas cactus graph being embedded. Since the path from the root to any leaf in the depth tree contains $O(\log n)$ heavy paths, our embedding has $O(\log n)$ of super levels. Between super levels we lay out the non-trivial heavy paths on *baby levels*.



(a) (b)

Figure 3: (a) A depth tree T with positive-length heavy paths highlighted, and (b) the new depth tree T' after the modification procedure.

To make our embedding scheme amenable to a proof by induction, we modify the input Christmas cactus graph. After constructing a greedy embedding of this modified graph, we use it to prove that we have a greedy embedding for the original graph.

4.3 Modifying the Input Christmas Cactus Graph

Given a Christmas cactus graph G on n vertices, we choose a depth tree T of G, and compute the heavy path decomposition of T. For a cycle C on a heavy path H, we define $\mathrm{relativeDepth}(C)$ to be $\mathrm{depth}(C) - \mathrm{depth}(\mathrm{head}(H))$. For each C_1, $C_2 = \mathrm{child}(C_1)$ forming a light edge in T, let $\{p\} = V(C_1) \cap V(C_2)$. Split p into two vertices p_1 and p_2 each on their own cycle, and connect p_1 to p_2 with a path of $n - 1 - \mathrm{relativeDepth}(C_1)$ edges. The new graph G' is also a Christmas cactus graph, and our new depth tree T' looks like T stretched out so that heads of heavy paths (from T) are at depths that are multiples of n. (See Fig. 3.) We continue to call the paths copied from T heavy paths (though they do not form a heavy path decomposition of T'), and the newly inserted edges are *dummy* edges.

4.4 Embedding the Modified Christmas Cactus Graph in $\mathbf{R}^2$

Given a Christmas cactus graph G on n vertices, run the modification procedure described above and get G' and T'. We embed G' in phases, and prove by induction that at the end of each phase we have a greedy embedding of an induced subgraph of G'.

Lemma 2 (Leighton and Moitra [32]). *If the coordinates*

$$\begin{aligned} c &= (0, 1 + z) \\ b &= (-\sin\beta, \cos\beta) \\ a &= (-(1+\epsilon)\sin(\beta-\alpha), (1+\epsilon)\cos(\beta-\alpha)) \end{aligned}$$

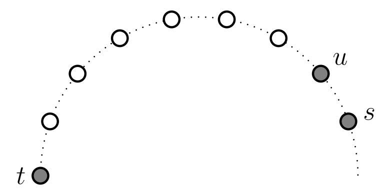


Figure 4: s, u and t form a lower bound for $\delta(G'_0)$.

are subject to the constraints

$$\begin{aligned} 0 <& \alpha, \beta \leq \pi/2 \\ 0 <& \epsilon \leq \frac{1-\cos\beta}{6} \\ 0 \leq& z \leq \epsilon \\ \sin\alpha \leq& \frac{\epsilon(1-\cos\beta)}{2(1+\epsilon)} \end{aligned}$$

then $d(a,c) - d(b,c) \geq \epsilon^2 > 0$.

We begin by embedding the root cycle, $C = (v_0, \ldots, v_{k-1})$, of T'. We trace out a semi-circle of radius $R_0 = 1$ centered at the origin and divide the perimeter of this semi-circle into $2n+1$ equal arcs. We allow vertices to be placed at the leftmost point of each arc, numbering these positions 0 to $2n$. We place vertices $v_0, \ldots, v_{k-1}$ clockwise into any k distinct positions, reserving position n for C's turnpike. If C does not have a turnpike, as is the case if C is a dummy edge or the tail of a heavy path, then position n remains empty. The embedding of C is greedy.

Proof. If C is a 2-cycle, then the embedding of C is greedy regardless of where the vertices are embedded. Otherwise, consider each segment $su \neq v_0v_{k-1}$. The perpendicular bisector to su does not intersect any of our embedded vertices. u is the neighbor of s that is closer to every vertex on the u side of the perpendicular bisector. Since all such segments have this property, the embedding of C is greedy. □

Inductive Step: Suppose we have a greedy embedding all cycles in T' up to depth i, call this induced subgraph G'_i. We show that the embedding can be extended to a greedy embedding of G'_{i+1}. Our proof relies on two values derived from the embedding of G'_i.

