# Evidence Algorithm and System for Automated Deduction: A Retrospective View

(In honor of 40 years of the EA announcement)

Alexander Lyaletski<sup>1</sup> and Konstantin Verchinine<sup>2</sup>

 Faculty of Cybernetics, Kiev National Taras Shevchenko University 2, Glushkov avenue, 03680 Kiev, Ukraine email: lav@unicyb.kiev.ua
<sup>2</sup> Math-Info Department, Paris 12 University 61, avenue du General De Gaulle, 94010 Creteil, France e-mail: verko@logique.jussieu.fr

**Abstract.** A research project aimed at the development of an automated theorem proving system was started in Kiev (Ukraine) in early 1960s. The mastermind of the project, Academician V.Glushkov, baptized it "Evidence Algorithm", EA<sup>3</sup>. The work on the project lasted, off and on, more than 40 years. In the framework of the project, the Russian and English versions of the System for Automated Deduction, SAD, were constructed. They may be already seen as powerful theorem-proving assistants. The paper<sup>4</sup> gives a retrospective view to the whole history of the development of the EA and SAD. Theoretical and practical results obtained on the long way are systematized. No comparison with similar projects is made.

### **1** Introduction

The research project entitled "Evidence Algorithm" was initiated by V.Glushkov in the early 60-s in Kiev. At that time, some fundamental facts concerning formal proof search and opportunities (potential in most cases) to use computers to find a proof, were already known. The domain that was called "automated theorem proving" (ATP or "machine reasoning" in the AI community) became a challenging one for logicians as well as for computer scientists (see e.g. [92] for short history). There were hopes! Recall the title of an early Hao Wang's paper: "Towards mechanical mathematics" [104].

V.Glushkov as he personally told us, was motivated by two main reasons:

(1) To get an aid while verifying long and routine algebraic transformation (as a working mathematician he obtained valuable results concerning Hilbert's 5th problem).

(2) To try the strength of the existent computers pushing them to run on the limits of their abilities.

V.Glushkov formulated the main question in a slightly unusual way.

Let us consider some relatively well formalized mathematical theory, e.g. Lie algebras. There are a small number of basic facts (axioms) which are considered to be

<sup>&</sup>lt;sup>3</sup> Below, we explain why this title was chosen; it was used first in [9].

<sup>&</sup>lt;sup>4</sup> The final publication of this paper is available at www.springerlink.com

evident even for beginners. Let's apply simple purely logical tools to obtain several consequences. They are also evident. Then one can apply the same logical tools to the conclusions and so on. Are the results still evident? If the conclusions were obtained by a programmed inference engine, the answer is "yes, they are". From the viewpoint of this engine. But probably not from the human point of view. Thus, provided the abovementioned engine, we would be able to prove/verify something that is not evident for humans. Further to that, this "evidence maintaining engine" may be reinforced with heuristics, proof methods, lemma application, definition expansion, and so on. In this way, we could enlarge the notion of "being evident" to the extent that might include nontrivial facts/theorems. Well, "now do it, guys!".

That is why the algorithmic part of the project (and afterwards the project as the whole) has got the name "Evidence Algorithm", EA, or  $\exists \forall$  for fun.

It was also already clear at that time that nobody would like to formalize the mathematical knowledge/reasoning in the usual first order language. Hence a formal but human-friendly language had to be developed to provide a possibility for the construction of a mathematical assistant system convenient for a wide range of scientists.

So, three major components of such a system should be:

(1) a powerful input language that must be close to the natural mathematical language and easy to use;

(2) an inference engine that implements the basic level of evidence (sometimes, we call it a "prover" below);

(3) an extensible collection of tools that reinforce the basic engine (sometimes, we call it a "reasoner" below).

In what follows, we give a short description in chronological order of what has been done in each of the above-mentioned directions.

Please note that our main goal is to trace the long path of the project development and to recall the results obtained. That is why the reference list is so long. For the same reason, we could neither make any comparison with similar existing systems, nor give an illustrative set of examples. Sorry for that. We frequently got an impression that the automated reasoning community is not sufficiently acquainted with EA project (for instance, SAD was not mentioned in F.Wiedijk's book [106]) though we think that some ideas and results might be useful to know. We hope that the text given below will partially meet the lack of such information.

Note on the bibliography. Almost all papers published before 1992 were written in Russian and therefore are hardly available now. We translated the titles and put them onto the list just to indicate what was done in the old time. All the papers are listed in chronological order.

