

Fast algorithms for min independent dominating set

N. Bourgeois B. Escoffier V. Th. Paschos

LAMSADE, CNRS FRE 3234 and Université Paris-Dauphine, France
 {bourgeois,escoffier,paschos}@lamsade.dauphine.fr

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Abstract

We first devise a branching algorithm that computes a minimum independent dominating set on any graph with running time $O^*(2^{0.424n})$ and polynomial space. This improves the $O^*(2^{0.441|V|})$ result by (S. Gaspers and M. Liedloff, *A branch-and-reduce algorithm for finding a minimum independent dominating set in graphs*, Proc. WG'06). We then show that, for every $r \geq 3$, it is possible to compute an $r - ((r-1)/r) \log_2 r$ -approximate solution for MIN INDEPENDENT DOMINATING SET within time $O^*(2^{n \log_2 r/r})$.

1 Introduction

An independent set in a graph $G(V, E)$ is a vertex subset $S \subseteq V$ such that for any $(v_i, v_j) \in S \times S$, $(v_i, v_j) \notin E$. An independent dominating set is an independent set that is maximal for inclusion. MIN INDEPENDENT DOMINATING SET is known to be **NP**-hard [2]. Consequently, it is extremely unlikely that a polynomial algorithm could ever be designed solving it to optimality. Unfortunately, this problem is also very badly approximable since no polynomial algorithm can approximately solve it within ratio $|V|^{1-\epsilon}$, for any $\epsilon > 0$, unless $\mathbf{P} = \mathbf{NP}$ [4].

Except polynomial approximation, another way to cope with intractability of MIN INDEPENDENT DOMINATING SET (as well as of any other **NP**-hard problem) is by designing algorithms able to solve it to optimality with worst-case exponential running time as low as possible. Since MIN INDEPENDENT DOMINATING SET can be trivially solved in $O(2^{|V|}p(|V|))$ by simply enumerating all the subsets of V , the stake of such a research issue is to solve it within $O(2^{c|V|}p(|V|))$, where c is a constant lower than 1 and p is some polynomial. Since in comparison with the slightest improvement of c , p is non relevant, we use from now on notation $O^*(2^{c|V|})$ to measure the complexity of an algorithm, this notation meaning that multiplicative polynomial factors are ignored.

For MIN INDEPENDENT DOMINATING SET, trivial $O^*(2^{|V|})$ bound has been initially broken by [5] down to $O^*(3^{|V|/3})$ using a result by [6], namely that the number of maximal (for inclusion) independent sets in a graph is at most $3^{|V|/3}$. Then, obviously, it is possible to compute a minimum independent dominating set in $O^*(3^{|V|/3}) = O^*(2^{0.529|V|})$ using polynomial space. This result has been dominated by [3] where using a branch & reduce technique an algorithm optimally solving MIN INDEPENDENT DOMINATING SET with running time $O^*(2^{0.441|V|})$ is proposed.

In this paper, we first devise a branching algorithm that can find a minimum independent dominating set on any graph with running time $O^*(2^{0.424|V|})$ and polynomial space. We then show that, for every $r \geq 3$, it is possible to compute an r -approximate solution for MIN INDEPENDENT DOMINATING SET within time $O^*(2^{|V| \log_2 r/r})$. Finally, we improve this ratio down to $r - ((r-1)/r) \log_2 r$ and prove that it can be achieved always in time $O^*(2^{|V| \log_2 r/r})$.

In what follows, given a graph $G(V, E)$ and a vertex $v \in V$, the neighborhood $N(v)$ of v is the set of vertices that are adjacent to v , and $N[v] = N(v) \cup \{v\}$ will be called the closed neighborhood of v . For the degree of v , we use the notation $d(v) = |N(v)|$. For any subset $H \subset V$, we denote by $G[H]$ the subgraph of G induced by H . For $v \in H$, for some subset H , we denote by $d'_H(v)$ the degree of v in $G[H]$ or, if it is clear by the context, we denote it by $d'(v)$. For convenience, we set $N[H] = \{N[v] : v \in H\}$. We use δ and Δ to denote the minimum and maximum degree of G , respectively. For simplicity, we set $n = |V|$ and $m = |E|$; $T(n)$ stands for the maximum running time an algorithm requires to solve MIN INDEPENDENT DOMINATING SET in a graph containing at most n vertices.

