

Polynomial-Space Approximation of No-Signaling Provers

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Abstract

In two-prover one-round interactive proof systems, no-signaling provers are those who are allowed to use arbitrary strategies, not limited to local operations, as long as their strategies cannot be used for communication between them. Study of multi-prover interactive proof systems with no-signaling provers is motivated by study of those with provers sharing quantum states. The relation between them is that no-signaling strategies include all the strategies realizable by provers sharing arbitrary entangled quantum states, and more.

This paper shows that two-prover one-round interactive proof systems with no-signaling provers only accept languages in PSPACE. Combined with the protocol for PSPACE by Ito, Kobayashi and Matsumoto (CCC 2009), this implies $\text{MIP}^{\text{ns}}(2, 1) = \text{PSPACE}$, where $\text{MIP}^{\text{ns}}(2, 1)$ is the class of languages having a two-prover one-round interactive proof system with no-signaling provers. This is proved by constructing a fast parallel algorithm which approximates within an additive error the maximum value of a two-player one-round game achievable by cooperative no-signaling players. The algorithm uses the fast parallel algorithm for the mixed packing and covering problem by Young (FOCS 2001).

1 Introduction

1.1 Background

Nonlocality [Bel64] is a peculiar property of quantum mechanics and has applications to quantum information processing. Following Cleve, Høyer, Toner and Watrous [CHTW04], quantum nonlocality can be naturally expressed in terms of cooperative two-player one-round game with imperfect information, which is a game played by two players and a referee as follows. The players are kept in separate rooms so that they cannot communicate with each other. The referee chooses a pair of questions according to some probability distribution, and sends one question to each player. Each player replies with an answer to the referee, and the referee declares whether the two players jointly win or jointly lose according to the questions and the answers. The players know the protocol used by the referee including the probability distribution of the pair of questions and how the referee determines the final outcome of the game, but none of the players knows the question sent to the other player. The aim of the players is to win the game with as high probability as possible, and the maximum winning probability is called the *value* of the game. In this framework, a Bell inequality is an inequality stating an upper bound of the value of a game of this kind when provers are not allowed to perform any quantum operations, and the violation of a Bell inequality means that the game value increases when provers are allowed to share a quantum state before the game starts.

The complexity of finding or approximating the value of a game has been one of the most fundamental problems in computational complexity theory. The computational model based on cooperative multi-player

games is called multi-prover interactive proof systems and were introduced by Ben-Or, Goldwasser, Kilian and Wigderson [BGKW88] for a cryptographic purpose.¹ It turned out that this computational model is extremely powerful: multi-prover interactive proof systems exactly characterize NEXP [FRS94, BFL91], even in the most restricted settings with two provers, one round and an exponentially small one-sided error [FL92]. In other words, given the description of a cooperative game, approximating the best strategy even in a very weak sense is notoriously difficult. These results were built on top of techniques developed in studies on (single-prover) interactive proof systems [Bab85, GMR89, LFKN92, Sha92] as well as multi-prover interactive proof systems with weaker properties [CCL94, Fei91, LS97]. It is noteworthy that the powerfulness of multi-prover one-round interactive proof systems has led to a successful study of probabilistically checkable proof systems [BFLS91, FGLSS96], which play a central role in proving NP-hardness of many approximation problems via the celebrated PCP theorem [AS98, ALMSS98].

Cleve, Høyer, Toner and Watrous [CHTW04] connected the computational complexity theory and the quantum nonlocality and raised the question on the complexity of approximating the value of a cooperative game with imperfect information in the case where the players are allowed to share quantum states or, in terms of interactive proof systems, the computational power of multi-prover interactive proof systems with entangled provers. Kobayashi and Matsumoto [KM03] considered another quantum variation of multi-prover interactive proof systems where the verifier can also use quantum information and can exchange quantum messages with provers, which is a multi-prover analogue of quantum interactive proof systems [Wat03]. In [KM03], it was shown that allowing the provers to share at most polynomially many qubits does not increase the power of multi-prover interactive proof systems beyond NEXP (even if the verifier is quantum). Although studied intensively [KKMTV08, CGJ09, IKPSY08, KRT08, KKMV09, Gut09, DLTW08, NPA08, BHP08, IKM09], the power of multi-prover interactive proof systems with provers allowed to share arbitrary quantum states has been still largely unknown.

The notion of no-signaling strategies was first studied in physics in the context of Bell inequalities by Khalfin and Tsirelson [KT85] and Rastall [Ras85], and it has gained much attention after reintroduced by Popescu and Rohrlich [PR94]. The acceptance probability of the optimal no-signaling provers is often useful as an upper bound of the acceptance probability of entangled provers (and even commuting-operator provers based on the notion of commuting-operator behaviors; see [Tsi06, NPA08, DLTW08, IKPSY08]) because no-signaling strategies have a simple mathematical characterization. Toner [Ton09] uses no-signaling provers to give the maximum acceptance probability of entangled provers in a certain game. Extreme points of the set of no-signaling strategies are also studied [BLMPPR05, AII06].

Kempe, Kobayashi, Matsumoto, Toner and Vidick [KKMTV08] prove, among other results, that every language in PSPACE has a two-prover one-round interactive proof system which has one-sided error $1 - 1/\text{poly}$ even if honest provers are unentangled and dishonest provers are allowed to have prior entanglement of any size (the proof is in [KKMTV07]). Ito, Kobayashi and Matsumoto [IKM09] improve their result to an exponentially small one-sided error by considering no-signaling provers; more specifically, they prove that the soundness of the protocol in [KKMTV08] actually holds against arbitrary no-signaling provers, then use the parallel repetition theorem for no-signaling provers [Hol09]. We note that the soundness analysis of [IKM09] is somewhat simpler than that of [KKMTV08].

Repeating the protocol of [KKMTV08] parallelly as is done in [IKM09] results in the protocol identical to the one used by Cai, Condon and Lipton [CCL94] to prove that every language in PSPACE has a two-prover one-round interactive proof system with an exponentially small one-sided error in the classical world. Therefore, an implication of [IKM09] is that the protocol in [CCL94] has an unexpected strong soundness property: the protocol remains to have an exponentially small error even if we allow the two provers to behave arbitrarily as long as they are no-signaling.

