# Randomized Online Algorithms for the Buyback Problem 

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#### Abstract

In the matroid buyback problem, an algorithm observes a sequence of bids and must decide whether to accept each bid at the moment it arrives, subject to a matroid constraint on the set of accepted bids. Decisions to reject bids are irrevocable, whereas decisions to accept bids may be canceled at a cost which is a fixed fraction of the bid value. We present a new randomized algorithm for this problem, and we prove matching upper and lower bounds to establish that the competitive ratio of this algorithm, against an oblivious adversary, is the best possible. We also observe that when the adversary is adaptive, no randomized algorithm can improve the competitive ratio of the optimal deterministic algorithm. Thus, our work completely resolves the question of what competitive ratios can be achieved by randomized algorithms for the matroid buyback problem.


## 1 Introduction

Imagine a seller allocating a limited inventory (e.g. impressions of a banner ad on a specified website at a specified time in the future) to a sequence of potential buyers who arrive sequentially, submit bids at their arrival time, and expect allocation decisions to be made immediately after submitting their bid. An informed seller who knows the entire bid sequence can achieve much higher profits than an uninformed seller who discovers the bids online, because of the possibility that a very large bid is received after the uninformed seller has already allocated the inventory. A number of recent papers $[1,2]$ have proposed a model that offsets this possibility by allowing the uninformed seller to cancel earlier allocation decisions, subject to a penalty which is a fixed fraction of the canceled bid value. This option of canceling an allocation and paying a penalty is referred to as buyback, and we refer to online allocation problems with a buyback option as buyback problems.

Buyback problems have both theoretical and practical appeal. In fact, Babaioff et al. [1] report that this model of selling was described to them by the ad marketing group at a major Internet software company. Constantin et al. [2] cite numerous other applications including allocation of TV, radio, and newsprint advertisements; they also observe that advance booking with cancellations is a common practice in the airline industry, where limited inventory is oversold and then, if necessary, passengers are "bumped" from flights and compensated with a penalty payment, often in the form of credit for future flights.

Different buyback problems are distinguished from each other by the constraints that express which sets of bids can be simultaneously accepted. In the simplest case, the only constraint is a fixed upper bound on the total number of accepted bids. Alternatively, there may be a bipartite graph whose two vertex sets are called bids and slots, and a set of bids may be simultaneously accepted if and only if each bid in the set can be matched to a different slot using edges of the bipartite graph. Both of these examples are special cases of the matroid buyback problem, in which there is a matroid structure on the bids, and a set of bids may be simultaneously accepted if and only if they constitute an independent set in this matroid. Other types of constraints (e.g. knapsack constraints) have also been studied in the context of buyback problems [1], but the matroid buyback problem has received the most study. This is partly because of its desirable theoretical properties

- the offline version of the problem is computationally tractable, and the online version admits an online algorithm whose payoff is identical to that of the omniscient seller when the buyback penalty is zero - and partly because of its well-motivated special cases, such as the problem of matching bids to slots described above.

As is customary in the analysis of online algorithms, we evaluate algorithms according to their competitive ratio: the worst-case upper bound on the ratio between the algorithm's (expected) payoff and that of an informed seller who knows the entire bid sequence and always allocates to an optimal feasible subset without paying any penalties. The problem of deterministic matroid buyback algorithms has been completely solved: a simple algorithm was proposed and analyzed by Constantin et al. [3, 2] and, independently, Babaioff et al. [4], and it was recently shown [1] that the competitive ratio of this algorithm is optimal for deterministic matroid buyback algorithms, even for the case of rank-one matroids (i.e., selling a single indivisible good). However, this competitive ratio can be strictly improved by using a randomized algorithm against an oblivious adversary. Babaioff et al. [1] showed that this result holds when the buyback penalty factor is sufficiently small, and they left open the question of determining the optimal competitive ratio of randomized algorithms - or even whether randomized algorithms can improve on the competitive ratio of the optimal deterministic algorithm when the buyback factor is large.