We conclude this introduction by a remark that has to be taken into account when operating sequentially several branchings, in order to get a sound bound on the complexity. Let us call a branching “single” if given a vertex v , one has to decide what vertex in $N[v]$ dominates v . A “multiple” branching is a decision tree, nodes of which correspond to simple branchings.

Branch & reduce-based algorithms have been used for decades, and a classical analysis of their running times leading to worst case complexity upper bounds is now well-known. Let $T(n)$ be an upper bound on the running time of the algorithm for an instance of size at most n . If we now that computing a solution on an instance of size n amounts to computation of a solution on a sequence of p instances of respective sizes $n - k_1, \dots, n - k_p$, we can write:

$$T(n) \leq \sum_{i \leq p} T(n - k_i) + q(n) \tag{1}$$

for some polynomial q . The running time $T(n)$ is bounded by $O^*(c^n)$, where c is the largest positive real root of: $1 = \sum_{i \leq p} x^{-k_i}$. This root is often called the contribution of the branching to overall complexity factor, or the complexity factor of the branching. Note that in the remainder of the article, for simplicity, we will omit to precise the additive polynomial term $q(n)$ in recurrence relations as 1

Now, it is possible that there is not only a single recurrence as in (1), but several ones, depending on the instance. For example, either $T(n) \leq 2T(n - 2)$ or $T(n) \leq 7T(n - 7)$. Fortunately, as far as single branchings are concerned, the analysis is very simple: the running time is never greater than what is needed to solve an instance where at every step we make a branching that has the highest possible complexity factor, i.e., the greatest solution of (1).

On the other hand, multi-branching analysis may encounter some problems. Indeed, this rule is not true any more when we make multiple branchings. Multiple branching is based upon a very simple idea: instead of choosing between some branchings, we choose between some sequences of branching, such as: “if one adds a to the solution, then one knows that b has degree 3 in the remaining graph and one can make a very good branching on it ...”. The efficiency of such a sequence can be measured with inequalities very close to (1). If for instance, the first branching allows us to consider two graphs with, say, 2 fewer vertices, and we know that one of these graphs is good enough to allow us to consider three graphs with 7 fewer vertices, we can state the following recurrence: $T(n) \leq T(n - 2) + 3T(n - 2 - 7)$.

The problem arises when we have several choices for one of these “elementary” branchings. Unfortunately, we cannot simply try the one with the higher complexity factor, because it is possible that a locally smaller factor leads to a globally higher one, see for example Figure 1. Although branching (a) has a higher complexity factor than branching (b) (1.443 instead of 1.415), the branching (c)+(a) is better than (c)+(b) (1.659 and 1.696, respectively).

In what follows, if a branching is never used in a combination with another one, we simply give the inequality with the highest factor. Otherwise, we give all the relevant inequalities.

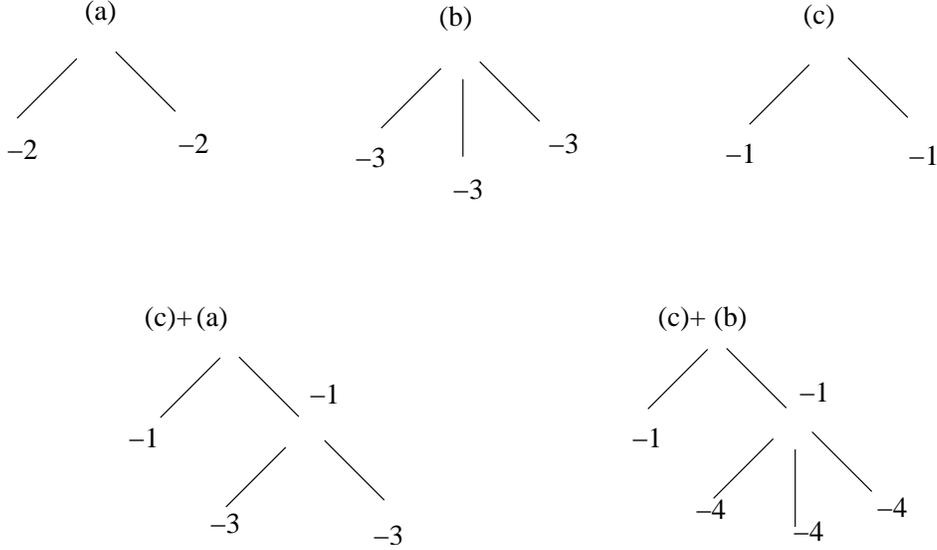


Figure 1: A multiple branching that can be misleading when dealing with the evaluation of the highest complexity factor.