¹Because of this connection, we use “player” and “prover” synonymously in this paper.

Given that the soundness analysis of protocols against no-signaling provers is perhaps easier than that against entangled provers, it is tempting to try to extend the result of [IKM09] to a class of languages larger than PSPACE. For example, is it possible to construct a two-prover one-round interactive proof system for NEXP which is sound against no-signaling provers? The answer is no unless $\text{EXP} = \text{NEXP}$ because two-prover one-round interactive proof systems with no-signaling provers can recognize at most EXP as pointed out by Preda [Pre]. Then what about EXP?

1.2 Our results

Let $\text{MIP}^{\text{ns}}(2, 1)$ be the class of languages having a two-prover one-round interactive proof system with no-signaling provers with bounded two-sided error. The abovementioned result in [IKM09] implies $\text{MIP}^{\text{ns}}(2, 1) \supseteq \text{PSPACE}$. Preda [Pre] shows $\text{MIP}^{\text{ns}}(2, 1) \subseteq \text{EXP}$.

Our main result is:

Theorem 1. $\text{MIP}^{\text{ns}}(2, 1) \subseteq \text{PSPACE}$.

An immediate corollary obtained by combining Theorem 1 with the abovementioned result in [IKM09] is the following exact characterization of the class $\text{MIP}^{\text{ns}}(2, 1)$:

Corollary 2. $\text{MIP}^{\text{ns}}(2, 1) = \text{PSPACE}$, and this is achievable with exponentially small one-sided error, even if honest provers are restricted to be unentangled.

This puts the proof system of [CCL94] in a rather special position: while other two-prover one-round interactive proof systems [BFL91, Fei91, FL92] work with the whole NEXP, the one in [CCL94] attains the best achievable by two-prover one-round interactive proof systems with two-sided bounded error that are sound against no-signaling provers, and at the same time, it achieves an exponentially small one-sided error.

At a lower level, our result is actually a parallel algorithm to approximately decide² the value of a two-player one-round game for no-signaling players as follows. For a two-player one-round game G , $w_{\text{ns}}(G)$ is the value of G for no-signaling provers and $|G|$ is the size of G , both of which will be defined in Section 2.1.

Theorem 3. *There exists a parallel algorithm which, given a two-player one-round game G and numbers $0 \leq s < c \leq 1$ such that either $w_{\text{ns}}(G) \leq s$ or $w_{\text{ns}}(G) \geq c$, decides which is the case. The parallel time of the algorithm is polynomial in $\log|G|$ and $1/(c - s)$. The total work is polynomial in $|G|$ and $1/(c - s)$.*

Theorem 1 follows by applying the algorithm of Theorem 3 to the exponential-size game naturally arising from a two-prover one-round interactive proof system. This approach is similar to that of the recent striking result on the PSPACE upper bound on QIP [JJUW09] as well as other complexity classes related to quantum interactive proof systems, i.e. QRG(1) [JW09] and QIP(2) [JUW09].³

The construction of the parallel algorithm in Theorem 3 is much simpler than those used in [JW09, JUW09, JJUW09] because our task can be formulated as solving a *linear* program of a certain special form approximately instead of a *semidefinite* program. This allows us to use the fast parallel algorithm for the mixed packing and covering problem by Young [You01].

²The algorithm stated in Theorem 3 can be converted to an algorithm to approximate $w_{\text{ns}}(G)$ within an additive error in a standard way. See Remark 2 in Section 4.

³Do not be confused by an unfortunate inconsistency as for whether the number in the parenthesis represents the number of *rounds* or *turns*, where one round consists of two turns. The “1” in QRG(1) and the “2” in QIP(2) represent the number of turns whereas the “1” in $\text{MIP}^{\text{ns}}(2, 1)$ represents the number of rounds just in the same way as the “1” in $\text{MIP}(2, 1)$.

1.3 Organization of the paper

The rest of this paper is organized as follows. Section 2 gives the definitions used later and states the result by Young [You01] about a fast parallel approximation algorithm for the mixed packing and covering problem. Section 3 proves Theorem 1 assuming Theorem 3. Section 4 proves Theorem 3 by using Young's fast parallel algorithm. Section 5 concludes the paper by discussing some natural open problems.

2 Preliminaries

We assume the familiarity with the notion of multi-prover interactive proof systems. Readers are referred to the textbook by Goldreich [Gol08].

2.1 Definitions on games

A protocol of a two-prover one-round interactive proof system defines an exponential-size game for each instance. Here we give a formal definition of *games*.

A *two-prover one-round game*, or simply a *game* in this paper, is played by two cooperative provers called the prover 1 and the prover 2 with help of a verifier who enforces the rule of the game. A game is formulated as $G = (Q_1, Q_2, A_1, A_2, \pi, R)$ by nonempty finite sets Q_1, Q_2, A_1 and A_2 , a probability distribution π over $Q_1 \times Q_2$, and a function $R: Q_1 \times Q_2 \times A_1 \times A_2 \rightarrow [0, 1]$. As is customary, we write $R(q_1, q_2, a_1, a_2)$ as $R(a_1, a_2 \mid q_1, q_2)$.

In this game, the verifier generates a pair of questions $(q_1, q_2) \in Q_1 \times Q_2$ according to the probability distribution π , and sends q_1 to the prover 1 and q_2 to the prover 2. Each prover ν ($\nu \in \{1, 2\}$) sends an answer $a_\nu \in A_\nu$ to the verifier without knowing the question sent to the other prover. Finally, the verifier accepts with probability $R(a_1, a_2 \mid q_1, q_2)$ and rejects with probability $1 - R(a_1, a_2 \mid q_1, q_2)$. The provers try to make the verifier accept with as high probability as possible.

The *size* $|G|$ of the game G is defined as $|G| = |Q_1||Q_2||A_1||A_2|$.

A *strategy* in a two-prover one-round game G is a family $p = (p_{q_1 q_2})$ of probability distributions on $A_1 \times A_2$ indexed by $(q_1, q_2) \in Q_1 \times Q_2$. As is customary, the probability $p_{q_1 q_2}(a_1, a_2)$ is written as $p(a_1, a_2 \mid q_1, q_2)$. A strategy p is said to be *no-signaling* if it satisfies the following *no-signaling conditions*:

- The marginal probability $p_1(a_1 \mid q_1) = \sum_{a_2} p(a_1, a_2 \mid q_1, q_2)$ does not depend on q_2 .
- The marginal probability $p_2(a_2 \mid q_2) = \sum_{a_1} p(a_1, a_2 \mid q_1, q_2)$ does not depend on q_1 .

