HEREDITARY ZERO-ONE LAWS FOR GRAPHS

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Abstract. We consider the random graph $M^n_{\bar{p}}$ on the set $[n]$, were the probability of $\{x, y\}$ being an edge is $p_{|x-y|}$, and $\bar{p} = (p_1, p_2, p_3, ...)$ is a series of probabilities. We consider the set of all $\bar{q}$ derived from $\bar{p}$ by inserting 0 probabilities to $\bar{p}$, or alternatively by decreasing some of the p_i. We say that $\bar{p}$ hereditarily satisfies the 0-1 law if the 0-1 law (for first order logic) holds in $M^n_{\bar{q}}$ for any $\bar{q}$ derived from $\bar{p}$ in the relevant way described above. We give a necessary and sufficient condition on $\bar{p}$ for it to hereditarily satisfy the 0-1 law.

1. Introduction

In this paper we will investigate the random graph on the set $[n] = \{1, 2, ..., n\}$ were the probability of a pair $i \neq j \in [n]$ being connected by an edge depends only on their distance $|i - j|$. Let us define:

Definition 1.1. *For a sequence $\bar{p} = (p_1, p_2, p_3, ...)$ where each p_i is a probability i.e. a real in $[0, 1]$, let $M^n_{\bar{p}}$ be the random graph defined by:*

- *The set of vertices is $[n] = \{1, 2, ..., n\}$.*
- *For $i, j \leq n$, $i \neq j$ the probability of $\{i, j\}$ being an edge is $p_{|i-j|}$.*
- *All the edges are drawn independently.*

If $\mathfrak{L}$ is some logic, we say that $M^n_{\bar{p}}$ satisfies the 0-1 law for the logic $\mathfrak{L}$ if for each sentence $\psi \in \mathfrak{L}$ the probability that ψ holds in $M^n_{\bar{p}}$ tends to 0 or 1, as n approaches ∞. The relations between properties of $\bar{p}$ and the asymptotic behavior of $M^n_{\bar{p}}$ were investigated in [1]. It was proved there that for L, the first order logic in the vocabulary with only the adjacency relation, we have:

Theorem 1.2. (1) *Assume $\bar{p} = (p_1, p_2, ...)$ is such that $0 \leq p_i < 1$ for all $i > 0$ and let $f_{\bar{p}}(n) := \log(\prod_{i=1}^{n}(1 - p_i))/\log(n)$. If $\lim_{n\to\infty} f_{\bar{p}}(n) = 0$ then $M^n_{\bar{p}}$ satisfies the 0-1 law for L.*

(2) *The demand above on $f_{\bar{p}}$ is the best possible. Formally for each $\epsilon > 0$, there exists some $\bar{p}$ with $0 \leq p_i < 1$ for all $i > 0$ such that $|f_{\bar{p}}(n)| < \epsilon$ but the 0-1 law fails for $M^n_{\bar{p}}$.*

Part (1) above gives a necessary condition on $\bar{p}$ for the 0-1 law to hold in $M^n_{\bar{p}}$, but the condition is not sufficient and a full characterization of $\bar{p}$ seems to be harder. However we give below a complete characterization of $\bar{p}$ in terms of the 0-1 law in $M^n_{\bar{q}}$ for all $\bar{q}$ "dominated by $\bar{p}$", in the appropriate sense. Alternatively one may ask which of the asymptotic properties of $M^n_{\bar{p}}$ are kept under some operations on $\bar{p}$. The notion of "domination" or the "operations" are taken from examples of the failure of the 0-1 law, and specifically the construction for part (2) above. Those

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are given in [1] by either adding zeros to a given sequence or decreasing some of the members of a given sequence. Formally define:

Definition 1.3. *For a sequence $\bar{p} = (p_1, p_2, ...)$:*

(1) *$Gen_1(\bar{p})$ is the set of all sequences $\bar{q} = (q_1, q_2, ...)$ obtained from $\bar{p}$ by adding zeros to $\bar{p}$. Formally $\bar{q} \in Gen_1(\bar{p})$ iff for some increasing $f : \mathbb{N} \to \mathbb{N}$ we have for all $l > 0$*

$$q_l = \begin{cases} p_i & F(i) = l \\ 0 & l \notin Im(f). \end{cases}$$

(2) *$Gen_2(\bar{p}) := \{\bar{q} = (q_1, q_2, ...) : l > 0 \Rightarrow q_l \in [0, p_l]\}$.*
(3) *$Gen_3(\bar{p}) := \{\bar{q} = (q_1, q_2, ...) : l > 0 \Rightarrow q_l \in \{0, p_l\}\}$.*