The rest of the paper contains three parts according to the three periods in the history of the EA project. They are as follows.

The first one: 1962 - 1970. We call it "Pre-EA Stage" below.

The second one: 1970 - 1992. It is called "EA and Russian SAD" below.

The third one: 1998 - nowadays. Below it is called "Post-EA Stage and English SAD".

Several final remarks conclude the paper.

# 2 Pre-EA Stage (1962–1970)

Few people remember now that the Soviet computer history began in Kiev. The first von Neuman computer was assembled and tested at the turn of 1950 in a small laboratory headed by the academician S.Lebedev. In 1955 S.Lebedev left for Moscow and the director of Kiev Institute for Mathematics, prominent mathematician B.Gnedenko invited V.Glushkov to take the supervision of the laboratory (which was transformed into the Institute of Cybernetics 5 years later).

On the other hand, at that time there was a powerful logic, linguistic and algebraic team at the mathematical department of the Kiev University. Professor L.Kaluzhnin who was the head and the heart of the team, invited V.Glushkov to join their efforts.

So the Kiev school in the ATP domain appeared really at the borderline of computer science and mathematical logic.

In 1962, V.Glushkov published a paper [1] where he analyzed several rather simple proofs in Group Theory and suggested that the proofs might be built automatically with the help of a not too complex procedure. The idea attracted three people who began their research on the subject: A.Letichevsky, one of the first Glushkov's disciples, (in 1962), F.Anufriev (in 1962) and V.Fedyurko (in 1963). A bit later, V.Kostyrko and Z.Aselderov had joined the team. The first-time approach to the problem was purely empirical – they analyzed a lot of proofs taken from textbooks, monographs and articles for trying to formalize all them and to find (almost by feeling) methods, heuristics and representation details that might help to construct a proof of a theorem under consideration automatically. As a result an algorithm of proof search in Group Theory was constructed and even implemented (the corresponding program run on the monstrous Ural-1 computer). The first communication about it was done at the First All-Union Symposium on the Machine Methods of Logical Inference Search, that took place in Lithuania in 1964 [97] (see also [93]). Later a paper on the subject was published [4] (and translated afterwards into English).

The algorithm, though being comparatively simple, contained nevertheless:

- a method of inference search for some class of first-order formulae;
- a reduction technique for simplifying search space;
- a collection of heuristics (e.g. the inclusion relation was exploited);
- special methods of equation solving.

So we can say that it was the first problem-oriented prover for Group Theory. Here is an example of proved theorem: "The centralizer Z of any subgroup P is a normal subgroup of the normalizer N of P".

The above-mentioned proof-search method resembled, in some sense, well known backward chaining, but some features were added to make it applicable to non-Horn formulae. Later on, the method was generalized by F.Anufriev and extended to the whole first-order classical logic without equality [5, 8]. It can be interpreted as a goaloriented sequent calculus not requiring skolemization and using an analog of Kanger's notion of substitution admissibility. Later, the method was transformed into a correct and complete sequent calculus [22] with skolemization. It had got the name "Auxiliary-Goals Search calculus" ("AGS calculus" below) and served as a prototype for various sequent-type inference engines of the EA project. Solving equations in free groups became the subject of Z.Aselderov's PhD thesis [7], which was defended in 1968.

Now let's cast a glance at the list of required components of the conceived EA system. No convenient input language was yet proposed at the time being. On the other hand, it was difficult to continue the project without it. To see why, try to convert the theorem above to the first-order language. On this subject, there were only two Kaluzhnin's papers: [2] and [3]. Some time later, V.Kostyrko made an attempt to solve the problem and after some period, a paper was published [10] where a contour of such a language was outlined. The main idea was as follows.

Let's consider an atomic first-order formula. It is always of the form  $R(t_1, t_2, ..., t_n)$  where *R* is an *n*-ary relation symbol, whereas a "natural" atomic statement is of the form  $\langle subject group \rangle \langle predicate group \rangle$ . Well, one can:

- select an argument among  $t_1, t_2, \ldots, t_n$ , say  $t_1$ ,

- consider it as the subject,

- "reduce" *R* to something (n-1)-ary  $N(t_2, \ldots, t_n)$ ,

- add a new connector to "attach"  $t_1$  to  $N(t_2,...,t_n)$  ( $\varepsilon$  was chosen in the original version).

