Definability of Combinatorial Functions and Their Linear Recurrence Relations

Extended Abstract

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Abstract. We consider functions of natural numbers which allow a combinatorial interpretation as density functions (speed) of classes of relational structures, such as Fibonacci numbers, Bell numbers, Catalan numbers and the like. Many of these functions satisfy a linear recurrence relation over $\mathbb{Z}$ or $\mathbb{Z}_m$ and allow an interpretation as counting the number of relations satisfying a property expressible in Monadic Second Order Logic (MSOL).
C. Blatter and E. Specker (1981) showed that if such a function f counts the number of binary relations satisfying a property expressible in MSOL then f satisfies for every $m \in \mathbb{N}$ a linear recurrence relation over $\mathbb{Z}_m$.
In this paper we give a complete characterization in terms of definability in MSOL of the combinatorial functions which satisfy a linear recurrence relation over $\mathbb{Z}$, and discuss various extensions and limitations of the Specker-Blatter theorem.

1 Introduction

1.1 The Speed of a Class of Finite Relational Structures

Let $\mathcal{P}$ be a graph property, and $\mathcal{P}^n$ be the set of graphs with vertex set $[n]$. We denote by $sp_{\mathcal{P}}(n) = |\mathcal{P}^n|$ the number of labeled graphs in $\mathcal{P}^n$. The function $sp_{\mathcal{P}}(n)$ is called the *speed of* $\mathcal{P}$, or in earlier literature the *density of* $\mathcal{P}$. Instead of graph properties we also study classes of finite relational structures $\mathcal{K}$ with relations $R_i : i = 1, \ldots, s$ of arity ρ_i. For the case of $s = 1$ and $\rho_1 = 1$ such classes can be identified with binary words over the positions $1, \ldots, n$.

The study of the function $sp_{\mathcal{K}}(n)$ has a rich literature concentrating on two types of behaviours of the sequence $sp_{\mathcal{K}}(n)$:

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– Recurrence relations
– Growth rate

Clearly, the existence of recurrence relations limits the growth rate.

(i) In formal language theory it was studied in N. Chomsky and M.P. Schuetzenberger [12] who proved that for $\mathcal{K} = L$, a regular language, the sequence $sp_L(n)$ satisfies a linear recurrence relation over $\mathbb{Z}$. This implies that the formal power series $\sum_n sp_L(n)X^n$ is rational. The paper [12] initiated the field Formal Languages and Formal Power Series.
Furthermore, it is known that L is regular iff L is definable in Monadic Second Order Logic MSOL, [11].
(ii) In C. Blatter and E. Specker [8] the case of $\mathcal{K}$ was studied, where $\rho_i \leq 2$ for all $i \leq s$ and $\mathcal{K}$ definable in MSOL. They showed that in this case for every $m \in \mathbb{N}$, the sequence $sp_{\mathcal{K}}(n)$ is ultimately periodic modulo m, or equivalently, that the sequence $sp_{\mathcal{K}}(n)$ satisfies a linear recurrence relation over $\mathbb{Z}_m$.
(iii) In E.R. Scheinerman and J. Zito [26] the function $sp_{\mathcal{P}}(n)$ was studied for *hereditary* graph properties $\mathcal{P}$, i.e., graph properties closed under induced subgraphs. They were interested in the growth properties of $sp_{\mathcal{P}}(n)$. The topic was further developed by J. Balogh, B. Bollobas and D. Weinreich in a sequence of papers, [10, 1, 2], which showed that only six classes of growth of $sp_{\mathcal{P}}(n)$ are possible, roughly speaking, constant, polynomial, or exponential growth, or growth in one of three factorial ranges. They also obtained similar results for monotone graph properties, i.e., graph properties closed under subgraphs, [3]. Early precursors of the study of $sp_{\mathcal{P}}(n)$ for monotone graph properties is [16], and for hereditary graph properties, [24].
We note that hereditary (monotone) graph properties $\mathcal{P}$ are characterized by a countable set $IForb(\mathcal{P})$ ($SForb(\mathcal{P})$) of forbidden induced subgraphs (subgraphs). In the case that $IForb(\mathcal{P})$ is finite, $\mathcal{P}$ is definable in First Order Logic, FOL, and in the case that $IForb(\mathcal{P})$ is MSOL-definable, also $\mathcal{P}$ is MSOL-definable. The same holds also for monotone properties.
(iv) The classification of the growth rate of $sp_{\mathcal{P}}(n)$ was extended to minor-closed classes in [6]. We note that minor-closed classes $\mathcal{P}$ are always MSOL-definable. This is due to the Robertson-Seymour Theorem, which states that they are characterized by a finite set $MForb(\mathcal{P})$ of forbidden minors.

One common theme of all the above cited papers is the connection between the definability properties of $\mathcal{K}$ and the arithmetic properties of the sequence $sp_{\mathcal{K}}(n)$. In this paper we concentrate on the relationship between definability of a class $\mathcal{K}$ of relational structures and the various linear recurrence relations $sp_{\mathcal{K}}(n)$ can satisfy.

1.2 Combinatorial Functions and Specker Functions

We would like to say that a function $f : \mathbb{N} \to \mathbb{N}$ is a combinatorial function if it has a combinatorial interpretation. One way of making this more precise is the following. We say that $\mathcal{K}$ is definable in $\mathcal{L}$ if there is a $\mathcal{L}$-sentence ϕ such that for every $\bar{R}$-structure $\mathfrak{A}$, $\mathfrak{A} \in \mathcal{K}$ iff $\mathfrak{A} \models \phi$. Then a function $f : \mathbb{N} \to \mathbb{N}$ is a combinatorial function if $f(n) = sp_{\mathcal{K}}(n)$ for some class of finite structures $\mathcal{K}$ definable in a suitable logical formalism $\mathcal{L}$. Here $\mathcal{L}$ could be FOL, MSOL or any interesting fragment of Second Order Logic, SOL. We assume the reader is familiar with these logics, cf. [14].

Definition 1 (Specker[1] function).
A function $f : \mathbb{N} \to \mathbb{N}$ is called a $\mathcal{L}^k$-Specker function *if there is a finite set of relation symbols $\bar{R}$ of arity at most k and a class of $\bar{R}$-structures $\mathcal{K}$ definable in $\mathcal{L}$ such that $f(n) = sp_{\mathcal{K}}(n)$.*

A typical non-trivial example is given by A. Cayley's Theorem from 1889, which says that $T(n) = n^{n-2}$ can be interpreted as the number of labeled trees on n vertices. Another example are the Bell numbers B_n which count the number of equivalence relations on n elements.

In this paper we study under what conditions the Specker function given by the sequence $sp_{\mathcal{K}}(n)$ satisfies a linear recurrence relation.