2 General recurrence

Following an idea by [3], we partition the graph into “marked” and “free” vertices. Marked vertices are those that have already been disqualified from belonging to optimum, but are not dominated yet. In other terms, we generalize the problem at hand in the following way: given a subset $W \subseteq V$ (W is the set of free vertices), find the minimum independent set in W that dominates V . Notice that, without further hypothesis on W , this problem may have no solution (for instance, when $V \setminus W$ contains a vertex and its whole neighborhood); in this case we consider $\text{opt}(G, W) = \infty$.

In what follows we say that two vertices v and u are equivalent if $N[u] = N[v]$. In this case, we can remove from the graph one of them, a marked one if any, otherwise at random.

Lemma 1. *Let δ be the minimum degree of $G(V, E)$, and r be the maximum number of marked vertices in $N[v]$, for any v such that $d(v) = \delta$. Then:*

$$T(n) \leq (\delta + 1 - r)T(n - \delta - 1)$$

Furthermore, when branching on some neighbor of v , we can mark all other neighbors that have already been examined.

Proof. We can always suppose that $\delta \geq 1$, since (not marked) isolated vertices must be added to the solution. Let v be a vertex such that $d(v) = \delta$. Our solution has to dominate v , so at least one vertex of $N[v]$ must belong to, and this vertex must be a free one. Furthermore, if some vertex u belongs to optimum, its neighbors do not so and, consequently, they are dominated. Hence, we get the following recurrence:

$$\text{opt}(G) = 1 + \min_{u \in W \cap N[v]} \{\text{opt}(G[V \setminus N[u]])\} \quad (2)$$

By hypothesis, $d(u) \geq \delta$, so we get the inequality claimed.

Remark that the order we use to examine neighbors u_i , $i = 1, \dots, \delta$ of v is important, since the sequence of choices is not “ $v; u_1; u_2; \dots; u_\delta$ ” but “ $v; \bar{v}u_1; \bar{v}\bar{u}_1u_2; \dots; \bar{v}\bar{u}_1 \dots u_{\delta-1}u_\delta$ ” where for

a vertex u , \bar{u} means “not u ” and we assume that u_i ’s are ordered in increasing degree order. Figure 2 illustrates the proof of the lemma. ■

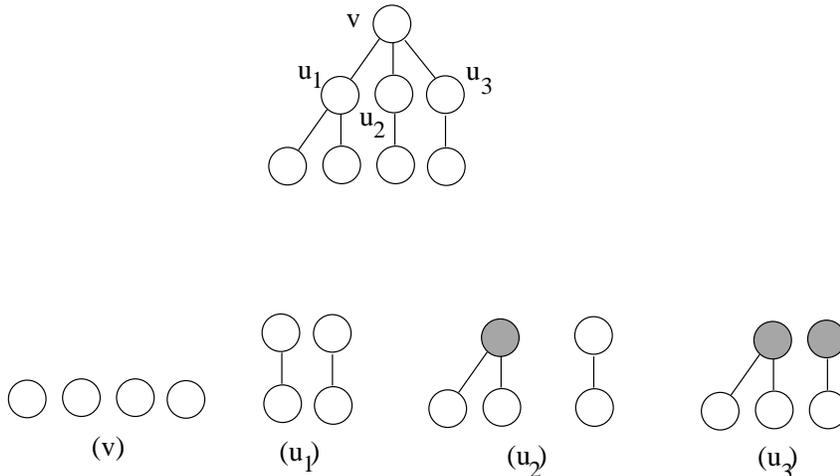


Figure 2: The four branches $v; \bar{v}u_1; \bar{v}\bar{u}_1u_2; \bar{v}\bar{u}_1\bar{u}_2u_3$.