Now  $R(t_1, t_2, ..., t_n)$  can be written as  $t_1 \in N(t_2, ..., t_n)$  and read as  $t_1$  is a  $N(t_2, ..., t_n)$ . For example, Subgroup(H, G) gives  $H \in Subgroup_of G$  and so on. Was it not more than syntactic "sugar" or one could gain something interesting with it? Below we demonstrate what was made in this direction later.

For the completeness of the description of that time, it may be needed to remind the last implementation of the propositional part of Anufriev's procedure, which was made by A. Malashonok on the BESM-2 computer at the beginning of 1970 [13].

# **3** EA and Russian SAD (1971 - 1992)

In 1970, V.Glushkov published one more paper on the subject [9]. At that time, he associated the progress in the domain of ATP with the general tendency to make computers more intelligent (see also [14]). As to the project in question (except the fact that it had got its name "Evidence Algorithm"), V.Glushkov emphasized the importance of a natural formal language for writing mathematical texts in [9]. We should also note that for the first time, the term "automated theorem proving" was used instead of "automatic theorem proving" and the problem of how to construct something like an "interactive proof environment" was explicitly formulated. In fact, a proof assistant was conceived at that time.

It seems that somewhen in the middle of 1970, V.Glushkov decided to add "young forces" to the existing EA team, and he charged one of his former pupil V.Bodnarchuk to became the leader of the new team. At that time, V.Bodnarchuk was the head of a computer department; its members had just finished their work under a specialized mini-computer for engineering computation with the language "Analitik" [11].

The input language "Analitik", being convenient for engineers and having its hardware implementation, was one of the distinguishing features of the computer, and V.Bodnarchuk was its main creator. Besides, V.Bodnarchuk was very intimate with L.Kaluzhnin, and these exerted great influence on the development of the EA. At the same time, four young people became the postgraduate students at the Institute of Cybernetics, both authors were among them.

Two of them, A.Degtyarev and K.Verchinin, were graduated from the Mechanical and Mathematical Faculty of the Moscow State University. Other two, A.Lyaletski and N.Malevanyi – from the Cybernetics Faculty of the Kiev State University. So, we had joined the EA team and V.Bodnarchuk became our "local supervisor" (the global one was V.Glushkov). We were young, full of energy and illusions...

At the very beginning, V.Bodnarchuk has formulated the following tasks:

- careful revision of everything that was done previously by the "old team";

- detailed analysis of mathematical texts in various domains;

- preparation of two surveys: (1) of combinatorial proof-search methods (published, see [16]) and (2) of using heuristics in proof search (published, see [18])

The revision of the existing version of the AGS method demonstrated that, first, the use of the Kanger's notion of substitution admissibility instead of skolemization complicates drastically an eventual implementation and, second, the method requires a special technique for equality handling. So, to advance the whole project, one needed:

- either to improve the AGS method paying special attention to redundancy avoidance and equality handling, or to adapt one of existing combinatorial methods of proof search for the role of inference engine in the EA project;

- to develop a practically usable version of the "mathematical language" along with the whole syntactical service around it;

- to find a convenient formalization of what is frequently used in mathematical texts to make them available for human reader – "proof method", "proof scheme", "lemma application", "definition dependency", etc.

- to find methods of what is called "knowledge management" now, e.g. to try to understand what the "relevancy relation" on mathematical facts might be;

- to develop an implementation base (it became clear at the very beginning that experimental work was strongly needed and it could not be done in the paper-and-pencil mode).

We began in quite favorable setting. Two circumstances should be especially noted. At that time, it was easy to establish scientific contacts in the ex-USSR and we have done that: with famous Leningrad logic school, with excellent Novosibirsk logic school (founded by A.Maltsev), with strong Moscow logic school, with linguists, psychologists, etc. The second point is that last-year students of the new Cybernetics department of the Kiev University used to pass their six month professional training at the Institute of Cybernetics. In this way the second EA team had got two very capable young researchers: A.Zhezherun (in 1973) and M.Morokhovets (in 1978).

#### 3.1 Theoretical work

Here is a brief description of research interests and results obtained by members of the second EA team.

At the beginning, A.Degtyarev studied the role of heuristics in formal proofs. He restricted himself with linear algebra and showed that for large class of theorems, the proof search (by resolution with paramodulation) may be controlled in a way and reduced to the problem of finding solution to a set of linear equations [17, 21]. It was quite

interesting result but A.Degtyarev did not continue that direction and devoted himself to the problem of equality handling in resolution-like methods. As a "side effect" he obtained an efficient unification algorithm (published later in [23, 42]) that was based on the same principles that the well-known Martelli and Montanary algorithm [95] formulated later.

