FPT-Algorithms for Connected Feedback Vertex Set

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Abstract. We study the recently introduced Connected Feedback Vertex Set (CFVS) problem from the view-point of parameterized algorithms. CFVS is the connected variant of the classical Feedback Vertex Set problem and is defined as follows: given a graph G = (V, E) and an integer k, decide whether there exists $F \subseteq V$, $|F| \le k$, such that $G[V \setminus F]$ is a forest and G[F] is connected. We show that Connected Feedback Vertex Set can be solved in time $O(2^{O(k)}n^{O(1)})$ on general graphs and in time $O(2^{O(\sqrt{k}\log k)}n^{O(1)})$ on graphs excluding a fixed graph H as a minor. Our result on general undirected graphs uses as subroutine, a parameterized algorithm for Group Steiner Tree, a well studied variant of Steiner Tree. We find the algorithm for Group Steiner Tree of independent interest and believe that it will be useful for obtaining parameterized algorithms for other connectivity problems.

1 Introduction

FEEDBACK VERTEX SET (FVS) is a classical NP-complete problem and has been extensively studied in all subfields of algorithms and complexity. In this problem we are given an undirected graph G=(V,E) and a positive integer k as input, and the goal is to check whether there exists a subset $F\subseteq V$ of size at most k such that $G[V\setminus F]$ is a forest. This problem originated in combinatorial circuit design and found its way into diverse applications such as deadlock prevention in operating systems, constraint satisfaction and Bayesian inference in artificial intelligence. We refer to the survey by Festa, Pardalos and Resende [14] for further details on the algorithmic study of feedback set problems in a variety of areas like approximation algorithms, linear programming and polyhedral combinatorics.

In this paper we focus on the recently introduced connected variant of FEEDBACK VERTEX SET, namely, Connected Feedback Vertex Set (CFVS). Here, given a graph G = (V, E) and a positive integer k, the objective is to check whether there exists a vertex-subset F of size at most k such that $G[V \setminus F]$ is a forest and G[F] is connected. Sitters and Grigoriev [32] recently introduced this problem and obtained a polynomial time approximation scheme (PTAS) for CFVS on planar graphs. We find it a bit surprising that the connected version of FVS was not considered in the literature until now. This is in complete contrast to the fact that the connected variant of other problems like Vertex Cover—Connected Vertex Cover [13, 18, 23, 27], Dominating Set—Connected Dominating Set [16, 20, 21, 25] are extremely well studied in the literature. In this paper, we continue the algorithmic study of CFVS from the view-point of parameterized algorithms.

Parameterized complexity is a two-dimensional generalization of "P vs. NP" where, in addition to the overall input size n, one studies how a secondary measurement that captures additional relevant information affects the computational complexity of the problem

in question. Parameterized decision problems are defined by specifying the input, the parameter and the question to be answered. The two-dimensional analogue of the class P is decidability within a time bound of $f(k)n^c$, where n is the total input size, k is the parameter, f is some computable function and c is a constant that does not depend on k or n. A parameterized problem that can be decided in such a time-bound is termed fixed-parameter tractable (FPT). For general background on the theory see the textbooks by Downey and Fellows [12], Flum and Grohe [15] and Niedermeier [29].

FVS has been extensively studied in parameterized algorithms. The earliest known FPT-algorithms for FVS go back to the late 80's and the early 90's [2,11] and used the seminal Graph Minor Theory of Robertson and Seymour (for an overview, see [26]). Raman et al. [30] designed an algorithm with run-time $O(2^{O(k \log \log k)} n^{O(1)})$ which basically branched on short cycles in a bounded search tree approach. Subsequently, several algorithms for FVS with run-time $O(2^{O(k)} n^{O(1)})$ were designed using a technique known as iterative compression. After several rounds of improvements, the current best FPT-algorithm for FVS runs in time $O(5^k kn^2)$ [5].