Definition 1.4. *Let $\bar{p} = (p_1, p_2, ...)$ be a sequence of probabilities and $\mathfrak{L}$ be some logic. For a sentence $\psi \in \mathfrak{L}$ denote by $Pr[M^n_{\bar{p}} \models \psi]$ the probability that ψ holds in $M^n_{\bar{p}}$.*

(1) *We say that $M^n_{\bar{p}}$ satisfies the 0-1 law for $\mathfrak{L}$, if for all $\psi \in \mathfrak{L}$ the limit $\lim_{n\to\infty} Pr[M^n_{\bar{p}} \models \psi]$ exists and belongs to $\{0, 1\}$.*
(2) *We say that $M^n_{\bar{p}}$ satisfies the convergence law for $\mathfrak{L}$, if for all $\psi \in \mathfrak{L}$ the limit $\lim_{n\to\infty} Pr[M^n_{\bar{p}} \models \psi]$ exists.*
(3) *We say that $M^n_{\bar{p}}$ satisfies the weak convergence law for $\mathfrak{L}$, if for all $\psi \in \mathfrak{L}$, $\limsup_{n\to\infty} Pr[M^n_{\bar{p}} \models \psi] - \liminf_{n\to\infty} Pr[M^n_{\bar{p}} \models \psi] < 1$.*
(4) *For $i \in \{1, 2, 3\}$ we say that $\bar{p}$ i-hereditarily satisfies the 0-1 law for $\mathfrak{L}$, if for all $\bar{q} \in Gen_i(\bar{p})$, $M^n_{\bar{q}}$ satisfies the 0-1 law for $\mathfrak{L}$.*
(5) *Similarly to (4) for the convergence and weak convergence law.*

The main theorem of this paper is the following strengthening of theorem 1.2:

Theorem 1.5. *Let $\bar{p} = (p_1, p_2, ...)$ be such that $0 \leq p_i < 1$ for all $i > 0$, and $j \in \{1, 2, 3\}$. Then $\bar{p}$ j-hereditarily satisfies the 0-1 law for L iff*

$$(*) \qquad \lim_{n\to\infty} \log(\prod_{i=1}^{n}(1 - p_i))/\log n = 0.$$

Moreover we may replace above the "0-1 law" by the "convergence law" or "weak convergence law".

Note that the 0-1 law implies the convergence law which in turn implies the weak convergence law. Hence it is enough to prove the "if" direction for the 0-1 law and the "only if" direction for the weak convergence law. Also note that the "if" direction is an immediate conclusion of Theorem 1.2 (in the case $j = 1$ it is stated in [1] as a corollary at the end of section 3). The case $j = 1$ is proved in section 2, and the case $j \in \{2, 3\}$ is proved in section 3. In section 4 we deal with the case $U^*(\bar{p}) := \{i : p_i = 1\}$ is not empty. We give an almost full analysis of the hereditary $0 - 1$ law in this case as well. The only case which is not fully characterized is the case $j = 1$ and $|U^*(\bar{p})| = 1$. We give some results regarding this case in section 5. The case $j = 1$ and $|U^*(\bar{p})| = 1$ and the case that the successor relation belongs to the dictionary, will be dealt with in [2]. The following table summarizes the results in this article regarding the j-hereditary laws.

<table>
<tr><th></th><th>$|U^*| = \infty$</th><th>$2 \leq |U^*| < \infty$</th><th>$|U^*| = 1$</th><th>$|U^*| = 0$</th></tr>
<tr><td>$j = 1$</td><td rowspan="3">The weak convergence law fails</td><td>The 0-1 law holds
$\Updownarrow$
$\{l : 0 < p_l < 1\} = \emptyset$</td><td>See section 5</td><td rowspan="3">$\lim_{n\to\infty} \frac{\log(\prod_{i=1}^{n}(1-p_i))}{\log n} = 0$
$\Updownarrow$
The 0-1 law holds
$\Updownarrow$
The convergence law holds
$\Updownarrow$
The weak convergence law holds</td></tr>
<tr><td>$j = 2$</td><td colspan="2">The 0-1 law holds
$\Updownarrow$
$|\{l : p_l > 0\}| \leq 1$</td></tr>
<tr><td>$j = 3$</td><td colspan="2">The 0-1 law holds
$\Updownarrow$
$\{l : 0 < p_l < 1\} = \emptyset$</td></tr>
</table>