His main results concern various paramodulation strategies and the problem of compatibility the paramodulation rule with term orderings. The most known is so called monotonic paramodulation [30, 31, 50] subsequently used in many other researchs on the subject.

A.Lyaletski occupied himself with the careful analysis of combinatorial proof search methods trying to put them in a common setting and find (or build) the best candidate for a resolution-type inference engine. He suggested a modification of the resolution rule which operated with more general objects than clauses – conjunctive clauses or c-clauses. (Later, V.Lifschitz [94] independently proposed something similar and called them "super-clauses"). Two different c-clause calculi were build [24, 25] which permitted to reformulate well-known Maslov's Inverse Method [96] in a resolution-like manner.

Another problem was the skolemization. Is it bad or not? Anufriev's method did not use skolemization, but it adds new entities as in the case of Kanger's method. On the other hand, skolemization simplifies the algorithmic part of proof search methods. A.Lyaletski found an original notion of admissible substitution that allowed him to get in some sense a compromise. He built a series of sequent calculi with resolution style inference rules, that, on one hand, don't require skolemization and, on the other hand, are not less efficient than the usual resolution calculus ([34, 35, 53]).

K.Verchinine was strongly involved in the language problem. We had to formalize mathematical texts, not only isolated statements. A text may be considered as a structured collection of sections: chapters, paragraphs, definitions, theorems, proofs, etc. So a part of the language was designed to represent this structure, its "semantics" was given by the "trip rules". Another part served to formalize a statement. New units were added to the standard first-order syntax which permitted to use nouns, adjectives, special quantifiers, etc. The language was developed and has got the name TL – Theory Language [15, 20]. Here is a formal TL phrase: "there is no remedy against all deseases but there is a desease against all remedies". (That time the vocabulary as well as the syntax was certainly Russian.)

Two kind of semantics were defined for that part: a transformational one (an algorithm to convert a TL statement into its first-order image) and another one – in the traditional set-theoretical style where  $\varepsilon$  was interpreted as the membership relation [19]. The last semantics permitted to define the "extension" of every notion (e.g. the extension of "subgroup\_of G" is the class of all subgroups of G) and to introduce a structure on the set of notions which restrict quantifiers in the given sentence. That structure was called "situation" and was used in attempts to formalize a relevancy relation.

At the beginning, A.Zhezherun took active part in the TL language development. He designed and implemented the whole syntactic service for the linguistic part of the future system. As usual, there were funny side effects of the work. For instance, computer linguists have always searched for some invariant (called profound semantic structure) that could be used in machine translation algorithms. A.Zhezherun and K.Verchinine showed that the first-order image of a TL statement can play the role of such invariant. So just changing the superficial decorations in some regular way, one can translate mathematical statements from Russian into English and vice versa (provided the dictionary). A.Zhezherun wrote a program to play with, and it worked surprisingly well! Besides, he studied the opportunity to formalize mathematical reasoning in a higher-order logic and proved in particular the decidability of the second-order monadic unification [39].

M.Morokhovets occupied herself with the problem of "reasoner" (see above). As the reasoner must have a prover to cooperate with, the last was badly needed. The AGS based prover didn't fit well to that purpose, so we decided to develop and implement a resolution-and-paramodulation based prover with a flexible architecture that could be adapted to various strategies and auxiliary inference rules. M.Morokhovets has done it. The first observation showed that some particular premises are strongly responsible for the search space explosion. The transitivity axiom clearly is among them. M.Morokhovets proved that for some large class of transitive relations, this axiom may be eliminated and replaced by a special inference rule which can be controlled to shorten the search space [57].

Another idea was to use the fact that all quantifiers in the TL statement are restricted (bounded). Is it possible to "forget" the restrictions, to find an inference and then just to verify that all substitutions are correct w.r.t. these restrictions (bounds). M.Morokhovets has found several classes of statements for which the answer is "yes", and has implemented corresponding procedure [56]. One more question was as follows. Let's suppose that a conjecture is proved and the resolution style inference is constructed. How to present it in a human readable form? The set of conversion rules that permit to do it (based on an early result of K.Verchinine), was designed and implemented by M.Morokhovets, too.

