

# Boxicity and Poset Dimension

Abhijin Adiga<sup>1</sup>, Diptendu Bhowmick<sup>1</sup>, L. Sunil Chandran<sup>1</sup>

Department of Computer Science and Automation, Indian Institute of Science, Bangalore-560012, India.  
 emails: abhijin@csa.iisc.ernet.in, diptendubhowmick@gmail.com, sunil@csa.iisc.ernet.in

**Abstract.** Let  $G$  be a simple, undirected, finite graph with vertex set  $V(G)$  and edge set  $E(G)$ . A  $k$ -dimensional box is a Cartesian product of closed intervals  $[a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_k, b_k]$ . The *boxicity* of  $G$ ,  $\text{box}(G)$  is the minimum integer  $k$  such that  $G$  can be represented as the intersection graph of  $k$ -dimensional boxes, i.e. each vertex is mapped to a  $k$ -dimensional box and two vertices are adjacent in  $G$  if and only if their corresponding boxes intersect. Let  $\mathcal{P} = (S, P)$  be a poset where  $S$  is the ground set and  $P$  is a reflexive, anti-symmetric and transitive binary relation on  $S$ . The dimension of  $\mathcal{P}$ ,  $\text{dim}(\mathcal{P})$  is the minimum integer  $t$  such that  $P$  can be expressed as the intersection of  $t$  total orders.

Let  $G_{\mathcal{P}}$  be the *underlying comparability graph* of  $\mathcal{P}$ , i.e.  $S$  is the vertex set and two vertices are adjacent if and only if they are comparable in  $\mathcal{P}$ . It is a well-known fact that posets with the same underlying comparability graph have the same dimension. The first result of this paper links the dimension of a poset to the boxicity of its underlying comparability graph. In particular, we show that for any poset  $\mathcal{P}$ ,  $\text{box}(G_{\mathcal{P}})/(\chi(G_{\mathcal{P}}) - 1) \leq \text{dim}(\mathcal{P}) \leq 2\text{box}(G_{\mathcal{P}})$ , where  $\chi(G_{\mathcal{P}})$  is the chromatic number of  $G_{\mathcal{P}}$  and  $\chi(G_{\mathcal{P}}) \neq 1$ . It immediately follows that if  $\mathcal{P}$  is a height-2 poset, then  $\text{box}(G_{\mathcal{P}}) \leq \text{dim}(\mathcal{P}) \leq 2\text{box}(G_{\mathcal{P}})$  since the underlying comparability graph of a height-2 poset is a bipartite graph.

The second result of the paper relates the boxicity of a graph  $G$  with a natural partial order associated with the *extended double cover* of  $G$ , denoted as  $G_c$ : Note that  $G_c$  is a bipartite graph with partite sets  $A$  and  $B$  which are copies of  $V(G)$  such that corresponding to every  $u \in V(G)$ , there are two vertices  $u_A \in A$  and  $u_B \in B$  and  $\{u_A, v_B\}$  is an edge in  $G_c$  if and only if either  $u = v$  or  $u$  is adjacent to  $v$  in  $G$ . Let  $\mathcal{P}_c$  be the natural height-2 poset associated with  $G_c$  by making  $A$  the set of minimal elements and  $B$  the set of maximal elements. We show that  $\frac{\text{box}(G)}{2} \leq \text{dim}(\mathcal{P}_c) \leq 2\text{box}(G) + 4$ .

These results have some immediate and significant consequences. The upper bound  $\text{dim}(\mathcal{P}) \leq 2\text{box}(G_{\mathcal{P}})$  allows us to derive hitherto unknown upper bounds for poset dimension such as  $\text{dim}(\mathcal{P}) \leq 2 \text{tree-width}(G_{\mathcal{P}}) + 4$ , since boxicity of any graph is known to be at most its tree-width + 2. In the other direction, using the already known bounds for partial order dimension we get the following: (1) The boxicity of any graph with maximum degree  $\Delta$  is  $O(\Delta \log^2 \Delta)$  which is an improvement over the best known upper bound of  $\Delta^2 + 2$ . (2) There exist graphs with boxicity  $\Omega(\Delta \log \Delta)$ . This disproves a conjecture that the boxicity of a graph is  $O(\Delta)$ . (3) There exists no polynomial-time algorithm to approximate the boxicity of a bipartite graph on  $n$  vertices with a factor of  $O(n^{0.5-\epsilon})$  for any  $\epsilon > 0$ , unless  $NP = ZPP$ .

**Keywords:** Boxicity, partial order, poset dimension, comparability graph, extended double cover.

## 1 Introduction

### 1.1 Boxicity

A  $k$ -box is a Cartesian product of closed intervals  $[a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_k, b_k]$ . A  $k$ -box representation of a graph  $G$  is a mapping of the vertices of  $G$  to  $k$ -boxes in the  $k$ -dimensional Euclidean space such that two vertices

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in  $G$  are adjacent if and only if their corresponding  $k$ -boxes have a non-empty intersection. The *boxicity* of a graph denoted  $\text{box}(G)$ , is the minimum integer  $k$  such that  $G$  has a  $k$ -box representation. Boxicity was introduced by Roberts [24]. Cozzens [9] showed that computing the boxicity of a graph is NP-hard. This was later strengthened by Yannakakis [31] and finally by Kratochvíl [22] who showed that determining whether boxicity of a graph is at most two itself is NP-complete.

It is easy to see that a graph has boxicity at most 1 if and only if it is an *interval graph*, i.e. each vertex of the graph can be associated with a closed interval on the real line such that two intervals intersect if and only if the corresponding vertices are adjacent. By definition, boxicity of a complete graph is 0. Let  $G$  be any graph and  $G_i$ ,  $1 \leq i \leq k$  be graphs on the same vertex set as  $G$  such that  $E(G) = E(G_1) \cap E(G_2) \cap \dots \cap E(G_k)$ . Then we say that  $G$  is the *intersection graph* of  $G_i$  s for  $1 \leq i \leq k$  and denote it as  $G = \bigcap_{i=1}^k G_i$ . Boxicity can be stated in terms of intersection of interval graphs as follows:

**Lemma 1.** Roberts [24]: *The boxicity of a non-complete graph  $G$  is the minimum positive integer  $b$  such that  $G$  can be represented as the intersection of  $b$  interval graphs. Moreover, if  $G = \bigcap_{i=1}^m G_i$  for some graphs  $G_i$  then  $\text{box}(G) \leq \sum_{i=1}^m \text{box}(G_i)$ .*

Roberts, in his seminal work [24] proved that the boxicity of a complete  $k$ -partite graph is  $k$ . Chandran and Sivadasan [6] showed that  $\text{box}(G) \leq \text{tree-width}(G) + 2$ . Chandran, Francis and Sivadasan [5] proved that  $\text{box}(G) \leq \chi(G^2)$  where,  $\chi(G^2)$  is the chromatic number of  $G^2$ . In [14] Esperet proved that  $\text{box}(G) \leq \Delta^2(G) + 2$ , where  $\Delta(G)$  is the maximum degree of  $G$ . Scheinerman [25] showed that the boxicity of outer planar graphs is at most 2. Thomassen [26] proved that the boxicity of planar graphs is at most 3. In [11], Cozzens and Roberts studied the boxicity of split graphs.