Definition 3. *Let s, t be any two distinct vertices in G'_i and fix $n_{s,t}$ to be a neighbor of s such that $d(s,t) > d(n_{s,t}, t)$. We define $\delta(G'_i) = \min_{s,t}\{d(s,t) - d(n_{s,t}, t)\}$.*

We refer to the difference $d(s,t) - d(n_{s,t}, t)$ as the *delta value* for distance-decreasing paths from s to t through $n_{s,t}$.

Definition 4. *Let $\beta(G'_i)$ to be the minimum (non-zero) angle that any two vertices in the embedding of G'_i form with the origin.*

Since we do not specify exact placement of all vertices, we cannot compute $\delta(G'_0)$ and $\beta(G'_0)$ exactly. We instead compute positive underestimates, δ_0 and β_0, by considering hypothetical vertex placements, and by invoking the following lemma.

Lemma 5. *Let s and u be two neighboring vertices embedded in the plane. If there exists a vertex t that is simultaneously closest to the perpendicular bisector of su (on the u side), and farthest from the line su, then the delta value for s to t through u is the smallest for any choice of t.*

Applying the above lemma to all hypothetical s, u, and t placements for the embedding of G_0' leads to the underestimate $\delta_0 = 2 - \sqrt{2 + 2\cos\frac{\pi}{2n+1}} < d(s,t) - d(u,t) \le \delta(G_0')$ where s, u, and t are shown in Fig. 4. Trivially, $\beta_0 = \frac{\pi}{2n+1} \le \beta(G_0')$.

We now show how to obtain a greedy embedding of G_{i+1}', given a greedy embedding of G_i' and values δ_i and β_i.

Let $\epsilon_i = \min\{\delta_i/3, R_i\frac{1-\cos\frac{2}{3}\beta_i}{6}\}$. Trace out a semi-circle of radius $R_{i+1} = R_i + \epsilon_i$ centered at the origin. Each cycle at depth $i+1$ of T' has the form $C = (v, x_1, \ldots, x_m)$ where v, the primary node of C, has already been embedded on the ith semi-circle. We embed vertices x_1 to x_m in two subphases:

Subphase 1 We first embed vertex x_1 from each C. Choose an orientation for C so that x_1 is not a turnpike.[1] We place x_1 where the ray beginning at the origin and passing through v meets semi-circle $i+1$. We now show that distance decreasing paths exist between all pairs of vertices embedded thus far.

Distance decreasing paths between vertices in G_i' are preserved by the induction hypothesis. For t placed during this subphase: t has a neighbor v embedded on semi-circle i. If $s = v$ then s's neighbor t is strictly closer to t. Otherwise if $s \in G_i'$ then since t is within distance $\delta_i/3$ of v, then s's neighbor u that is closer to v is also closer to t. By definition of δ_i, $d(s,v) \ge d(u,v) + \delta_i$.

Since t is in the $\delta_i/3$-ball around v, $d(s,t) \ge d(s,v) - \delta_i/3$, and $d(u,t) \le d(u,v) + \delta_i/3$.

Then,

$$\begin{aligned} d(s,t) &\ge d(s,v) - \delta_i/3 \\ &\ge d(u,v) + \delta_i - \delta_i/3 \\ &\ge d(u,v) - \delta_i/3 + \delta_i - \delta_i/3 \\ &= d(u,t) + \delta_i/3 \\ &> d(u,t) \end{aligned}$$

Therefore, s's neighbor u that is closer to the primary node v is also closer to t. If s was placed during this subphase then s is within distance $R_i\frac{1-\cos\frac{2}{3}\beta_i}{6}$ from its neighbor v, and the perpendicular bisector of sv contains s on one side and every other vertex placed on the other side. Therefore s's neighbor v is closer to t.