Note that complexity of branching is decreasing with δ , for $\delta \geq 2$. So, a straightforward idea is to perform a fine analysis on graphs of low minimum degree. Formally, the algorithm we propose works as follows:

- if there exists a marked vertex of degree 3 or less, or a vertex which is adjacent to only one free vertex, make a branching according to what is described in Section 3;
- otherwise, pick a vertex of minimum degree, and branch as described in Section 4.

In our analysis of the running time, we adopt a measure and conquer approach. More precisely, we do not count in the measure the marked vertices of degree at most 2 (they receive weight 0), and we count with a weight $w = 0.2$ the marked vertices of degree 3. The other vertices receive weight 1. We get recurrences on the time $T(p)$ required to solve instances of weight p , where the weight of an instance is total weight of the vertices in the graph. Since initially $p = n$ we get the running time as a function of n . This is valid since when the total weight is $p = 0$, there are only marked vertices in the graph and we can solve the problem (there is no solution). Note that this way of measuring progress is introduced in order to simplify the branching analysis (we could obtain a similar result without measure and conquer but with a deeper analysis).

3 Branching on marked vertices or vertices which are adjacent to only one free vertex

As a preliminary remark note that if there is an edge between two adjacent marked vertices, we can remove this edge. This removal does not increase our complexity measure p .

We branch as follows: either there is a marked vertex of degree at most 2 (Lemma 2), or a (free) vertex which is adjacent to only one free vertex (Lemma 3) or a marked vertex of degree 3 (Lemma 4). If there are no such vertices, then go to Section 4.

Lemma 2. *Assume some vertex of degree at most 2 is marked. Then either the algorithm ends (there is no solution), or we can remove at least one vertex without branching, or $T(p) \leq$*

$T(p-2) + T(p-4)$; this branching contributes to the overall complexity with a factor $\lambda \leq 1.2721 = 2^{0.348}$.

Proof. If there is a marked vertex v of degree 0, then $\text{opt}(G) = \infty$.

If v has degree 1 we add its neighbor u to the solution and we reduce the current instance's weight by 1 without branching.

Suppose now that v is marked and adjacent to u_1, u_2 (which are free). Then:

$$\text{opt}(G) = 1 + \min \{ \text{opt}(G \setminus N[u_1]), \text{opt}(G \setminus N[u_2]) \}$$

If both u_1 and u_2 are adjacent to at least 2 free vertices, then $T(p) \leq 2T(p-3)$, that is better than the result claimed. If some u_i is adjacent only to marked vertices, we must add it to the optimum, decreasing p by 1 without branching. Otherwise, u_1 is adjacent to only one free vertex t_1 and u_2 is adjacent to at least one free vertex t_2 . One of the following situations occurs:

- If u_1 and u_2 are adjacent: if u_2 is adjacent to two other free vertices t_2 and t_3 , then if we take u_1 we reduce p by 2, if we take u_2 we reduce it by 4. If u_2 is adjacent to only one vertex t_2 , then taking u_1 is interesting only if we take t_1 . Hence, either we take u_1 (and t_1) and p reduces by 3, or we take u_2 and p reduces by 3.
- Otherwise, if $t_1 = t_2$, the only possibility to have both u_1 and v dominated is to add u_1 to the solution; thus we reduce the current instance without branching.
- Finally, if $N[u_1] \cap N[u_2] = \{v\}$, we branch on v ; when we add u_2 to the solution, we must add t_1 too, in order to dominate u_1 ; this leads to $T(p) \leq T(p-2) + T(p-4)$. ■

Now, we suppose that the graph does not contain any marked vertex of degree at most 2.

Lemma 3. *If there exists $v \in V$ such that $N(v) \cap W = \{u\}$, then $T(p) \leq 2T(p - (2 + 2w))$, and the complexity factor induced is $\lambda \leq 1.3349 = 2^{0.417}$.*

Proof. Let u be the only free neighbor of v . If $d(u) = 1$, we can add v to the solution and discard u without branching. Otherwise, if u or v is adjacent to at least two vertices of weight 1, removing $N[u]$ (or $N[v]$) reduces p by at least 3 vertices. Taking either v or u gives $T(p) \leq T(p-2) + T(p-3)$. The only remaining situation occurs when all the other neighbors of both u and v are marked and of degree 3. If there is only one such vertex, then u and v are equivalent and we don't need to branch. Otherwise, when taking u or v these marked vertices of degree 3 either are removed or become of degree 2, hence $T(p) \leq 2T(p - (2 + 2w))$. ■

