

On rationality of the intersection points of a line with a plane quartic

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Abstract. We study the rationality of the intersection points of certain lines and smooth plane quartics C defined over \mathbb{F}_q . For $q \geq 127$, we prove the existence of a line such that the intersection points with C are all rational. Using another approach, we further prove the existence of a tangent line with the same property as soon as $q \geq 66^2 + 1$. Finally, we study the probability of the existence of a rational flex on C and exhibit a curious behavior when $\text{char } \mathbb{F}_q = 3$.

Key words: smooth plane quartics, rationality, intersection points, tangent line, flex.

Subject Classifications: 11G20, 14G05, 14G15, 14H45, 14N10.

1 Introduction

In computational arithmetic geometry, it is an important task to develop an efficient group law for the Jacobian variety of algebraic curves defined over a finite field. One of the most important and recent application of such efficient arithmetic comes from cryptography [17, 18, 4]. In [7], the authors introduced an efficient algorithm to perform arithmetic in the Jacobian of smooth plane quartics. The presented algorithm depends on the existence of a rational line l intersecting the quartic in rational points only. Moreover, the more special l is (for instance, tangent, tangent at a flex, ...), the better is the complexity of the algorithm. Motivated by the above efficiency argument, we prove here the following theorems.

Theorem 1. *Let C be a smooth plane quartic over the finite field \mathbb{F}_q with q elements. If $q \geq 127$, then there exists a line l which intersects C at rational points only.*

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Theorem 2. *Let C be a smooth plane quartic over \mathbb{F}_q . If $q \geq 66^2 + 1$, then there exists a tangent to C which intersects C at rational points only.*

In [7], the authors gave heuristic arguments and computational evidences that the probability for a plane smooth quartic over a finite field to have a rational flex is about 0.63. In this article, we present a ‘proof’ which depends on a conjectural analogue over finite fields of a result of [13] on the Galois group G of the 24 flexes of a general quartic. Harris proved that, over \mathbb{C} , the group G is as big as possible, namely the symmetric group S_{24} . Unfortunately, Harris’s proof uses monodromy arguments which cannot be adapted so easily in positive characteristic. Worse, we surprisingly found that Harris’s result is not valid over fields of characteristic 3 and we prove that for this field the Galois group is S_8 . This is a consequence of the peculiar fact that, in characteristic 3, a smooth plane quartic C has generically only 8 flexes (with multiplicity 3), which belong to a conic. We suspect that this is the only exceptional case.

The methods used for these three problems are various and can be generalized or adapted to other questions. This was our principal motivation to write down our approaches in the case of quartics. It illustrates also the very unusual behavior of special points and lines in small characteristics.

Coming back to our initial motivation, it appears nowadays unlikely (due to recent progress in index calculus [5]) that smooth plane quartics may be used for building discrete logarithm cryptosystems. However, it is interesting to mention that the complexity analysis of the index calculus attack of [5] uses an asymptotic bound for the number of lines intersecting a smooth plane quartic in four distinct rational points, in the spirit of Theorem 1.

The paper is organized as follows: Section 2 gives a brief overview on the possible geometric intersections of a line and a smooth plane quartic. In Section 3, we give a proof of Theorem 1 using Chebotarev density theorem for covers of curves. In Section 4, we prove Theorem 2 using the tangential correspondence and its associated curve $X_C \subset C \times C$. When $\text{char } \mathbb{F}_q \neq 2$, the crucial point is to prove that X_C is geometrically irreducible in order to apply Hasse-Weil bound for (possibly singular) geometrically irreducible curves. For the characteristic 2 case, X_C is not always absolutely irreducible. However, if it is not, we prove it is reducible (over \mathbb{F}_q) and we can apply Hasse-Weil bound on each component. Finally,

in Section 5, we also address the question of the probability of existence of a rational flex and show that in characteristic 3, the flexes are on a conic.

Conventions and notation. In the following, we denote by $(x : y : z)$ the coordinates in \mathbb{P}^2 , and by (x, y) the coordinates in \mathbb{A}^2 . Let p be a prime or 0 and $n \geq 1$ an integer. We use the letter K for an arbitrary field of characteristic p and let $k = \mathbb{F}_q$ be a finite field with $p \neq 0$ and $q = p^n$ elements. When C is a smooth geometrically irreducible projective curve, we denote by κ_C its canonical divisor. Operators such as Hom, End or Aut applied to varieties over a field K will always refer to K -rational homomorphisms and endomorphisms.