The next subphase requires new underestimates, which we call δ_i^1 and β_i^1. By construction, $\beta_i^1 = \beta_i$. No s–t paths within G_i' decrease the delta value. Paths from $s \in G_i'$ to t placed in this subphase have delta value at least $\delta_i/3$ by design. This follows directly from the proof of greediness of this subphase.For paths from s placed in this subphase, s's neighbor v is the closest vertex to the perpendicular bisector of sv on the v side. If we translate v along the perpendicular bisector of sv to a distance of R_{i+1} from sv, this hypothetical point allows us to invoke Lemma 5 to get an underestimate for the delta value of all paths beginning with s. Therefore, our new underestimate is: $\delta_i^1 = \min\{\delta_i/3, \sqrt{R_{i+1}^2 + \epsilon_i^2} - R_{i+1}\}$.

Subphase 2 We now finish embedding each cycle $C = (v, x_1, \ldots x_m)$. Let the value $\alpha = \min\{\beta_i^1/3, \delta_i^1/(3R_{i+1})\}$, s.t. $\sin\alpha \le \frac{\epsilon_i(1-\cos\frac{2}{3}\beta_i^1)}{2(1+\epsilon_i)}$. Trace out an arc of length $R_{i+1}\alpha$ from the embedding of

[1] For the case where C is a 2-cycle and x_1 is a turnpike we insert a temporary placeholder vertex p into C with edges to v and x_1, and treat p as the new x_1. We can later remove this placeholder by transitivity.

x_1, clockwise along semi-circle $i+1$. We evenly divide this arc into $2n+1$ positions, numbered 0 to $2n$. Position 0 is already filled by x_1. We embed vertices in clockwise order around the arc in $m-1$ distinct positions; reserving position n for C's turnpike. If there is no such node, position n remains empty.

This completes the embedding of G'_{i+1}. We show that the embedding of G'_{i+1} is greedy. We only need to consider distance decreasing paths that involve a vertex placed during this subphase. For t placed during this subphase, t is within distance $\delta_i^1/3$ from an x_1, therefore, all previously placed $s \neq x_1$ have a neighbor u that is closer to t. If $s = x_1$ the s's neighbor closer to t is x_2. Finally, for s placed during this subphase, let the cycle that s is on be $C = (v, x_1, \ldots, x_m)$. For $s = x_i \neq x_m$, since $\alpha \leq \beta_i^1/3$, the interior of the sector formed by x_1, x_m and the origin is empty, therefore t is either on the x_{i-1} side of the perpendicular bisector to $x_{i-1}x_i$ or on the x_{i+1} side of the perpendicular bisector to $x_i x_{i+1}$. If $s = x_m$ If t is embedded to the left s, the closer neighbor is x_{m-1}. Otherwise, applying Lemma 2, our choice of $\sin \alpha \leq \frac{\epsilon_i(1-\cos\frac{2}{3}\beta_i^1)}{2(1+\epsilon_i)}$ forces the perpendicular bisector to sv to have s on one side, and all nodes to the right of s on the other side. All cases are considered, so the embedding of G'_{i+1} is greedy.

To complete the inductive proof, we must compute δ_{i+1} and β_{i+1}. Trivially, $\beta_{i+1} = \frac{\alpha}{2n} \leq \beta(G'_{i+1})$. Distance decreasing paths between vertices placed before this subphase will not update the delta value. Therefore, we only evaluate paths with s or t embedded during this subphase. By design, paths from s previously placed to t placed during this subphase have a delta value $\geq \delta_i^1/3$. Distance-decreasing paths from s placed in this subphase to $t \in G'_{i+1}$ take two different directions. If s's neighbor u which is closer to t is on semi-circle $i+1$ then points that are closest to the perpendicular bisector to su are along the perimeter of the sector formed by s, u, and the origin. The point closest to the perpendicular bisector is where the first semi-circle intersects the sector. We translate this point down $R_{i+1}+2$ units along the perpendicular bisector, and we have an underestimate for the delta value for any path beginning with a left/right edge. If s's neighbor that is closer to t is on the ith semi-circle, then a down edge is followed. To finish, we evaluate down edges su added during the second subphase. The closest vertex to the perpendicular bisector to su on the u side is either u, or the vertex placed in the next clockwise position the $i+1$th semi-circle. Translating this point $2R_{i+1}$ units away from su along the perpendicular bisector gives us the an underestimate for paths beginning with su.