Convention 1.6. *Formally speaking Definition 1.1 defines a probability on the space of subsets of $G^n := \{G : G$ is a graph with vertex set $[n]\}$. If H is a subset of G^n we denote its probability by $Pr[M^n_{\bar{p}} \in H]$. If ϕ is a sentence in some logic we write $Pr[M^n_{\bar{p}} \models \phi]$ for the probability of $\{G \in G^n : G \models \phi\}$. Similarly if A_n is some property of graphs on the set of vertexes $[n]$, then we write $Pr[A_n]$ or $Pr[A_n$ holds in $M^n_{\bar{p}}]$ for the probability of the set $\{G \in G^n : G$ has the property $A_n\}$.*

Notation 1.7.
1. $\mathbb{N}$ *is the set of natural numbers (including* 0*).*
2. n, m, r, i, j *and* k *will denote natural numbers.* l *will denote a member of* $\mathbb{N}^*$ *(usually an index).*
3. p, q *and similarly* p_l, q_l *will denote probabilities i.e. reals in* $[0, 1]$.
4. ϵ, ζ *and* δ *will denote positive reals.*
5. $L = \{\sim\}$ *is the vocabulary of graphs i.e* $\sim$ *is a binary relation symbol. All* L*-structures are assumed to be graphs i.e.* $\backsim$ *is interpreted by a symmetric non-reflexive binary relation.*
6. *If* $x \sim y$ *holds in some graph* G*, we say that* $\{x, y\}$ *is an edge of* G *or that* x *and* y *are "connected" or "neighbors" in* G.

2. Adding zeros

In this section we prove theorem 1.5 for $j = 1$. As the "if" direction is immediate from Theorem 1.2 it remains to prove that if $(*)$ of 1.5 fails then the 0-1 law for L fails for some $\bar{q} \in Gen_1(\bar{p})$. In fact we will show that it fails "badly" i.e. for some $\psi \in L$, $Pr[M^n_{\bar{q}} \models \psi]$ approaches both 0 and 1 simultaneously. Formally:

Definition 2.1.
1. *Let* ψ *be a sentence in some logic* $\mathfrak{L}$*, and* $\bar{q} = (q_1, q_2, ...)$ *be a series of probabilities. We say that* ψ *holds infinitely often in* $M^n_{\bar{q}}$ *if* $\limsup_{n\to\infty} Prob[M^n_{\bar{q}} \models \psi] = 1$.
2. *We say that the 0-1 law for* $\mathfrak{L}$ *strongly fails in* $M^n_{\bar{q}}$*, if for some* $\psi \in \mathfrak{L}$ *both* ψ *and* $\neg\psi$ *hold infinitely often in* $M^n_{\bar{q}}$.

Obviously the 0-1 law strongly fails in some $M^n_{\bar{q}}$ iff $M^n_{\bar{q}}$ does not satisfy the weak semi 0-1 law. Hence in order to prove Theorem 1.5 for $j = 1$ it is enough if we prove:

Lemma 2.2. *Let $\bar{p} = (p_1, p_2, ...)$ be such that $0 \leq p_i < 1$ for all $i > 0$, and assume that $(*)$ of 1.5 fails. Then for some $\bar{q} \in Gen_1(\bar{p})$ the 0-1 law for L strongly fails in $M^n_{\bar{q}}$.*

In the remainder of this section we prove Lemma 2.2. We do so by inductively constructing $\bar{q}$, as the limit of a series of finite sequences. Let us start with some basic definitions:

Definition 2.3. (1) *Let $\mathfrak{P}$ be the set of all, finite or infinite, sequences of probabilities. Formally each $\bar{p} \in \mathfrak{P}$ has the form $\langle p_l : 0 < l < n_{\bar{p}} \rangle$ where each $p_l \in [0,1]$ and $n_{\bar{p}}$ is either ω (the first infinite ordinal) or a member of $\mathbb{N} \setminus \{0,1\}$. Let $\mathfrak{P}^{inf} = \{\bar{p} \in \mathfrak{P} : n_{\bar{p}} = \omega\}$, and $\mathfrak{P}^{fin} := \mathfrak{P} \setminus \mathfrak{P}^{inf}$.*

(2) *For $\bar{q} \in \mathfrak{P}^{fin}$ and increasing $f : [n_{\bar{q}}] \to \mathbb{N}$, define $\bar{q}^f \in \mathfrak{P}^{fin}$ by $n_{\bar{q}^f} = f(n_{\bar{q}})$, $(\bar{q}^f)_l = q_i$ if $f(i) = l$ and $(\bar{q}^f)_l = 0$ if $l \notin Im(f)$.*