Lemma 4. *Assume some vertex v of degree 3 is marked. Then in the worst case $T(p) \leq 2T(p - (3 + w)) + T(p - (5 + w))$, and the complexity factor induced is $\lambda \leq 1.3409 = 2^{0.424}$.*

Proof. Let $\{u_1, u_2, u_3\}$ the three neighbors of v . One of the following situations occur:

1. If each u_i is adjacent to at least 3 free vertices, then, by taking either u_1, u_2 or u_3 we get three branches of weight at most $p - (4 + w)$.
2. If say u_1 is adjacent to two free vertices, then we branch on u_1 : if we take u_1 we reduce p by at least $3 + w$. Otherwise, we do not take u_1 . In this case u_1 is marked and we can remove the edges between u_1 and the marked vertices. Hence, u_1 and v are marked and have degree at most 2. Then, p reduces by $1 + w$. But a further branching on a marked vertex of degree at most 2 (see Lemma 2) allows either to reduce p by 1, or creates two branches of weight at most $1 + w + 2 = 3 + w$ or $1 + w + 4 = 5 + w$. In the worst case, $T(p) \leq 2T(p - (3 + w)) + T(p - (5 + w))$. ■

4 Branching on vertices of minimum degree

Now, we suppose that the graph does not contain any marked vertex of degree at most 3, and that every vertex is adjacent to at least two free vertices. Then, we branch on the vertex of minimum degree. If this minimum degree is at least 6, then the branching given in Section 2 gives a sufficiently low running time. We distinguish in the following lemmas the different possible values of the minimum degree. Let us start with two preliminary remarks.

Remark 1. When branching on a vertex of minimum degree δ , we can always assume that it is adjacent to at least one vertex of degree at least $\delta + 1$. Notice that the degree of a vertex never increases. Then, the situation where the graph is δ -regular arrives at most once (even in case of disconnection). Thus, we make only a finite number of “bad” branchings, fact that may increase the global running time only by a constant factor. In particular, if $\delta = 5$, the branching given in Section 2 gives $T(p) \leq 5T(p - 6) + T(p - 7)$ leading to a complexity factor $1.3384 = 2^{0.421}$. ■

Remark 2. Suppose that we branch on a vertex v which has a neighbor u_1 adjacent to at most 3 free neighbors. Let us consider the branch where we take u_k not adjacent to u_1 (for $2 \leq k \leq d(v)$). In this branch u_1 is marked.

- If $N_W(u_1) \subseteq N_W(u_k)$, then we cannot take u_k and this branch is useless.
- If there is only one vertex t in $N_W(u_1) \setminus N_W(u_k)$: in this branch we have to take t and then we remove at least $d(u_k) + 3$ vertices ($d(u_k) + 1$ by taking u_k , and t and u_1 by taking t).
- Otherwise u_1 has two other free neighbors t_1, t_2 which are not in $N(u_k)$: in this branch we create a marked vertex (u_1) of degree 2 and hence reduce p by $d(u_k) + 2$. Thanks to Lemma 2, a further branching on the marked vertex of degree at most 2 created gives two branches where p reduces by at least $d(u_k) + 2 + 2$ and $d(u_k) + 2 + 4$.

In all, either we have one branch with a reduction of $d(u_k) + 3$, or two branches with $d(u_k) + 4$ and $d(u_k) + 6$ (the latter will always be the worst case). ■

We are ready now to state the following theorem, that is the main result of the paper.

Theorem 1. MIN INDEPENDENT DOMINATING SET *can be solved in $O^*(2^{0.424n}) = O^*(1.3413^n)$, using polynomial space.*

The proof of Theorem 1 is immediate consequence of putting together Lemmata 5, 6 and 7 below that deal with the cases of minimum degree 2, 3 and 4, respectively.