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2 Structure of the canonical divisor

In this section, we recall some geometric facts about special points and lines on a plane smooth quartic. Let K be an algebraically closed field of characteristic p and C be a smooth (projective) plane quartic defined over K . The curve C is a non hyperelliptic genus 3 curve which is canonically embedded. Hence the intersection of C with a line l are the positive canonical divisors of C . There are 5 possibilities for the intersection divisor of l and C denoted $(l \cdot C) = P_1 + P_2 + P_3 + P_4$:

- case 1 The four points are pairwise distinct. This is the generic position.
- case 2 $P_1 = P_2$, then l is tangent to C at P_1 .
- case 3 $P_1 = P_2 = P_3$. The point P_1 is then called a *flex*. As a linear intersection also represents the canonical divisor κ_C , these points are exactly the ones where a regular differential has a zero of order 3. The curve C has infinitely many flexes if and only if $p = 3$ and C is isomorphic to $x^4 + y^3z + yz^3 = 0$ which is also isomorphic to the Fermat quartic $x^4 + y^4 + z^4 = 0$ and to the Klein quartic $x^3y + y^3z + z^3x = 0$. This is a *funny curve* in the sense of [14, Ex.IV.2.4] or a *non classical* curve in the sense of [27, p.28]. On the contrary, if C has finitely many flexes, then these points are the Weierstrass points of C and the sum of their weights is 24.

- case 4 $P_1 = P_2$ and $P_3 = P_4$. The line l is called a *bitangent* of the curve C and the points P_i *bitangency points*. If $p \neq 2$, then C has exactly 28 bitangents (see for instance [22, Sec.3.3.1]). If $p = 2$, then C has respectively 7, 4, 2, or 1 bitangents, if the 2-rank of its Jacobian is respectively 3, 2, 1 or 0 [26]. Recall that the p -rank γ of an abelian variety A/K is defined by $\#A[p](K) = p^\gamma$.
- case 5 $P_1 = P_2 = P_3 = P_4$. The point P_1 is called a *hyperflex*. Generically, such a point does not exist. More precisely, the locus of quartics with at least one hyperflex is of codimension one in the moduli space \mathbf{M}_3 (see [28, Prop.4.9,p.29]). If $p = 3$ and C is isomorphic to the Fermat quartic then the number of hyperflexes of C is equal to 28. Otherwise, the weight of a hyperflex is greater than or equal to 2, so there are less than 12 of them [25]. Note that the weight of a hyperflex is exactly 2 when $p > 3$ or 0 and that the weight of a flex which is not a hyperflex is 1 if $p \neq 3$. See also [29] for precisions when $p = 2$ and Section 5 when $p = 3$.

A point P can even be more special. Let $P \in C$ be a point and let us denote $\phi_P : C \rightarrow |\kappa_C - P| = \mathbb{P}^1$ the degree three map induced by the linear system $|\kappa_C - P|$. If this cover is Galois, such a point P is called a (inner) *Galois point* and we denote $\text{Gal}(C)$ the set of Galois points of C .

Lemma 1 ([8–11]). *Let C be a smooth plane quartic defined over K . The number of Galois points is at most 4 if $p \neq 3$ and at most 28 if $p = 3$. Moreover, the above bounds are reached, respectively by the curve $yz^3 + x^4 + z^4 = 0$ and the Fermat quartic.*

In the sequel, we will need the first item of the following lemma.

Lemma 2. *Let C be a smooth plane quartic defined over K . There is always a bitangency point which is not a hyperflex unless*

- $p = 3$ and C is isomorphic to the Fermat quartic,
- $p = 2$ and $\text{Jac } C$ is supersingular,
- $p = 2$ and C is isomorphic to a 2-rank one quartic

$$(ax^2 + by^2 + cz^2 + dxy)^2 + xy(y^2 + xz) = 0$$

with $ac \neq 0$,

- $p = 2$ and C is isomorphic to a 2-rank two quartic

$$(ax^2 + by^2 + cz^2)^2 + xyz(y + z) = 0$$

with $abc \neq 0$ and $b + c \neq 0$.

Proof. According to Section 2 case 5, the number of hyperflexes when C is not isomorphic to the Fermat quartic and $p = 3$, is less than 12. On the other hand, if $p \neq 2$, a curve C has 28 bitangents, and thus there are at least $28 \cdot 2 - 12 \cdot 2 = 32$ bitangency points which are not hyperflexes. However, for the Fermat quartic in characteristic 3, all bitangency points are hyperflexes.