This completes the proof for the greedy embedding of G'. We call the levels where the on-ramps to heavy paths are embedded *super levels*, and all other levels are *baby levels*. There are $n-1$ baby levels between consecutive super levels and, since any path from root to leaf in a depth tree travels through $O(\log n)$ different heavy paths, there are $O(\log n)$ super levels.

4.5 Obtaining a Greedy Embedding of G

Let G' be a modified Christmas cactus graph greedily embedded using the procedure discussed above. We now show that collapsing the dummy edges leaves us with a graph G and a greedy embedding of G.

Let C_1, C_2 be any two cycles with a path of dummy edges between them. We show that collapsing this path down to a single vertex gives us new graph that is also greedily embedded.

Proof. Assume, without loss of generality, that C_2 is a descendant of C_1. Let P be the path of dummy edges between C_1 and C_2. Let p_1 be the vertex that cycle C_1 shares with P, let p_2 be the vertex that cycle C_2 shares with P.

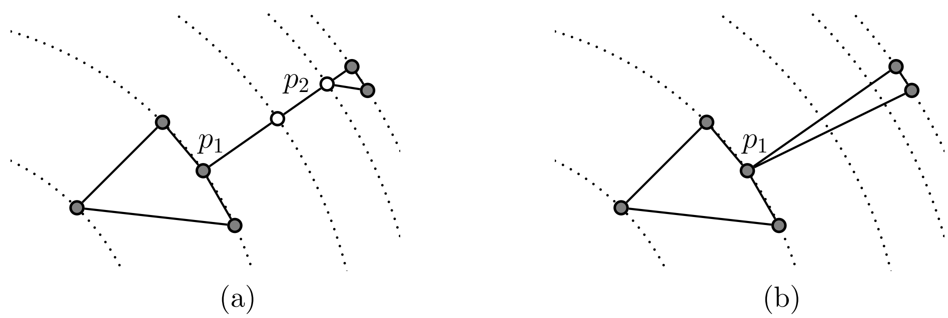


Figure 5: (a) Before removal of dummy nodes and (b) after removal.

Collapse the path P down to the vertex p_1, call this new graph G''. We assign vertices in G'' the same coordinates in $\mathbf{R}^2$ that they are assigned in the embedding of G'. (See Fig. 5.) We show that distance-decreasing paths exist between all pairs of vertices in the embedding of G'', using the greediness of the embedding of G'.

Consider any two vertices s and t in G''. There are four cases:

1. If a distance-decreasing path from s to t in G' involves both p_1 and p_2, then there is a distance-decreasing path in G'' by transitivity.

2. If a distance-decreasing path from s to t in G' involves p_1 and not p_2, then the same distance-decreasing path exists in G'' since no vertices or edges on this path were modified.

3. If a distance-decreasing path from s to t in G' involves p_2 and not p_1, then either s or t is not in G''. Therefore, this case is irrelevant.

4. If a distance-decreasing path from s to t in G' involves neither p_1 nor p_2, then the same distance-decreasing path exists in G'' since no vertices or edges on this path were modified.

Therefore, since there are distance decreasing paths between all s and t in our embedding of G', there are distance-decreasing paths between all s and t in the embedding of our new graph as well. $\square$

Furthermore, every distance-decreasing path in G'' looks like the same path from G', but with vertices in $P \setminus \{p_1\}$ removed.

We apply the above modification algorithm to G' repeatedly, until all dummy edges are removed. After removing all of the dummy edges in this way, we have our original graph G and a greedy embedding of G.

4.6 Our Coordinate System

Let v be a vertex in G. We define level(v) to be the number of baby levels between v and the previous super level (zero if v is on a super level) and cycle(v) to be the position, 0 to $2n$, where v

is placed when its cycle is embedded. These values can be assigned to vertices without performing the embedding procedure.