The *acceptance probability* of a strategy p is given by

$$\sum_{q_1 \in Q_1, q_2 \in Q_2} \pi(q_1, q_2) \sum_{a_1 \in A_1, a_2 \in A_2} R(a_1, a_2 \mid q_1, q_2) p(a_1, a_2 \mid q_1, q_2).$$

The *no-signaling value* $w_{\text{ns}}(G)$ of G is the maximum of the acceptance probability over all no-signaling strategies.

2.2 Definitions on interactive proof systems

Let $\Sigma = \{0, 1\}$. A *two-prover one-round interactive proof system* is defined by a polynomial $l: \mathbb{Z}_{\geq 0} \rightarrow \mathbb{Z}_{\geq 0}$, a polynomial-time computable mapping $M_\pi: \Sigma^* \times \Sigma^* \rightarrow \Sigma^* \times \Sigma^*$ such that $x \in \Sigma^n$ and $r \in \Sigma^{l(n)}$ imply $M_\pi(x, r) \in \Sigma^{l(n)} \times \Sigma^{l(n)}$, and a polynomial-time decidable predicate $M_R: \Sigma^* \times \Sigma^* \times \Sigma^* \times \Sigma^* \rightarrow \{0, 1\}$. On receiving an input string $x \in \Sigma^*$, the verifier prepares an $l(|x|)$ -bit string r uniformly at random and computes

$(q_1, q_2) = M_\pi(x, r)$. Then he sends each string q_ν ($\nu \in \{1, 2\}$) to the prover ν and receives an $l(|x|)$ -bit string a_ν from each prover ν . Finally he accepts if and only if $M_R(x, r, a_1, a_2) = 1$. This naturally defines a game $G^{(x)} = (Q_1^{(x)}, Q_2^{(x)}, A_1^{(x)}, A_2^{(x)}, \pi^{(x)}, R^{(x)})$ for each input string x , where $Q_1^{(x)} = Q_2^{(x)} = A_1^{(x)} = A_2^{(x)} = \Sigma^{l(|x|)}$,

$$\pi^{(x)}(q_1, q_2) = 2^{-l(|x|)} \cdot \#\{r \in \Sigma^{l(|x|)} \mid M_\pi(x, r) = (q_1, q_2)\},$$

$$R^{(x)}(a_1, a_2 \mid q_1, q_2) = \frac{\#\{r \in \Sigma^{l(|x|)} \mid M_\pi(x, r) = (q_1, q_2) \wedge M_R(x, r, a_1, a_2) = 1\}}{\#\{r \in \Sigma^{l(|x|)} \mid M_\pi(x, r) = (q_1, q_2)\}}.$$

Let $c, s: \mathbb{Z}_{\geq 0} \rightarrow [0, 1]$ be functions such that $c(n) > s(n)$ for every n . The two-prover one-round interactive proof system is said to *recognize a language*⁴ L with *completeness acceptance probability at least $c(n)$ and soundness error at most $s(n)$ with no-signaling provers* when the following conditions are satisfied.

Completeness $x \in L \implies w_{\text{ns}}(G^{(x)}) \geq c(|x|)$.

Soundness $x \notin L \implies w_{\text{ns}}(G^{(x)}) \leq s(|x|)$.

In particular, the proof system is said to *recognize L with bounded errors with no-signaling provers* if the binary representations of $c(n)$ and $s(n)$ are computable in time polynomial in n and there exists a polynomial $f: \mathbb{Z}_{\geq 0} \rightarrow \mathbb{Z}_{\geq 1}$ such that for every n , it holds $c(n) - s(n) > 1/f(n)$. We denote by $\text{MIP}^{\text{ns}}(2, 1)$ the class of languages L which are recognized by a two-prover one-round interactive proof system with bounded errors with no-signaling provers.

2.3 Mixed packing and covering problem

The *mixed packing and covering problem* is the linear feasibility problem of the form

$$\begin{aligned} \text{Find} \quad & x \in \mathbb{R}^N, \\ \text{Such that} \quad & Ax \leq b, \\ & Cx \geq d, \\ & x \geq 0, \end{aligned}$$

where matrices A, C and vectors b, d are given and the entries of A, b, C, d are all nonnegative. For $r \geq 1$, an *r -approximate solution* is a vector $x \geq 0$ such that $Ax \leq rb$ and $Cx \geq d$.

Theorem 4 (Young [You01]). *There exists a parallel algorithm which, given an instance (A, b, C, d) of the mixed packing and covering problem and a number $\varepsilon > 0$, either:*

- *claims that the given instance does not have a feasible solution, or*
- *finds a $(1 + \varepsilon)$ -approximate solution.*

If the size of A and C are $M_1 \times N$ and $M_2 \times N$, respectively, then the algorithm runs in parallel time polynomial in $\log M_1, \log M_2, \log N$ and $1/\varepsilon$ and total work polynomial in M_1, M_2, N and $1/\varepsilon$.

⁴Although we define $\text{MIP}^{\text{ns}}(2, 1)$ as a class of languages in this paper to keep the notations simple, we could alternatively define $\text{MIP}^{\text{ns}}(2, 1)$ as the class of *promise problems* [ESY84, Gol05] recognized by a two-prover one-round interactive proof system with no-signaling provers. A generalization of Theorem 1 to the case of promise problems is straightforward.

3 Proof of Theorem 1

Theorem 1 follows from Theorem 3 by a standard argument using the polynomial equivalence between space and parallel time [Bor77].

Let $L \in \text{MIP}^{\text{ns}}(2, 1)$, and fix an two-prover one-round interactive proof system which recognizes L with bounded errors with no-signaling provers. Let $c(n)$ and $s(n)$ be the completeness acceptance probability and the soundness error of this proof system, respectively. We construct a polynomial-space algorithm which recognizes L .