There remains to look at the case $p = 2$ for which we use the classification of [30], [21]. The quartic C falls into four categories according to its number of bitangents:

1. if C has only one bitangent, then C is isomorphic to a model of the form $Q^2 = x(y^3 + x^2z)$ where $Q = ax^2 + by^2 + cz^2 + dxy + eyz + fzx$ and $c \neq 0$. The unique bitangent $x = 0$ intersects C at points $(x : y : z)$ satisfying $by^2 + cz^2 + eyz = 0$. Therefore, C has a hyperflex if and only if $e = 0$, *i.e.* C falls into the subfamily \mathcal{S} [21, p.468] of curves whose Jacobian is supersingular. Conversely, any curve in \mathcal{S} has a hyperflex.
2. if C has two bitangents, then C is isomorphic to a model of the form $Q^2 = xy(y^2 + xz)$ where $Q = ax^2 + by^2 + cz^2 + dxy + eyz + fzx$ and $ac \neq 0$. All bitangency points are then hyperflexes if and only if $e = f = 0$.
3. if C has four bitangents, then C is isomorphic to $Q^2 = xyz(y + z)$ with

$$Q = ax^2 + by^2 + cz^2 + dxy + eyz + fzx \text{ and } abc \neq 0, b + c + e \neq 0.$$

All bitangency points are hyperflexes if and only if $d = e = f = 0$.

4. if C has seven bitangents, then C is isomorphic to $Q^2 = xyz(x + y + z)$ with $Q = ax^2 + by^2 + cz^2 + dxy + eyz + fzx$ and some open conditions on the coefficients [21, p.445]. The bitangents are

$$\{x, y, z, x + y + z, x + y, y + z, x + z\}.$$

Suppose that the intersection points of $x = 0$, $y = 0$ and $z = 0$ with C are hyperflexes, then $d = e = f = 0$. Moreover $x + y = 0$ gives a hyperflex if and only if $(a + b)y^2 + yz + cz^2$ is a perfect square, which is never possible. \square

3 Proof of Theorem 1

Let $q \geq 127$ be a prime power. Note that, as an easy consequence of Serre-Weil bound, we know that

$$\#C(k) \geq q + 1 - 3 \cdot \lfloor 2\sqrt{q} \rfloor = 62.$$

For the proof, we follow the same strategy as [5, p.604]. The lines intersecting C at P are in bijection with the divisors in the complete linear system $|\kappa_C - P|$. We wish to estimate the number of completely split divisors in this linear system, since such a divisor defines a line solution of Theorem 1. To get the existence of such a divisor, we will use an effective Chebotarev density theorem for function fields, as in [20, Th.1]. This theorem assumes that the cover is Galois but we can reduce to this case thanks to the following lemma.

Lemma 3. *Let K/F be a finite separable extension of function fields over a finite field. Let L be the Galois closure of K/F . A place of F splits completely in K if and only if it splits completely in L .*

Proof. It is clear that, if a place $P \in F$ splits completely in L , it splits completely in K . Conversely, let G be the Galois group of L/F and H be the Galois subgroup of L/K . By construction (see [3, A.V.p.54]), L is the compositum of the conjugates K^σ with $\sigma \in G/H$. If a place $P \in F$ splits completely in K , it splits completely in each of the K^σ . It is then enough to apply [24, Cor.III.8.4] to conclude. \square

We consider the separable geometric cover $\phi_P : C \rightarrow |\kappa_C - P| = \mathbb{P}^1$ of degree 3 induced by the linear system $|\kappa_C - P|$. We may assume that no rational point in \mathbb{P}^1 is ramified for ϕ_P . Otherwise, it is easy to see that the fiber of ϕ_P above this point has only rational points and the line defined by these points intersects the quartic in rational points only. Theorem [20, Th.1] boils down to the following proposition.

Proposition 1. *a) If the cover ϕ_P has a non-trivial automorphism, then the number N of completely split divisors in $|\kappa_C - P|$ satisfies*

$$\left| N - \frac{q+1}{3} \right| \leq 2\sqrt{q} + |D|$$

where $|D| = \sum_{y \in \mathbb{P}^1, \text{ramified}} \deg y$.

- b) *If the cover ϕ_P has a non-trivial \bar{k} -automorphism not defined over k , then there are no completely split divisors in $|\kappa_C - P|$.*
- c) *If the cover ϕ_P has no non-trivial \bar{k} -automorphism, then the number N of completely split divisors in $|\kappa_C - P|$ satisfies*

$$\left| N - \frac{q+1}{6} \right| \leq \sqrt{q} + |D|.$$

Proof (of Theorem 1). To avoid case (b) of Proposition 1, it is enough that P is not a Galois point (and we will avoid case (a) as well). By Lemma 1, we know that the number of such P is less than 28. So let the point $P \in C(k) \setminus (\text{Gal}(C) \cap C(k))$. Then the cover ϕ_P has a completely split divisor if

$$\frac{q+1}{6} > \sqrt{q} + |D|. \quad (1)$$

Using Riemann-Hurwitz formula we get

$$|D| \leq (2 \cdot 3 - 2) - 3 \cdot (0 - 2) = 10.$$

Hence the inequality (1) is satisfied as soon as $q \geq 127$. \square

Remark 1. We do not pretend that our lower bound 127 is optimal. In [2], the converse problem is considered (*i.e.* the existence of a plane (not necessarily smooth) curve with no line solution of Theorem 1) but their bound, 3, is also far from being optimal in the case of quartics. Indeed, by [16], for $q = 32$, there still exists a pointless smooth plane quartic for which of course there is no line satisfying Theorem 1.