Lemma 5. *If there exists $v \in V$ such that $N(v) = \{u_1, u_2\}$, then in the worst case we get $T(p) \leq T(p - 3) + T(p - 4) + T(p - 6) + T(p - 8)$ and the complexity factor induced is $\lambda \leq 1.3384 = 2^{0.421}$.*

Proof. One of the following situations occur (note that either u_1 or u_2 has degree at least 3):

1. If $d(u_2) \geq 4$ and $d(u_1) \geq 3$ then by taking either v , or u_1 , or u_2 , we get three branches of weight at most $p - 3$, $p - 4$ and $p - 5$.
2. If $d(u_2) \geq 4$ and $d(u_1) = 2$ then, when we take u_2 (u_1 having already been discarded), we must also add the only remaining neighbor t of u_1 to the solution. Thus, thanks to Remark 2 (first and second item), we get $T(p) \leq 2T(p - 3) + T(p - 7)$.

3. If u_1 and u_2 have degree 3 and if there are adjacent: let t the third neighbor of u_1 (t is not adjacent to u_2 otherwise u_1 and u_2 are equivalent and we can remove one of them). When taking v , we can take also t since if we don't take t it is useless to take v . Hence, we get three branches, each of weight at most $p - 4$ (even better on the first branch actually).
4. If $d(u_1) = 3$, then u_1 and u_2 are not adjacent (either because the case has been dealt before, or because $d(u_2) = 2$ and u_2 would be equivalent to v), then by branching on v , either we take v (weight at most $p - 3$) or we take u_1 (weight at most $p - 4$) or we take u_2 and we don't take v and u_1 . According to Remark 2, this last choice reduces p either by $d(u_2) + 3 = 5$, or gives birth to two branches of weight at most $p - 6$ and $p - 8$. ■

Lemma 6. *If there exists $v \in V$ such that $N(v) = \{u_1, u_2, u_3\}$, then in the worst case we get $T(p) \leq T(p - 4) + 3T(p - 5) + T(p - 6) + T(p - 8)$ and the complexity factor induced is $\lambda \leq 1.3413 = 2^{0.424}$.*

Proof. If there exists such a vertex v which is marked, then we only have to consider 3 branches where p reduces by at least 4: $T(p) \leq 3T(p - 4)$. This is the same if one of the neighbors of v is marked.

Now we consider that neither v nor the u_i 's are marked. If $4 \leq d(u_i)$, $i = 1, 2, 3$, by branching on v we get one branch of weight $p - 4$ and 3 branches of weight at most $p - 5$. Now, we consider that u_1 has degree 3 and u_3 has degree at least 4. Note that u_1 cannot be adjacent to u_2 and u_3 otherwise it is equivalent to v . We consider the three following cases: either there are two edges in $N(v)$, or 1 or zero.

1. There are two edges in $N(v)$. Then wlog., u_3 is adjacent to both u_1 and u_2 (and u_1 is not adjacent to u_2). We get four branches of weight at most $p - 4$, $p - 4$, $p - (d(u_2) + 2)$ (since u_1 becomes marked and of degree at most 2) and $p - (d(u_3) + 1)$. In the third branch, thanks to Remark 2, either we remove one more vertex or we get two branches of weight at most $p - (d(u_2) + 4) \leq p - 7$ and $p - (d(u_2) + 6) \leq p - 9$.
 - If $d(u_3) \geq 5$: in the worst case $T(p) \leq 2T(p - 4) + T(p - 7) + T(p - 9) + T(p - 6)$.
 - Otherwise $d(u_3) = 4$. Let t be the fourth neighbor of u_3 . In the branch we take v we can take t (indeed, if we don't take t it is useless to take v , taking u_3 is always better). Hence, we get one more vertex deleted when taking v (even more actually) and get the following recurrence: $T(p) \leq T(p - 5) + T(p - 4) + 2T(p - 5)$.
2. Otherwise, there is at most one edge in $N(v)$.
 - (a) If there is a triangle of vertices of degree 3 v, u_1, u_2 , let t_1 and t_2 be the third neighbors of u_1 and u_2 . If two vertices among u_3, t_1 and t_2 are equal or adjacent, either two vertices in the triangle are equivalent (case equal) or one vertex in the triangle v, u_1, u_2 must belong to the solution, otherwise one of them would not be dominated (case adjacent); hence, $T(p) \leq 3T(p - 4)$. Finally, if t_1, t_2 and u_3 are distinct and non adjacent, set $\Gamma' = N(t_1) \cup N(t_2) \cup N(t_3) \setminus \{v, t_1, t_2, u_1, u_2, u_3\}$; either we take one vertex of the triangle, or we have to take t_1, t_2 and u_3 : $T(p) \leq 3T(p - 4) + T(p - 6 - |\Gamma'|)$. If $|\Gamma'| \geq 4$, $T(p) \leq 3T(p - 4) + T(p - 10)$ leading to $\lambda \leq 2^{0.417}$. If $|\Gamma'| = 3$, in this case u_3 has degree 4 and $\Gamma' \subset N(u_3)$. Then, we branch as follows: either we take u_3 and remove 9 vertices, or we take v and remove 4 vertices, or we mark u_3 and v . By removing the edge between them, they have respectively degree 3 and 2. Hence $T(p) \leq T(p - 9) + T(p - 4) + T(p - (2 - w))$.