## 1.2 Poset Dimension

A *partially ordered set* or *poset*  $\mathcal{P} = (S, P)$  consists of a non empty set  $S$ , called the *ground set* and a reflexive, antisymmetric and transitive binary relation  $P$  on  $S$ . A *total order* is a partial order in which every two elements are comparable. It essentially corresponds to a permutation of elements of  $S$ . A *height-2 poset* is one in which every element is either a minimal element or a maximal element. A *linear extension*  $L$  of a partial order  $P$  is a total order which satisfies  $(x \leq y \text{ in } P \Rightarrow x \leq y \text{ in } L)$ . A *realizer* of a poset  $\mathcal{P} = (S, P)$  is a set of linear extensions of  $P$ , say  $\mathcal{R}$  which satisfy the following condition: for any two distinct elements  $x$  and  $y$ ,  $x < y$  in  $P$  if and only if  $x < y$  in  $L$ ,  $\forall L \in \mathcal{R}$ . The *poset dimension* of  $\mathcal{P}$  (sometimes abbreviated as dimension of  $\mathcal{P}$ ) denoted by  $\text{dim}(\mathcal{P})$  is the minimum integer  $k$  such that there exists a realizer of  $P$  of cardinality  $k$ . Poset dimension was introduced by Dushnik and Miller [12]. Clearly, a poset is one-dimensional if and only if it is a total order. Pnueli et al. [23] gave a polynomial time algorithm to recognize dimension 2 posets. In [31] Yannakakis showed that it is NP-complete to decide whether the dimension of a poset is at most 3. For more references and survey on dimension theory of posets see Trotter [28,29]. Recently, Hegde and Jain [21] showed that it is hard to design an approximation algorithm for computing the dimension of a poset.

A simple undirected graph  $G$  is a comparability graph if and only if there exists some poset  $\mathcal{P} = (S, P)$ , such that  $S$  is the vertex set of  $G$  and two vertices are adjacent in  $G$  if and only if they are comparable in  $\mathcal{P}$ . We will call such a poset an *associated poset* of  $G$ . Likewise, we refer to  $G$  as the *underlying comparability graph* of  $\mathcal{P}$ . Note that for a height-2 poset, the underlying comparability graph is a bipartite graph with partite sets  $A$  and  $B$ , with say  $A$  corresponding to minimal elements and  $B$  to maximal elements. For more on comparability graphs see [19]. It is easy to see that there is a unique comparability graph associated with a poset, whereas, there can be several posets with the same underlying comparability graph. However, Trotter, Moore and Sumner [30] proved that posets with the same underlying comparability graph have the same dimension.

## 2 Our Main Results

The results of this paper are the consequence of our attempts to bring out some connections between boxicity and poset dimension. As early as 1982, Yannakakis had some intuition regarding a possible connection between these problems when he established the NP-completeness of both poset dimension and boxicity in [31]. But interestingly, no results were discovered in the last 25 years which establish links between these two notions. Perhaps the researchers were misled by some deceptive examples such as the following one: Consider a complete graph  $K_n$  where  $n$  is even and remove a perfect matching from it. The resulting graph is a comparability graph and the dimension of any of its associated posets is 2, while its boxicity is  $n/2$ . In this context it may be worth recalling a result from [16] which relates the poset dimension to another parameter namely the dimension of box orders. A poset  $\mathcal{P} = (S, P)$  is said to be a box order in  $m$  dimensions if there exists a mapping of its elements to  $m$ -dimensional axis-parallel boxes such that  $x < y$  in  $P$  if and only if the box of  $y$  strictly contains the box of  $x$ . Box order is a particular type of geometrical containment order (See [16,28]). The result is as follows: the dimension of  $\mathcal{P}$  is at most  $2m$  if and only if it is a box order in  $m$  dimensions [18,20]. But note that boxicity is fundamentally different from box orders. As in the case of the above example, we can demonstrate families of posets of constant dimension whose underlying comparability graphs have arbitrarily high boxicity, which is in contrast with the above result on box orders.

First we state an upper bound and a lower bound for the dimension of a poset in terms of the boxicity of its underlying comparability graph.

**Theorem 1.** *Let  $\mathcal{P} = (V, P)$  be a poset such that  $\dim(\mathcal{P}) > 1$  and  $G_{\mathcal{P}}$  its underlying comparability graph. Then,  $\dim(\mathcal{P}) \leq 2\text{box}(G_{\mathcal{P}})$ .*

**Theorem 2.** *Let  $\mathcal{P} = (V, P)$  be a poset and let  $\chi(G_{\mathcal{P}})$  be the chromatic number of its underlying comparability graph  $G_{\mathcal{P}}$  such that  $\chi(G_{\mathcal{P}}) > 1$ . Then,  $\dim(\mathcal{P}) \geq \frac{\text{box}(G_{\mathcal{P}})}{\chi(G_{\mathcal{P}})-1}$ .*

Note that if  $\mathcal{P}$  is a height-2 poset, then  $G_{\mathcal{P}}$  is a bipartite graph and therefore  $\chi(G_{\mathcal{P}}) = 2$ . Thus, from the above results we have the following:

**Corollary 1.** *Let  $\mathcal{P} = (V, P)$  be a height-2 poset and  $G_{\mathcal{P}}$  its underlying comparability graph. Then,  $\text{box}(G_{\mathcal{P}}) \leq \dim(\mathcal{P}) \leq 2\text{box}(G_{\mathcal{P}})$ .*

The *double cover* and *extended double cover* of a graph are popular notions in graph theory. They provide a natural way to associate a bipartite graph to the given graph. In this paper we make use of the latter construction.

**Definition 1.** *The extended double cover of  $G$ , denoted as  $G_c$  is a bipartite graph with partite sets  $A$  and  $B$  which are copies of  $V(G)$  such that corresponding to every  $u \in V(G)$ , there are two vertices  $u_A \in A$  and  $u_B \in B$  and  $\{u_A, v_B\}$  is an edge in  $G_c$  if and only if either  $u = v$  or  $u$  is adjacent to  $v$  in  $G$ .*

We prove the following lemma relating the boxicity of  $G$  and  $G_c$ .