Example 1

(i) *The number of binary relations on $[n]$ is 2^{n^2}, and the number of linear orders on $[n]$ is $n!$. Both are FOL^2-Specker functions. $n!$ satisfies the linear recurrence relation $n! = n \cdot (n-1)!$. We note the coefficient in the recurrence relation is not constant.*

(ii) *The Stirling numbers of the first kind denoted $\left[{n \atop k}\right]$ are defined as the number of ways to arrange n objects into k cycles. It is well known that for $n > 0$ we have $\left[{n \atop 1}\right] = (n-1)!$. Specker functions are functions in one variable. For fixed k, $\left[{n \atop k}\right]$ is a FOL^2-Specker function. Using our main results, we shall discuss Stirling numbers in more detail in Section 4, Proposition 20 and Corollary 21.*

(iii) *For the functions $2^{n^2}, n^{n-2}$ and $n!$ no linear recurrence relation with constant coefficients exists, because functions defined by linear recurrence relations with constant coefficients grow not faster than $2^{O(n)}$. However, for every $m \in \mathbb{N}$ we have that 2^{n^2} satisfies a linear recurrence relation over $\mathbb{Z}_m$, where the coefficients depend on m.*

(iv) *The Catalan numbers C_n count the number of valid arrangements of n pairs of parentheses. C_n is even iff n is not of the form $n = 2^k - 1$ for some $k \in \mathbb{N}$ ([20]). Therefore, the sequence C_n cannot be ultimately periodic modulo 2. We discuss the Catalan numbers in Section 4.*

For R unary we can interpret $\langle [n], R \rangle$ as a binary word where position i is occupied by letter 1 if $i \in R$ and by letter 0 otherwise. Similarly, For $\bar{R} = (R_1, \ldots, R_s)$ which consists of unary relations only we can interpret $\langle [n], R_1, \ldots, R_s \rangle$ as a word over an alphabet of size 2^s. With this way of viewing

[1] E. Specker studied such functions in the late 1970ties in his lectures on topology at ETH-Zurich.

languages we have the celebrated theorem of R. Büchi (and later but independently of C. Elgot and B. Trakhtenbrot), cf. [21, 13] states:

Theorem 2 *Let $\mathcal{K}$ be a language. Then $\mathcal{K}$ is regular iff $\mathcal{K}'$ is definable in* MSOL *given the natural order $<_{nat}$ on $[n]$.*

From Theorem 2 and [12] we get immediately:

Proposition 3 *If $f(n) = sp_{\mathcal{K}}(n)$ is definable in* MSOL^1, MSOL *with unary relation symbols only, given the natural order $<_{nat}$ on $[n]$, then it satisfies a linear recurrence relation over $\mathbb{Z}$*

$$sp_{\mathcal{K}}(n) = \sum_{j=1}^{d} a_j \cdot sp_{\mathcal{K}}(n-j)$$

with constant coefficients,

We say a function $f : \mathbb{N} \to \mathbb{N}$ is ultimately periodic over $\mathcal{R} = \mathbb{Z}$ or over $\mathcal{R} = \mathbb{Z}_m$ if there exist $i, n_0 \in \mathbb{N}$ such that for every $n \geq n_0$, $f(n+i) = f(n)$ over $\mathcal{R}$. It is well-known that f is ultimately periodic over $\mathbb{Z}_m$ iff it satisfies a linear recurrence relation with constant coefficients over $\mathbb{Z}_m$. We note that if f satisfies a linear recurrence over $\mathbb{Z}$ then it also satisfies a linear recurrence over $\mathbb{Z}_m$ for every m. C. Blatter and E. Specker proved the following remarkable but little known theorem in [8],[9],[29].

Theorem 4 (Specker-Blatter Theorem) *If $f(n) = sp_{\mathcal{K}}(n)$ is definable in* MSOL^2, MSOL *with unary and binary relation symbols only, then for every $m \in \mathbb{N}$, $f(n)$ satisfies a linear recurrence relation with constant coefficients*

$$sp_{\mathcal{K}}(n) \equiv \sum_{j=1}^{d_m} a_j^{(m)} sp_{\mathcal{K}}(n-j) \ (mod\, m)$$

and hence is ultimately periodic over $\mathbb{Z}_m$.

In [18] it was shown that in Proposition 3 and in Theorem 4 the logic MSOL can be augmented by modular counting quantifiers.

Furthermore, E. Fischer showed in [17]

Theorem 5 *For every prime $p \in \mathbb{N}$ there is an* FOL^4*-definable function $sp_{\mathcal{K}_p}(n)$, where $\mathcal{K}_p$ consists of finite (E, R)-structures with E binary and R quaternary, which is not ultimately periodic modulo p.*

The definability status of various combinatorial functions from the literature will be discussed in Section 4.

1.3 Formal Power Series

Our main result can be viewed as related to the theory of generating functions for formal languages, cf. [25, 7] Let A be a commutative semi-ring with unity and denote by $A\langle\langle x\rangle\rangle$ the semi-ring of formal power series F in one variable over A

$$F = \sum_{n=0}^{\infty} f(n)x^n$$

where f is a function from $\mathbb{N}$ to A. A power series F in on variable is an *A-rational series* if it is in the closure of the polynomials over A by the sum, product and star operations, where the star operation F^* is defined as $F^* = \sum_{i=0}^{\infty} F^i$.

We say a function $f : \mathbb{N} \to A$ is A-rational if $\{f(n)\}_{n=0}^{\infty}$ is the sequence of coefficients of an A-rational series F. We will be interested in the cases of $A = \mathbb{N}$ and $A = \mathbb{Z}$. It is trivial that every $\mathbb{N}$-rational function is a $\mathbb{Z}$-rational function. It is well-known that $\mathbb{Z}$-rational functions are exactly those functions $f : \mathbb{N} \to \mathbb{Z}$ which satisfy linear recurrence relations over $\mathbb{Z}$. Furthermore, $\mathbb{Z}$-rational functions can also be characterized as those functions f which are the coefficents of the power series of $P(x)/Q(X)$, where $P, Q \in \mathbb{Z}[x]$ are polynomials and $Q(0) = 1$.

We aim to study Specker functions, which are by definition functions over $\mathbb{N}$. Clearly, every $\mathbb{N}$-rational function is over $\mathbb{N}$, while the $\mathbb{Z}$-rational functions may take negative values. Those non-negative $\mathbb{Z}$-rational functions which are $\mathbb{N}$-rational were characterized by Soittola, cf. [28]. However, there are non-negative $\mathbb{Z}$-rational series which are not $\mathbb{N}$-rational, cf. [15, 4].

There are strong ties between regular languages and rational series. From Theorem 2 and [25, Thm II.5.1] it follows that:

Proposition 6 *Let $\mathcal{K}$ be a language. If $\mathcal{K}$ is definable in* MSOL *given the natural order $<_{nat}$ on $[n]$, then $sp_{\mathcal{K}}$ is $\mathbb{N}$-rational.*

1.4 Extending MSOL and Order Invariance

In this paper we investigate the existence of linear and modular linear recurrence relations of Specker functions for the case where $\mathcal{K}$ is definable in logics $\mathcal{L}$ which are sublogics of SOL and extend MSOL.