4 Proof of Theorem 2

Let C be a smooth plane quartic defined over a field K . Let the map $T : C \rightarrow \text{Sym}^2(C)$ be the *tangential correspondence* which sends a point P of C to the divisor $(T_P(C) \cdot C) - 2P$. We associate to T its correspondence curve

$$X_C = \{(P, Q) \in C \times C : Q \in T(P)\}$$

which is defined over K . Our goal is to show that when $K = k = \mathbb{F}_q$ with $q > 66^2$, then X_C has a rational point, *i.e.* there is $(P, Q) \in C(\mathbb{F}_q)^2$ such that $(T_P(C) \cdot C) = 2P + Q + R$ for some point R , necessarily in $C(\mathbb{F}_q)$. Thus, the tangent $T_P(C)$ is a solution of Theorem 2.

4.1 Study of the geometric cover $X_C \rightarrow C$

Let $\pi_i : X_C \rightarrow C$, $i = 1, 2$, be the projections on the first and second factor. The morphism π_1 is a 2-cover between these two projective curves.

Lemma 4. *Let K be an algebraically closed field of characteristic p . The projection $\pi_1 : X_C \rightarrow C$ has the following properties:*

1. *The ramification points of π_1 are the bitangency points of C ,*

2. π_1 is separable,
3. The point $(P, Q) \in X_C$ such that P, Q are bitangency points and P is not a hyperflex (i.e. $P \neq Q$) is a regular point if and only if $p \neq 2$,
4. If $p \neq 2$, the only possible singular points of X_C are the points (P, P) where P is a hyperflex of C .

Proof. The first property is an immediate consequence of the definition of a bitangent.

If π_1 is not separable then $p = 2$ and π_1 is purely inseparable. Thus $\#\pi_1^{-1}(P) = 1$ for all $P \in C$, i.e. all P are bitangency points. This is impossible since the number of bitangents is finite (less than or equal to 7).

Let $F(x, y, z) = 0$ be an equation of C . Let $Q \neq P$ be a point of C defining a point (P, Q) in $X_C \setminus \Delta$ where Δ is the diagonal of $C \times C$. For such points, it is easy to write local equations as follows. We can suppose that $P = (0 : 0 : 1) = (0, 0)$, $Q = (1 : 0 : 1) = (1, 0)$ and assume that $f(x, y) = F(x, y, 1) = 0$ is an equation of the affine part of C . Then, if we consider the curve Y_C in $\mathbb{A}^4(x, y, z, t)$ defined by

$$\begin{cases} f(x, y) &= 0, \\ f(z, t) &= 0, \\ \frac{\partial f}{\partial x}(x, y)(z - x) + \frac{\partial f}{\partial y}(x, y)(t - y) &= 0, \end{cases}$$

$Y_C \setminus \Delta$ is an open subvariety of X_C containing (P, Q) . The Jacobian matrix at the point $(P, Q) = ((0, 0), (1, 0))$ is equal to

$$\begin{pmatrix} \frac{\partial f}{\partial x}(0, 0) & 0 & \frac{\partial^2 f}{\partial x^2}(0, 0) - \frac{\partial f}{\partial x}(0, 0) \\ \frac{\partial f}{\partial y}(0, 0) & 0 & \frac{\partial^2 f}{\partial x \partial y}(0, 0) - \frac{\partial f}{\partial y}(0, 0) \\ 0 & \frac{\partial f}{\partial x}(1, 0) & \frac{\partial f}{\partial x}(0, 0) \\ 0 & \frac{\partial f}{\partial y}(1, 0) & \frac{\partial f}{\partial y}(0, 0) \end{pmatrix}.$$

Now, if P and Q are bitangency points, then the tangent at these points is $y = 0$, so $\frac{\partial f}{\partial x}(0, 0) = \frac{\partial f}{\partial x}(1, 0) = 0$. The only non-trivially zero minor determinant of the matrix is then

$$\frac{\partial f}{\partial y}(1, 0) \cdot \frac{\partial f}{\partial y}(0, 0) \cdot \frac{\partial^2 f}{\partial x^2}(0, 0).$$