**Lemma 2.** *Let  $G$  be any graph and  $G_c$  its extended double cover. Then,*

$$\frac{\text{box}(G)}{2} \leq \text{box}(G_c) \leq \text{box}(G) + 2.$$

Let  $\mathcal{P}_c$  be the natural height-2 poset associated with  $G_c$ , i.e. the elements in  $A$  are the minimal elements and the elements in  $B$  are the maximal elements. Combining Corollary 1 and Lemma 2 we have the following theorem:

**Theorem 3.** *Let  $G$  be a graph and  $\mathcal{P}_c$  be the natural height-2 poset associated with its extended double cover. Then,  $\frac{\dim(\mathcal{P}_c)}{2} - 2 \leq \text{box}(G) \leq 2 \dim(\mathcal{P}_c)$  and therefore  $\text{box}(G) = \Theta(\dim(\mathcal{P}_c))$ .*

## 2.1 Consequences

**New upper bounds for poset dimension:** Our results lead to some hitherto unknown bounds for poset dimension. Some general bounds obtained in this manner are listed below:

1. It is proved in [6] that for any graph  $G$ , boxicity of  $G$  is at most  $\text{tree-width}(G) + 2$ . For more information on *tree-width* see [2]. Applying this bound in Theorem 1 it immediately follows that, for a poset  $\mathcal{P}$ ,  $\dim(\mathcal{P}) \leq 2 \text{tree-width}(G_{\mathcal{P}}) + 4$ .
2. The *threshold dimension* of a graph  $G$  is the minimum number of *threshold graphs* such that  $G$  is the edge union of these graphs. For more on threshold graphs and threshold dimension see [19]. Cozzens and Halsey [10] proved that  $\text{box}(G) \leq \text{threshold-dimension}(\overline{G})$ , where  $\overline{G}$  is the complement of  $G$ . From this it follows that  $\dim(\mathcal{P}) \leq 2 \text{threshold-dimension}(\overline{G_{\mathcal{P}}})$ .
3. In [3] it is proved that  $\text{box}(G) \leq \left\lfloor \frac{\text{MVC}(G)}{2} \right\rfloor + 1$ , where  $\text{MVC}(G)$  is the cardinality of the *minimum vertex cover* of  $G$ . Therefore, we have  $\dim(\mathcal{P}) \leq \text{MVC}(G_{\mathcal{P}}) + 2$ .

Some more interesting results can be obtained if we restrict  $G_{\mathcal{P}}$  to belong to certain subclasses of graphs. Note that there are several research papers in the partial order literature which study the dimension of posets whose underlying comparability graph has some special structure – interval order, semi order and crown posets are some examples.

4. Scheinerman [25] proved that the boxicity of outer planar graphs is at most 2 and later Thomassen [26] proved that the boxicity of planar graphs is at most 3. Therefore, it follows that  $\dim(\mathcal{P}) \leq 4$  if  $G_{\mathcal{P}}$  is outer planar and  $\dim(\mathcal{P}) \leq 6$  if  $G_{\mathcal{P}}$  is planar.
5. Bhowmick and Chandran [1] proved that boxicity of AT-free graphs is at most  $\chi(G_{\mathcal{P}})$ . Hence,  $\dim(\mathcal{P}) \leq 2\chi(G_{\mathcal{P}})$ , if  $G_{\mathcal{P}}$  is AT-free.
6. If  $G_{\mathcal{P}}$  is an interval graph, then, we get from Theorem 1,  $\dim(\mathcal{P}) \leq 2$ , since  $\text{box}(G_{\mathcal{P}}) = 1$ . However, observing that interval graphs are co-comparability graphs this result would follow also as a consequence of a result by Dushnik and Miller [12]:  $\dim(\mathcal{P}) \leq 2$  if and only if  $G_{\mathcal{P}}$  is a co-comparability graph.
7. The boxicity of a  $d$ -dimensional hypercube is  $O(d/\log(d))$  [7]. Therefore, if  $G_{\mathcal{P}}$  is a height-2 poset which corresponds to a  $d$ -dimensional hypercube, then from Corollary 1 we have  $\dim(\mathcal{P}) = O(d/\log(d))$ .
8. Chandran et al. [4] recently proved that chordal bipartite graphs have arbitrarily high boxicity. From Corollary 1 it follows that height-2 posets whose underlying comparability graph are chordal bipartite graphs can have arbitrarily high dimension.

**Improved upper bound for boxicity based on maximum degree:** Füredi and Kahn [17] proved the following

**Lemma 3.** *Let  $\mathcal{P}$  be a poset and  $\Delta$  be the maximum degree of  $G_{\mathcal{P}}$ . Then, there exists a constant  $c$  such that  $\dim(\mathcal{P}) < c\Delta(\log \Delta)^2$ .*

From Lemma 2 and Corollary 1 we have  $\text{box}(G) \leq 2\text{box}(G_c) \leq 2\dim(\mathcal{P}_c)$ , where  $G_c$  is the extended double cover of  $G$ . Note that by construction  $\Delta(G_c) = \Delta(G) + 1$ . On applying the above lemma, we have

**Theorem 4.** *For any graph  $G$  having maximum degree  $\Delta$  there exists a constant  $c'$  such that  $\text{box}(G) < c'\Delta(\log \Delta)^2$ .*

This result is an improvement over the previous upper bound of  $\Delta^2 + 2$  by Esperet [14].

**Counter examples to the conjecture of [5]:** Chandran et al. [5] conjectured that boxicity of a graph is  $O(\Delta)$ . We use a result by Erdős, Kierstead and Trotter [13] to show that there exist graphs with boxicity  $\Omega(\Delta \log \Delta)$ , hence disproving the conjecture. Let  $\mathbb{P}(n, p)$  be the probability space of height-2 posets with  $n$  minimal elements forming set  $A$  and  $n$  maximal elements forming set  $B$ , where for any  $a \in A$  and  $b \in B$ ,  $\text{Prob}(a < b) = p(n) = p$ . They proved the following:

**Theorem 5.** [13] *For every  $\epsilon > 0$ , there exists  $\delta > 0$  so that if  $(\log^{1+\epsilon} n)/n < p < 1 - n^{-1+\epsilon}$ , then,  $\dim(\mathcal{P}) > (\delta p n \log(pn))/(1 + \delta p \log(pn))$  for almost all  $\mathcal{P} \in \mathbb{P}(n, p)$ .*

When  $p = 1/\log n$ , for almost all posets  $\mathcal{P} \in \mathbb{P}(n, 1/\log n)$ ,  $\Delta(G_{\mathcal{P}}) < \delta_1 n / \log n$  and by the above theorem  $\dim(\mathcal{P}) > \delta_2 n$ , where  $\delta_1$  and  $\delta_2$  are some positive constants (see [29] for a discussion on the above theorem). From Theorem 1, it immediately implies that for almost all  $\mathcal{P} \in \mathbb{P}(n, 1/\log n)$ ,  $\text{box}(G_{\mathcal{P}}) \geq \frac{\dim(\mathcal{P})}{2} > \delta' \Delta(G_{\mathcal{P}}) \log \Delta(G_{\mathcal{P}})$  for some positive constant  $\delta'$ , hence proving the existence of graphs with boxicity  $\Omega(\Delta \log \Delta)$ .