#### 3.2 Experimental work

Certainly, some computer experiments have been done from the very beginning of EA project development (it was one of Glushkov's ideas – to be permanently accompanied with computers while doing theoretical research). Still in 1971 K.Verchinine used the syntactic tools taken from another system (developed in the same department) to implement a part of TL grammar. A.Malashonok have programmed AGS prover to make local experiments with. Also local experiments with paramodulation strategies were maid by A.Degtyarev. N.Malevanyi began to prepare something like a specialized library for future experiments on the BESM-6 machine – another Lebedev's creation – one of the most powerful computer in the ex-USSR.

Systematic programming was initiated after A.Zhezherun appeared. He became the main designer and programmer of the system for mathematical text processing. But no doubt, we all were involved in programming. At that time, the IBM System 360/370 (cached under the name "ES Line Computer") was admitted in the ex-USSR as the main platform. With the native operating system and the PL/1 as the main programming language – what a hell!!!

Nevertheless the work advanced and the first experiments with the whole system were done in 1976/1977. The main task was formulated as mathematical text verification and may be presented as follows.

Let a TL text be given. The system can:

- parse the text informing the user about syntactic errors (if any);

- convert the text to some tree-like internal form;

- run the main loop: choose a goal sentence to verify and find its logical predecessors;

- construct an initial proof environment for one of available provers<sup>5</sup>;

- start the prover and wait;

- if the prover fails then ask to help;

- if the prover succeeds then output the proof, choose the next goal and repeat the main loop until the end of the text be reached.

The first public presentation of the system in question was made at the All-Union symposium "Artificial intelligence and automated research in Mathematics" (Kiev, Ukraine, 28-30 November 1978). It worked!

In 1980, V.M. Glushkov gave the name "System for Automated Deduction" (SAD) to the implemented system and it has this name now.

The further work consisted in improving the system and adding new features to it. We extended the mathematical texts library and developed a conception of further extension of TL language with "imperative" (algorithmic) constructions. A method of using auxiliary statements in proof search (based on the notion of situation) was implemented by V.Atayan [47]. Efficient paramodulation strategies were added and tested by A.Degtyarev. A resolution-based prover was implemented by M.Morokhovets.

In the meantime four PhD thesis were defended at the Institut for Cybernetics: A.Zhezherun has got his PhD in 1980 [46], A.Lyaletski [53], A.Degtyarev [51] and K.Verchinine [54] – in 1982. M.Morokhovets' thesis was in preparation.

We understood that to advance the project we need to try the SAD system in some more or less practical applications. One possible application was the automated program synthesis and we established a contact with professor Enn Tyugu (Tallinn, Estonia) and his team. Another interesting application was the deduction tool for expert systems. The problem is that classical logic is rarely used in this domain. So, the question appeared: is it possible to adapt SAD for the inference search problem in non-classical logics?<sup>6</sup>.

But everithing comes to its end. Sooner or later.

<sup>&</sup>lt;sup>5</sup> At that time, the SAD system prover was constructed and implemented on the base of an original sequent-type calculus [48]. It had the following features: it was goal-oriented, skolemization was not obligatory, and equality handling was separated from deduction. Now, the native prover of the current (English) SAD possesses the same features.

<sup>&</sup>lt;sup>6</sup> Later, a theoretical answer on this question was obtained in a number of papers of A.Lyaletski (see, for example, [77, 86]); from this point of view, some researches on Herbrand theorems ([78, 79, 82, 90]) also may seem to be interesting.

#### 3.3 Team evolution (or the sad part of the SAD history)

Already in the end of 1972, V.Bodnarchuk falled seriously ill and, actually, he abandoned the research activity for a long time. From 1973 to 1975 F. Anufriev, Z. Aselderov, V. Kostyrko, and A. Malashonok left the team because of various reasons, they never came back to the subject area afterwards. In 1982, V.Glushkov was dead. The administration style in the Institut for Cybernetics changed and we were not the favorit director's team any more. In the middle of 1983, A. Lyaletski and A. Zhezherun left for the Kiev University. In 1984, K. Verchinine moved to another department and changed his research area. Finally, in 1987, A. Degtyarev left for the Kiev University, too. M.Morochovets stayed at our former department of the Institute of Cybernetics. The EA team did no more exist...