Let x be an input string and $n = |x|$. Let $G^{(x)} = (Q_1^{(x)}, Q_2^{(x)}, A_1^{(x)}, A_2^{(x)}, \pi^{(x)}, R^{(x)})$ be the game naturally arising from the proof system on input x . The size of $Q_1^{(x)}, Q_2^{(x)}, A_1^{(x)}, A_2^{(x)}$ is at most exponential in n . For each q_1, q_2, a_1, a_2 , it is possible to compute $\pi^{(x)}(q_1, q_2)$ and $R^{(x)}(a_1, a_2 \mid q_1, q_2)$ in space polynomial in n by simulating every choice of randomness of the verifier. By Theorem 4 of Borodin [Bor77], the parallel algorithm of Theorem 3 can be converted to a sequential algorithm which runs in space polynomial in $\log|G|$ and $1/(c-s)$. By applying this algorithm to the game $G^{(x)}$ and the threshold values $c(|x|)$ and $s(|x|)$, we decide whether $w_{\text{ns}}(G^{(x)}) \geq c(|x|)$ or $w_{\text{ns}}(G^{(x)}) \leq s(|x|)$, or equivalently whether $x \in L$ or $x \notin L$, in space polynomial in $\log|G^{(x)}| = \text{poly}(n)$ and $1/(c(|x|) - s(|x|)) = \text{poly}(n)$.

Note that the composition of two functions computable in space polynomial in $|x|$ is also computable in space polynomial in $|x|$, which can be proved in the same way as Proposition 8.2 of [Pap94].

4 Formulating no-signaling value by mixed packing and covering problem

This section proves Theorem 3.

Let $G = (Q_1, Q_2, A_1, A_2, \pi, R)$ be a game. Let $\pi_1(q_1) = \sum_{q_2 \in Q_2} \pi(q_1, q_2)$ and $\pi_2(q_2) = \sum_{q_1 \in Q_1} \pi(q_1, q_2)$ be the marginal distributions. Without loss of generality, we assume that every question in Q_1 and Q_2 is used with nonzero probability, i.e. $\pi_1(q_1) > 0$ for every $q_1 \in Q_1$ and $\pi_2(q_2) > 0$ for every $q_2 \in Q_2$.

By definition, the no-signaling value $w_{\text{ns}}(G)$ of G is equal to the optimal value of the following linear program:

$$\text{Maximize} \quad \sum_{q_1, q_2} \pi(q_1, q_2) \sum_{a_1, a_2} R(a_1, a_2 \mid q_1, q_2) p(a_1, a_2 \mid q_1, q_2), \quad (1a)$$

$$\text{Subject to} \quad \sum_{a_2} p(a_1, a_2 \mid q_1, q_2) = p_1(a_1 \mid q_1), \quad \forall q_1 \in Q_1, q_2 \in Q_2, a_1 \in A_1, \quad (1b)$$

$$\sum_{a_1} p(a_1, a_2 \mid q_1, q_2) = p_2(a_2 \mid q_2), \quad \forall q_1 \in Q_2, q_2 \in Q_2, a_2 \in A_2, \quad (1c)$$

$$\sum_{a_1, a_2} p(a_1, a_2 \mid q_1, q_2) = 1, \quad \forall q_1 \in Q_1, q_2 \in Q_2, \quad (1d)$$

$$p(a_1, a_2 \mid q_1, q_2) \geq 0, \quad \forall q_1 \in Q_1, q_2 \in Q_2, a_1 \in A_1, a_2 \in A_2. \quad (1e)$$

We transform this linear program (1) successively without changing the optimal value. First, we replace the constraint (1d) by two constraints:

$$\sum_{a_1} p_1(a_1 \mid q_1) = 1, \quad \forall q_1 \in Q_1, \quad (2a)$$

$$\sum_{a_2} p_2(a_2 \mid q_2) = 1, \quad \forall q_2 \in Q_2. \quad (2b)$$

It is clear that this rewriting does not change the optimal value.

Next, we relax the constraints (1b) and (1c) to inequalities:

$$\sum_{a_2} p(a_1, a_2 \mid q_1, q_2) \leq p_1(a_1 \mid q_1), \quad \forall q_1 \in Q_1, q_2 \in Q_2, a_1 \in A_1, \quad (3a)$$

$$\sum_{a_1} p(a_1, a_2 \mid q_1, q_2) \leq p_2(a_2 \mid q_2), \quad \forall q_1 \in Q_1, q_2 \in Q_2, a_2 \in A_2. \quad (3b)$$

Claim 1. *The optimal value w of the linear program (1) is equal to the maximum value w' of (1a) subject to the constraints (1e), (2a), (2b), (3a) and (3b).*

Proof. Since we only relaxed the constraints, $w \leq w'$ is obvious. To prove $w \geq w'$, let (\tilde{p}, p_1, p_2) be a solution satisfying the constraints (1e), (2a), (2b), (3a) and (3b). We will construct p such that (p, p_1, p_2) is a feasible solution of the linear program (1) and $p(a_1, a_2 \mid q_1, q_2) \geq \tilde{p}(a_1, a_2 \mid q_1, q_2)$ for every q_1, q_2, a_1, a_2 .

Fix any $q_1, q_2 \in Q$. Let

$$s_{q_1 q_2}(a_1) = p_1(a_1 \mid q_1) - \sum_{a_2 \in A_2} \tilde{p}(a_1, a_2 \mid q_1, q_2), \quad \forall a_1 \in A_1,$$

$$t_{q_1 q_2}(a_2) = p_2(a_2 \mid q_2) - \sum_{a_1 \in A_1} \tilde{p}(a_1, a_2 \mid q_1, q_2), \quad \forall a_2 \in A_2.$$

The following relations are easy to verify:

$$s_{q_1 q_2}(a_1) \geq 0, \quad \forall a_1 \in A_1, \quad (4a)$$

$$t_{q_1 q_2}(a_2) \geq 0, \quad \forall a_2 \in A_2, \quad (4b)$$

$$\sum_{a_1 \in A_1} s_{q_1 q_2}(a_1) = \sum_{a_2 \in A_2} t_{q_1 q_2}(a_2) (= F_{q_1 q_2}). \quad (4c)$$