Our work resolves this open question by supplying a randomized algorithm whose competitive ratio (against an oblivious adversary) is optimal for all values of the buyback penalty factor. We present the algorithm and the upper bound on its competitive ratio in Section 3 and the matching lower bound in Section 4. Our algorithm is also much simpler than the randomized algorithm of [1], avoiding the use of stationary renewal processes. It may be viewed as an online randomized reduction that transforms an arbitrary instance of the matroid buyback problem into a specially structured instance on which deterministic algorithms are guaranteed to perform well. Our matching lower bound relies on defining and analyzing a suitable continuous-time analogue of the single-item buyback problem.

Adaptive adversaries. In this paper we analyze randomized algorithms with an oblivious adversary. If the adversary is adaptive ${ }^{1}$, then no randomized algorithm can achieve a better competitive ratio than that achieved by the optimal deterministic algorithm. This fact is a direct consequence of a more general theorem asserting the same equivalence for the class of request answer games (Theorem 2.1 of [5] or Theorem 7.3 of [6]), a class of online problems that includes the buyback problem. ${ }^{2}$

Strategic considerations. In keeping with $[4,1]$, we treat the buyback problem as a pure online optimization with non-strategic bidders. For an examination of strategic aspects of the buyback problem, we refer the reader to [2].

Related work. The buyback model was first investigated by Constantin et al. $[3,2]$ and Babaioff et al. $[4,1]$. The optimal deterministic algorithm for the matroid buyback problem was presented in [4, $3,2]$ and a proof of its optimality appeared in [4, 1]. Constantin et al. also investigated strategic aspects of the matroid buyback problem in $[3,2]$; this research was featured in a recent survey of theory research at Google in ACM SIGACT News [7]. Babaioff et al. presented algorithms for the

[^0]knapsack buyback problem $[4,1]$ and designed a randomized algorithm for the matroid buyback problem that strictly improves the competitive ratio of the optimal deterministic algorithm when the adversary is oblivious and the buyback penalty factor is sufficiently small [1].

Prior to the aforementioned work on the buyback problem, several earlier papers considered models in which allocations, or other commitments, could be cancelled at a cost. Biyalogorsky et al. [8] studied such "opportunistic cancellations" in the setting of a seller allocating $N$ units of a good in a two-period model, demonstrating that opportunistic cancellations could improve allocative efficiency as well as the seller's revenue. Sandholm and Lesser [9] analyzed a more general model of "leveled commitment contracts" and proved that leveled commitment never decreases the expected payoff to either contract party. However, to the best of our knowledge, the buyback problem studied in this paper and its direct precursors $[4,1,3,2]$ is the first to analyze commitments with cancellation costs in the framework of worst-case competitive analysis rather than average-case Bayesian analysis.

## 2 Preliminaries

First we define the problem in the setting of single item and then generalize the definition in the case of matroids.

### 2.1 Single Item Case

The seller has a single item to allocate. The bids $v_{1}, v_{2}, \ldots, v_{n}$ come in a sequence and when bid $v_{i}$ arrives the seller must either commit or reject the bid immediately. When the seller commits, the previous commitment must be revoked by paying a penalty of $f \cdot v_{j}$, where $v_{j}$ is the bid being revoked and $f \geq 0$ is a fixed number called the buyback factor. This implies that at the end of processing the bid sequence, the seller's payoff is equal to the final committed bid minus $f$ times the sum of all revoked bids. The customer with the final accepted bid gets the item.

### 2.2 General model for matroids

Consider a matroid ${ }^{3}(\mathcal{U}, \mathcal{I})$ where $\mathcal{U}$ is the ground set and $\mathcal{I}$ is the set of independent subsets of $\mathcal{U}$. We describe the problem abstractly and then relate it to the single item case. We will assume that the ground set $\mathcal{U}$ is identified with the set $\{1, \ldots, n\}$. There is a bid value $v_{i} \geq 0$ associated to each element $i \in \mathcal{U}$. The information available to the algorithm at time $k(1 \leq k \leq n)$ consists of the first $k$ elements of the bid sequence - i.e. the subsequence $v_{1}, v_{2}, \ldots, v_{k}$ - and the restriction of the matroid structure to the first $k$ elements. (In other words, for every subset $S \subseteq\{1,2, \ldots, k\}$, the algorithm knows at time $k$ whether $S \in \mathcal{I}$.)