$C_{a,b}$MSOL is the extension of MSOL with modular counting quantifiers "the number of elements x satisfying $\phi(x)$ equals a modulo b". $C_{a,b}$MSOL is a fragment of *SOL* since the modular counting quantifiers are definble in SOL using a linear order of the universe which is existentially quantifed.

Example 7 *The Specker function which counts the number of Eulerian graphs with n vertices is not* MSOL*-definable. It is definable in* $C_{a,b}$MSOL *and indeed $b = 2$ suffices.*

We now look at the case where $[n]$ is equipped with a linear order.

Definition 2 (Order invariance).

(i) *A class* $\mathcal{D}$ *of* ordered $\bar{R}$-structures *is a class of* $\bar{R} \cup \{<_1\}$*-structures, where for every* $\mathfrak{A} \in \mathcal{D}$ *the interpretation of the relation symbol* $<_1$ *is always a linear order of the universe of* $\mathfrak{A}$*.*

(ii) *An* $\mathcal{L}$ *formula* $\phi(\bar{R}, <_1)$ *for ordered* $\bar{R}$*-structures is* truth-value order invariant (t.v.o.i.) *if for any two structures* $\mathfrak{A}_i = \langle [n], <_i, \bar{R} \rangle$ *(*$i = 1, 2$*) we have that* $\mathfrak{A}_1 \models \phi$ *iff* $\mathfrak{A}_2 \models \phi$*. Note* $\mathfrak{A}_1$ *and* $\mathfrak{A}_2$ *differ only in the linear orders* $<_1$ *and* $<_2$ *of* $[n]$*. We denote by* $\mathrm{TV}\mathcal{L}$ *the set of* $\mathcal{L}$*-formulas for ordered* $\bar{R}$*-structures which are t.v.o.i. We consider* $\mathrm{TV}\mathcal{L}$ *formulas as formulas for* $\bar{R}$*-structures.*

(iii) *For a class of ordered structures* $\mathcal{D}$*, let* $osp_{\mathcal{D}}(n, <_1) =$

$$|\{(R_1, \ldots, R_s) \subseteq [n]^{\rho(1)} \times \ldots \times [n]^{\rho(s)} : \langle [n], <_1, R_1, \ldots, R_s \rangle \in \mathcal{D}\}|$$

A function $f : \mathbb{N} \to \mathbb{N}$ *is called an* $\mathcal{L}^k$-ordered Specker function *if there is a class of ordered* $\bar{R}$*-structures* $\mathcal{D}$ *of arity at most* k *definable in* $\mathcal{L}$ *such that* $f(n) = osp_{\mathcal{D}}$*.*

(iv) *A function* $f : \mathbb{N} \to \mathbb{N}$ *is called a* counting order invariant (c.o.i.) $\mathcal{L}^k$-Specker function *if there is a finite set of relation symbols* $\bar{R}$ *of arity at most* k *and a class of ordered* $\bar{R}$*-structures* $\mathcal{D}$ *definable in* $\mathcal{L}$ *such that for all linear orders* $<_1$ *and* $<_2$ *of* $[n]$ *we have* $f(n) = osp_{\mathcal{D}}(n, <_1) = osp_{\mathcal{D}}(n, <_2)$*.*

Example 8

(i) *Every formula* $\phi(\bar{R}, <_1) \in \mathrm{TVSOL}^k$ *is equivalent to the formula* $\psi(\bar{R}) = \exists <_1 \phi(\bar{R}, <_1) \wedge \phi_{linOrd}(<_1) \in \mathrm{SOL}^k$*, where* $\phi_{linOrd}(<_1)$ *says* $<_1$ *is a linear ordering of the universe.*

(ii) *Every* TVMSOL^k*-Specker function is also a counting order invariant* MSOL^k*-Specker function.*

(iii) *We shall see in Section 3 that there are counting order invariant* MSOL^2*-definable Specker functions which are not* TVMSOL^2*-definable.*

The following proposition is folklore:

Proposition 9 *Every formula in* $C_{a,b}\mathrm{MSOL}^k$ *is equivalent to a formula in* TVMSOL^k*.*

Proof. We give a sketch of the proof for the $C_{a,b}\mathrm{MSOL}$ formula $\phi_{even} = C_{0,2}(x = x)$, which says the size of the universe is even. The general proof is similar. ϕ_{even} can be written as $\phi(\bar{R}, <_1) = \exists U \phi_{min}(U) \wedge \forall x \forall y (\phi_{succ}(x, y) \to (x \in U \leftrightarrow y \notin U)) \wedge \forall x (x \in U \to \exists y \; x <_1 y)$ where $\phi_{min}(U) = \forall x \left((\neg \exists y \; y <_1 x) \to x \in U \right)$ says the minimal element x in the order $<_1$ belongs to U, and $\phi_{succ}(x, y) = (x <_1 y) \wedge \neg \exists z (x <_1 z \wedge z <_1 y)$ says y is the successor of x in $<_1$.

1.5 Main Results

Our first result is a characterization of functions over the natural numbers which satisfy a linear recurrence relation over $\mathbb{Z}$.

Theorem 10 *Let f be a function over $\mathbb{N}$. Then f satisfies a linear recurrence relation over $\mathbb{Z}$ iff $f = f_1 - f_2$ is the difference of two counting order invariant* MSOL^1*-Specker functions.*

In the terminology of rational functions we get the following corollary:

Corollary 11 *Let f be a function $\mathbb{N} \to \mathbb{N}$. Then f is $\mathbb{Z}$-rational iff f is the difference of two $\mathbb{N}$-rational functions.*

In the proof of Theorem 10 we introduce the notion of Specker polynomials, which can be thought of as a special case of graph polynomials where graphs are replaced by linear orders.

Next we show that the Specker-Blatter Theorem cannot be extended to counting order invariant Specker functions which are definable in MSOL^2. More precisely:

Proposition 12 *Let $E_{2,=}(n)$ be the number of equivalence relations with two equal-sized equivalence classes. Then $E_{2,=}(2n) = \frac{1}{2}\binom{2n}{n}$, and $E_{2,=}(2n+1) = 0$. $E_{2,=}$ is a counting order invariant* MSOL^2*-definable. However, it does not satisfy a linear recurrence relation over $\mathbb{Z}_2$, since it is not ultimately periodic modulo 2. To see this note that $E_{2,=}(2n) = 0 \pmod 2$ iff n is an even power of 2.*

In Section 4 we shall show in Corollary 22 the same also for the Catalan number.

However, if we require that the defining formula ϕ of a Specker function is itself order invariant, i.e. $\phi \in \mathrm{TVMSOL}^2$, then the Specker-Blatter Theorem still holds.