We define $p(a_1, a_2 \mid q_1, q_2)$ by

$$p(a_1, a_2 \mid q_1, q_2) = \begin{cases} \tilde{p}(a_1, a_2 \mid q_1, q_2) + \frac{1}{F_{q_1 q_2}} s_{q_1 q_2}(a_1) t_{q_1 q_2}(a_2), & \text{if } F_{q_1 q_2} > 0, \\ \tilde{p}(a_1, a_2 \mid q_1, q_2), & \text{if } F_{q_1 q_2} = 0. \end{cases}$$

Then it is clear from Eqs. (4a) and (4b) that $p(a_1, a_2 \mid q_1, q_2) \geq \tilde{p}(a_1, a_2 \mid q_1, q_2)$ for every q_1, q_2, a_1, a_2 . Eqs. (1b) and (1c) follow from Eq. (4c). \square

Replace the variables $p(a_1, a_2 \mid q_1, q_2)$ by $x(a_1, a_2 \mid q_1, q_2) = \pi(q_1, q_2)p(a_1, a_2 \mid q_1, q_2)$. The resulting linear program is as follows.

$$\text{Maximize} \quad \sum_{a_1, a_2, q_1, q_2} R(a_1, a_2 \mid q_1, q_2) x(a_1, a_2 \mid q_1, q_2), \quad (5a)$$

$$\text{Subject to} \quad \sum_{a_2} x(a_1, a_2 \mid q_1, q_2) \leq \pi(q_1, q_2) p_1(a_1 \mid q_1), \quad \forall q_1 \in Q_1, q_2 \in Q_2, a_1 \in A_1, \quad (5b)$$

$$\sum_{a_1} x(a_1, a_2 \mid q_1, q_2) \leq \pi(q_1, q_2) p_2(a_2 \mid q_2), \quad \forall q_1 \in Q_1, q_2 \in Q_2, a_2 \in A_2, \quad (5c)$$

$$\sum_{a_1} p_1(a_1 \mid q_1) = 1, \quad \forall q_1 \in Q_1, \quad (5d)$$

$$\sum_{a_2} p_2(a_2 \mid q_2) = 1, \quad \forall q_2 \in Q_2, \quad (5e)$$

$$x(a_1, a_2 \mid q_1, q_2) \geq 0, \quad \forall q_1 \in Q_1, q_2 \in Q_2, a_1 \in A_1, a_2 \in A_2. \quad (5f)$$

By the strong duality theorem of linear programming, the linear program (5) has the same objective value as the following:

$$\text{Minimize} \quad \sum_{q_1} z_1(q_1) + \sum_{q_2} z_2(q_2), \quad (6a)$$

$$\text{Subject to} \quad y_1(q_1, q_2, a_1) + y_2(q_1, q_2, a_2) \geq R(a_1, a_2 \mid q_1, q_2), \quad \forall q_1 \in Q_1, q_2 \in Q_2, a_1 \in A_1, a_2 \in A_2, \quad (6b)$$

$$z_1(q_1) \geq \sum_{q_2} \pi(q_1, q_2) y_1(q_1, q_2, a_1), \quad \forall q_1 \in Q_1, a_1 \in A_1, \quad (6c)$$

$$z_2(q_2) \geq \sum_{q_1} \pi(q_1, q_2) y_2(q_1, q_2, a_2), \quad \forall q_2 \in Q_2, a_2 \in A_2, \quad (6d)$$

$$y_1(q_1, q_2, a_1) \geq 0, \quad \forall q_1 \in Q_1, q_2 \in Q_2, a_1 \in A_1, \quad (6e)$$

$$y_2(q_1, q_2, a_2) \geq 0, \quad \forall q_1 \in Q_1, q_2 \in Q_2, a_2 \in A_2. \quad (6f)$$

Note that the constraints (6c)–(6f) imply $z_1(q_1) \geq 0$ and $z_2(q_2) \geq 0$.

Let (z_1, z_2, y_1, y_2) be a feasible solution of the linear program (6). If $y_1(q_1, q_2, a_1) > 1$ for some q_1, q_2, a_1 , we can replace $y_1(q_1, q_2, a_1)$ by 1 without violating any constraints or increasing the objective value. The same holds for $y_2(q_1, q_2, a_2)$. Therefore, adding the constraints

$$\begin{aligned} y_1(q_1, q_2, a_1) &\leq 1, \quad \forall q_1 \in Q_1, q_2 \in Q_2, a_1 \in A_1, \\ y_2(q_1, q_2, a_2) &\leq 1, \quad \forall q_1 \in Q_1, q_2 \in Q_2, a_2 \in A_2 \end{aligned}$$

does not change the optimal value.

Replacing the variables $y_1(q_1, q_2, a_1)$ by $1 - \bar{y}_1(q_1, q_2, a_1)$ and $y_2(q_1, q_2, a_2)$ by $1 - \bar{y}_2(q_1, q_2, a_2)$, the following claim is immediate.

Claim 2. *The no-signaling value $w_{\text{ns}}(G)$ is equal to the optimal value of the following linear program.*

$$\text{Minimize} \quad \sum_{q_1} z_1(q_1) + \sum_{q_2} z_2(q_2), \quad (7a)$$

$$\text{Subject to} \quad \bar{y}_1(q_1, q_2, a_1) + \bar{y}_2(q_1, q_2, a_2) \leq 2 - R(a_1, a_2 \mid q_1, q_2), \quad \forall q_1 \in Q_1, q_2 \in Q_2, a_1 \in A_1, a_2 \in A_2, \quad (7b)$$

$$z_1(q_1) + \sum_{q_2} \pi(q_1, q_2) \bar{y}_1(q_1, q_2, a_1) \geq \pi_1(q_1), \quad \forall q_1 \in Q_1, a_1 \in A_1, \quad (7c)$$

$$z_2(q_2) + \sum_{q_1} \pi(q_1, q_2) \bar{y}_2(q_1, q_2, a_2) \geq \pi_2(q_2), \quad \forall q_2 \in Q_2, a_2 \in A_2, \quad (7d)$$

$$\bar{y}_1(q_1, q_2, a_1) \leq 1, \quad \forall q_1 \in Q_1, q_2 \in Q_2, a_1 \in A_1, \quad (7e)$$