So $(P, Q) \in X_C$ is not singular if and only if $\frac{\partial^2 f}{\partial x^2}(0, 0) \neq 0$. This can never be the case if $p = 2$, so we now suppose that $p \neq 2$. We can always assume that the point $(0 : 1 : 0) \notin C$ and we write

$$f(x, y) = x^4 + x^3 h_1(y) + x^2 h_2(y) + x h_3(y) + h_4(y),$$

where h_i are polynomials (in one variable) over K of degree $\leq i$. Since $y = 0$ is a bitangent at P and Q , we have

$$f(x, 0) = x^2(x - 1)^2 = x^4 - 2x^3 + x^2,$$

and thus $h_2(0) = 1$. Now

$$\frac{\partial^2 f}{\partial x^2}(0, 0) = 2h_2(0) \neq 0.$$

Finally if $(P, Q) \in X_C$ is not ramified for π_1 , it is a smooth point. This proves the last assertion. \square

We want to apply the following version of Hasse-Weil bound to the curve X_C .

Proposition 2 ([1]). *Let X be a geometrically irreducible curve of arithmetic genus π_X defined over \mathbb{F}_q . Then*

$$|\#X(\mathbb{F}_q) - (q + 1)| \leq 2\pi_X \sqrt{q}.$$

In particular if $q \geq (2\pi_X)^2$ then X has a rational point.

Hence, to finish the proof, we need to show that X_C is geometrically irreducible and then to compute its arithmetic genus. For the first point, we use the following easy lemma for which we could not find a reference.

Lemma 5. *Let $\phi : X \rightarrow Y$ be a separable morphism of degree 2 between two projective curves defined over an algebraically closed field K such that*

1. *Y is smooth and irreducible,*
2. *there exists a point $P_0 \in Y$ such that ϕ is ramified at P_0 and $\phi^{-1}(P_0)$ is not singular.*

Then X is irreducible.

Proof. Let $s : \tilde{X} \rightarrow X$ be the normalization of X and $\tilde{\phi} = \phi \circ s : \tilde{X} \rightarrow Y$. Due to the second hypothesis, $\tilde{\phi} : \tilde{X} \rightarrow Y$ is a separable, ramified 2-cover. Clearly, X is geometrically irreducible if and only if \tilde{X} is.

Let us assume that \tilde{X} is not irreducible. There exist smooth projective curves \tilde{X}_1 and \tilde{X}_2 such that $\tilde{X} = \tilde{X}_1 \cup \tilde{X}_2$. Then, consider for $i = 1, 2$, $\tilde{\phi}_i = \tilde{\phi}|_{\tilde{X}_i} : \tilde{X}_i \rightarrow Y$. Each of these morphisms is of degree 1 and since the curves are projective and smooth, they define an isomorphism between \tilde{X}_i and Y .

Since $P_0 \in Y$ is a ramified point, $\tilde{\phi}_1^{-1}(P_0) = \tilde{\phi}_2^{-1}(P_0)$. It follows that

$$\tilde{\phi}^{-1}(P_0) \in \tilde{X}_1 \cap \tilde{X}_2$$

so that $\tilde{\phi}^{-1}(P_0)$ is singular, which contradicts the hypothesis. \square

4.2 Proof of Theorem 2: the case $p \neq 2$

Let us first start with C \bar{k} -isomorphic to the Fermat quartic in characteristic 3. By Section 2 case 3, all its points are flexes. So if there exists $P \in C(k)$, then the tangent at P cuts C at P and at another unique point which is again rational over k . Now when $q > 23$ and $q \neq 29, 32$, it is proved in [16] that a genus 3 non-hyperelliptic curve over \mathbb{F}_q has always a rational point and the result follows.

We suppose that C is a smooth plane quartic not \bar{k} -isomorphic to the Fermat quartic if $p = 3$. As we assumed that $p \neq 2$, by Lemma 5, Lemma 2 and Lemma 4, we conclude that X_C is an geometrically irreducible projective curve. Moreover, if we assume that C has no hyperflex, then X_C is smooth and we can compute its genus g_{X_C} using Riemann-Hurwitz formula for the 2-cover $\pi_1 : X_C \rightarrow C$ ramified over the $2 \cdot 28$ bitangency points. In fact,

$$2g_{X_C} - 2 = 2(2 \cdot 3 - 2) + 56,$$

and thus $g_{X_C} = 33$. The family of curves X_C is flat over the locus of smooth plane quartics C by [6, Prop.II.32]. As the arithmetic genus π_{X_C} is constant in flat families [14, Cor.III.9.10] and equal to g_{X_C} for smooth X_C [14, Prop.IV.1.1], we get that $\pi_{X_C} = 33$ for any curve C . We can now use Proposition 2 to get the bound $q > (2 \cdot 33)^2$.