We show that CFVS can be solved in time $O(2^{O(k)}n^{O(1)})$ on general graphs and in time $O(2^{O(\sqrt{k}\log k)}n^{O(1)})$ on graphs excluding a fixed graph H as a minor. Most of the known FPT-algorithms for connectivity problems enumerate all minimal solutions and then try to connect each solution using an algorithm for the STEINER TREE problem. This is the case with the existing FPT-algorithms for CONNECTED VERTEX COVER [23, 27]. The crucial observation which the algorithms for CONNECTED VERTEX COVER rely on is that there are at most 2^k minimal vertex covers of size at most k. However, this approach fails for CFVS as the number of minimal feedback vertex sets of size at most k is $\Omega(n^k)$ (consider a graph that is a collection of k vertex-disjoint cycles each of length approximately n/k). To circumvent this problem, we make use of "compact representations" of feedback vertex sets. A compact representation is simply a collection of families of mutually disjoint sets, where each family represents a number of different feedback vertex sets. This notion was defined by Guo et al. [22] who showed that the set of all minimal feedback vertex sets of size at most k can be represented by a collection of set-families of size $O(2^{O(k)})$.

We use compact representations to obtain an FPT-algorithm for CFVS in Section 3. But in order to do that we need an FPT-algorithm for a general version of STEINER TREE, namely GROUP STEINER TREE (GST), which is defined as follows. Given a graph G = (V, E), subsets $T_i \subseteq V$, $1 \le i \le l$, and an integer p, does there exist a subgraph of G on p vertices that is a tree T and includes at least one vertex from each T_i . Observe that when the T_i 's are each of size one then GST is the STEINER TREE problem.

We find it mildly surprising that GST can be solved in time $O(2^l n^{O(1)})$ using polynomial space just as STEINER TREE because this is in sharp contrast to their behaviour in terms of approximability. While STEINER TREE admits a 1.55-factor approximation algorithm [31], GST is as hard as SET COVER [19] and hence has an $\Omega(\log n)$ lower bound on the approximation factor [1]. Our FPT-algorithm for GST uses a Turing-reduction to a directed version of STEINER TREE, called DIRECTED STEINER OUT-TREE, which we first show to be fixed-parameter tractable. We note that GST is known to be of interest to database theorists, and it has has been studied in [9], where an algorithm with runtime $O(3^l \cdot n + 2^l \cdot (n+m))$ (that uses exponential space) is discussed.

We also note that CFVS does not admit a polynomial kernel on general graphs but has a quadratic kernel on the class of graphs that exclude a fixed graph H as minor. Finally, in Section 4 we design a subexponential time algorithm for CFVS on graphs excluding some fixed graph H as a minor using the theory of bidimensionality. This algorithm is obtained

using an $O^*(w^{O(w)})$ -algorithm that computes an optimal connected feedback vertex set in graph of treewidth at most w.

2 Preliminaries

This section contains some basic definitions related to parameterized complexity and graph theory as well as commentary on the notation used in this paper. To describe running times of algorithms we sometimes use the O^* notation. Given $f: \mathbb{N} \to \mathbb{N}$, we define $O^*(f(n))$ to be $O(f(n) \cdot p(n))$, where $p(\cdot)$ is some polynomial function. That is, the O^* notation suppresses polynomial factors in the running-time expression.

A parameterized problem Π is a subset of $\Gamma^* \times \mathbb{N}$, where Γ is a finite alphabet. An instance of a parameterized problem is a tuple (x,k), where k is called the parameter. A central notion in parameterized complexity is fixed-parameter tractability (FPT) which means, for a given instance (x,k), decidability in time $f(k) \cdot p(|x|)$, where f is an arbitrary function of k and p is a polynomial in the input size. The notion of kernelization is formally defined as follows.

Definition 1. [Kernelization] A kernelization algorithm, or in short, a kernel for a parameterized problem $\Pi \subseteq \Gamma^* \times \mathbb{N}$ is an algorithm that given $(x,k) \in \Gamma^* \times \mathbb{N}$ outputs in time polynomial in |x| + k a pair $(x',k') \in \Gamma^* \times \mathbb{N}$ such that (a) $(x,k) \in \Pi$ if and only if $(x',k') \in \Pi$ and (b) $|x'|,k' \leq g(k)$, where g is some computable function. The function g is referred to as the size of the kernel. If $g(k) = k^{O(1)}$ or g(k) = O(k) then we say that Π admits a polynomial kernel and linear kernel respectively.