$$\bar{y}_2(q_1, q_2, a_2) \leq 1, \quad \forall q_1 \in Q_1, q_2 \in Q_2, a_2 \in A_2, \quad (7f)$$

$$\bar{y}_1(q_1, q_2, a_1) \geq 0, \quad \forall q_1 \in Q_1, q_2 \in Q_2, a_1 \in A_1, \quad (7g)$$

$$\bar{y}_2(q_1, q_2, a_2) \geq 0, \quad \forall q_1 \in Q_1, q_2 \in Q_2, a_2 \in A_2, \quad (7h)$$

$$z_1(q_1) \geq 0, \quad \forall q_1 \in Q_1, \quad (7i)$$

$$z_2(q_2) \geq 0, \quad \forall q_2 \in Q_2. \quad (7j)$$

Lemma 5. *Let $G = (Q_1, Q_2, A_1, A_2, \pi, R)$ be a game and $0 \leq s < c \leq 1$. Consider the instance of the mixed packing and covering problem consisting of a constraint $\sum_{q_1} z_1(q_1) + \sum_{q_2} z_2(q_2) \leq s$ and the constraints (7b)–(7j). Let $\varepsilon = (c - s)/4$. Then,*

- (i) If $w_{\text{ns}}(G) \leq s$, this instance has a feasible solution.
- (ii) If $w_{\text{ns}}(G) \geq c$, this instance does not have a $(1 + \varepsilon)$ -approximate solution.

Proof. (i) Clear from Claim 2.

(ii) We prove the contrapositive. Assume that $(\bar{y}_1, \bar{y}_2, z_1, z_2)$ is a $(1 + \varepsilon)$ -approximate solution, and let

$$\begin{aligned}\bar{y}'_1(q_1, q_2, a_1) &= \frac{1}{1 + \varepsilon} \bar{y}_1(q_1, q_2, a_1), \\ \bar{y}'_2(q_1, q_2, a_2) &= \frac{1}{1 + \varepsilon} \bar{y}_2(q_1, q_2, a_2), \\ z'_1(q_1) &= z_1(q_1) + \varepsilon \pi_1(q_1), \\ z'_2(q_2) &= z_2(q_2) + \varepsilon \pi_2(q_2).\end{aligned}$$

Then $(\bar{y}'_1, \bar{y}'_2, z'_1, z'_2)$ satisfies (7b)–(7j), and

$$\sum_{q_1} z'_1(q_1) + \sum_{q_2} z'_2(q_2) = \sum_{q_1} z_1(q_1) + \varepsilon \sum_{q_1} \pi_1(q_1) + \sum_{q_2} z_2(q_2) + \varepsilon \sum_{q_2} \pi_2(q_2) \leq s + 3\varepsilon < c.$$

Therefore, $w_{\text{ns}}(G)$, or the optimal value of the linear program (7), is less than c . \square

Proof of Theorem 3. Apply Theorem 4 to the instance of the mixed packing and covering problem in Lemma 5. \square

Remark 1. It is easy to see that adding the constraints $z_1(q_1) \leq \pi_1(q_1)$ for $q_1 \in Q_1$ and $z_2(q_2) \leq \pi_2(q_2)$ for $q_2 \in Q_2$ to the instance of the mixed packing and covering problem in Lemma 5 does not change the feasibility or approximate feasibility. The resulting linear program has a constant “width” in the sense stated in Theorem 2.12 of Plotkin, Shmoys and Tardos [PST95] with a suitable tolerance vector. See [PST95] for relevant definitions. This gives an alternative proof of Theorem 3 which uses the algorithm of [PST95] instead of the algorithm of [You01].

Remark 2. Given Theorem 3, it is easy to approximate $w_{\text{ns}}(G)$ within additive error ε (rather than deciding whether $w_{\text{ns}}(G) \leq s$ or $w_{\text{ns}}(G) \geq c$) in parallel time polynomial in $\log|G|$ and $1/\varepsilon$ and total work polynomial in $|G|$ and $1/\varepsilon$. This can be done by trying all the possibilities of $s = k\varepsilon$ and $c = (k + 1)\varepsilon$ for integers k in the range $0 \leq k \leq 1/\varepsilon$ in parallel, or by using the binary search.

5 Concluding remarks

This paper gave the exact characterization of the simplest case of multi-prover interactive proof systems with no-signaling provers: $\text{MIP}^{\text{ns}}(2, 1) = \text{PSPACE}$. A natural direction seems to be to extend this result to show a PSPACE upper bound on a class containing $\text{MIP}^{\text{ns}}(2, 1)$. Below we discuss some hurdles in doing so.

- *More than two provers.* In the completely classical case, a many-prover one-round interactive proof system can be transformed to a two-prover one-round interactive proof system by using the oracularization technique, and therefore $\text{MIP}(\text{poly}, 1) \subseteq \text{MIP}(2, 1)$. The same transformation is not known to preserve soundness in the case of no-signaling provers even when the original proof system uses three provers.⁵

⁵The Magic Square game in [CHTW04] is a counterexample which shows that this transformation cannot be used alone to reduce the number of provers from three to two in the case of *entangled* provers because it sometimes transforms a three-prover game whose entangled value is less than 1 to a two-prover game whose entangled value is equal to 1 [IKPSY08]. The situation might be different in the case of no-signaling provers.

As a result, whether or not $\text{MIP}^{\text{ns}}(3, 1) \subseteq \text{MIP}^{\text{ns}}(2, 1)$ is unknown, and our result does not imply $\text{MIP}^{\text{ns}}(3, 1) \subseteq \text{PSPACE}$.

To extend the current proof to $\text{MIP}^{\text{ns}}(3, 1)$, the main obstacle is to extend Claim 1, which replaces equations by inequalities. It does not seem that an analogous claim can be proved for three provers by a straightforward extension of the current proof of Claim 1.