4.3 Proof of Theorem 2: the case $p = 2$

We first need to compute the arithmetic genus of X_C . It is not so easy in this case, as wild ramification occurs. We therefore suggest another point of view, which can actually be used in any characteristic.

Lemma 6. *The arithmetic genus of X_C is 33.*

Proof. In order to emphasize how general the method is, we will denote by $g = 3$ the genus of C and by $d = 4$ its degree. The map T is a correspondence with *valence* $\nu = 2$, *i.e.* the linear equivalence class of $T(P) + \nu P$ is independent of P . Let denote by E (resp. F) a fiber of π_1 (resp. π_2) and Δ the diagonal of $C \times C$. We get, as in the proof of [12, p.285], that X_C is linearly equivalent to $aE + bF - \nu\Delta$ for some $a, b \in \mathbb{Z}$ to be determined. Then, one computes the arithmetic genus π_{X_C} of X_C thanks to the adjunction formula [14, Ex.V.1.3.a]

$$2\pi_{X_C} - 2 = X_C \cdot (X_C + \kappa_{C \times C})$$

where $\kappa_{C \times C}$ is the canonical divisor on $C \times C$. Using that (see for instance [12, p.288], [14, Ex.V.1.6])

$$\begin{cases} E.E = F.F & = 0, \\ E.F = \Delta.E = \Delta.F & = 1, \\ \kappa_{C \times C} & \equiv_{\text{num}} (2g - 2)E + (2g - 2)F, \\ \Delta^2 & = (2 - 2g), \end{cases}$$

we find

$$\pi_{X_C} = ab - 15.$$

Now, we determine the values of a and b . One has

$$X_C.E = b - \nu = \deg \pi_1 = d - 2 = 2$$

so $b = 4$. Also, $X_C.F = a - \nu = \deg \pi_2$. The degree of π_2 is equal to the degree of the dual map $C \rightarrow C^*$, where C^* is the dual curve, which is $d(d - 1)$. In the case $d = 4$, we then get $a = 12$. Plugging the values of a and b , we find that $\pi_{X_C} = 33$. \square

Remark 2. In a previous version of this article, we claimed that $a = 6$ when $p = 2$. This was due to a confusion between the degree of the dual curve and the degree of the dual map, which is in this case inseparable.

We can now come back to the proof of Theorem 2.

If X_C is absolutely irreducible, then we can argue as in Section 4.2 and get the bound $q > (2 \cdot 33)^2$.

If X_C is not absolutely irreducible, since $\deg \pi_1 = 2$, X_C is the union over \mathbb{F}_{q^2} of two absolutely irreducible curves X_1, X_2 such that X_1 and X_2 are birationally equivalent to C .

Since C is smooth, there are two \mathbb{F}_{q^2} -rational morphisms $s_i : C \rightarrow X_i$ such that $\pi_1|_{X_i} \circ s_i = 1$. Hence $\deg s_i = 1$ and the X_i are geometrically isomorphic to C and smooth. If the X_i are defined over \mathbb{F}_q they have a rational point because $q > 32$ and $g_{X_i} = 3$. This concludes the proof in this case.

It is then sufficient to prove that the X_i are defined over \mathbb{F}_q . Otherwise, we would have $X_2 = X_1^\sigma$, where σ is the Frobenius automorphism of $\text{Gal}(\mathbb{F}_{q^2}/\mathbb{F}_q)$. Let us consider the morphisms $t_i = \pi_2|_{X_i} \circ s_i : C \rightarrow C$ which map a point $P \in C$ to one of the two points in the support of $T(P)$. Since $g_C > 1$, these morphisms have to be purely inseparable. Let P be the generic point on C and $Q_1, Q_2 = Q_1^\sigma$ the two points in the support of $T(P)$. Since $\deg \pi_2 = 10 = \deg \pi_2|_{X_1} + \deg \pi_2|_{X_2}$ and π_2 is

purely inseparable, we have, say, $\deg \pi_2|_{X_1} = 2$ and $\deg \pi_2|_{X_1} = 8$. We then get the following commutative diagram

$$\begin{array}{ccc} C & \xrightarrow{s_1} & X_1 \xrightarrow{\pi_2|_{X_1}} C \\ & \searrow s_2 & \downarrow F_q \downarrow F_q \\ & & X_2 \xrightarrow{\pi_2|_{X_2}} C \end{array} \quad \begin{array}{ccccc} P & \xrightarrow{1} & (P, Q_1) & \xrightarrow{2} & Q_1 \\ & \searrow 1 & \downarrow q & & \downarrow q \\ & & (P, Q_2) & \xrightarrow{8} & Q_2 \end{array}$$

where the labelling indicates the degree of the maps and F_q is the \mathbb{F}_q -Frobenius morphism. We hence get a contradiction.