At any step the algorithm can choose a subset $S^{k} \subseteq S^{k-1} \cup\{k\}$. This set $S^{k}$ must be an independent set, i.e $S^{k} \in \mathcal{I}$. Hence the final set held by the algorithm is $R=S^{n}$. The algorithm must perform a buyback for every element of $B=\left(\cup_{i=1}^{n} S^{i}\right) \backslash S^{n}$. For any set $S \subseteq \mathcal{U}$ let $\operatorname{val}(S)=\sum_{i \in S} v_{i}$. Finally we define the payoff of the algorithm as $\operatorname{val}(R)-f \cdot \operatorname{val}(B)$. This definition generalizes the single item case, which corresponds to the case in which $\mathcal{I}$ consists of all one-element subsets of $\mathcal{U}$.

[^1]
## 3 Randomized algorithm against oblivious adversary

In this section we give a randomized algorithm with competitive ratio $-W\left(\frac{-1}{e(1+f)}\right)$ against an oblivious adversary. Here $W$ is Lambert's $W$ function ${ }^{4}$, defined as the inverse of the function $z \mapsto z e^{z}$. The design of our randomized algorithm is based on two insights:

1. Although the standard greedy online algorithm for picking a maximum-weight basis of a matroid can perform arbitrarily poorly on a worst-case instance of the buyback problem, it performs well when the ratios between values of different matroid elements are powers of some scalar $r>1+f$. (We call such instances " $r$-structured.")
2. There is a randomized reduction from arbitrary instances of the buyback problem to instances that are $r$-structured.

### 3.1 The greedy algorithm and $r$-structured instances

Definition 1. Let $r>1$ be a constant. An instance of the matroid buyback problem is $r$-structured if for every pair of elements $i, j$, the ratio $v_{i} / v_{j}$ is equal to $r^{l}$ for some $l \in \mathbb{Z}$.

```
Algorithm 1 Greedy Matroid Algorithm (GMA)
    Initialize \(S=\emptyset\).
    for all elements \(i\), in order of arrival, do
        if \(S \cup\{i\} \in \mathcal{I}\) then
            Sell to \(i\).
        else
            Let \(j\) be an element of smallest value such that \(S \cup\{i\} \backslash\{j\} \in \mathcal{I}\).
            if \(v_{j}<v_{i}\) then
                    Sell to \(i\) and buy back \(j\).
            end if
        end if
    end for
```

Lemma 1. For $r>1+f$, when the greedy matroid algorithm is executed on an $r$-structured instance of the matroid buyback problem, its competitive ratio is at most $\frac{r-1}{r-1-f}$.

Proof. As is well known, at termination the set $S$ selected by GMA is a maximum-weight basis of the matroid. To give an upper bound on the total buyback payment, we define a set $B(i)$ for each $i \in \mathcal{U}$ recursively as follows: if GMA never sold to $i$, or sold to $i$ in step 4 , then $B(i)=\emptyset$. If GMA sold to $i$ in step 8 while buying back $j$, then $B(i)=\{j\} \cup B(j)$. By induction on the cardinality of $B(i)$, we find that the set $\left\{v_{x} / v_{i} \mid x \in B(i)\right\}$ consists of distinct negative powers of $r$, so

$$
\sum_{x \in B(i)} v_{x} \leq v_{i} \cdot \sum_{i=1}^{\infty} r^{-i}=\frac{v_{i}}{r-1}
$$

[^2]By induction on the number of iterations of the main loop, the set $\bigcup_{i \in S} B(i)$ consists of all the elements ever bought back by GMA; consequently, the total buyback payment is bounded by

$$
f \cdot \sum_{i \in S} \sum_{x \in B(i)} v_{x} \leq \frac{f}{r-1} \sum_{i \in S} v_{i}
$$

Thus, the algorithm's net payoff is at least $1-\frac{f}{r-1}$ times the value of the maximum weight basis.