**Approximation hardness for the boxicity of bipartite graphs:** Hegde and Jain [21] proved the following

**Theorem 6.** *There exists no polynomial-time algorithm to approximate the dimension of an  $n$ -element poset within a factor of  $O(n^{0.5-\epsilon})$  for any  $\epsilon > 0$ , unless  $NP = ZPP$ .*

This is achieved by reducing the *fractional chromatic number problem* on graphs to the poset dimension problem. In addition they observed that a slight modification of their reduction will imply the same result for even height-2 posets. From Corollary 1, it is clear that for any height-2 poset  $\mathcal{P}$ ,  $\dim(\mathcal{P}) = \Theta(\text{box}(G_{\mathcal{P}}))$ . Suppose there exists an algorithm to compute the boxicity of bipartite graphs with approximation factor  $O(n^{0.5-\epsilon})$ , for some  $\epsilon > 0$ , then, it is clear that the same algorithm can be used to compute the dimension of height-2 posets with approximation factor  $O(n^{0.5-\epsilon})$ , a contradiction. Hence,

**Theorem 7.** *There exists no polynomial-time algorithm to approximate the boxicity of a bipartite graph on  $n$ -vertices with a factor of  $O(n^{0.5-\epsilon})$  for any  $\epsilon > 0$ , unless  $NP = ZPP$ .*

### 3 Notations

Let  $[n]$  denote  $\{1, 2, \dots, n\}$  where  $n$  is a positive integer. For any graph  $G$ , let  $V(G)$  and  $E(G)$  denote its vertex set and edge set respectively. If  $G$  is undirected, for any  $u, v \in V(G)$ ,  $\{u, v\} \in E(G)$  means  $u$  is adjacent to  $v$  and if  $G$  is directed,  $(u, v) \in E(G)$  means there is a directed edge from  $u$  to  $v$ . Whenever we refer to an  $AB$  bipartite (or co-bipartite) graph, we imply that its vertex set is partitioned into non-empty sets  $A$  and  $B$  where both these sets induce independent sets (cliques respectively).

In a poset  $\mathcal{P} = (S, P)$ , the notations  $aPb$ ,  $a \leq b$  in  $P$  and  $(a, b) \in P$  are equivalent and are used interchangeably.  $G_{\mathcal{P}}$  denotes the underlying comparability graph of  $\mathcal{P}$ . A subset of  $\mathcal{P}$  is called a *chain* if each pair of distinct elements is comparable. If each pair of distinct elements is incomparable, then it is called an *antichain*. Given an  $AB$  bipartite graph  $G$ , the natural poset associated with  $G$  with respect to the bipartition is the poset obtained by taking  $A$  to be the set of minimal elements and  $B$  to be the set of maximal elements. In particular, if  $G_c$  is the extended double cover of  $G$ , we denote by  $\mathcal{P}_c$  the natural associated poset of  $G_c$ .

Suppose  $I$  is an interval graph. Let  $f_I$  be an *interval representation* for  $I$ , i.e. it is a mapping from the vertex set to closed intervals on the real line such that for any two vertices  $u$  and  $v$ ,  $\{u, v\} \in E(I)$  if and only if  $f_I(u) \cap f_I(v) \neq \emptyset$ . Let  $l(u, f_I)$  and  $r(u, f_I)$  denote the left and right end points of the interval corresponding to the vertex  $u$  respectively. In this paper, we will never consider more than one interval representation for an interval

graph. Therefore, we will simplify the notations to  $l(u, I)$  and  $r(u, I)$ . Further, when there is no ambiguity about the graph under consideration and its interval representation, we simply denote the left and right end points as  $l(u)$  and  $r(u)$  respectively. Note that for any interval graph there exists an interval representation with all end points distinct. Such a representation is called a *distinguishing* interval representation. It is an easy exercise to derive such a distinguishing interval representation starting from an arbitrary interval representation of the graph.

## 4 Proof of Theorem 1

Let  $\text{box}(G_{\mathcal{P}}) = k$ . Note that since  $\dim(\mathcal{P}) > 1$ ,  $G_{\mathcal{P}}$  cannot be a complete graph and therefore  $k \geq 1$ . Let  $\mathcal{I} = \{I_1, I_2, \dots, I_k\}$  be a set of interval graphs such that  $G_{\mathcal{P}} = \bigcap_{i=1}^k I_i$ . Now, corresponding to each  $I_i$  we will construct two total orders  $L_i^1$  and  $L_i^2$  such that  $\mathcal{R} = \{L_i^j | i \in [k] \text{ and } j \in [2]\}$  is a realizer of  $\mathcal{P}$ .

Let  $I \in \mathcal{I}$  and  $f_I$  be an interval representation of  $I$ . We will define two partial orders  $P_I$  and  $\overline{P}_I$  as follows:  $\forall a \in V$ ,  $(a, a)$  belongs to  $P_I$  and  $\overline{P}_I$  and for every non-adjacent pair of vertices  $a, b \in V$  with respect to  $I$ ,

$$\left. \begin{array}{l} (a, b) \in P_I \\ (b, a) \in \overline{P}_I \end{array} \right\} \text{ if and only if } r(a, f_I) < l(b, f_I).$$

Partial orders constructed in the above manner from a collection of closed intervals are called *interval orders* (See [29] for more details). It is easy to see that  $\overline{I}$  (the complement of  $I$ ) is the underlying comparability graph of both  $P_I$  and  $\overline{P}_I$ .

Let  $G_1$  and  $G_2$  be two directed graphs with vertex set  $V$  and edge set  $E(G_1) = (P \cup P_I) \setminus \{(a, a) | a \in V\}$  and  $E(G_2) = (P \cup \overline{P}_I) \setminus \{(a, a) | a \in V\}$  respectively. Note that from the definition it is obvious that there are no directed loops in  $G_1$  and  $G_2$ .

**Lemma 4.**  *$G_1$  and  $G_2$  are acyclic directed graphs.*

*Proof.* We will prove the lemma for  $G_1$  – a similar proof holds for  $G_2$ . First of all, since  $G_{\mathcal{P}}$  is not a complete graph  $P_I \neq \emptyset$ . Suppose  $P_I$  is a total order, i.e. if  $P$  is an antichain, then it is clear that  $E(G_1) = P_I$  and therefore  $G_1$  is acyclic. Henceforth, we will assume that  $P_I$  is not a total order.

Suppose  $G_1$  is not acyclic. Let  $C = \{(a_0, a_1), (a_1, a_2), \dots, (a_{t-2}, a_{t-1}), (a_{t-1}, a_1)\}$  be a shortest directed cycle in  $G_1$ .