We say that a graph G (undirected or directed) contains a graph H if H is a subgraph of G. Given a directed graph (digraph) D = (V, A), we let V(D) and A(D) denote the vertex and arc set of D, respectively. A vertex $u \in V(D)$ is an in-neighbor (out-neighbor) of $v \in V(D)$ if $uv \in A$ ($vu \in A$, respectively). The in- and out-neighborhood of a vertex v are denoted by $N^-(v)$ and $N^+(v)$, respectively. The in-degree $d^-(v)$ (resp. out-degree $d^+(v)$) of a vertex v is $|N^-(v)|$ (resp. $|N^+(v)|$). We say that a subdigraph T of D with vertex set $V_T \subseteq V(D)$ is an out-tree if T is an oriented tree with only one vertex v of in-degree zero (called the root). The vertices of T of out-degree zero are called leaves and every other vertex is called an internal vertex.

3 Connected Feedback Vertex in General Graphs

In this section we give an FPT-algorithm for CFVS on general graphs. Recall the problem definition.

Input: An undirected graph G = (V, E) and an integer k.

Parameter: The integer k.

Question: Does there exist $S \subseteq V$ of size at most k such that $G \setminus S$ is acyclic

and G[S] is connected?

We start by describing an FPT-algorithm for the GROUP STEINER TREE problem which is crucially used in our algorithm for CFVS.

3.1 Group Steiner Tree

The Group Steiner Tree (GST) problem is defined as follows:

Input: An undirected graph G = (V, E); vertex-disjoint subsets $S_1, \ldots, S_l \subseteq V$;

and an integer p.

Parameter: The integer l.

Question: Does G contain a tree on at most p vertices that includes at least one

vertex from each S_i ?

Our fixed-parameter algorithm for GST first reduces it to DIRECTED STEINER OUT-TREE (defined below) which we then show to be fixed-parameter tractable.

Input: A directed graph D = (V, A); a distinguished vertex $r \in V$; a set of

terminals $S \subseteq V$; and an integer p.

Parameter: The integer l = |S|.

Question: Does D contain an out-tree on at most p vertices that is rooted at r and

that contains all the vertices of S?

Lemma 1. The GST problem Turing-reduces to the DIRECTED STEINER OUT-TREE problem.

Proof. Given an instance $(G = (V, E), S_1, \ldots, S_l, p)$ of GST, construct an instance of DIRECTED STEINER OUT-TREE as follows. Let $S = \{s_1, s_2, \ldots, s_l\}$ be a set of l new vertices, that is, $s_i \notin V$ for $1 \le i \le l$. Let $V' = V \cup S$ and $A = \{uv, vu : \{u, v\} \in E\} \cup \bigcup_{i=1}^{l} \{xs_i : x \in S_i\}$. Finally, let D = (V', A). It is easy to see that G contains a tree on at most p vertices that includes at least one vertex from each S_i if and only if there exists a vertex $r \in V'$ and an out-tree in D rooted at r on at most p+l vertices containing all vertices of S.

We now show that DIRECTED STEINER OUT-TREE is fixed-parameter tractable. The algorithm outlined here is essentially the same as that for the STEINER TREE problem due to Nederlof [28]. We give an outline for the sake of completeness. First recall the well-known Inclusion-Exclusion (IE) formula. Let U be a finite universe and $A_1, \ldots, A_l \subseteq U$. Then

$$\left| \bigcap_{i=1}^{l} A_i \right| = |U| + \sum_{\emptyset \neq X \subset [l]} (-1)^{|X|} \left| \bigcap_{i \in X} \bar{A}_i \right|. \tag{1}$$

Now note that if for all $X \subseteq [l]$, one can evaluate $|\bigcap_{i \in X} \bar{A}_i|$ in time polynomial in the input size n, then one can evaluate $|\bigcap_{i=1}^l A_i|$ in time $O(2^l \cdot n^{O(1)})$ and using space polynomial in n.

Given a directed graph D=(V,A), define a branching walk B in D to be a pair $(T_B=(V_B,E_B),\phi)$, where T_B is a rooted ordered out-tree and $\phi:V_B\to V$ is a homomorphism from T_B to D. The length of B, denoted by |B|, is $|E_B|$. For a node $s\in V$, B is from s if the root of T_B is mapped to s by ϕ . We let $\phi(V_B)$ denote $\{\phi(u):u\in V_B\}$ and $\phi(E_B)$ denote $\{(\phi(u),\phi(b)):(a,b)\in E_B\}$. Let (D,r,S,p) be an instance of the DIRECTED STEINER TREE problem. As in [28], one can show that there exists an out-tree T=(V',E') of D rooted at r such that $S\subseteq V'$ and $|V'|\leq p$ if and only if there exists a branching walk $B=(T_B=(V_B,E_B),\phi)$ from r such that $S\subseteq \phi(V_B)$ and $|B|\leq p-1$. We now

frame the problem as an IE-formula. Let U be the set of all branching walks from r of length p-1. For each $v \in S$, let A_v be the set of all elements of U that contain v. Then $|\bigcap_{v \in S} A_v|$ is the number of all branching walks that contain all the vertices of S and this number is larger than zero if and only if the instance is a yes-instance.