Remark 3. Using the first line of the diagram, we see that if X_C is not absolutely irreducible, then there is a purely inseparable degree 2 morphism from C to C . This implies that there is a geometric isomorphism $\gamma : C^{(2)} \rightarrow C$, where $C^{(2)}$ is the conjugate of C by the Frobenius automorphism of $\text{Gal}(\mathbb{F}_q/\mathbb{F}_2)$. Hence, the moduli point corresponding to C is defined over \mathbb{F}_2 . As the field of moduli is a field of definition over finite fields, C descends over \mathbb{F}_2 , *i.e.* there exists a curve \tilde{C}/\mathbb{F}_2 and a geometric isomorphism $\psi : \tilde{C} \rightarrow C$. More specifically, we have the following commutative diagram

$$\begin{array}{ccccc} & & C & \xrightarrow{t_1} & C \\ & \nearrow \psi & \searrow F_2 & \nearrow \gamma & \searrow \psi^{-1} \\ \tilde{C} & & C^{(2)} & & \tilde{C} \\ & & \searrow (\psi^{-1})^{(2)} & & \end{array}$$

Hence the map $\psi^{-1} \circ t_1 \circ \psi : \tilde{C} \rightarrow \tilde{C}$ is the \mathbb{F}_2 -Frobenius morphism F_2 . Therefore, the Frobenius morphism maps a point $P \in \tilde{C}$ to a point on the tangent at P . Such a (plane) curve \tilde{C} is called Frobenius non-classical. Using [15, Th.1], we see that \tilde{C} has no rational point. It is then easy to run a program on plane smooth quartics over \mathbb{F}_2 with no rational points and to check if X_C is irreducible or not. See for instance

<http://iml.univ-mrs.fr/~ritzenth/programme/tangent-char2.mag>.

Surprisingly, we find that X_C is reducible if and only if C/\mathbb{F}_2 is isomorphic to

$$x^4 + x^2y^2 + x^2yz + x^2z^2 + xy^2z + xyz^2 + y^4 + y^2z^2 + z^4$$

or

$$x^4 + x^3z + xy^3 + xyz^2 + y^4 + yz^3 + z^4.$$

These two curves are twists of the Klein quartic. Hence, the curves C/\mathbb{F}_q such that X_C is reducible are the ones isomorphic to the Klein quartic.

5 The case of flexes

The heuristic results and computations of [7] tend to suggest that a random plane smooth quartic over \mathbb{F}_q has an asymptotic probability of about 0.63 to have at least one rational flex when q tends to infinity. We describe how to turn the heuristic strategy into a proof.

Let \mathbb{P}^{14} be the linear system of all plane quartic curves over a field K and $I_0 = \{(P, l), P \in l\} \subset \mathbb{P}^2 \times \mathbb{P}^{2*}$. Let $I_4 \subset \mathbb{P}^{14} \times I_0$ be the locus

$$I_4 = \{(C, (P, l)), C \text{ is smooth and } P \text{ is a flex of } C \text{ with tangent line } l\}.$$

Harris proved the following result using monodromy arguments.

Theorem 3 ([13, p.698]). *The Galois group of the cover $I_4 \rightarrow \mathbb{P}^{14}$ over \mathbb{C} is the full symmetric group S_{24} .*

Let us assume for a moment that this result is still valid over finite fields. Then, using a general Chebotarev density theorem for function fields like in [23, Th.7], this would mean that the probability of finding a rational flex is within $O(1/\sqrt{q})$ of the probability that a random permutation of 24 letters has a fixed point, which is

$$p_{24} := 1 - \frac{1}{2!} + \frac{1}{3!} - \dots - \frac{1}{24!} \approx 1 - \exp(-1) \approx 0.63.$$

Unfortunately, it is not easy to transpose Harris's proof over any field. And actually, Harris's result is not true in characteristic 3, as the following proposition implies.

Proposition 3. *Let $C : f(x_1, x_2, x_3) = 0$ be a smooth plane quartic defined over an algebraically closed field K of characteristic 3. The flexes of C are the intersection points of C with a certain curve $H_C : h_C = 0$ of degree less than or equal to 2. The curve H_C can be degenerate as in the case of the Fermat quartic where $h_C = 0$.*

Proof. We use the method to compute flexes of a plane curve of degree d described in the appendix of [7] (see also [25, Th.0.1]). Indeed, when the characteristic of the field divides $2(d-1)$, one cannot use the usual

Hessian and one should proceed as follows.