First we will show that  $t > 2$  ( $t$  is the length of  $C$ ). If  $t = 2$ , then there should be  $a, b \in V$  such that  $(a, b), (b, a) \in E(G_1)$ . Since  $P$  is a partial order,  $(a, b)$  and  $(b, a)$  cannot be simultaneously present in  $P$ . The same holds for  $P_I$ . Thus, without loss of generality we can assume that  $(a, b) \in P$  and  $(b, a) \in P_I$ . But if  $(a, b) \in P$ , then,  $a$  and  $b$  are adjacent in  $G_{\mathcal{P}}$  and thus adjacent in  $I$ . Then clearly the intervals of  $a$  and  $b$  intersect and therefore  $(b, a) \notin P_I$ , a contradiction.

Now, we claim that two consecutive edges in  $C$  cannot belong to  $P$  (or  $P_I$ ). Suppose there do exist such edges, say  $(a_i, a_{i+1})$  and  $(a_{i+1}, a_{i+2})$  which belong to  $P$  (or  $P_I$ ) (note that the addition is modulo  $t$ ). Since  $P$  (or  $P_I$ ) is a partial order, it implies that  $(a_i, a_{i+2}) \in P$  (or  $P_I$ ) and as a result we have a directed cycle of length  $t - 1$ , a contradiction to the assumption that  $C$  is a shortest directed cycle. Therefore, the edges of  $C$  alternate between  $P$  and  $P_I$ . It also follows that  $t \geq 4$ .

Without loss of generality we will assume that  $(a_1, a_2), (a_3, a_4) \in P_I$ . We claim that  $\{(a_1, a_2), (a_3, a_4)\}$  is an induced poset of  $P_I$ . First of all  $a_2$  and  $a_3$  are not comparable in  $P_I$  as they are comparable in  $P$ . If either  $\{a_1, a_3\}$  or  $\{a_2, a_4\}$  are comparable, then we can demonstrate a shorter directed cycle in  $G_1$ , a contradiction. Finally we consider the pair  $\{a_1, a_4\}$ . If  $t = 4$ , then they are not comparable as they are comparable in  $P$  while if  $t \neq 4$  and if they are comparable, then, it would again imply a shorter directed cycle, a contradiction. Hence,

$\{(a_1, a_2), (a_3, a_4)\}$  is an induced subposet. In the literature such a poset is denoted as  $\mathbf{2} + \mathbf{2}$  where  $+$  refers to *disjoint sum* and  $\mathbf{2}$  is a two-element total order. Fishburn [15] has proved that a poset is an interval order if and only if it does not contain a  $\mathbf{2} + \mathbf{2}$ . This implies that  $P_I$  is not an interval order, a contradiction.

We have therefore proved that there cannot be any directed cycles in  $G_1$ . In a similar way we can show that  $G_2$  is an acyclic directed graph.  $\square$

Since  $G_1$  and  $G_2$  are acyclic, we can construct total orders, say  $L^1$  and  $L^2$  using *topological sort* on  $G_1$  and  $G_2$  such that  $P \cup P_I \subseteq L^1$  and  $P \cup \overline{P}_I \subseteq L^2$  (For more details on topological sort, see [8] for example).

For each  $I_i$ , we create linear extensions  $L_i^1$  and  $L_i^2$  as described above. We claim that  $\mathcal{R} = \{L_i^j | i \in [k], j \in [2]\}$  is a realizer of  $\mathcal{P}$ . For each  $L_i^j$ , it is clear from construction that  $P \subseteq L_i^j$ . If  $a$  and  $b$  are not comparable in  $P$ , then  $\{a, b\} \notin E(G_{\mathcal{P}})$ , and therefore there exists some interval graph  $I_q \in \mathcal{I}$  such that  $\{a, b\} \notin E(I_q)$ . Assuming that the interval for  $a$  occurs before the interval for  $b$  in the interval representation of  $I_q$ , it follows by construction that  $(a, b) \in P_{I_q}$  and  $(b, a) \in \overline{P}_{I_q}$  and therefore  $(a, b) \in L_q^1$  and  $(b, a) \in L_q^2$ . Hence proved.

#### 4.1 Tight Example for Theorem 1

Consider the *crown* poset  $S_n^0$ : a height-2 poset with  $n$  minimal elements  $a_1, a_2, \dots, a_n$  and  $n$  maximal elements  $b_1, b_2, \dots, b_n$  and  $a_i < b_j$ , for  $j = i+1, i+2, \dots, i-1$ , where the addition is modulo  $n$ . Its underlying comparability graph is the bipartite graph obtained by removing a perfect matching from the complete bipartite graph  $K_{n,n}$ . The dimension of this poset is  $n$  (see [27,29]) while the boxicity of the graph is  $\lceil \frac{n}{2} \rceil$  [3].

## 5 Proof of Theorem 2

We will prove that  $\text{box}(G_{\mathcal{P}}) \leq (\chi(G_{\mathcal{P}}) - 1) \dim(\mathcal{P})$ . Let  $(\chi(G_{\mathcal{P}}) - 1) = p$ ,  $\dim(\mathcal{P}) = k$  and  $\mathcal{R} = \{L_1, \dots, L_k\}$  a realizer of  $\mathcal{P}$ . Now we color the vertices of  $G_{\mathcal{P}}$  as follows: For a vertex  $v$ , if  $\gamma$  is the length of a longest chain in  $\mathcal{P}$  such that  $v$  is its maximum element, then we assign color  $\gamma$  to it. This is clearly a proper coloring scheme since if two vertices  $x$  and  $y$  are assigned the same color, say  $\gamma$  and  $x < y$ , then it implies that the length of a longest chain in which  $y$  occurs as the maximum element is at least  $\gamma + 1$ , a contradiction. Also, this is a minimum coloring because the maximum number that gets assigned to any vertex equals the length of a longest chain in  $\mathcal{P}$ , which corresponds to the clique number of  $G_{\mathcal{P}}$ .

Now we construct  $pk$  interval graphs  $\mathcal{I} = \{I_{ij} | i \in [p], j \in [k]\}$  and show that  $G_{\mathcal{P}}$  is an intersection graph of these interval graphs. Let  $\Pi_j$  be the *permutation induced* by the total order  $L_j$  on  $[n]$ , i.e.  $xL_jy$  if and only if  $\Pi_j^{-1}(x) < \Pi_j^{-1}(y)$ . The following construction applies to all graphs in  $\mathcal{I}$  except  $I_{pk}$ . Let  $I_{ij} \in \mathcal{I} \setminus \{I_{pk}\}$ . We assign the point interval  $[\Pi_j^{-1}(v), \Pi_j^{-1}(v)]$  for all vertices  $v$  colored  $i$ . For all vertices  $v$  colored  $i' < i$ , we assign  $[\Pi_j^{-1}(v), n + 1]$  and for those colored  $i' > i$ , we assign  $[0, \Pi_j^{-1}(v)]$ . The interval assignment for the last interval graph  $I_{pk}$  is as follows: for all vertices  $v$  colored  $p + 1 = \chi(G_{\mathcal{P}})$  we assign the point interval  $[\Pi_k^{-1}(v), \Pi_k^{-1}(v)]$  and for the rest of the vertices we assign the interval  $[\Pi_k^{-1}(v), n + 1]$ . Next, we will show that  $G_{\mathcal{P}} = \bigcap_{I \in \mathcal{I}} I$ .