Theorem 13 *Let f be a* TVMSOL^2*-Specker function. Then, for all $m \in \mathbb{N}$ the function f satisfies a modular linear recurrence relation modulo m.*

Table 1 summarizes the relationship between definablity of a $\mathcal{L}^k$-Specker function $f(n)$ and existance of linear recurrence. We denote by MLR that $f(n)$ has a modular linear recurrence (for every $m \in \mathbb{N}$) and by LR that $f(n)$ satisfies a linear recurrence over $\mathbb{Z}$. We write $NO\ LR$ (respectively $NO\ MLR$) to indicate that there is some $\mathcal{L}^k$-Specker function without a linear recurrence over $\mathbb{Z}$ (respectively $\mathbb{Z}_m$, for some $m \in \mathbb{N}$). The entries in bold face are new.

k	MSOL^k	$C_{a,b}\mathrm{MSOL}^k$	TVMSOL^k	c.o.i.MSOL^k
4	$No\ MLR$	$No\ MLR$	$No\ MLR$	$No\ MLR$
3	?	?	?	?
2	MLR (**No LR**)	MLR (**No LR**)	MLR (**No LR**)	**No MLR**
1	**All functions with LR**			

Table 1. Linear recurrences and definability of $\mathcal{L}^k$-Specker functions

2 Linear Recurrence Relations for $\mathcal{L}^k$-Specker Functions

To prove Theorem 10 we first introduce Specker polynomials and prove a generalized version of one direction of the theorem in subsection 2.1. We finish this direction of the proof of Theorem 10 in subsection 2.2. The other direction of Theorem 10 is easy and is also given in subsection 2.2.

2.1 $\mathcal{L}^k$-Specker polynomials

Definition 3.

(i) A $\mathcal{L}^k$-Specker polynomial $A(n,\bar{x})$ in indeterminate set $\bar{x}$ has the form

$$\sum_{R_1:\Phi_1(R_1)} \cdots \sum_{R_t:\Phi_t(R_1,\dots,R_t)} \left(\prod_{v_1,\dots,v_k:\Psi_1(\bar{R},\bar{v})} x_{m_1} \cdots \prod_{v_1,\dots,v_k:\Psi_l(\bar{R},\bar{v})} x_{m_l} \right)$$

where $\bar{v}$ stands for $(v_1,\dots,v_k)$, $\bar{R}$ stands for $(R_1,\dots,R_t)$ and the R_i's are relation variables of arity ρ_i at most k. The R_i's range over relations of arity k over $[n]$ and the v_i range over elements of $[n]$ satisfying the iteration formulas $\Phi_i, \Psi_i \in \mathcal{L}$.

(ii) Simple ordered $\mathcal{L}^k$-Specker polynomials and order invariance thereof are defined analogously to Specker functions.

Every Specker function can be viewed as a Specker polynomial in zero indeterminates. Conversely, if we evaluate a Specker polynomial at $x = 1$ we get a Specker function.

In this subsection we prove a stronger version of Theorem 10.

Lemma 14 *Let $A(n,\bar{z})$ be a c.o.i.* MSOL^1*-Specker polynomial with indeterminates $\bar{z} = (z_1,\dots,z_s)$ and let $h_1(\bar{w}),\dots,h_s(\bar{w}) \in \mathbb{Z}[\bar{w}]$. Let $A(n,(h_1(\bar{w}),\dots,h_s(\bar{w})))$ denote the variable subtitution in $A(n,\bar{z})$ where for $i \in [s]$, z_i is substituted to $h_i(\bar{w})$. Then $A\left(n,\bar{h}\right)$ is an integer evaluation of a c.o.i.* MSOL^1*-Specker polynomial.*

Proof. We look at $A(n,\bar{z})$ with z_1 substituted to the polynomial

$$h_1(\bar{w}) = \sum_{j=1}^{d} c_j w_1^{\alpha_{j1}} \cdots w_t^{\alpha_{jt}}$$

where $d, \alpha_{11},\dots,\alpha_{dt} \in \mathbb{N}$ and $c_1,\dots,c_d \in \mathbb{Z}$. The c.o.i. MSOL$^1-$Specker polynomial $A(n,\bar{z})$ is given by

$$\sum_{R_1:\Phi_1(R_1)} \cdots \sum_{R_m:\Phi_m(R_1,\dots,R_{m-1})} \left(\prod_{v_1:\Psi_1(\bar{R},v_1)} z_1 \cdots \prod_{v_1:\Psi_s(\bar{R},v_1)} z_s \right)$$

so substituting z_1 to $h_1(\bar{w})$ we get $A(n, (h_1(w), z_2, \ldots, z_s)) =$

$$\sum_{R_1:\Phi_1(R_1)} \cdots \sum_{R_m:\Phi_m(R_1,\ldots,R_{m-1})} \left(\prod_{v_1:\Psi_1(\bar{R},v_1)} h_1(\bar{w}) \cdots \prod_{v_1:\Psi_s(\bar{R},v_1)} z_s \right).$$

We note that for every $\alpha(v) \in$ MSOL we can define an MSOL formula with d unary relation variables $\phi_{Part(\alpha)}(U_1 \ldots, U_d)$ which holds iff $U_1, \ldots, U_d$ are a partition of the set of elements of $[n]$ which satisfy $\alpha(v)$. Then
$A\left(n, (h_1(\bar{w}), z_2, \ldots, z_s)\right) =$

$$\sum_{R_1:\Phi_1(R_1)} \cdots \sum_{R_m:\Phi_m(R_1,\ldots,R_{m-1})} \sum_{U_1,\ldots,U_d:\phi_{Part(\Psi_1)}(\bar{U})} \left(\prod_{v_1:\Psi_2(\bar{R},v_1)} z_2 \cdots \right.$$
$$\left. \prod_{v_1:\Psi_s(\bar{R},v_1)} z_s \prod_{v_1:v_1\in U_1} c_1 w_1^{\alpha_{11}} \cdots w_t^{\alpha_{1t}} \cdots \prod_{v_1:v_1\in U_d} c_d w_1^{\alpha_{d1}} \cdots w_t^{\alpha_{dt}} \right)$$

Next, we note for any formula θ,

$$\prod_{v_1:\theta} c_j w_1^{\alpha_{j1}} \cdots w_t^{\alpha_{jt}} = \prod_{v_1:\theta} c_j \overbrace{\prod_{v_1:\theta} w_1 \cdots \prod_{v_1:\theta} w_1}^{\alpha_{j1} \text{ times}} \cdots \overbrace{\prod_{v_1:\theta} w_t \cdots \prod_{v_1:\theta} w_t}^{\alpha_{jt} \text{ times}}$$

We now replace all c_j with new indeterminates w'_j and thus obtain that
$A\left(n, (h_1(\bar{w}), z_2, \ldots, z_s)\right)$ is an evaluation of an c.o.i. MSOL^1-Specker polynomial.