### 3.2 The random filtering reduction

Consider two instances $\mathbf{v}, \mathbf{w}$ of the matroid buyback problem, consisting of the same matroid $(\mathcal{U}, \mathcal{I})$, with its elements presented in the same order, but with different values: element $i$ has values $v_{i}, w_{i}$ in instances $\mathbf{v}, \mathbf{w}$, respectively. Assume furthermore that $v_{i} \geq w_{i}$ for all $i$, and that both values $v_{i}, w_{i}$ are revealed to the algorithm at the time element $i$ arrives. Given a (deterministic or randomized) algorithm ALG which achieves expected payoff $P$ on instance $\mathbf{w}$, we present here an algorithm Filter(ALG) which achieves expected payoff $P$ on instance $\mathbf{v}$.

```
Algorithm 2 Random Filtering Algorithm Filter(ALG)
    Initialize \(S=\emptyset\)
    for all elements \(i\), in order of arrival, do
        Observe \(v_{i}, w_{i}\).
        Randomly sample \(x_{i}=1\) with probability \(w_{i} / v_{i}\), else \(x_{i}=0\).
        Present element \(i\) with value \(w_{i}\) to ALG.
        if ALG sells to \(i\) and \(x_{i}=1\) then
            Sell to \(i\).
        end if
        if ALG buys back an element \(j\) and \(x_{j}=1\) then
            Buy back \(j\).
        end if
    end for
```

Lemma 2. The expected payoff of Filter(ALG) on instance $\mathbf{v}$ equals the expected payoff of ALG on instance $\mathbf{w}$.

Proof. For each element $i \in \mathcal{U}$, let $\sigma_{i}=1$ if ALG sells to $i$, and let $\beta_{i}=1$ if ALG buys back $i$. Similarly, let $\sigma_{i}^{\prime}=1$ if Filter(ALG) sells to $i$, and let $\beta_{i}^{\prime}=1$ if Filter (ALG) buys back $i$. Observe that $\sigma_{i}^{\prime}=\sigma_{i} x_{i}$ and $\beta_{i}^{\prime}=\beta_{i} x_{i}$ for all $i \in \mathcal{U}$, and that the random variable $x_{i}$ is independent of $\left(\sigma_{i}, \beta_{i}\right)$. Thus,

$$
\begin{aligned}
\mathbf{E}\left[\sum_{i \in \mathcal{U}} \sigma_{i}^{\prime} v_{i}-(1+f) \beta_{i}^{\prime} v_{i}\right] & =\mathbf{E}\left[\sum_{i \in \mathcal{U}} \sigma_{i} x_{i} v_{i}-(1+f) \beta_{i} x_{i} v_{i}\right] \\
& =\sum_{i \in \mathcal{U}} \mathbf{E}\left[\sigma_{i}-(1+f) \beta_{i}\right] \mathbf{E}\left[x_{i} v_{i}\right] \\
& =\sum_{i \in \mathcal{U}} \mathbf{E}\left[\sigma_{i}-(1+f) \beta_{i}\right] w_{i} \\
& =\mathbf{E}\left[\sum_{i \in \mathcal{U}} \sigma_{i} w_{i}-(1+f) \beta_{i} w_{i}\right]
\end{aligned}
$$

The left side is the expected payoff of Filter(ALG) on instance $\mathbf{v}$ while the right side is the expected payoff of ALG on instance $\mathbf{w}$.

### 3.3 A randomized algorithm with optimal competitive ratio

In this section we put the pieces together, to obtain a randomized algorithm with competitive ratio $-W\left(\frac{-1}{e(1+f)}\right)$ against oblivious adversary ${ }^{5}$.

```
Algorithm 3 Randomized Algorithm RandAlg(r)
    Given: a parameter \(r>1+f\).
    Sample \(u \in[0,1]\) uniformly at random.
    for all elements \(i\) do
        Let \(z_{i}=u+\left\lfloor\ln _{r}\left(v_{i}\right)-u\right\rfloor\).
        Let \(w_{i}=r^{z_{i}}\).
    end for
    Run Filter(GMA) on instances \(\mathbf{v}, \mathbf{w}\).
```