**Claim 1.** If  $u$  and  $v$  are adjacent in  $G_{\mathcal{P}}$ , then they are adjacent in all  $I \in \mathcal{I}$ .

*Proof.* Let  $u$  be colored  $i$  and  $v$  be colored  $i'$ . It is clear that  $i \neq i'$  and without loss of generality we will assume that  $i < i'$ . By the way we have colored, it implies that  $u < v$  in  $P$  and therefore  $\Pi_j^{-1}(u) < \Pi_j^{-1}(v)$ ,  $\forall j \in [k]$ . Let  $I_{hj}$ ,  $h \in [p]$  and  $j \in [k]$  be the interval graph under consideration. There are 5 possible cases which we consider one by one:

*Case 1: ( $h < i, i'$ )* By construction in  $I_{hj}$ ,  $u$  and  $v$  are assigned intervals  $[0, \Pi_j^{-1}(u)]$  and  $[0, \Pi_j^{-1}(v)]$  respectively and therefore  $u$  and  $v$  are adjacent in  $I_{hj}$ ,  $\forall j \in [k]$ .

*Case 2: ( $i, i' < h$ )*  $u$  and  $v$  are assigned intervals  $[\Pi_j^{-1}(u), n+1]$  and  $[\Pi_j^{-1}(v), n+1]$  respectively and therefore are adjacent in  $I_{hj}$ ,  $\forall j \in [k]$ .

*Case 3: ( $i < h < i'$ )*  $u$  is assigned interval  $[\Pi_j^{-1}(u), n+1]$  and  $v$  is assigned interval  $[0, \Pi_j^{-1}(v)]$ . Since  $0 < \Pi_j^{-1}(u) < \Pi_j^{-1}(v) < n+1$ , it follows that  $u$  is adjacent to  $v$  in  $I_{hj}$ ,  $\forall j \in [k]$ .

*Case 4: ( $h = i$ )* If  $h = p$  and  $j = k$ , then  $i' = p+1$  and therefore  $u$  is assigned  $[\Pi_k^{-1}(u), n+1]$  and  $v$  is assigned  $[\Pi_k^{-1}(v), \Pi_k^{-1}(v)]$ . If not, then  $u$  is assigned the point interval  $[\Pi_j^{-1}(u), \Pi_j^{-1}(u)]$  and  $v$  is assigned  $[0, \Pi_j^{-1}(v)]$ . In either case, since  $\Pi_j^{-1}(u) < \Pi_j^{-1}(v)$ , the two vertices are adjacent.

*Case 5: ( $h = i'$ )* Since  $h \leq p = \chi(G_{\mathcal{P}}) - 1$ , it implies that  $i, i' \leq p$ . Therefore, if  $h = p$  and  $j = k$ , then  $u$  and  $v$  are assigned  $[\Pi_j^{-1}(u), n+1]$  and  $[\Pi_j^{-1}(v), n+1]$  respectively. If not, then  $v$  is assigned the point interval  $[\Pi_j^{-1}(v), \Pi_j^{-1}(v)]$  and  $u$  is assigned  $[\Pi_j^{-1}(u), n+1]$ . Again, since  $\Pi_j^{-1}(u) < \Pi_j^{-1}(v)$ , in either case the two vertices are adjacent. Hence proved.  $\blacksquare$

**Claim 2.** If  $u$  and  $v$  are not adjacent in  $G_{\mathcal{P}}$ , then there exists some  $I \in \mathcal{I}$  such that  $\{u, v\} \notin E(I)$ .

*Proof.* Again let  $u$  be colored  $i$  and  $v$  be colored  $i'$ . Recall that  $k \geq 2$ . If  $i = i'$ , then by construction it is clear that  $u$  and  $v$  are not adjacent in  $I_{i1}$  if  $i \neq p+1$  and when  $i = p+1$ , then they are not adjacent in  $I_{pk}$ . Therefore, without loss of generality we will assume that  $i < i'$ . Since  $u$  and  $v$  are not adjacent in  $G_{\mathcal{P}}$ , they are incomparable in  $P$  and therefore, there exists some  $l \in [k]$  such that  $u > v$  in  $L_l$  which in turn implies that  $\Pi_l^{-1}(u) > \Pi_l^{-1}(v)$ . There are 2 possible cases:

*Case 1: ( $i < p$ )* Since  $i < i'$ , in  $I_{il}$ ,  $u$  and  $v$  are assigned intervals  $[\Pi_l^{-1}(u), \Pi_l^{-1}(u)]$  and  $[0, \Pi_l^{-1}(v)]$  respectively and therefore, since  $\Pi_l^{-1}(u) > \Pi_l^{-1}(v)$   $u$  and  $v$  are not adjacent in  $I_{il}$ .

*Case 2: ( $i = p$ )* Clearly  $i' = p+1$ . If  $l < k$ , then it is similar to the previous case. If  $l = k$ , then, in  $I_{pk}$ ,  $u$  and  $v$  are assigned  $[\Pi_k^{-1}(u), n+1]$  and  $[\Pi_k^{-1}(v), \Pi_k^{-1}(v)]$  respectively. Since  $\Pi_l^{-1}(u) > \Pi_l^{-1}(v)$ ,  $u$  and  $v$  are not adjacent in  $I_{pk}$ .  $\blacksquare$

Hence we have proved Theorem 2.

Consider a complete  $k$ -partite graph  $G$  on  $n = qk$  vertices where  $q, k > 1$ , i.e.  $V(G) = V_1 \uplus V_2 \uplus \dots \uplus V_k$  is a partition of  $V(G)$  where  $|V_i| = q$ . For any two vertices  $x \in V_i$  and  $y \in V_j$ ,  $\{x, y\} \in E(G)$  if and only if  $i \neq j$ .  $G$  is a comparability graph and here is one transitive orientation of  $G$ : for every pair of adjacent vertices  $u \in V_i$  and  $v \in V_j$ , where  $u, v \in [k]$  and  $i \neq j$ , make  $u < v$  if and only if  $i < j$ . Let  $\mathcal{P}$  be the resulting poset. It is an easy exercise to show that  $\dim(\mathcal{P}) = 2$ . The chromatic number of  $G$  is  $k$  and Roberts [24] showed that its boxicity is  $k$ . From Theorem 2 it follows that  $\dim(\mathcal{P}) \geq \frac{k}{k-1}$ . Therefore, the complete  $k$ -partite graph serves as a tight example for Theorem 2.

However, it would be interesting to see if there are posets of higher dimension for which Theorem 2 is tight.

## 6 Boxicity of the extended double cover

In this section, we will prove Lemma 2. But first, we will need some definitions and lemmas.