- (b) Otherwise, if the only edge in $N(v)$ is (u_2, u_3) we get $T(p) \leq T(p-4) + T(p-4) + T(p-5) + T(p-6)$, but in the two last branches, thanks to Remark 2, either we remove one more vertex or we get two branches of weight reduced by 2 and 4. In the worst case, we get: $T(p) \leq 2T(p-4) + T(p-7) + T(p-9) + T(p-8) + T(p-10)$.
- (c) If there is an edge u_1, u_2 with $d(u_2) \geq 4$ (otherwise this is the triangle case) we get $T(p) \leq T(p-4) + T(p-4) + T(p-5) + T(p-6)$, but in the last branch, again thanks to Remark 2, we get in the worst case $T(p) \leq 2T(p-4) + T(p-5) + T(p-8) + T(p-10)$.
3. Finally, if there is no edge in $N(v)$: if two u_2 and u_3 have degree at least 4, then we get $T(p) \leq T(p-4) + T(p-4) + T(p-6) + T(p-6)$ (indeed in the last two branches u_1 is marked and of degree at most 2). If say u_2 has degree 3, then we get $T(p) \leq T(p-4) + T(p-4) + T(p-5) + T(p-6)$, but in the two last branches, thanks to Remark 2, either we remove one more vertex or we get two branches of weight reduced by 2 and 4. In the worst case, we get: $T(p) \leq 2T(p-4) + T(p-7) + T(p-9) + T(p-8) + T(p-10)$. ■

Lemma 7. *If there exists $v \in V$ such that $N(v) = \{u_1, u_2, u_3, u_4\}$, then in the worst case we get $T(p) \leq 4T(p-5) + T(p-9)$ with a contribution to the overall complexity factor bounded above by $1.3394 = 2^{0.422}$.*

Proof. We consider that u_4 has degree at least 5. If at least 3 u_i 's have degree at least 5, then $T(p) \leq 2T(p-5) + 3T(p-6)$. Now, we consider that u_1 and u_2 have degree 4.

Suppose first that the u_i 's of degree 4 are not adjacent. In the branch we take u_2 , p reduces by $6-w$ (since u_1 is marked and of degree at most 3). Then either u_3 has degree at least 5 and then $T(p) \leq 2T(p-5) + T(p-(6-w)) + 2T(p-6)$, or u_3 has degree 4 and in this case, in the branch we take u_3 , p reduces by $5+2(1-w)$: $T(p) \leq 2T(p-5) + T(p-(6-w)) + T(p-(7-2w)) + T(p-6)$.

If u_1 and u_2 are adjacent. When v, u_1 and u_2 are marked, they become of degree 2. Then:

- Either u_1 and u_2 are not adjacent to u_3 and we get $T(p) \leq 3T(p-5) + T(p-7) + T(p-6)$.
- Or u_1 is not adjacent to u_3 and u_2 is not adjacent to u_4 and we get $T(p) \leq 3T(p-5) + T(p-6) + T(p-7)$.
- Otherwise both u_1 and u_2 are adjacent to u_3 and not to u_4 . If u_3 has degree at least 5, $T(p) \leq 3T(p-5) + T(p-6) + T(p-7)$. Otherwise, u_4 is not adjacent to any of the u_i 's (if not, one would be equivalent to v). Then, we get 4 branches of weight $p-5$ and in the last branch the u_i 's are marked and are of degree at most 2, hence p reduces by 9: $T(p) \leq 4T(p-5) + T(p-9)$. ■