For $X \subseteq S$ define $X' = X \cup (V \setminus S)$, and define $b_j^X(r)$ to be the number of branching walks from r of length j in the graph G[X']. Then $|\bigcap_{v \in X} \bar{A}_v| = b_c^{S \setminus X}(r)$. Now $b_j^X(r)$ can be computed in polynomial time:

$$b_{j}^{X}(r) = \begin{cases} 1 & \text{if } j = 0; \\ \sum_{s \in N^{+}(r) \cap X'} \sum_{j_{1} + j_{2} = j - 1} b_{j_{1}}^{X}(s) \cdot b_{j_{2}}^{X}(r) & \text{otherwise.} \end{cases}$$

The proof of this again follows from [28]. Now for each $X \subseteq \{1, ..., l\}$, we can compute the term $|\bigcap_{v \in X} \bar{A}_v|$ in polynomial time and hence by identity (1), we can solve the problem in time $O(2^l \cdot n^{O(1)})$ using polynomial space.

Lemma 2. Directed Steiner Out-Tree can be solved in $O(2^l \cdot n^{O(1)})$ time using polynomial space.

Lemmas 1 and 2 together imply:

Lemma 3. The Group Steiner Tree problem can be solved in $O(2^l \cdot n^{O(1)})$ time using polynomial space.

One can also solve the Group Steiner Tree problem in time $O(2^p \cdot n^{O(1)})$ by reducing it to a weighted version of the Steiner Tree problem as done in [24] and then using the polynomial-space algorithm of Nederlof for solving Steiner Tree [28]. We believe that Directed Steiner Out-Tree problem will find various other applications similar to the Steiner Tree problem.

3.2 An FPT-Algorithm for CFVS

Our FPT-algorithm for CFVS uses as subroutine an algorithm for enumerating an efficient representation of minimal feedback vertex sets of size at most k due to Guo et al. [22]. Strictly speaking, the subroutine actually enumerates all compact representations of minimal feedback sets. A compact representation for a set of minimal feedback sets of a graph G = (V, E) is a set C of pairwise disjoint subsets of V such that choosing exactly one vertex from every set in C results in a minimal feedback set for G. Call a compact representation a k-compact representation if the number of sets in the representation is at most k. Clearly, any connected feedback set of size at most k must necessarily pick vertices from the sets of some k-compact representation. Given a graph G = (V, E) and a k-compact representation S_1, \ldots, S_r , where $r \leq k$, the problem of deciding whether there exists a connected feedback vertex set with at least one vertex from each set S_i reduces to the Group Steiner Tree problem where the Steiner groups are the sets of the compact representation.

Our algorithm therefore cycles through all k-compact representations and for each such representation uses the algorithm for Group Steiner Tree to check if there is a tree on at most k vertices that includes one vertex from each set S_i of the compact representation. If it fails to do this for all k-compact representations, it reports that the given instance is a NO-instance. If it succeeds on some compact representation, it reports the solution. Since one can enumerate all compact representations in time $O(c^k \cdot m)$ [22] we have:

Theorem 1. Given a graph G = (V, E) and an integer k, one can decide whether G has a connected feedback set of size at most k in time $O(c^k \cdot n^{O(1)})$, for some constant c.

Although CFVS is fixed-parameter tractable, it is unlikely to admit a polynomial kernel as the following theorem shows. This is in contrast to FEEDBACK VERTEX SET which admits a quadratic kernel [33].

Theorem 2. The CFVS problem does not admit a polynomial kernel unless the Polynomial Hierarchy collapses to Σ_3 .