Lemma 3. For all $i \in \mathcal{U}$, we have $v_{i} \geq w_{i}$ and $\mathbf{E}\left[w_{i}\right]=\frac{r-1}{r \ln (r)} v_{i}$.
Proof. The random variable $\ln _{r}\left(v_{i}\right)-z_{i}$ is equal to the fractional part of the number $\ln _{r}\left(v_{i}\right)-u$, which is uniformly distributed in $[0,1]$ since $u$ is uniformly distributed in $[0,1]$. It follows that $w_{i} / v_{i}$ has the same distribution as $r^{-u}$, which proves that $v_{i} \geq w_{i}$ and also that

$$
\mathbf{E}\left[\frac{w_{i}}{v_{i}}\right]=\int_{0}^{1} r^{-u} d u=-\left.\frac{1}{\ln (r)} \cdot r^{-u}\right|_{0} ^{1}=\frac{r-1}{r \ln (r)} .
$$

Theorem 1. The competitive ratio of RandAlg against an oblivious adversary is $\frac{r \ln (r)}{r-1-f}$.
Proof. Let $S^{*} \subseteq \mathcal{U}$ denote the maximum-weight basis of $(\mathcal{U}, \mathcal{I})$ with respect to the weights $\mathbf{v}$. Since the mapping from $v_{i}$ to $w_{i}$ is monotonic (i.e., $v_{i} \geq v_{j}$ implies $w_{i} \geq w_{j}$ ), we know that $S^{*}$ is also a maximum-weight basis of $(\mathcal{U}, \mathcal{I})$ with respect to the weights $\mathbf{w}^{6}$. Let $v\left(S^{*}\right)=\sum_{i \in S^{*}} v_{i}$ and let $w\left(S^{*}\right)=\sum_{i \in S^{*}} w_{i}$.

The input instance $\mathbf{w}$ is $r$-structured, so the payoff of GMA on instance $\mathbf{w}$ is at least $\frac{r-1-f}{r-1} w\left(S^{*}\right)$. The modified weights $w_{i}$ satisfy two properties that allow application of algorithm Filter(ALG): the value of $w_{i}$ can be computed online when $v_{i}$ is revealed at the arrival time of element $i$, and it satisfies $w_{i} \leq v_{i}$. By Lemma 2, the expected payoff of Filter(GMA) on instance $\mathbf{v}$, conditional on the values $\left\{w_{i}: i \in \mathcal{U}\right\}$, is at least $\left(\frac{r-1-f}{r-1}\right) \cdot w\left(S^{*}\right)$. Finally, by Lemma 3 and linearity of expectation, $\mathbf{E}\left[w\left(S^{*}\right)\right] \geq\left(\frac{r-1}{r \ln (r)}\right) \cdot v\left(S^{*}\right)$. The theorem follows by combining these bounds.

The function $f(r)=\frac{r \ln (r)}{r-1-f}$ on the interval $r \in(1+f, \infty)$ is minimized when $-\frac{r}{1+f}=W\left(\frac{-1}{e(1+f)}\right)$ and $f(r)=-W\left(\frac{-1}{e(1+f)}\right)$. This completes our analysis of the randomized algorithm RandAlg $(r)$.

[^3]
## 4 Lower Bound

We prove the lower bound on the competitive ratio of randomized algorithms for online algorithms with buyback against an oblivious adversary. The proof is by first reducing to a continuous version of the problem and then applying Yao's Principle [11]. As noted in the introduction, the lower bound in the case of an adaptive adversary matches the lower bound for deterministic algorithms. Both of these lower bounds are for the single item case and hence are also applicable for the general matroid case.

### 4.1 Reduction to continuous version

Consider the following continuous version of the problem for the single item case. Time starts at $t=1$ and stops at some time $t=x$. The value of $x$ is not known to the algorithm. The algorithm at any instant in time can make a mark. The final payoff of the algorithm is equal to the time at which it made its final mark minus $f$ times the sum of times of marks before the final mark. There is a clear relationship between this problem and the single item buyback problem. In particular, we can transform any algorithm for the single item buyback problem with competitive ratio $c$ to an algorithm for the continuous case with competitive ratio $c \times(1+\epsilon)$ for arbitrarily small $\epsilon>0$. This transformation works by running the single item buyback algorithm on the input sequence $1,1+\delta,(1+\delta)^{2},(1+\delta)^{3}, \ldots$ for sufficiently small $\delta>0$, and making marks at the times $t$ corresponding to the values accepted in the execution of the single item buyback algorithm.