Let $C : f = 0$ be the generic plane quartic over K

$$\begin{aligned} f(x, y, z) := & a_{00}y^4 + y^3(a_{10}x + a_{01}z) + y^2(a_{20}x^2 + a_{11}xz + a_{02}z^2) \\ & + y(a_{30}x^3 + a_{21}x^2z + a_{12}xz^2 + a_{03}z^3) \\ & + (a_{40}x^4 + a_{31}x^3z + a_{22}x^2z^2 + a_{13}xz^3 + a_{04}z^4), \end{aligned}$$

We define as in [7]

$$\begin{aligned} 2\bar{h} &= 2f_1f_2f_{12} - f_1^2f_{22} - f_2^2f_{11}, \\ &= f_1(f_2f_{12} - f_1f_{22}) + f_2(f_1f_{12} - f_2f_{11}) \end{aligned}$$

where f_i or f_{ij} are the partial derivatives with respect to i th variable (or to i th and j th variables). It is then easy to check via a computer algebra system, see

<http://iml.univ-mrs.fr/~ritzenth/programme/flex-char3.mw>,

that

$$2\bar{h} - a_{20}f^2 - f \cdot (ax^3 + by^3 + cz^3)z = \tilde{h}_C \cdot z^2,$$

with

$$\begin{aligned} a &:= a_{40}a_{11} + a_{21}a_{30} - 2a_{20}a_{31}, \\ b &:= a_{10}a_{11} + a_{00}a_{21} - 2a_{20}a_{01}, \\ c &:= 2a_{12}^2 + a_{13}a_{11} + a_{20}a_{04} + a_{03}a_{21} + a_{02}a_{22}, \end{aligned}$$

where \tilde{h}_C is a homogeneous polynomial in $K[x, y, z]$ of degree 6, for which nonzero coefficients appear only for the monomials $x^6, y^6, z^6, x^3y^3, x^3z^3$ and y^3z^3 . Since the map $u \mapsto u^3$ is an isomorphism of K , there is a polynomial $h_C \in K[x, y, z]$ satisfying $\tilde{h}_C = h_C^3$. If we suppose that there is no flex at infinity, the flexes are the intersection points of $\bar{h} = 0$ and $f = 0$, so they are also the intersection points of $h_C = 0$ and $f = 0$ and $h_C = 0$ is the equation of a (possibly degenerate) conic H_C . \square

Remark 4. As $\tilde{h}_C = h_C^3$, we see that the weight of a flex which is not a hyperflex is 3 in characteristic 3.

Corollary 1. *The Galois group of the cover $I_4 \rightarrow \mathbb{P}^{14}$ over $\bar{\mathbb{F}}_3$ is the full symmetric group S_8 .*

Proof. Note that the Galois group G of the cover is the Galois group of the x -coordinate of the 8 flexes of the general quartic. Hence, G is included in S_8 . To show that G is exactly S_8 , we are going to specialize the general quartic to smooth quartics over finite fields with 8 distinct flexes having different arithmetic patterns. More precisely, to generate S_8 we need to produce (see [19, Lem.4.27]):

- a smooth quartic with 8 Galois conjugate flexes over \mathbb{F}_3 :

$$2x^4 + 2x^3y + x^3z + 2x^2z^2 + xy^3 + 2xy^2z + y^3z + yz^3 = 0;$$

- a smooth quartic with one rational flex and 7 Galois conjugate flexes over \mathbb{F}_3 :

$$x^3y + x^2z^2 + 2xy^3 + xy^2z + 2xyz^2 + 2xz^3 + y^4 + 2yz^3 = 0;$$

- a smooth quartic with two quadratic conjugate flexes and 6 rational flexes over \mathbb{F}_9 :

$$\begin{aligned} & a^6x^4 + ax^3y + a^7x^3z + a^6x^2y^2 + a^2x^2z^2 + a^7xy^3 + a^7xy^2z \\ & + xyz^2 + a^5xz^3 + a^5y^4 + a^3y^3z + a^5y^2z^2 + 2yz^3 + a^7z^4 = 0, \end{aligned}$$

where $a^2 - a - 1 = 0$. □

Corollary 2. *Let C be a smooth plane quartic over \mathbb{F}_{3^n} . The probability that C has a rational flex tends to*

$$p_8 := 1 - \frac{1}{2!} + \frac{1}{3!} - \dots - \frac{1}{8!} \approx 0.63$$

when n tends to infinity.

Remark 5. The fact that the Galois group G in characteristic 3 is S_8 and not S_{24} was unnoticed in our computations in [7] because $|p_{24} - p_8| \leq 10^{-5}$.

General reduction arguments show that the Galois group G remains S_{24} almost all p . We conjecture that $p = 3$ is the only exceptional case.

Conjecture 1. The Galois group of the cover $I_4 \rightarrow \mathbb{P}^{14}$ over $\bar{\mathbb{F}}_p$ is the full symmetric group S_{24} if $p \neq 3$ and S_8 otherwise.

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