Proof. The proof follows from a polynomial-time parameter-preserving reduction from Connected Vertex Cover, which does not admit a polynomial kernel unless the Polynomial Hierarchy collapses to the third level [10]. This would prove that CFVS too does not admit a polynomial kernel [4]. Given an instance (G = (V, E), k) of the Connected Vertex Cover problem, construct a new graph G' as follows: $V(G') = V(G) \cup \{x_{uv} \notin V(G) : \{u, v\} \in E(G)\}$; if $\{u, v\} \in E(G)$ then add the edges $\{u, v\}, \{u, x_{uv}\}, \{x_{uv}, v\}$ to E(G'). This completes the construction of G'. It is easy to see that G has a connected vertex cover of size at most k if and only if G' has a connected feedback vertex set of size at most k. This completes the proof of the theorem.

Interestingly, the results from [17] imply that CFVS has polynomial kernel on a graph class \mathcal{C} which excludes a fixed graph H as a minor.

We note in passing that the algorithm for enumerating compact representations can be improved using results from [6]. The authors of [6] describe a set of reduction rules such that if a YES-instance of the FOREST BIPARTITION problem (defined below) is reduced with respect to this set of rules then the instance has size at most 5k + 1.

FOREST BIPARTITION

Input: An undirected graph G = (V, E), possibly with multiple edges and loops

and a set $S \subseteq V$ such that |S| = k + 1 and $G \setminus S$ is acyclic.

Parameter: The integer k.

Question: Does G have a feedback vertex set of size at most k contained in $V \setminus S$?

Thus in a YES-instance of FOREST BIPARTITION that is reduced with respect to the rules in [6], we have $|V \setminus S| \leq 4k$. Using this bound in the algorithm described by Guo et al. [22], one obtains a $O^*(c^k)$ -time algorithm for enumerating compact representations of minimal feedback vertex sets of size at most k, where c = 52. The constant c in [22] is more than 160.

Theorem 3. [6,22] Given a graph G = (V, E) and an integer k, the compact representations of all minimal feedback vertex sets of G of size at most k can be enumerated in time $O(52^k \cdot |E|)$.

4 Subexponential Algorithm for CFVS on H-Minor-Free Graphs

In the last section, we obtained an $O^*(c^k)$ algorithm for CFVS on general graphs. In this section we show that CFVS on the class of H-minor-free graphs admits a sub-exponential time algorithm with run-time $O(2^{O(\sqrt{k}\log k)}n^{O(1)})$. This section is divided into three parts. In the first part we give essential definitions from topological graph theory, the second part shows that CFVS can be solved in time $O(w^{O(w)}n^{O(1)})$ on graphs with treewidth bounded by w. In the last part we give the desired algorithm for CFVS on H-minor-free graphs by bounding the treewidth of the input graph using the known "grid theorems".

4.1 Definitions and Terminology

Given an edge e in a graph G, the *contraction* of e is the result of identifying its endpoints in G and then removing all loops and duplicate edges. A *minor* of a graph G is a graph H that can be obtained from a subgraph of G by contracting edges. A graph class C is *minor-closed* if any minor of any graph in C is also an element of C. A minor-closed graph class C is C is

A tree decomposition of a graph G = (V, E) is a pair $(T = (V_T, E_T), \mathcal{X} = \{X_t\}_{t \in V_T})$ where T is a tree and the X_t are subsets of V such that:

- 1. $\bigcup_{u \in V_T} X_t = V;$
- 2. for each edge $e = \{u, v\} \in E$ there exists $t \in V_T$ such that $u, v \in X_t$; and
- 3. for each vertex $v \in V$, the subgraph $T[\{t \mid v \in X_t\}]$ is connected.

The width of a tree decomposition is $\max_{t \in V_T} |X_t| - 1$ and the treewidth of G = (V, E) is the minimum width over all tree decompositions of G.

A tree decomposition is called a *nice tree decomposition* if the following conditions are satisfied:

- Every node of the tree T has at most two children;
- if a node t has two children t_1 and t_2 , then $X_t = X_{t_1} = X_{t_2}$; and
- if a node t has one child t_1 , then either $|X_t| = |X_{t_1}| + 1$ and $X_{t_1} \subset X_t$ or $|X_t| = |X_{t_1}| 1$ and $X_t \subset X_{t_1}$.

It is possible to transform a given tree decomposition into a nice tree decomposition in time O(|V| + |E|) [3].

4.2 Connected FVS and Treewidth

In this section we show that the CONNECTED FEEDBACK VERTEX SET problem is FPT with the treewidth of the input graph as the parameter. Specifically, we show that the following problem is FPT:

Input: An undirected graph G = (V, E); an integer k; and a nice tree decom-

position of G of width w.