Let s be v's ancestor on the first super level. The path from s to v passes through $O(\log(n))$ heavy paths, entering each heavy path at an on-ramp, and leaving at an off-ramp. We define v's coordinate to be a $O(\log n)$-tuple consisting of the collection of $(\text{level}(\cdot), \text{cycle}(\cdot))$ pairs for each off-ramp where a change in heavy paths occurs on the path from s to v, and the pair $(\text{level}(v), \text{cycle}(v))$, which is either an off-ramp or a turnpike. Using the coordinate for v and the parameter n, we can compute the Euclidean coordinates for all the turnpikes and off-ramps where a change in heavy path occurs on the path from s to v, including the coordinate for v. Thus, we have defined a coordinate system for the Euclidean plane.

Using a straightforward encoding scheme, each level-cycle pair is encoded using $O(\log n)$ bits. Since a coordinate contains $O(\log n)$ of these pairs, we encode each coordinate using $O(\log^2 n)$ bits.

4.7 Greedy Routing with Coordinate Representations

Although contrived, it is possible to perform greedy geometric routing by converting our coordinates to Euclidean points and using the Euclidean L_2 metric whenever we need to make a comparison along the greedy route. Alternatively, we can define a comparison rule, which can be used for greedy routing in our coordinate system, and which evaluates consistently with the L_2 metric for all vertices on the path from start to goal.

By design, the routing scheme discussed in Sect. 3 is greedy for our embedding. We develop a comparison rule using the potential number of edges that may be traversed on a specific path from s to t.

Let s_i be the vertex between super levels i and $i+1$, whose level-cycle pair is in position i of s's coordinate. We define t_i similarly. Let superlevel(s) be the position that contains the level-cycle pair for s itself. Let h be the smallest integer such that s_h and t_h differ. Using the level-cycle pairs for s_h and t_h, we can compute the level-cycle pair for the off-ramps on the least common ancestor C that diverge toward s and t, which we call s_C and t_C. That is, if $\text{level}(s_h) = \text{level}(t_h)$ then $s_C = s_h$ and $t_C = t_h$. Otherwise, assume without loss of generality that $\text{level}(s_h) < \text{level}(t_h)$, then s_C's pair is $(\text{level}(s_h), \text{cycle}(s_h))$ and t_C is a turnpike with the pair $(\text{level}(s_h), n)$.

We define l, r, d, u be the potential number of left, right, down, and up edges that may be traversed from s to t. Values d and u are simply the number of semi-circles passed through by down and up hops, respectively. That is,

$$d = (\text{superlevel}(s) \cdot n + \text{level}(s)) - (hn + \text{level}(s_C))$$
$$u = (\text{superlevel}(t) \cdot n + \text{level}(t)) - (hn + \text{level}(t_C)).$$

If $\text{cycle}(t_C) < \text{cycle}(s_C)$, then we count the maximum number left edges on the path from s to t_C, and the maximum number of right edges from t_C to t. That is,

$$l = \begin{cases} \text{cycle}(s) + 2n(d-1) + \text{cycle}(s_C) - \text{cycle}(t_C) & \text{if } s \neq s_C, \\ \text{cycle}(s_C) - \text{cycle}(t_C) & \text{if } s = s_C. \end{cases}$$

$$r = \begin{cases} 2n(u-1) + \text{cycle}(t) & \text{if } t \neq t_C, \\ 0 & \text{if } t = t_C. \end{cases}$$

If $\text{cycle}(t_C) \geq \text{cycle}(s_C)$, then we count the maximum number of right edges on the path from s to t_C, and the maximum number of right edges from t_C to t. That is,

$$l = 0$$

$$r = r_1 + r_2, \text{ where}$$

$$r_1 = \begin{cases} 2n - \text{cycle}(s) + 2n(d-1) + \text{cycle}(t_C) - \text{cycle}(s_C) & \text{if } s \neq s_C, \\ \text{cycle}(t_C) - \text{cycle}(s_C) & \text{if } s = s_C. \end{cases}$$

$$r_2 = \begin{cases} 2n(u-1) + \text{cycle}(t) & \text{if } t \neq t_C, \\ 0 & \text{if } t = t_C. \end{cases}$$

Our comparison rule is:

$$D(s,t) = l + r + (2n+1)u + d.$$

Following the routing scheme from Sect. 3, any move we make toward the goal will decrease $D(\cdot,\cdot)$, and all other moves will will increase $D(\cdot,\cdot)$ or leave it unchanged. Therefore, we can use this comparison rule to perform greedy routing in our embedding efficiently, and comparisons made along the greedy route will evaluate consistently with the corresponding Euclidean coordinates under the L_2 metric.