Doing the same for the other z_i we get that $A\left(n, (h_1(\bar{w}), \ldots, h_s(\bar{w}))\right)$ is an evaluations of an o.i. MSOL^1-definable Specker polynomial, as required.

Theorem 15 *Let $A_n(\bar{x})$ be a sequence of polynomials with a finite indeterminate set $\bar{x} = (x_1, \ldots, x_s)$ which satisfies a linear recurrences over $\mathbb{Z}$. Then there exists a c.o.i* MSOL^1*-Specker polynomial $A'(n, \bar{x}, \bar{y})$ such that $A_n(\bar{x}) = A'(n, \bar{x}, \bar{a})$ where $\bar{a} = (a_1, \ldots, a_l)$ and $a_i \in \mathbb{Z}$ for $i = 1, \ldots, l$.*

Proof. Let $A_n(\bar{x})$ be given by a linear recurrence

$$A_n(\bar{x}) = \sum_{i=1}^{r} f_i(\bar{x}) \cdot A_{n-i}(\bar{x}),$$

where $f_i(\bar{x}) \in \mathbb{Z}[\bar{x}]$ and initial conditions $A_1(\bar{x}), \ldots, A_r(\bar{x}) \in \mathbb{Z}[\bar{x}]$. To write $A_n(\bar{x})$ as a c.o.i. MSOL^1-Specker polynomials, we sum over the paths of the recurrence tree. A path in the recurrence tree corresponds to the successive application of the recurrence $A_n(\bar{x}) \to A_{n-i_1}(\bar{x}) \to A_{n-i_1-i_2}(\bar{x}) \to \cdots \to A_{n-i_1-\ldots-i_l}(\bar{x})$ where $i_1, \ldots, i_l \in [r]$ and $A_{n-i_1-\ldots-i_l}(\bar{x})$ is an initial condition.

In the following, the U_i for $i \in [r]$ stand for the vertices in the path, I_i for $i \in [r]$ stand for initial conditions $A_i(\bar{x})$, and S stands for all those elements of $[n]$ skipped by the recurrence. We may write $A_n(\bar{x})$ as

$$A_n(\bar{x}) = \sum_{\bar{U},\bar{I},S:\phi_{rec}(\bar{U},\bar{I},S)} \prod_{v:v\in U_1} f_1(\bar{x}) \cdots \prod_{v:v\in U_r} f_r(\bar{x}) \prod_{v:v\in I_1} A(1,\bar{x}) \cdots \prod_{v:v\in I_r} A(r,\bar{x})$$

where $\phi_{rec}(\bar{U},\bar{I},S)$ says

- $\phi_{Part}(\bar{U},\bar{I},S)$ holds, i.e. $\bar{U},\bar{I},S$ is a partition of $[n]$,
- $n \in \bigcup U_i$, i.e. the path in the recurrence tree starts from n,
- $|\bigcup_{i=1}^{r} I_i| = 1$, i.e. the path reaches exactly one initial condition
- if $v \in [n] - [r]$, then $v \notin \bigcup_{i=1}^{r} I_i$, i.e. the path may not reach an initial condition until $v \in [r]$,
- if $v \in [r]$, then $v \notin \bigcup_{i=1}^{r} U_i$, i.e. the path ends when reaching the initial conditions, and
- for every $v \in U_i$, $\{v-1,\ldots,v-(i-1)\} \subseteq S$ and $v - i \in \bigcup_{i=1}^{r} (U_i \cup I_i)$, i.e. the next element in the path is $v - i$.

The formula ϕ_{rec} is MSOL definable using the given order. Let $B(n,\bar{x})$ be

$$B(n,\bar{z}) = \sum_{\bar{U},\bar{I},S:\phi_{rec}(\bar{U},\bar{I},S)} \prod_{v:v\in U_1} z_1 \cdots \prod_{v:v\in U_r} z_r \prod_{v:v\in I_1} z_{r+1} \cdots \prod_{v:v\in I_r} z_{2r}$$

then $B(n,\bar{z})$ is a c.o.i. MSOL^1-Specker polynomial. By Lemma 14, substituting z_i to $f_i(\bar{x})$ for $i \in [r]$ and to $A_{i-r}(\bar{x})$ for $i \in [2r]\backslash[r]$, we have that $B\left(n, (f_1(\bar{x}),\ldots,f_r(\bar{x}), A(1,\bar{x}),\ldots,A(r,\bar{x}))\right) = A_n(\bar{x})$ is an evaluation in $\mathbb{Z}$ of a c.o.i. MSOL^1-Specker polynomial.

2.2 Proof of Theorem 10

Let $f = f_1 - f_2$ and f_1 and f_2 be c.o.i MSOL^1-Specker functions. By Proposition 3 together with Theorem 2 we have that f_1 and f_2 satisfy linear recurrence relations over $\mathbb{Z}$. It is well known that finite sums, differences and products of functions satisfying a linear recurrence relation again satisfy a linear recurrence relation, cf. [22, Chapter 8] or [27, Chapter 6]. Thus, $f = f_1 - f_2$ satisfies a linear recurrence relation over $\mathbb{Z}$.

Conversely, if f satisfies a linear recurrence relation over $\mathbb{Z}$, then by Theorem 15, f is given by an evaluation $\bar{a} = (a_1,\ldots,a_l)$ where $a_i \in \mathbb{Z}$ for $i = 1,\ldots,l$ of a c.o.i. MSOL^1 Specker polynomial $A(n,\bar{y})$ in variables y_i. We have to show that f is a difference of two c.o.i. MSOL^1-Specker functions. For the sake of simplicity we will show this only for the case of a MSOL^1-Specker polynomial in one indeterminate,

$$A(n,y) = \sum_{R:\Phi(R)} \prod_{v_1:\Psi(R,v_1)} y$$

The general case is similar. We may write $A(n,y)$ as

$$A(n,y)=\sum_{R,Y:\Phi(R)\wedge\Psi'(R,Y)}\prod_{v:v\in Y}y$$

where $\Psi'(Y)=\forall v\,(v\in Y\leftrightarrow\Psi(R,v))$. For $a>0$ we can write $\prod_{v:v\in Y}a$ as

$$\prod_{v:v\in Y}a=a^{|Y|}=|\{Z_1,\dots Z_a\mid Z_1,\dots,Z_a \text{ form a partition of } Y\}|\,.$$

So,

$$A(n,a)=\sum_{R,Y,\bar{Z}:\beta_a(R,Y,\bar{Z})}1=\left|\left\{R,Y,\bar{Z}\mid\beta_a(R,Y,\bar{Z})\right\}\right|$$

where $\beta_a(R,Y,\bar{Z})=\Phi(R)\wedge\Psi'(R,Y)\wedge\phi_{part}(Y,\bar{Z})$ and $\phi_{part}(Y,Z_1,\dots,Z_a)$ says $Z_1,\dots.Z_a$ form a partition of Y. We note that ϕ_{part} is definable in MSOL. For $a=0$,

$$A(n,a)=\sum_{R:\gamma(R)}1=|\{R\mid\gamma(R)\}|\,.,$$

where $\gamma(R)=\Phi(R)\wedge\forall v_1\neg\Psi(R,v_1)$. Thus, since the constant function 0 is definable in MSOL, we get that if $a\geq 0$ then $A(n,a)$ is the difference of two c.o.i MSOL^1-Specker functions.