- *More than one round.* The proof of Claim 1 seems to work in the case of two-prover systems with polynomially many rounds. However, in a linear program corresponding to (6), an upper bound on the variables y_1 and y_2 becomes exponentially large and the current proof does not work even in the case of two-prover two-round systems with adaptive questions or two-prover $\omega(\log n)$ -round systems with non-adaptive questions.
- *Quantum verifier and quantum messages.* The notion of no-signaling strategies can be extended to the case of quantum messages [BGNP01, Gut09] ([BGNP01] uses the term “causal” instead of “no-signaling”). This allows us to define e.g. the class $\text{QMIP}^{\text{ns}}(2, 2)$ of languages having a *quantum* two-prover one-round (two-turn) interactive proof system with no-signaling provers. The class $\text{QMIP}^{\text{ns}}(2, 2)$ contains both $\text{MIP}^{\text{ns}}(2, 1)$ and $\text{QIP}(2)$, and it would be nice if the method of [JW09] and ours can be unified to give $\text{QMIP}^{\text{ns}}(2, 2) = \text{PSPACE}$. One obvious obstacle is how to extend the fast parallel algorithm in [JW09] for the special case of semidefinite programming to the case of $\text{QMIP}^{\text{ns}}(2, 2)$. Another obstacle is again Claim 1; the current proof of Claim 1 essentially constructs a joint probability distribution over (q_1, q_2, a_1, a_2) from its marginal distributions over (q_1, q_2, a_1) and (q_1, q_2, a_2) , and this kind of *state extension* is not always possible in the quantum case [Wer89, Wer90].

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References

- [AII06] David Avis, Hiroshi Imai, and Tsuyoshi Ito. On the relationship between convex bodies related to correlation experiments with dichotomic observables. *Journal of Physics A: Mathematical and General*, 39(36):11283–11299, September 2006.
- [ALMSS98] Sanjeev Arora, Carsten Lund, Rajeev Motwani, Madhu Sudan, and Mario Szegedy. Proof verification and the hardness of approximation problems. *Journal of the ACM*, 45(3):501–555, May 1998.
- [AS98] Sanjeev Arora and Shmuel Safra. Probabilistic checking of proofs: A new characterization of NP. *Journal of the ACM*, 45(1):70–122, January 1998.
- [Bab85] László Babai. Trading group theory for randomness. In *Proceedings of the Seventeenth Annual ACM Symposium on Theory of Computing*, pages 496–505, May 1985.
- [Bel64] J. S. Bell. On the Einstein-Podolsky-Rosen paradox. *Physics*, 1:195–200, 1964.
- [BFL91] László Babai, Lance Fortnow, and Carsten Lund. Non-deterministic exponential time has two-prover interactive protocols. *Computational Complexity*, 1(1):3–40, March 1991.

- [BFLS91] László Babai, Lance Fortnow, Leonid A. Levin, and Mario Szegedy. Checking computations in polylogarithmic time. In *Proceedings of the Twenty-Third Annual ACM Symposium on Theory of Computing*, pages 21–32, May 1991.
- [BGKW88] Michael Ben-Or, Shafi Goldwasser, Joe Kilian, and Avi Wigderson. Multi-prover interactive proofs: How to remove intractability assumptions. In *Proceedings of the Twentieth Annual ACM Symposium on Theory of Computing*, pages 113–131, May 1988.
- [BGNP01] David Beckman, Daniel Gottesman, M. A. Nielsen, and John Preskill. Causal and localizable quantum operations. *Physical Review A*, 64(052309), October 2001.
- [BHP08] Michael Ben-Or, Avinatan Hassidim, and Haran Pilpel. Quantum multi prover interactive proofs with communicating provers. In *Proceedings: Forty-Ninth Annual IEEE Symposium on Foundations of Computer Science (FOCS 2008)*, pages 467–476, October 2008.
- [BLMPPR05] Jonathan Barrett, Noah Linden, Serge Massar, Stefano Pironio, Sandu Popescu, and David Roberts. Nonlocal correlations as an information-theoretic resource. *Physical Review A*, 71(022101), February 2005.
- [Bor77] Allan Borodin. On relating time and space to size and depth. *SIAM Journal on Computing*, 6(4):733–744, December 1977.
- [CCL94] Jin-Yi Cai, Anne Condon, and Richard J. Lipton. PSPACE is provable by two provers in one round. *Journal of Computer and System Sciences*, 48(1):183–193, February 1994.
- [CGJ09] Richard Cleve, Dmitry Gavinsky, and Rahul Jain. Entanglement-resistant two-prover interactive proof systems and non-adaptive PIRs. *Quantum Information and Computation*, 9(7–8):648–656, July 2009.
- [CHTW04] Richard Cleve, Peter Høyer, Benjamin Toner, and John Watrous. Consequences and limits of nonlocal strategies. In *Proceedings: Nineteenth Annual IEEE Conference on Computational Complexity (CCC 2004)*, pages 236–249, June 2004.
- [DLTW08] Andrew C. Doherty, Yeong-Cherng Liang, Ben Toner, and Stephanie Wehner. The quantum moment problem and bounds on entangled multi-prover games. In *Proceedings: Twenty-Third Annual IEEE Conference on Computational Complexity (CCC 2008)*, pages 199–210, June 2008.
- [ESY84] Shimon Even, Alan L. Selman, and Yacov Yacobi. The complexity of promise problems with applications to public-key cryptography. *Information and Control*, 61(2):159–173, May 1984.
- [Fei91] Uriel Feige. On the success probability of the two provers in one-round proof systems. In *Proceedings of the Sixth Annual Structure in Complexity Theory Conference*, pages 116–123, June–July 1991.
- [FGLSS96] Uriel Feige, Shafi Goldwasser, László Lovász, Shmuel Safra, and Mario Szegedy. Interactive proofs and the hardness of approximating cliques. *Journal of the ACM*, 43(2):268–292, March 1996.
- [FL92] Uriel Feige and László Lovász. Two-prover one-round proof systems: Their power and their problems. In *Proceedings of the Twenty-Fourth Annual ACM Symposium on Theory of Computing*, pages 733–744, May 1992.