5 An Optimal Succinct Greedy Embedding

Conceptually, the level$(\cdot)$ and cycle$(\cdot)$ values used in the previous section are encoded as integers whose binary representation corresponds to a path from root to a leaf in a full binary tree with n leaves. Instead of encoding with a static $O(\log n)$ bits per integer, we will modify our embedding procedure so we can further exploit the heavy path decomposition of the dual tree T, using *weight-balanced binary trees* [18, 27].

Definition 6. *A* ***weight-balanced binary tree*** *is a binary tree which stores weighted items from a total order in its leaves. If item i has weight w_i, and all items have a combined weight of W then item i is stored at depth $O(\log W/w_i)$. An inorder listing of the leaves outputs the items in order.*

By using appropriate weight functions with our weight-balanced binary trees, we will be able to get telescoping sums for the lengths of the codes for the level$(\cdot)$ and cycle$(\cdot)$ values, giving us $O(\log n)$ bits per coordinate, which is optimal.

5.1 Encoding the Level Values

As in the $O(\log^2 n)$ embedding, we will lay the heavy paths between super levels. However, we no longer require the on-ramps of heavy paths to be embedded on super levels, nor do we require adjacent cycles on the same heavy path to be embedded on consecutive levels; instead, cycles will be assigned to baby levels by an encoding derived from a weight-balanced binary tree.

We will have a different weight-balanced binary tree for each heavy path in our depth tree. The items that we store in the tree are the cycles on the heavy path. The path in the weight-balanced binary tree from the root to the leaf containing a cycle gives us an encoding for the level that the cycle should be embedded on between super levels.

Suppose we have a depth tree T for G, and a heavy path decomposition of T. Let C be a simple cycle in G on some heavy path H and let C_{next} be the next cycle on the heavy path H, if it exists. Let $n(C)$ be the number of vertex descendants of C in G. We define a weight function $\gamma(\cdot)$ on the cycles in G as follows:

$$\gamma(C) = \begin{cases} n(C) & \text{if } C = \text{tail}(H), \\ n(C) - n(C_{\text{next}}) & \text{if } C \neq \text{tail}(H). \end{cases}$$

That is, $\gamma(C)$ is the number of descendants of cycle C in G excluding the descendants of the next cycle on the heavy path with C.

For each heavy path H, create a weight-balanced binary tree B_H containing each cycle C in H as an item with weight $\gamma(C)$, and impose a total order so that cycles are in their path order from head(H) to tail(H).

Let v be a vertex whose coordinate we wish to encode, and suppose v is located between super levels l and $l+1$. Let v_i be the vertex whose level-cycle pair is in position i of v's coordinate. Let v_i be contained in cycle C_i (such that v_i is not C_i's primary node) on heavy path H_i. Then the coordinate for v will contain the collection of level$(\cdot)$ values for each off-ramp v_i on the path to v, and the level$(\cdot)$ value for v itself. Let C_i be the cycle containing vertex v_i, such that v_i is not the primary node for $C_i \in H_i$. The code for level(v_i) is a bit-string representing the path from root to the leaf for C_i in the weight-balanced binary tree B_{H_i}. Let W_i be the combined weight of the items in B_{H_i}. Since C_i is at a depth of $O(\log W_i/\gamma(C_i))$, this is length of the code. Thus, the level values in v's coordinate are encoded with $O(\sum_{0\le i\le l} \log W_i/\gamma(C_i))$ bits total. We now show that this is a telescoping sum, giving us $O(\log n)$ bits total. All descendants counted in W_i are counted in $\gamma(C_{i-1})$, therefore, we have that $\gamma(C_{i-1}) \geq W_i$. By subtracting off descendants that are further along the heavy path, we ensure that $W_0 = n$. Thus, $\sum_{0\le i\le l} \log W_i/\gamma(C_i) \leq \log W_0/\gamma(C_l) \leq \log n$.