Parameter: The treewidth w of the graph G.

Question: Does there exist $S \subseteq V$ such that $G \setminus S$ is acyclic, G[S] is connected,

and |S| < k?

We design a dynamic programming algorithm on the nice tree decomposition with run-time $O(w^{O(w)} \cdot n^{O(1)})$ for this problem. Let $(T = (I, F), \{X_i | i \in I\})$ be a nice tree decomposition of the input graph G of width w and rooted at $r \in I$. We let T_i denote the subtree of T rooted at $i \in I$, and $G_i = (V_i, E_i)$ denote the subgraph of G induced on all the vertices of G in the subtree T_i , that is, $G_i = G[\bigcup_{i \in V(T_i)} X_j]$.

For each node $i \in I$ we compute a table A_i , the rows of which are 4-tuples [S, P, Y, val]. Table A_i contains one row for each combination of the first three components which denote the following:

- -S is a subset of X_i .
- -P is a partition of S into at most |S| labelled pieces.
- Y is a partition of $X_i \setminus S$ into at most $|X_i \setminus S|$ labelled pieces.

We use P(v) (resp. Y(v)) to denote the piece of the partition P (resp. Y) that contains the vertex v. We let |P| (resp. |Y|) denote the number of pieces in the partition P (resp. Y). The last component val, also denoted as $A_i[S, P, Y]$, is the size of a smallest feedback vertex set $F_i \subseteq V(G_i)$ of G_i which satisfies the following properties:

- If $S = \emptyset$, then F_i is connected in G_i .
- If $S \neq \emptyset$, then
 - $F_i \cap X_i = S$.
 - All vertices of S that are in any one piece of P are in a single connected component of $G_i[F_i]$. Moreover $G_i[F_i]$ has exactly |P| connected components.
 - All vertices of $X_i \setminus S$ that are in the same piece of Y are in a single connected component (a tree) of $G_i[V_i \setminus F_i]$. Moreover $G_i[V_i \setminus F_i]$ has at least |Y| connected components.

If there is no such set F_i , then the last component of the row is set to ∞ .

We fix an arbitrary ordering of the vertices of X_i , and compute the table A_i for each node $i \in I$ of the tree decomposition. Since there are at most w+1 vertices in each bag X_i , there are no more than

$$\sum_{i=0}^{w+1} {w+1 \choose i} i^i \cdot (w+1-i)^{w+1-i} \le (2w+2)^{2w+2}$$

rows in any table A_i . We compute the tables A_i starting from the leaf nodes of the tree decomposition and going up to the root.

Leaf Nodes. Let i be a leaf node of the tree decomposition. We compute the table A_i as follows. For each triple (S, P, Y) where S is a subset of X_i , P a partition of S, and Y a partition of $X_i \setminus S$:

- Set $A_i[S, P, Y] = \infty$ if at least one of the following holds:
 - $G_i \setminus S$ contains a cycle (i.e., S is not an FVS of G_i).
 - At least one piece of P is not connected in $G_i[S]$ or if $G_i[S]$ has less than |S| connected components.
 - At least one piece of Y is not connected in $G_i[V_i \setminus S]$ or if $G_i[V_i \setminus S]$ has less than |Y| connected components.
- In all other cases, set $A_i[S, P, Y] = |S|$.

It is easy to see that this computation correctly determines the last component of each row of A_i for a leaf node i of the tree decomposition.

Introduce Nodes. Let i be an introduce node and j its unique child. Let $x \in X_i \setminus X_j$ be the introduced vertex. For each triple (S, P, Y), we compute the entry $A_i[S, P, Y]$ as follows.

Case 1. $x \in S$. Check whether $N(x) \cap S \subseteq P(x)$; if not, set $A_i[S, P, Y] = \infty$.

- Subcase 1. $P(x) = \{x\}$. Set $A_i[S, P, Y] = A_j[S \setminus \{x\}, P \setminus P(x), Y] + 1$.
- Subcase 2: $|P(x)| \ge 2$ and $N(x) \cap P(x) = \emptyset$. Set $A_i[S, P, Y] = \infty$, as no extension of S to an fvs for G_i can make P(x) connected.
- Subcase 3: $|P(x)| \ge 2$ and $N(x) \cap P(x) \ne \emptyset$. Let \mathcal{A} be the set of all rows [S', P', Y] of the table A_i that satisfy the following conditions:
 - $\bullet S' = S \setminus \{x\}.$
 - $P' = (P \setminus P(x)) \cup Q$, where Q is a partition of $P(x) \setminus \{x\}$ such that each piece of Q contains an element of $N(x) \cap P(x)$.