For $a<0$ we have

$$A(n,a)=\sum_{R,Y:\Phi(R)\wedge\Psi'(R,Y)}\prod_{v:v\in Y}|a|\prod_{v:v\in Y}(-1).$$

As above, we may write $A(n,a)$ as

$$A(n,a)=\sum_{R,Y,\bar{Z}:\beta_{|a|}(R,Y,\bar{Z})}\prod_{v:v\in Y}(-1)$$

and we have

$$A(n,a)=\sum_{R,Y,\bar{Z}:\alpha_{Even}(Y)\wedge\beta_{|a|}(R,Y,\bar{Z})}1-\sum_{R,Y,\bar{Z}:\neg\alpha_{Even}(Y)\wedge\beta_{|a|}(R,Y,\bar{Z})}1,$$

where $\alpha_{Even}(Y)$ says $|Y|$ is even. Thus, $A(n,a)$ is given by $A(n,a)=$

$$\left|\left\{R,Y,\bar{Z}\mid\alpha_{Even}(Y)\wedge\beta_{|a|}(R,Y,\bar{Z})\right\}\right|-\left|\left\{R,Y,\bar{Z}\mid\neg\alpha_{Even}(Y)\wedge\beta_{|a|}(R,Y,\bar{Z})\right\}\right|$$

Since α_{Even} is definable in MSOL given an order, as discussed in example 8, we get that $A(n,a)$ is a difference of two c.o.i MSOL^1-Specker functions for $a<0$.

3 Modular Linear Recurrence Relations

In this section we prove Theorem 13, the extension of the Specker-Blatter Theorem to TVMSOL^2-Specker functions. We also prove Proposition 12, which shows Theorem 13 cannot be extended to c.o.i. MSOL^2-Specker functions.

3.1 Specker Index

We say a structure $\mathcal{A} = \left\langle [n], \bar{R}, a \right\rangle$ is a pointed $\bar{R}$-structure if is consists of a universe $[n]$, relations $R_1, \ldots, R_k$, and an element $a \in [n]$ of the universe. We now define a binary operation on pointed structures. Given two pointed structures $\mathcal{A}_1 = \left\langle [n_1], \bar{R}^1, a_1 \right\rangle$ and $\mathcal{A}_2 = \left\langle [n_2], \bar{R}^2, a_2 \right\rangle$, let $Subst\left(\mathcal{A}_1, \mathcal{A}_2\right)$ be a new pointed structure $Subst(\mathcal{A}_1, \mathcal{A}_2) = \mathcal{B}$ where $\mathcal{B} = \left\langle [n_1] \sqcup [n_2] - \{a_1\}, \bar{R}, a_2 \right\rangle$, such that the relations in $\bar{R}$ are defined as follows. For every $R_i \in \bar{R}$ of arity r, $R_i = \left(R_i^1 \cap \left([n_1] - \{a_1\}\right)^r\right) \cup R_i^2 \cup I$, where I contains all possibilities of replacing occurrences of a_1 in R_i^1 with members of $[n_2]$.

Similarly, we define $Subst(\mathcal{A}_1, \mathcal{A}_2)$ for a pointed structure $\mathcal{A}_1$ and a structure $\mathcal{A}_2 = \left\langle [n], \bar{R} \right\rangle$ (which is not pointed). Let $\mathcal{C}$ be a class of possibly pointed $\bar{R}-$structures. We define an equivalence relation between $\bar{R}-$structures:

- We say $\mathcal{A}_1$ and $\mathcal{A}_2$ are equivalent, denoted $\mathcal{A}_1 \sim_{Su(\mathcal{C})} \mathcal{A}_2$ if for every pointed structure $\mathcal{D}$ we have that $Subst(\mathcal{D}, \mathcal{A}_1) \in \mathcal{C}$ if and only if $Subst\left(\mathcal{D}, \mathcal{A}_2\right) \in \mathcal{C}$.
- The *Specker index* of $\mathcal{C}$ is the number of equivalence classes of $\sim_{Su(\mathcal{C})}$.

We use in the next subsection the following lemmas by Specker [29]:

Lemma 16 *Let $\mathcal{C}$ be a class of $\bar{R}-$structures of finite Specker index with all relation symbols in $\bar{R}$ at most binary. Then $f_{\mathcal{C}}(n)$ satisfies modular linear recurrence relations for every $m \in \mathbb{N}$.*

Lemma 17 *If $\mathcal{C}$ is a class of $\bar{R}$-structures which is* MSOL^2-*definable, then $\mathcal{C}$ has finite Specker index.*

3.2 Proof of Theorem 13

We prove the following lemma:

Lemma 18 *If $\mathcal{C}$ is a class of $\bar{R}$-structures which is* TVMSOL^2-*definable, then $\mathcal{C}$ has finite Specker index.*

Proof. Let $\mathcal{C}$ be a set of $\bar{R}$-structures defined by the $\mathrm{TVMSOL}(\bar{R})$ formula ϕ. Let $\mathcal{C}'$ be the class of all $\bar{R} \cup R_{<}$-structures $\langle \mathcal{A}, R_{<} \rangle$ such that $\mathcal{A} \in \mathcal{C}$ and $R_{<}$ is a linear ordering of the universe of $\mathcal{A}$. Let ϕ' be the $\mathrm{MSOL}(\bar{R} \cup \{R_{<}\})$ formula obtained from ϕ by the following changes:

(i) the order used in ϕ, $a <_1 b$, is replaced with the new relation symbol $R_{<}$
(ii) it is required that $R_{<}$ is a linear ordering of $[n]$.

We note that ϕ' defines $\mathcal{C}'$, since ϕ is truth-value order invariant and that ϕ' is an MSOL^2-formula.