5.2 Encoding the Cycle Values

For a node v in G we define a weight function $\mu(v)$ to be the number of descendants of v in G.

Let $C = (p, x_1, x_2, \ldots, x_m)$ be a cycle in G, where p is the primary node of C. Let x_h be the turnpike that connects C to the next cycle on the heavy path, if it exists. Let x_i have weight $\mu(x_i)$ and impose a total order so $x_j < x_k$ if $j < k$. For each cycle C, we create a weight-balanced binary tree B_C containing nodes x_1 to x_m as follows. We first create two weight-balanced binary trees B_C^1 and B_C^2 where B_C^1 contains x_j for $j < h$ and B_C^2 contains x_k for $k > h$. If no such x_h exists, then choose an integer $1 \leq k \leq m$ and insert items x_j for $j < k$ into B_C^1 and insert the remaining items into B_C^2. We form our single weight-balanced binary tree B_C in two steps: (1) create a tree B_C^3 with B_C^1 as a left subtree and a node for x_h as a right subtree, and (2) form B_C with B_C^3 as a left subtree and B_C^2 as a right subtree. We build B_C in this way to ensure that every turnpike is given the same path within its tree, and hence the same cycle code and value.

The code for cycle(v_i) is a bit-string representing the path from root to the leaf for v_i in the weight-balanced binary tree B_{C_i}. Let W_i be the combined weight of the items in B_{C_i}. Since v_i is at a depth of $O(\log W_i/\mu(v_i))$, this is length of the code. Thus, the cycle values in v's coordinate are encoded with $O(\sum_{0\le i\le l} \log W_i/\mu(v_i))$ bits total. We now show that this is a telescoping sum, giving us $O(\log n)$ bits total.

Every descendant counted in W_i is also counted in $\mu(r_{i-1})$, thus $\mu(r_{i-1}) \geq W_i$. By design, $W_0 = n$. Hence $\sum_{0\le i\le l} \log W_i/\mu(r_i) \leq \log W_0/w(r_l) \leq \log n$.

5.3 Interpreting the Codes

Let c be the smallest integer constant such that item i stored in the weight-balanced binary tree is at depth $\leq c \log W/w_i$. We can treat the position of i in the weight-balanced binary tree as a position in a full binary tree of height $c \log n$. We interpret this code to be the number of tree nodes preceding i in an in-order traversal of the full binary tree. Using our codes as described, we require $2n^c - 2$ baby levels between each super level and $8n^c - 1$ cycle positions.

5.4 An Overview of the Optimal Embedding

Let T be the depth tree for our Christmas cactus graph G. We create weight-balanced binary trees on the heavy paths in T and on each of the cycles in G, giving us the level and cycle codes for every vertex. We adjust the graph modification procedure so that adjacent cycles on heavy paths are spaced out according to the level codes. That is, adjacent cycles on the same heavy path have heavy dummy edges (dummy edges that are considered to be on the heavy path) inserted between them so that they are placed on the appropriate baby levels. For cycles on different heavy paths, we insert dummy edges to pad out to the next superlevel, and heavy dummy edges to pad out to the appropriate baby level.

We embed the modified graph analogously to our $O(\log^2 n)$ embedding, except that the cycle codes dictate vertex placements. We augment our coordinate system to store the level value for elements on the root cycle, otherwise it is not possible to compute the corresponding Euclidean point from our succinct representation. The same comparison rule applies to our new coordinate system, with little change to account for the new range of level and cycle values. Using this embedding scheme and coordinate system we achieve optimal $O(\log n)$ bits per coordinate.

6 Conclusion

We have provided a succinct coordinate-based representation for the vertices in 3-connected planar graphs so as to support greedy routing in $\mathbf{R}^2$. Our method uses $O(\log n)$ bits per vertex and allows greedy routing to proceed using only our representation, in a way that is consistent with the Euclidean metric. For future work, it would be interesting to design an efficient distributed algorithm to perform such embeddings.

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