### 4.2 Lower bound against oblivious adversaries

Theorem 2. Any randomized algorithm for the continuous version of the single item buyback problem has competitive ratio at least $-W\left(\frac{-1}{e(1+f)}\right)$.

The proof is an application of Yao's Principle [11]. We give a one-parameter family of input distributions (parametrized by a number $y>1$ ) for the continuous version and prove that any deterministic algorithm for the continuous version of the problem must have a competitive ratio which tends to $-W\left(\frac{-1}{e(1+f)}\right)$ as $y \rightarrow \infty$. It is easy to note that an input to the continuous version is completely specified by the time $x$ at which the input stops, and hence the input distribution is just a distribution on $x$. For a given $y>1$, let the probability density for the stopping times be defined as follows.

$$
\begin{align*}
& \mathbf{f}(x)=1 / x^{2} \text { if } x<y \\
& \mathbf{f}(x)=0 \quad \text { if } x>y \tag{1}
\end{align*}
$$

Note that the above definition is not a valid probability density function, so we place a point mass at $x=y$ of probability $\frac{1}{y}$. Hence our distribution is a mixture of discrete and continuous probability. For notational convenience let $d(F(x))=\mathbf{f}(x)$ where $F$ is the cumulative distribution function. Also let $G(x)=1-F(x)$. Any deterministic algorithm is defined by a set $T=\left\{u_{1}, u_{2}, \ldots, u_{k}\right\}$ of times at which it makes a mark(Given that it does not stop before that time).

Lemma 4. There exists an optimal deterministic algorithm for the distribution described by $T=$ $\left\{1, w, w^{2}, \ldots, w^{k-1}\right\}$ for some $w, k$.

Proof. Let $T=\left\{u_{1}, u_{2}, \ldots, u_{k}\right\}$. We prove that $u_{i}=u_{i+1}^{(i-1) / i}$ for $i \in[k-1]$ by induction and it is easy to see that the claim follows from this. For lack of space we just prove the inductive case. Please refer to the appendix for the base case. Let $u_{0}=0$ and $u_{k+1}=\infty$.

It is easy to see that the algorithm's expected payoff, $P$, is $\sum_{i=1}^{k} \int_{u_{i}}^{u_{i+1}}\left(u_{i}-f \cdot \sum_{j=1}^{i-1} u_{j}\right) d(F(y))$. We simplify this expression as follows.

$$
\begin{align*}
P & =\sum_{i=1}^{k} \int_{u_{i}}^{u_{i+1}}\left(u_{i}-f \cdot \sum_{j=1}^{i-1} u_{j}\right) d(F(y)) \\
& =\sum_{i=1}^{k} \int_{u_{i}}^{\infty}\left(u_{i}-(1+f) \cdot u_{i-1}\right) d(F(y)) \\
& =\sum_{i=1}^{k}\left(u_{i}-(1+f) \cdot u_{i-1}\right) \cdot G\left(u_{i}\right) \tag{2}
\end{align*}
$$

Now we rewrite this equation to express the right side as a function of $u_{i}$, using $\rho_{i}$ to denote the sum of all terms on the right side except for the $i, i+1$ terms. Crucially, $\rho_{i}$ is independent of $u_{i}$.

$$
\begin{align*}
P & =\left(u_{i}-(1+f) \cdot u_{i-1}\right) \cdot G\left(u_{i}\right)+\left(u_{i+1}-(1+f) \cdot u_{i}\right) \cdot G\left(u_{i+1}\right)+\rho_{i} \\
& =\quad\left(u_{i}-(1+f) \cdot u_{i-1}\right) \cdot \frac{1}{u_{i}}+\left(u_{i+1}-(1+f) \cdot u_{i}\right) \cdot \frac{1}{u_{i+1}}+\rho_{i} \tag{3}
\end{align*}
$$