- [FRS94] Lance Fortnow, John Rompel, and Michael Sipser. On the power of multi-prover interactive protocols. *Theoretical Computer Science*, 134(2):545–557, November 1994.
- [GMR89] Shafi Goldwasser, Silvio Micali, and Charles Rackoff. The knowledge complexity of interactive proof systems. *SIAM Journal on Computing*, 18(1):186–208, 1989.
- [Gol05] Oded Goldreich. On promise problems (a survey in memory of Shimon Even [1935–2004]). Technical Report TR05-018, Electronic Colloquium on Computational Complexity, February 2005.
- [Gol08] Oded Goldreich. *Computational Complexity: A Conceptual Perspective*. Cambridge University Press, 2008.
- [Gut09] Gus Gutoski. Properties of local quantum operations with shared entanglement. *Quantum Information and Computation*, 9(9–10):739–764, September 2009.
- [Hol09] Thomas Holenstein. Parallel repetition: Simplifications and the no-signaling case. *Theory of Computing*, 5(Article 8):141–172, July 2009.
- [IKM09] Tsuyoshi Ito, Hirotada Kobayashi, and Keiji Matsumoto. Oracularization and two-prover one-round interactive proofs against nonlocal strategies. In *Proceedings: Twenty-Fourth Annual IEEE Conference on Computational Complexity (CCC 2009)*, pages 217–228, July 2009.
- [IKPSY08] Tsuyoshi Ito, Hirotada Kobayashi, Daniel Preda, Xiaoming Sun, and Andrew C.-C. Yao. Generalized Tsirelson inequalities, commuting-operator provers, and multi-prover interactive proof systems. In *Proceedings: Twenty-Third Annual IEEE Conference on Computational Complexity (CCC 2008)*, pages 187–198, June 2008.
- [JJUW09] Rahul Jain, Zhengfeng Ji, Sarvagya Upadhyay, and John Watrous. QIP = PSPACE. arXiv:0907.4737v2 [quant-ph], August 2009. Manuscript.
- [JUW09] Rahul Jain, Sarvagya Upadhyay, and John Watrous. Two-message quantum interactive proofs are in PSPACE. In *Proceedings: Fiftieth Annual IEEE Symposium on Foundations of Computer Science (FOCS 2009)*, October 2009. To appear. Manuscript at arXiv:0905.1300v1 [quant-ph].
- [JW09] Rahul Jain and John Watrous. Parallel approximation of non-interactive zero-sum quantum games. In *Proceedings: Twenty-Fourth Annual IEEE Conference on Computational Complexity (CCC 2009)*, pages 243–253, July 2009.
- [KKMTV07] Julia Kempe, Hirotada Kobayashi, Keiji Matsumoto, Ben Toner, and Thomas Vidick. Entangled games are hard to approximate. arXiv:0704.2903v2 [quant-ph], November 2007. Manuscript.
- [KKMTV08] Julia Kempe, Hirotada Kobayashi, Keiji Matsumoto, Ben Toner, and Thomas Vidick. Entangled games are hard to approximate. In *Proceedings: Forty-Ninth Annual IEEE Symposium on Foundations of Computer Science (FOCS 2008)*, pages 447–456, October 2008.
- [KKMV09] Julia Kempe, Hirotada Kobayashi, Keiji Matsumoto, and Thomas Vidick. Using entanglement in quantum multi-prover interactive proofs. *Computational Complexity*, 18(2):273–307, June 2009.
- [KM03] Hirotada Kobayashi and Keiji Matsumoto. Quantum multi-prover interactive proof systems with limited prior entanglement. *Journal of Computer and System Sciences*, 66(3):429–450, May 2003.

- [KRT08] Julia Kempe, Oded Regev, and Ben Toner. The unique games conjecture with entangled provers is false. In *Proceedings: Forty-Ninth Annual IEEE Symposium on Foundations of Computer Science (FOCS 2008)*, pages 457–466, October 2008.
- [KT85] Leonid A. Khalfin and Boris S. Tsirelson. Quantum and quasi-classical analogs of Bell inequalities. In *Symposium on the Foundations of Modern Physics*, pages 441–460, June 1985.
- [LFKN92] Carsten Lund, Lance Fortnow, Howard J. Karloff, and Noam Nisan. Algebraic methods for interactive proof systems. *Journal of the ACM*, 39(4):859–868, October 1992.
- [LS97] Dror Lapidot and Adi Shamir. Fully parallelized multi-prover protocols for NEXP-time. *Journal of Computer and System Sciences*, 54(2):215–220, April 1997.
- [NPA08] Miguel Navascués, Stefano Pironio, and Antonio Acín. A convergent hierarchy of semidefinite programs characterizing the set of quantum correlations. *New Journal of Physics*, 10(073013), July 2008.
- [Pap94] Christos H. Papadimitriou. *Computational Complexity*. Addison-Wesley, 1994.
- [PR94] Sandu Popescu and Daniel Rohrlich. Quantum nonlocality as an axiom. *Foundations of Physics*, 24(3):379–385, March 1994.
- [Pre] Daniel Preda. Private communication.
- [PST95] Serge A. Plotkin, David B. Shmoys, and Éva Tardos. Fast approximation algorithms for fractional packing and covering problems. *Mathematics of Operations Research*, 20(2):257–301, May 1995.
- [Ras85] Peter Rastall. Locality, Bell’s theorem, and quantum mechanics. *Foundations of Physics*, 15(9):963–972, September 1985.
- [Sha92] Adi Shamir. $IP = PSPACE$. *Journal of the ACM*, 39(4):869–877, October 1992.
- [Ton09] Ben Toner. Monogamy of nonlocal quantum correlations. *Proceedings of the Royal Society A*, 465(2101), January 2009.
- [Tsi06] Boris S. Tsirelson. Bell inequalities and operator algebras (Problem 33). In *Quantum Information: Open Problems*. July 2006. | <http://www.imaph.tu-bs.de/qi/problems/33.html> |.
- [Wat03] John Watrous. $PSPACE$ has constant-round quantum interactive proof systems. *Theoretical Computer Science*, 292(3):575–588, January 2003.
- [Wer89] Reinhard F. Werner. An application of Bell’s inequalities to a quantum state extension problem. *Letters in Mathematical Physics*, 14(4):359–363, May 1989.
- [Wer90] Reinhard F. Werner. Remarks on a quantum state extension problem. *Letters in Mathematical Physics*, 19(4):319–326, May 1990.
- [You01] Neal E. Young. Sequential and parallel algorithms for mixed packing and covering. In *Proceedings of the Forty-Second IEEE Symposium on Foundations of Computer Science (FOCS’01)*, pages 538–546, October 2001.