Set $A_i[S, P, Y] = \min_{A} \{ A_i[S', P', Y] \} + 1.$

Case 2. $x \notin S$. Check whether $N(x) \cap (X_i \setminus S) \subseteq Y(x)$; if not, set $A_i[S, P, Y] = \infty$.

- Subcase 1: $Y(x) = \{x\}$. Set $A_i[S, P, Y] = A_i[S, P, Y \setminus Y(x)]$.
- Subcase 2: $|Y(x)| \ge 2$ and $N(x) \cap Y(x) = \emptyset$. Set $A_i[S, P, Y] = \infty$, as no extension of S to an fvs F_i for G_i can make Y(x) a connected component in $G_i[V_i \setminus F_i]$.
- Subcase 3: $|Y(x)| \ge 2$ and $N(x) \cap Y(x) \ne \emptyset$. Let \mathcal{A} be the set of all rows [S, P, Y'] of the table A_j where $Y' = (Y \setminus Y(x)) \cup Q$, and Q is a partition of $Y(x) \setminus \{x\}$ such that each piece of Q contains exactly one element of $N(x) \cap Y(x)$. Set $A_i[S, P, Y] = \min_{\mathcal{A}} \{A_j[S, P, Y']\}$.

Forget Nodes. Let i be a forget node and j its unique child node. Let $x \in X_j \setminus X_i$ be the forgotten vertex. For each triple (S, P, Y) in the table A_i , let A be the set of all rows [S', P', Y] of the table A_j that satisfy the following conditions:

- $-S' = S \cup \{x\}$, and
- $-P'(x) = P(y) \cup \{x\}$ for some $y \in S$.

Let \mathcal{B} be the set of all rows [S, P, Y'] of the table A_j such that $Y'(x) = Y(z) \cup \{x\}$ for some $z \in S$. Set

$$A_i[S, P, Y] = \min \left\{ \min_{\mathcal{A}} A_j[S', P', Y], \min_{\mathcal{B}} A_j[S, P, Y'] \right\}.$$

Join Nodes. Let i be a join node and j and l its children. For each triple (S, P, Y) we compute $A_i[S, P, Y]$ as follows.

- Case 1. $S = \emptyset$. If both $A_j[\emptyset, P, Y]$ and $A_l[\emptyset, P, Y]$ are positive finite, then set $A_i[\emptyset, P, Y] = \infty$. Otherwise, set $A_i[\emptyset, P, Y] = \max\{A_j[\emptyset, P, Y], A_l[\emptyset, P, Y]\}$.
- Case 2. $S \neq \emptyset$. Let \mathcal{A} denote the set of all pairs of triples $\langle (S, P_1, Y_1), (S, P_2, Y_2) \rangle$, where $(S, P_1, Y_1) \in A_j$ and $(S, P_2, Y_2) \in A_l$ with the following property: Starting with the partitions $Q_p = P_1$ and $Q_y = Y_1$ and repeatedly applying the following set of operations, we reach stable partitions that are identical to P and Y. The first operation that we apply is:

If there exist vertices $u, v \in S$ such that they are in different pieces of Q_p but are in the same piece of P_2 , delete $Q_p(u)$ and $Q_p(v)$ from Q_p and add $Q_p(u) \cup Q_p(v)$.

To describe the second set of operations, we need some notation. Let $Z = X_i \setminus S$ and let the connected components of $G_i[Z]$ be C_1, \ldots, C_q . First contract each connected component C_i to a vertex c_i , the representative of that component, and let $C = \{c_1, \ldots, c_q\}$. Note that for each $1 \leq i \leq q$, the component C_i is not split across pieces in either Y_1 or Y_2 . Denote by Y'_1 and Y'_2 the partitions obtained from Y_1 and Y_2 , respectively, be replacing each connected component C_i by its representative vertex c_i . Let $Q_y = Y'_1$. Repeat until no longer possible:

If there exist $c_a, c_b \in \mathcal{C}$ that are in different pieces of Q_y but in the same piece of Y_2 then delete $Q_y(c_a), Q_y(c_b)$ from Q_y and add $Q_y(c_a) \cup Q_y(c_b)$ provided the following condition holds: for all $c_e \in \mathcal{C} \setminus \{c_a, c_b\}$ either $Y_2(c_e) \cap Q_y(c_a) = \emptyset$ or $Y_2(c_e) \cap Q_y(c_b) = \emptyset$.