If we differentiate $P$ with respect to $u_{i}$, equate to 0 , and solve, then we obtain the equation $u_{i}^{2}=u_{i-1} \cdot u_{i+1}$. By induction we know that $u_{i-1}=u_{i}^{(i-2) /(i-1)}$. Substituting and solving we get the necessary equation.
Lemma 5. For any algorithm described by $T=\left\{1, w, w^{2}, \ldots, w^{k-1}\right\}$, the competitive ratio is bounded below by a number which tends to $-W\left(\frac{-1}{e(1+f)}\right)$ as $y$ tends to $\infty$.
Proof. It is easy to see that the expected payoff, $V$, of a prophet who knows the stopping time $x$ is given by the following equation.

$$
\begin{equation*}
V=\int_{1}^{y} \frac{1}{x^{2}} \cdot x d x+\frac{1}{y} \cdot y=1+\ln (y) \tag{4}
\end{equation*}
$$

Now we compute the payoff for any algorithm described by $T=\left\{1, w, w^{2}, \ldots, w^{k-1}\right\}$.

$$
\begin{align*}
P & =1 \cdot G(1)+\sum_{i=1}^{k-1}\left(w^{i}-(1+f) w^{i-1}\right) \cdot G\left(w^{i}\right) \\
& =\quad 1 \cdot 1+\sum_{i=1}^{k-1}\left(w^{i}-(1+f) w^{i-1}\right) \cdot \frac{1}{w^{i}} \\
& =\quad 1+(k-1) \cdot \frac{w-1-f}{w} \tag{5}
\end{align*}
$$

Hence if $c$ is the competitive ratio we have that.

$$
\begin{align*}
\frac{1}{c}=\frac{P}{V} & =\frac{1+(k-1) \cdot(w-1-f) / w}{1+\ln (y)} \\
& <\frac{1}{\ln (y)}+\frac{(k-1) \cdot(w-1-f) / w}{(k-1) \cdot \ln (w)} \\
& \leq \frac{1}{\ln (y)}+\max _{u}\left(\frac{u-1-f}{u \cdot \ln (u)}\right) \\
& \leq \frac{1}{\ln (y)}-\frac{1}{W\left(\frac{-1}{e(1+f)}\right)} \tag{6}
\end{align*}
$$

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## A Base case

We prove here the base case in the inductive hypothesis of proof of lemma 4 . Consider the payoff of the algorithm.

$$
\begin{equation*}
P=\sum_{i=1}^{k}\left(u_{i}-(1+f) \times u_{i-1}\right) \times G\left(u_{i}\right) \tag{7}
\end{equation*}
$$

Similar to the inductive case we rewrite the equation as a function of $u_{1}$, using $\rho_{1}$ to denote the sum of all terms on the right side except for the $1^{\text {st }}, 2^{\text {nd }}$ terms.

$$
\begin{array}{rlrl}
P & =u_{1} \times G\left(u_{1}\right)+\left(u_{2}-(1+f) \times u_{1}\right) \times G\left(u_{2}\right)+\rho_{1} \\
& = & u_{1} \times \frac{1}{u_{1}}+\left(u_{2}-(1+f) \times u_{1}\right) \times \frac{1}{u_{2}}+\rho_{1} \\
& = & -(1+f) \times \frac{u_{1}}{u_{2}}+1+1+\rho_{1}
\end{array}
$$

It is easy to see that $P$ is a decreasing function of $u_{1}$. Hence $u_{1}=1=u_{2}^{\frac{1-1}{1}}$.


[^0]:    ${ }^{1}$ A distinction between adaptive offline and adaptive online adversaries is made in $[5,6]$. When we refer to an adaptive adversary in this paper, we mean an adaptive offline adversary.
    ${ }^{2}$ The definition of request answer games in [6] requires that the game must have a minimization objective, whereas ours has a maximization objective. However, the proof of Theorem 7.3 in [6] goes through, with only trivial modifications, for request answer games with a maximization objective.

[^1]:    ${ }^{3}$ See [10] for the definition of a matroid.

[^2]:    ${ }^{4}$ Lambert's $W$ function is multivalued for our domain. We restrict to the case where $W\left(\frac{-1}{e(1+f)}\right) \leq-1$.

[^3]:    ${ }^{5}$ Note that the algorithm is written in an offline manner just for convenience and can be implemented as an online algorithm
    ${ }^{6}$ There may be other maximum-weight basis of $\mathbf{w}$ which were not maximum-weight basis of $\mathbf{v}$.