**Definition 2.** Let  $H$  be an  $AB$  bipartite graph. The associated co-bipartite graph of  $H$ , denoted by  $H^*$  is the graph obtained by making the sets  $A$  and  $B$  cliques, but keeping the set of edges between vertices of  $A$  and  $B$  identical to that of  $H$ , i.e.  $\forall u \in A, v \in B, \{u, v\} \in E(H^*)$  if and only if  $\{u, v\} \in E(H)$ .

The associated co-bipartite graph  $H^*$  is not to be confused with the complement of  $H$  (i.e.  $\overline{H}$ ) which is also a co-bipartite graph.

**Definition 3.** (Canonical interval representation of a co-bipartite interval graph:) Let  $I$  be an  $AB$  co-bipartite interval graph. A canonical interval representation of  $I$  satisfies:  $\forall u \in A, l(u) = l$  and  $\forall u \in B, r(u) = r$ , where the points  $l$  and  $r$  are the leftmost and rightmost points respectively of the interval representation.

We claim that such a representation exists for every  $AB$  co-bipartite interval graph. Note that if  $I$  is a complete graph, the claim is trivially true. Therefore we take  $I$  to be non-complete. Consider any interval representation of  $I$ . Since  $A$  is a clique there exists a point, say  $l$  which is contained in all intervals corresponding to vertices in  $A$ . Similarly, let  $r$  be a point in the intersection of intervals corresponding to vertices of  $B$ . Since  $I$  is non-complete, it is clear that  $l \neq r$ . By definition of  $l$  and  $r$  we have  $l(u) \leq l \leq r(u), \forall u \in A$  and  $l(u) \leq r \leq r(u), \forall u \in B$ . Without loss of generality we can assume that  $l < r$  and as a result  $r(u) \geq l$  and  $l(u) \leq r$  for all vertices  $u$ . This means no interval ends before the point  $l$  and no interval starts after the point  $r$ . Hence, it follows that for any interval containing  $l$ , we can make  $l$  its left end point and for an interval containing  $r$ , we can make  $r$  its right end point. Therefore, we have a canonical interval representation of  $I$ .

The following lemma is easy to verify.

**Lemma 5.** Consider two closed intervals on the real line with left end points  $l_1, l_2$  and right end points  $r_1, r_2$ . Then, the two intervals intersect if and only if  $l_1 \leq r_2$  and  $l_2 \leq r_1$ . In other words, the two intervals do not intersect if and only if  $r_1 < l_2$  or  $r_2 < l_1$ .

**Lemma 6.** Let  $H$  be an  $AB$  bipartite graph and  $H^*$  its associated co-bipartite graph. If  $H^*$  is a non-interval graph, then

$$\frac{\text{box}(H^*)}{2} \leq \text{box}(H) \leq \text{box}(H^*).$$

If  $H^*$  is an interval graph, then  $\text{box}(H) \leq 2$ .

*Proof.* We first show that  $\text{box}(H) \leq \text{box}(H^*)$ . Let  $\text{box}(H^*) = k \geq 2$  and  $H^* = I_1 \cap I_2 \cap \dots \cap I_k$ , where  $I_i$  are interval graphs. Note that since  $I_i$  is a supergraph of a co-bipartite graph, it is a co-bipartite interval graph. Let us consider a canonical interval representation for each  $I_i$  and further assume that the right end points of all vertices in  $A$  and left end points of all vertices in  $B$  are distinct. Let  $I'_1$  be the interval graph obtained by making  $r(u, I'_1) = l(u, I'_1) = l(u, I_1) \forall u \in B$  and keeping the rest of the intervals unchanged. Similarly, let  $I'_2$  be the interval graph obtained by making  $l(u, I'_2) = r(u, I'_2) = r(u, I_2) \forall u \in A$ . Due to our assumption of distinct end points it is clear that  $A$  and  $B$  are independent sets in  $I'_1$  and  $I'_2$  respectively. Suppose  $u \in A$  and  $v \in B$ . For  $i \in [2]$ :

$$\begin{aligned} \{u, v\} \in E(I'_i) &\iff r(u, I'_i) \geq l(v, I'_i) \\ \text{(by construction of } I' \text{ from } I) &\iff r(u, I_i) \geq l(v, I_i) \\ &\iff \{u, v\} \in E(I_i) \end{aligned}$$

From this, we immediately see that  $H = I'_1 \cap I'_2 \cap I_3 \cap \dots \cap I_k$ .

Now suppose  $\text{box}(H^*) = 1$ , i.e.  $H^*$  is an interval graph. Then we set  $I_1 = I_2 = H^*$  and proceed as in the previous case. Hence,  $\text{box}(H) \leq 2$ . Note that this inequality is tight: take for example  $H = C_4$ , the cycle of length 4.  $H^*$  is  $K_4$  and therefore an interval graph, but  $C_4$  is not.

Now we show that  $\text{box}(H^*) \leq 2\text{box}(H)$ . Let  $\text{box}(H) = l$  and  $H = I_1 \cap I_2 \cap \dots \cap I_l$ , where  $I_i$  are interval graphs. For each  $I_i$ , we create two interval graphs  $I'_{2i-1}$  and  $I'_{2i}$  as follows: Consider an interval representation of  $I_i$ . Let  $l_i = \min_{u \in V} l(u, I_i)$  and  $r_i = \max_{u \in V} r(u, I_i)$ , the leftmost and rightmost points in the interval representation respectively.  $I'_{2i-1}$  and  $I'_{2i}$  are defined as follows:

$$\begin{aligned} l(u, I'_{2i-1}) &= l_i \text{ and } r(u, I'_{2i-1}) = r(u, I_i), \quad \forall u \in A, \\ r(u, I'_{2i-1}) &= r_i \text{ and } l(u, I'_{2i-1}) = l(u, I_i), \quad \forall u \in B, \\ l(u, I'_{2i}) &= l_i \text{ and } r(u, I'_{2i}) = r(u, I_i), \quad \forall u \in B, \\ r(u, I'_{2i}) &= r_i \text{ and } l(u, I'_{2i}) = l(u, I_i), \quad \forall u \in A. \end{aligned}$$

Now we show that  $H^* = \bigcap_{i=1}^{2l} I'_i$ . From the definitions it is clear that in each  $I'_i$ ,  $A$  and  $B$  are cliques— for example, the interval corresponding to every vertex in  $A$  in  $I'_{2i-1}$  contains  $l_i$ . Therefore we will assume that  $u \in A$  and  $v \in B$ .

$$\begin{aligned} \{u, v\} \in E(H^*) &\implies \{u, v\} \in E(H) \\ &\implies \{u, v\} \in E(I_i), \quad \forall i = 1, 2, \dots, l \\ \text{(From Lemma 5)} &\implies l(u, I_i) \leq r(v, I_i) \text{ and } l(v, I_i) \leq r(u, I_i) \end{aligned}$$