# 4 Post-EA Stage and English SAD (1998-nowadays)

In 1998, the Evidence Algorithm project moved into a new stage. That year the IN-TAS project 96-0760 "Rewriting techniques and efficient theorem proving" started and brought financial support for resumption of work on SAD. The new working group included Alexander Lyaletski at Kiev National University (KNU), Marina Morokhovets at the Institute of Cybernetics in Kiev, Konstantin Verchinine at Paris 12 University in France, and Andrei Paskevich, fourth-year undergraduate student of KNU.

The work started in 1999, with re-implementation of the TL language on IBM PC. The programs were written in C on the Linux platform. In a year, towards March 2000, parsing and translation of TL sentences into a first-order language was implemented. The English-based version of TL had been given the name ForTheL, an acronym for "FORmal THEory Language" (also a Russian word meaning "trick" or "stunt"). The language was presented firstly at the Fifth International Conference "Information Theories and Applications" in September 2000 in Varna, Bulgaria [67].

The same summer the work started on re-implementation of the deductive tools of SAD. By January 2001, A.Paskevich created the first prototype of the prover (the prover had gotten the name "Moses"). A bit later the technique of admissible substitutions by A. Lyaletski which permitted to dispense with skolemization and preserve the initial signature of a proof task, was also implemented. Later, the equality elimination procedure by Brand [91] was added to handle the problems with equality. By June 2001, the complete "workflow" of the initial SAD: from ForTheL text to first-order representation to proof task to proof tree, was reestablished. Of course, a lot of functionality of the previous implementation has not been transferred into the new system.

In September 2001, A. Paskevich started his doctoral study under the joint supervision of Konstantin Verchinine and Alexander Lyaletski. His work aimed at the development of a new, two-level architecture of a mathematical assistant.

In the first prototype of the English SAD system, the reasoner was virtually nonexistent. The theoretical development of the reasoner started with the work on "local validity", which allowed to perform sound logical inferences inside a formula, possibly under quantifiers. This technique could provide a basis for in-place transformations (such as definition expansions) as well as for handling of partial functions and predicates [71]. By the end of 2003, tools for supporting proofs by case analysis and by general induction (with respect to some well-found ordering) were implemented in the SAD. In 2004, an experimental support for binding constructions, such as summation and limit, was also added [81].

An algorithm for generation of atomic local lemmas was constructed and implemented: these lemmas help to prove a lot of simple statements without using a prover at all.

An interesting feature of the SAD is that the prover does not depend on the rest of the system. It means that various provers can be used as the system inference engine (provided the interface be written). The following ones were used in our experiments: SPASS [105], Otter [98], E Prover [103], Vampire [100] and Prover9 [99].

In July, 2007, the "enriched" SAD system was presented at the 21st Conference on Automated Deduction in Bremen, Germany [85]. A. Paskevich has made several improvements since then. The current version of the system is freely available at http://www.nevidal.org. Here is a short list of texts (proofs) that were successfully verified by the SAD: Tarski's Fixed Point theorem, Newman's lemma, Chinese Remainder theorem, Infinite Ramsey theorem, "The square root of a prime number is not rational", Cauchy-Bouniakowsky-Schwartz inequality for real vectors, Fuerstenberg's proof of the infinitude of primes.

Finally, note that the EA project leaded to the carrying out of new investigations in automated reasoning (see the last publications in the reference list).

# 5 Conclusion

Let's imagine an ideal Mathematical Assistant. What its architecture might be from the EA position?

A user communicates with the system with the help of texts written in a high-level formal input language close to the natural one. She or he submits a problem like "verify whether the given text is correct" or "how to prove the following statement", or "what is the given text about" and so on. The text, provided being syntactically correct, is treated by the part of the system that we call "reasoner". The reasoner analyzes the problem and formulates a series of tasks that it submits to the inference engine, a prover. If the prover succeeds, the resulting conclusion (e.g. human-readable proof) is given to the user and the game is over. If it fails then a kind of "morbid anatomist" makes a diagnosis and supplies it to the reasoner who tries to repair the situation. In particular, the reasoner can decide that an auxiliary statement (lemma) might be useful and start the search for those in the mathematical archives. To do that it submits a request to the archive service, we call it "librarian". After getting an answer, the reasoner begins a new proof search cycle with the modified problem and the process goes on.

The user can interact with the system by playing for the reasoner, librarian, for the morbid anatomist (provided that she or he understands the internal prover's life) or for the prover itself, deciding whether a given conjecture should be considered as valid.

Where we are with respect to the ideal? Optimistic answers are welcome.

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