We will now prove that $\mathcal{C}$ has finite Specker index, by showing that if it does not, then $\mathcal{C}'$ also has infinite Specker index, in contradiction to Lemma 17. Assume $\mathcal{C}$ has infinite Specker index. Then there is an infinite set W of

$\bar{R}$−structures, such that for every distinct $\mathcal{A}_1, \mathcal{A}_2 \in W$, $\mathcal{A}_1 \not\sim_{Su(C)} \mathcal{A}_2$. So, for every $\mathcal{A}_1, \mathcal{A}_2 \in W$ there is $\langle \mathcal{G}, s \rangle$ such that

$$Subst(\langle \mathcal{G}, s \rangle, \mathcal{A}_1) \in \mathcal{C} \text{ iff } Subst(\langle \mathcal{G}, s \rangle, \mathcal{A}_2) \notin \mathcal{C}.$$

Now look at $W' = \{\langle \mathcal{A}, R_< \rangle \mid \mathcal{A} \in W, R_< \text{ linear order of } [n]\}$, where $[n]$ is the universe of $\mathcal{A}$. We note $Subst(\langle \mathcal{G}, R_{<_\mathcal{G}}, s \rangle, \langle \mathcal{A}_1, R_{<_{\mathcal{A}_1}} \rangle) = \langle Subst(\mathcal{G}, \mathcal{A}_1), R_{<'} \rangle$, where $R_{<'}$ a linear ordering of the universe of $Subst(\mathcal{G}, \mathcal{A}_1)$ which extends $R_{\mathcal{A}_1}$ and $R_\mathcal{G}$, and similarly for $\mathcal{A}_2$. Therefore,

$$Subst(\langle \mathcal{G}, R_{<_\mathcal{G}}, s \rangle, \langle \mathcal{A}_1, R_{<_{\mathcal{A}_1}} \rangle) \in \mathcal{C}' \text{ iff } Subst(\langle \mathcal{G}, R_{<_\mathcal{G}}, s \rangle, \langle \mathcal{A}_2, R_{<_{\mathcal{A}_2}} \rangle) \notin \mathcal{C}'.$$

So the Specker index of $\mathcal{C}'$ is infinite, in contradiction. $\square$

Theorem 13 now follows from lemma 16.

3.3 Counting Order Invariant MSOL^2

Here we show the Specker-Blatter Theorem does not hold for c.o.i. MSOL^2-definable Specker functions. We have two such examples, the function $E_{2,=}$, as defined in Proposition 12, and the Catalan numbers, which we discuss in Section 4.

More precisely, here we show:

Proposition 19 *$E_{2,=}$, as defined in Proposition 12 is a c.o.i.* MSOL^2*-Specker function.*

Proof. Let $\mathcal{C}$ be defined as follows:

$$\mathcal{C} = \{\langle U, R, F \rangle \mid \langle [n], <_1, U, R, F \rangle \models \Phi\},$$

where U and R are unary and F is binary, $<_1$ is a linear order of $[n]$, and Φ is says

(i) F is a function,
(ii) U is the domain of F,
(iii) R is the range of F,
(iv) U and R form a partition of $[n]$,
(v) the first element of $[n]$, is in U,
(vi) $F : U \to R$ is a bijection, and
(vii) F is monotone with respect to $<_1$.

We note $\mathcal{C}$ is MSOL^2 definable. We note also that $osp_\mathcal{C}(n, <_1)$ is counting order invariant. $osp_\mathcal{C}(n, <_1)$ counts the number of partitions of $[n]$ into two equal parts, because there is exactly one monotone bijection between any two subsets of $[n]$ of equal size. The condition that $1 \in U$ assures that we do not count the same partition twice. So $osp_\mathcal{C}(n, <_1) = E_{2,=}(n)$. $\square$

We know that $E_{2,=}$ is not ultimately periodic modulo 2 and hence the Specker-Blatter theorem cannot be extended to c.o.i. MSOL^2-Specker functions.

4 Examples

4.1 Examples of MSOL^k-Specker functions

Fibonacci and Lucas Numbers The Fibonacci numbers F_n satisfy the linear recurrence $F_n = F_{n-1} + F_{n-2}$ for $n > 1$, $F_0 = 0$ and $F_1 = 1$. The Lucas numbers L_n, a variation of the Fibonacci numbers, satisfy the same recurrence for $n > 1$, $L_n = L_{n-1} + L_{n-2}$, but have different initial conditions, $L_1 = 1$ and $L_0 = 2$.

It follows from the proof of Theorem 10 that a function which satisfies a linear recurrence relation over $\mathbb{N}$ is a c.o.i MSOL^1-Specker function. Thus. The Fibonacci and Lucas numbers are natural examples of c.o.i-MSOL^1-Specker functions.

Stirling Numbers The Stirling numbers of the first kind, denoted $\left[{n \atop r}\right]$ are defined as the number of ways to arrange n objects into r cycles. For fixed r, this is an MSOL^2-Specker function, since for $E \subseteq [n]^2$ and $U \subseteq E$, the property that U is a cycle in E and the property that E is a disjoint union of cycles are both MSOL^2-definable. Using again the growth argument from Example 1(iii), we can see that the Stirling numbers of the first kind do not satisfy a linear recurrence relation, because $\left[{n \atop 1}\right]$ grows like the factorial $(n-1)!$. However, from the Specker-Blatter Theorem it follows that they satisfy a modular linear recurrence relation for every m.

The Stirling numbers of the second kind, denoted $\left\{{n \atop r}\right\}$, count the number of partitions of a set $[n]$ into r many non-empty subsets. For fixed r, this is MSOL^2-definable: We count the number of equivalence relations with r non-empty equivalence classes. From the Specker-Blatter Theorem it follows that they satisfy a modular linear recurrence relation for every m. We did not find in the literature a linear recurrence relation for the Stirling numbers of the second kind which fits our context. But we show below that such a recurrence relation exists.

Proposition 20 *For fixed r, the Stirling numbers of the second kind are c.o.i.* MSOL^1*-Specker functions.*

Proof. We use r unary relations $U_1, \ldots, U_r$ and say that they partition the set $[n]$ into non-empty sets. However, when we permute the indices of the U_i's we count two such partitions twice. To avoid this we use a linear ordering on $[n]$ and require that, with respect to this ordering, the minimal element in U_i is smaller than all the minimal elements in U_j for $j > i$. □

Corollary 21 *For every r there exists a linear recurrence relation with constant coefficients for the Stirling numbers of the second kind $\left\{{n \atop r}\right\}$. Further more there are constants c_r such that $\left\{{n \atop r}\right\} \leq 2^{c_r \cdot n}$.*

Our proof is not constructive, and we did not bother here to calculate the explicit linear recurrence relations or the constants c_r for each r.

Catalan Numbers Catalan numbers were defined in Section 1 Example 1. We already noted that they do not satisfy any modular linear recurrence relation. However, like the example $E_{2,=}$, the functions $f_c(n) = C_n$ is a c.o.i. MSOL^2-Specker function. To see this we use the following interpretation of Catalan numbers given in [19].