If this latter condition does not hold, move on to the next pair of triples. Finally expand each c_i to the connected component it represents. Set

$$A_i[S, P, Y] = \min_{\mathcal{A}} \{ A_j[S, P_1, Y_1] + A_l[S, P_2, Y_2] - |S| \}.$$

The stated conditions ensure that $u, v \in S$ are in the same piece of P if and only if for each $\langle (S, P_1, Y_1), (S, P_2, Y_2) \rangle \in \mathcal{A}$, they are in the same piece of P_1 or of P_2 (or both). Similarly, the stated conditions ensure that merging solutions at join nodes do not create new cycles. Given this, it is easy to verify that the above computation correctly determines $A_i[S, P, Y]$.

Root Node. We compute the size of a smallest CFVS of G from the table A_r for the root node r as follows. Find the minimum of $A_r[S, P, Y]$ over all triples (S, P, Y), where $S \subseteq X_r$, P a partition of S such that P consists of a single (possibly empty) piece and Y is a partition of $X_r \setminus S$. This minimum is the size of a smallest CFVS of G.

This concludes the description of the dynamic programming algorithm for CFVS when the treewidth of the input graph is bounded by w. From the above description and the size of tables being bounded by $(2w+2)^{2w+2}$, we obtain the following result.

Lemma 4. Given a graph G = (V, E), a tree-decomposition of G of width w, one can compute the size of an optimum connected feedback vertex set of G (if it exists) in time $O((2w+2)^{2w+2} \cdot n^{O(1)})$.

4.3 FPT Algorithms for *H*-Minor Free Graphs

We first bound the treewidth of the yes instance of input graphs by $O(\sqrt{k})$.

Lemma 5. If (G,k) is a yes-instance of CFVS where G excludes a fixed graph H as a minor, then $\mathbf{tw}(G) \leq c_H \sqrt{k}$, where c_H is a constant that depends only on the graph H.

Proof. By [7], for any fixed graph H, every H-minor-free graph G that does not contain a $(w \times w)$ -grid as a minor has treewidth at most $c'_H w$, where c'_H is a constant that depends only on the graph H. Clearly a $(w \times w)$ -grid has a feedback vertex set of size at least $c_1 w^2$, where c_1 is a constant independent of w. Therefore if G has a connected feedback vertex set of size at most k, it cannot have a $(w \times w)$ -grid minor, where $w > \sqrt{k/c_1}$. Therefore $\mathbf{tw}(G) \leq c'_H w \leq c'_H \cdot (\sqrt{k/c_1} + 1) \leq c_H \sqrt{k}$, where $c_H = (c'_H + 1)/\sqrt{c_1}$.

Theorem 4. CFVS can be solved in time $O(2^{O(\sqrt{k}\log k)} + n^{O(1)})$ on H-minor-free graphs.

Proof. Given an instance (G, k) of CFVS, we first find a tree-decomposition of G using the polynomial-time constant-factor approximation algorithm of Demaine et al. [8]. If $\mathbf{tw}(G) > c_H \sqrt{k}$, then the given instance is a no-instance; else, use Lemma 4 to find an optimal CFVS for G. All this can be done in $O(2^{O(\sqrt{k}\log k)} \cdot n^{O(1)})$. To obtain the claimed run-time bound we first apply the results from [17] and obtain an $O(k^2)$ kernel for the problem in polynomial time and then apply the algorithm described.

5 Conclusion

We conclude with some open problems. The obvious question is to obtain an $O^*(c^k)$ algorithm for CFVS in general graphs with a smaller value of c. Also the approximability of CFVS in general graphs is unknown. Is there a constant-factor approximation algorithm for CFVS? If not, what is the limit of approximation? Is there an $O^*(c^w)$ algorithm for CFVS, for a constant c, for graphs of treewidth at most w? Note that this question is open in the context of finding a feedback vertex set in graphs of bounded treewidth.

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