In  $I'_{2i-1}$ ,  $l(u, I'_{2i-1}) = l_i \leq r_i \leq r(v, I'_{2i-1})$  and  $l(v, I'_{2i-1}) = l(v, I_i) \leq r(u, I_i) = r(u, I'_{2i-1})$  and in  $I'_{2i}$ ,  $l(v, I'_{2i}) = l_i \leq r_i \leq r(u, I'_{2i})$  and  $l(u, I'_{2i}) = l(u, I_i) \leq r(v, I_i) = r(v, I'_{2i})$ . Therefore  $u$  and  $v$  are adjacent in both  $I'_{2i-1}$  and  $I'_{2i}$ . Now suppose

$$\begin{aligned} \{u, v\} \notin E(H^*) &\implies \{u, v\} \notin E(H) \\ &\implies \exists I_j \text{ such that } \{u, v\} \notin E(I_j) \end{aligned}$$

In the interval representation of  $I_j$ , if  $r(u, I_j) < l(v, I_j)$ , then, by definition  $r(u, I'_{2j-1}) < l(v, I'_{2j-1})$  and hence,  $\{u, v\} \notin E(I'_{2j-1})$ . If  $r(v, I_j) < l(u, I_j)$ , then,  $r(v, I'_{2j}) < l(u, I'_{2j})$  and therefore,  $\{u, v\} \notin E(I'_{2j})$ . Hence proved.  $\square$

## 6.1 Proof of Lemma 2

$\text{box}(G_c) \leq \text{box}(G) + 2$ : Let  $\text{box}(G) = k$  and  $G = I_1 \cap I_2 \cap \dots \cap I_k$  where  $I_i$ s are interval graphs. For each  $I_i$ , we construct interval graphs  $I'_i$  with vertex set  $V(G_c)$  as follows: Consider an interval representation for  $I_i$ . For every vertex  $u$  in  $I_i$ , we assign the interval of  $u$  to  $u_A$  and  $u_B$  in  $I'_i$ . Let  $I'_{k+1}$  and  $I'_{k+2}$  be interval graphs where (1) all vertices in  $A$  are adjacent to all the vertices in  $B$  (2) In  $I'_{k+1}$   $A$  induces a clique and  $B$  induces an independent set while in  $I'_{k+2}$  it is the other way round. Now we show that  $G_c = I'_1 \cap I'_2 \cap \dots \cap I'_{k+2}$ . It is very easy to see that  $\{u_A, u_B\} \in E(I'_i) \quad \forall i \in [k+2]$ . Suppose  $u$  and  $v$  are distinct vertices in  $G$ .

$$\begin{aligned} \{u_A, v_B\} \in E(G_c) &\implies \{u, v\} \in E(G) \\ &\implies \{u, v\} \in E(I_i), i \in [k] \\ &\implies \{u_A, v_B\} \in E(I'_i), i \in [k]. \end{aligned}$$

Also, by definition it is clear that  $\{u_A, v_B\}$  is an edge in both  $I'_{k+1}$  and  $I'_{k+2}$ . Therefore,  $I'_i$ s are all supergraphs of  $G_c$ .

$$\begin{aligned} \{u_A, v_B\} \notin E(G_c) &\implies \{u, v\} \notin E(G) \\ &\implies \exists I_j, j \in [k] \text{ such that } \{u, v\} \notin E(I_j) \\ &\implies \{u_A, v_B\} \notin E(I'_j). \end{aligned}$$

$A$  and  $B$  induce independent sets in  $I'_{k+2}$  and  $I'_{k+1}$  respectively. Hence,  $G_c = I'_1 \cap I'_2 \cap \dots \cap I'_k \cap I'_{k+1} \cap I'_{k+2}$  and therefore  $\text{box}(G_c) \leq \text{box}(G) + 2$ .

$\text{box}(G) \leq 2\text{box}(G_c)$ : We will assume without loss of generality that  $|V(G)| > 1$ . This implies  $G_c$  is not a complete graph and therefore  $\text{box}(G_c) > 0$ . Let us consider the associated co-bipartite graph of  $G_c$ , i.e.  $G_c^*$ . We will show that  $\text{box}(G) \leq \text{box}(G_c^*)$  and the required result follows from Lemma 6. Let  $\text{box}(G_c^*) = p$  and  $G_c^* = J_1 \cap J_2 \cap \dots \cap J_p$  where  $J_i$ s are interval graphs. Let us assume canonical interval representation for each  $J_i$  (recall Definition 3). Corresponding to each  $J_i$ , we construct an interval graph  $J'_i$  with vertex set  $V(G)$  as follows: The interval for any vertex  $u$  is the intersection of the intervals of  $u_A$  and  $u_B$ , i.e.  $l(u, J'_i) = l(u_B, J_i)$  and  $r(u, J'_i) = r(u_A, J_i)$ . Note that since  $u_A$  and  $u_B$  are adjacent in  $J_i$ , their intersection is non-empty.

Now we show that  $G = \bigcap_{i=1}^p J'_i$ . First we consider two adjacent vertices  $u$  and  $v$ .

$$\begin{aligned} \{u, v\} \in E(G) &\implies \{u_A, v_B\}, \{u_B, v_A\} \in E(G_c^*) \\ &\implies \{u_A, v_B\}, \{u_B, v_A\} \in E(J_i), \forall i \in [p] \\ \text{(From Lemma 5)} &\implies l(v_B, J_i) \leq r(u_A, J_i) \text{ and } l(u_B, J_i) \leq r(v_A, J_i), \forall i \in [p] \\ \text{(By definition of } J'_i) &\implies l(v, J'_i) \leq r(u, J'_i) \text{ and } l(u, J'_i) \leq r(v, J'_i), \forall i \in [p] \\ \text{(From Lemma 5)} &\implies \{u, v\} \in E(J'_i), \forall i \in [p] \end{aligned}$$

Therefore, each  $J'_i$  is a supergraph of  $G$ . Now, suppose  $u$  and  $v$  are not adjacent.

$$\begin{aligned} \{u, v\} \notin E(G) &\implies \{u_A, v_B\} \notin E(G_c^*) \\ &\implies \exists J_j \text{ such that } \{u_A, v_B\} \notin E(J_j) \\ \text{(From Lemma 5)} &\implies r(u_A, J_j) < l(v_B, J_j) \text{ or } r(u_B, J_j) < l(v_A, J_j) \\ \text{(Since } J_j \text{ has a canonical interval representation)} &\implies r(u_A, J_j) < l(v_B, J_j) \\ \text{(By definition of } J'_j) &\implies r(u, J'_j) < l(v, J'_j) \\ \text{(From Lemma 5)} &\implies \{u, v\} \notin E(J'_j) \end{aligned}$$

Hence,  $G = J'_1 \cap J'_2 \cap \dots \cap J'_p$  and from Lemma 6 we have  $\text{box}(G) \leq \text{box}(G_c^*) \leq 2\text{box}(G_c)$ .

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