C_n counts the number of tuples $\bar{a} = (a_0, \ldots, a_{2n-1}) \in [n]^{2n}$ such that

(i) $a_0 = 1$
(ii) $a_{i-1} - a_i \in \{1, -1\}$ for $i = 1, \ldots, 2n-2$
(iii) $a_{2n-1} = 0$

We can express this in MSOL^2 using a linear order and two unary functions. The two functions F_1 and F_2 are used to describe $a_0, \ldots, a_{n-1}$ and $a_n, \ldots, a_{2n-1}$ respectively. Let $\Phi_{Catalan}$ be the formula that says:

(i) $F_1, F_2 : [n] \to [n]$
(ii) $F_i(x+1) = F_i(x) \pm 1$ for $i = 1, 2$ and there exists $y = x + 1 \in [n]$
(iii) $F_1(n-1) = F_2(0) \pm 1$.
(iv) $F_1(0) = 1$
(v) $F_2(n-1) = 0$

The resulting formula is not t.vo.i., but $C_n = sp_{\mathcal{C}}(n)$ where

$$\mathcal{C} = |\{(F_1, F_2) \mid \langle [n], <_1, F_1, F_2 \rangle \models \Phi_{Catalan}\}|$$

is a c.o.i MSOL^2-Specker function.

Corollary 22 *The function $f(n) = C_n$ is a c.o.i* MSOL^2*-Specker function and does not satisfy a modular linear recurrence relation modulo* 2*.*

Bell Numbers The Bell numbers B_n count the number of equivalence relations on n elements. We note $f(n) = B_n$ is a MSOL^2-Specker function. However, B_n is not c.o.i MSOL^1-definable due to a growth argument.

4.2 Examples of MSOL^k-Specker Polynomials

Our main interest are $\mathcal{L}^k$-Specker functions, and the $\mathcal{L}^k$-Specker polynomials were introduced as an auxiliary tool. However, there are natural examples in the literature of $\mathcal{L}^k$-Specker polynomials.

Fibonacci, Lucas and Chebyshev Polynomials The recurrence $F_n(x) = x \cdot F_{n-1}(x) + F_{n-2}(x)$, $F_1(x) = 1$ and $F_2(x) = x$ defines the Fibonacci polynomials. The Fibonacci numbers F_n can be obtained as an evaluation of the Fibonacci polynomial for $x = 1$, $F_n(1) = F_n$. The Lucas polynomials are defined analogously.

The Chebyshev polynomials of the first kind (see [23]) are defined similarly by the recurrence relation $T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$, $T_0(x) = 1$, and $T_1(x) = x$. The Fibonacci, Lucas and Chebyshev polynomials are natural examples of Specker polynomials. As they are defined by linear recurrence relations, they are c.o.i MSOL^1-definable.

Touchard Polynomials The Touchard polynomials are defined

$$T_n(x) = \sum_{k=1}^{n} \left\{ {n \atop k} \right\} x^k$$

where $\left\{ {n \atop k} \right\}$ is the Stirling number of the second kind. $T_n(x)$ is c.o.i MSOL^2-definable; To see this we note that it is defined by

$$T_n(x) = \sum_{E:\Phi_{cliques}(E)} \quad \prod_{u:\Phi_{first-in-cc}(E,u)} x$$

where $\Phi_{cliques}(E)$ says E is a disjoint union of cliques and where

$$\Phi_{first-in-cc}(E, u) = \forall v \left((v <_1 u \wedge v \neq u) \rightarrow (v, u) \notin E \right),$$

i.e. it says u is the first vertex in its connected component, with respect to the order (less or equal) of $[n]$. Clearly, $\Phi_{cliques}(E)$ and $\Phi_{first-in-cc}(E, u)$ are in MSOL^2. We note that $\Phi_{first-in-cc}(E, u)$ is not invariant under the order $<_1$. The Bell numbers B_n are given as an evaluation of $T_n(x)$, $B_n = T_n(1)$, which implies $T_n(x)$ is not co.i MSOL^1-definable due to a growth argument.

Mittag-Leffler Polynomials The Mittag-Leffler polynomial (see [5]) is given by

$$M_n(x) = \sum_{k=0}^{n} \binom{n}{k} (n-1)^{\underline{n-k}} 2^k x^{\underline{k}}$$

It holds that

$$M_n(x) = \sum_{U \subseteq [n]} (n-1) \cdots k \cdot 2^k \cdot x \cdots (x - (k-1)) = \sum_{U,F,T,S:\Phi_M} 1.$$

where $\Phi_M(U, F, T, S)$ says $U \subseteq [n]$, F is an injective function from $[n] - \{n\}$ to $[n]$, T is an injective function from U to $[x]$, and S is a function from U to $\{1, n\}$. So, every evaluation of $M_n(x)$ where $x = m$, $m \in \mathbb{N}$, is a c.o.i MSOL^2-Specker function.

Note that

$$M_{n+1}(x) = \frac{1}{2} x \left[M_n(x+1) + 2M_n(x) + M_n(x-1) \right] \tag{1}$$

This looks almost like a linear recurrence relation combined with an interpolation formula, and is not of the kind we are discussing here.

5 Conclusions and Open Problems

We have introduced the notion of one variable $\mathcal{L}^k$-Specker functions $f : \mathbb{N} \rightarrow \mathbb{N}$ as the speed of a $\mathcal{L}^k$-definable class of relational structures $\mathcal{K}$, i.e. $f(n) = sp_{\mathcal{K}}(n)$.

We have investigated for which fragments $\mathcal{L}$ of SOL the $\mathcal{L}^k$-Specker functions satisfy linear recurrence relations over $\mathbb{Z}$ or $\mathbb{Z}_m$.

We have used order invariance, definability criteria and limitation on the vocabulary to continue the line of study, initiated by C. Blatter and E. Specker [8],[9],[29], what type of linear recurrence relations one can expect from Specker functions. We have completely characterized (Theorem 10) the combinatorial functions which satisfy linear recurrence relations with constant coefficients, and we have discussed (Table 1 in Section 1) how far one can extend the Specker-Blatter Theorem in terms of order invariance and MSOL-definability. As a consequence, we obtained (Corollary 11) a new characterization of the $\mathbb{Z}$-rational functions $f : \mathbb{N} \to \mathbb{N}$ as the difference of $\mathbb{N}$-rational functions.

We have not studied many variables $\mathcal{L}^k$-Specker functions arising from many-sorted structures, although this is a natural generalizations: For a class of graphs $\mathcal{K}$, $sp_{\mathcal{K}}(n, m)$ couns the number of graphs with n vertices and m edges which are in $\mathcal{K}$. Even for functions in one variable the following remain open:

(i) Can one prove similar theorems for linear recurrence relations where the coefficients depend on n?
(ii) Can one characterize the $\mathcal{L}^k$-Specker functions which satisfy modular recurrence relations with constant coefficients for each modulus m, i.e., is there some kind of a converse to Theorem 13?
(iii) Does Theorem 13 hold for TVMSOL^3?

Finally, for many-sorted $\mathcal{L}^k$-Specker functions studying both growth rate and recurrence relations seems a promising topic of further research.

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