5 Approximation of min independent dominating set by moderately exponential algorithms

As we have mentioned in Section 1, for any $\varepsilon > 0$, MIN INDEPENDENT DOMINATING SET is inapproximable within ratio $O(n^{1-\varepsilon})$ unless $\mathbf{P} = \mathbf{NP}$. On the other hand, it is easy to see that any maximal independent set guarantees a ratio at most $\Delta + 1$. In this section, we devise algorithms achieving ratios much better than $O(n)$, i.e., “forbidden” in polynomial time, and with running times that, although exponential, are better than the running time of exact computation for MIN INDEPENDENT DOMINATING SET. Our results are based upon the following lemma by [1].

Lemma 8. *([1]) For any $k \geq 3$, it is possible to compute any independent dominating set (i.e., maximal independent set) of size at most n/k with running time $O^*(k^{n/k})$. This bound is tight.*

Proposition 1. For any $r \geq 3$, it is possible to compute an r -approximation of MIN INDEPENDENT DOMINATING SET with running time $O^*(2^{n \log_2 r/r})$.

Proof. We run the branching algorithm leading to Lemma 8. If it finds some minimum independent dominating set, our algorithm returns it; otherwise, $\text{opt}(G) > n/r$, where $\text{opt}(G)$ denotes the size of a minimum independent dominating set, and the algorithm returns some arbitrary independent dominating set. In the first case, the algorithm needs time $O^*(r^{n/r}) = O^*(2^{n \log_2 r/r})$ and computes an optimal solution; in the second case, any maximal independent set is an r -approximation and such a set is computed in polynomial time. ■

The following proposition further improves the result of Proposition 1.

Proposition 2. For any $r \geq 3$, it is possible to compute an approximation of MIN INDEPENDENT DOMINATING SET with running time $O^*(2^{n \log_2 r/r})$ and approximation ratio $r - ((r - 1)/r) \log_2 r$.

Proof. As previously, we first compute every independent dominating set of size n/r or less. If such sets exist, we return one of those with minimum size. Otherwise, we partition V into $l = r/\log_2 r$ subsets V_1, \dots, V_l , of size $n \log_2 r/r$, and we initialize S with some independent dominating set. Then, for $j \leq l$, we run the following procedure:

- for any $H \subset V_j$: if H is an independent set, compute an independent dominating set S_H in $G[V \setminus N[H]]$; if $|S| \geq |H \cup S_H|$, set $S = H \cup S_H$;
- return S .

Obviously, S is an independent dominating set. The algorithm has examined $l \times 2^{n/l}$ subsets, that concludes the running time claimed.

Fix some optimal solution S^* for MIN INDEPENDENT DOMINATING SET. Since S^* is maximally independent, $\cup_{i \leq l} N(V_i \cap S^*) = V \setminus S^*$ and we get (I.S. stands for independent set):

$$\begin{aligned} \frac{|S|}{\text{opt}(G)} &\leq \min_{j \leq l} \min_{H \text{ I.S. of } V_j} \left\{ \frac{n - |N(H)|}{\text{opt}(G)} \right\} \leq \min_{j \leq l} \left\{ \frac{n - |N(V_j \cap S^*)|}{\text{opt}(G)} \right\} \leq \frac{n - \frac{n - \text{opt}(G)}{l}}{\text{opt}(G)} \\ &\leq \frac{\log_2 r}{r} + r - \log_2 r \end{aligned}$$

that completes the proof of the proposition. ■

Ratio	2	3	4	5	10	20	50
Proposition 1		1.4423^n	1.4143^n	1.3798^n	1.2590^n	1.1616^n	1.0814^n
Proposition 2	1.4403^n	1.3870^n	1.3419^n	1.3077^n	1.2130^n	1.1398^n	1.0749^n

Table 1: Tradeoffs between running times and ratios derived by Propositions 1 and 2.

Tradeoffs between running times and ratios for some values of the ratios are displayed, for both propositions in Table 1. Recall that the exact algorithm given above runs in $O^*(1.3416^n)$.

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