(3) *For $\bar{p} \in \mathfrak{P}^{inf}$ and $r > 0$, let $Gen_1^r(\bar{p}) := \{\bar{q} \in \mathfrak{P}^{fin} :$ for some increasing $f : [r+1] \to \mathbb{N}, (\bar{p}|_{[r]})^f = \bar{q}\}$.*

(4) *For $\bar{p}, \bar{p}' \in \mathfrak{P}$ denote $\bar{p} \lhd \bar{p}'$ if $n_{\bar{p}} < n_{\bar{p}'}$ and for each $l < n_{\bar{p}}$, $p_l = p'_l$.*

(5) *If $\bar{p} \in \mathfrak{P}^{fin}$ and $n > n_{\bar{p}}$, we can still consider $M^n_{\bar{p}}$ by putting $p_l = 0$ for all $l \geq n_{\bar{p}}$.*

Observation 2.4. (1) *Let $\langle \bar{p}_i : i \in \mathbb{N} \rangle$ be such that each $\bar{p}_i \in \mathfrak{P}^{fin}$, and assume that $i < j \in \mathbb{N} \Rightarrow \bar{p}_i \lhd \bar{p}_j$. Then $\bar{p} = \cup_{i \in \mathbb{N}} \bar{p}_i$ (i.e. $p_l = (p_i)_l$ for some $\bar{p}_i$ with $n_{\bar{p}_i} > l$) is well defined and $\bar{p} \in \mathfrak{P}^{inf}$.*

(2) *Assume further that $\langle r_i : i \in \mathbb{N} \rangle$ is non-decreasing and unbounded, and that $\bar{p}_i \in Gen_1^{r_i}(\bar{p}')$ for some fixed $\bar{p}' \in \mathfrak{P}^{inf}$, then $\cup_{i \in \mathbb{N}} \bar{p}_i \in Gen_1(\bar{p}')$.*

We would like our graphs $M^n_{\bar{q}}$ to have a certain structure, namely that the number of triangles in $M^n_{\bar{q}}$ is $o(n)$ rather then say $o(n^3)$. we can impose this structure by making demands on $\bar{q}$. This is made precise by the following:

Definition 2.5. *A sequence $\bar{q} \in \mathfrak{P}$ is called proper (for l^*), if:*

(1) *l^* and $2l^*$ are the first and second members of $\{0 < l < n_{\bar{q}} : q_l > 0\}$.*

(2) *Let $l^{**} = 3l^* + 2$. If $l < n_{\bar{q}}$, $l \notin \{l^*, 2l^*\}$ and $q_l > 0$, then $l \equiv 1 \pmod{l}^{**}$.*

For $\bar{q}, \bar{q}' \in \mathfrak{P}$ we write $\bar{q} \lhd^{prop} \bar{q}'$ if $\bar{q} \lhd \bar{q}'$, and both $\bar{q}$ and $\bar{q}'$ are proper.

Observation 2.6. (1) *If $\langle \bar{p}_i : i \in \mathbb{N} \rangle$ is such that each $\bar{p}_i \in \mathfrak{P}$, and $i < j \in \mathbb{N} \Rightarrow \bar{p}_i \lhd^{prop} \bar{p}_j$, then $\bar{p} = \cup_{i \in \mathbb{N}} \bar{p}_i$ is proper.*

(2) *Assume that $\bar{q} \in \mathfrak{P}$ is proper for l^* and $n \in \mathbb{N}$. Then the following event holds in $M^n_{\bar{q}}$ with probability 1:*

$(*)_{\bar{q},l^*}$ *If $m_1, m_2, m_3 \in [n]$ and $\{m_1, m_2, m_3\}$ is a triangle in $M^n_{\bar{q}}$, then $\{m_1, m_2, m_3\} = \{l, l + l^*, l + 2l^*\}$ for some $l > 0$.*

We can now define the sentence ψ for which we have failure of the 0-1 law.

Definition 2.7. *Let k be an even natural number. Let ψ_k be the L sentence "saying": There exists $x_0, x_1, ..., x_k$ such that:*

- *$(x_0, x_1, ..., x_k)$ is without repetitions.*
- *For each even $0 \leq i < k$, $\{x_i, x_{i+1}, x_{i+2}\}$ is a triangle.*
- *The valency of x_0 and x_k is 2.*
- *For each even $0 < i < k$ the valency of x_i is 4.*
- *For each odd $0 < i < k$ the valency of x_i is 2.*

If the above holds (in a graph G) we say that $(x_0, x_1, ..., x_k)$ is a chain of triangles (in G).

Definition 2.8. *Let $n \in \mathbb{N}$, $k \in \mathbb{N}$ be even and $l^* \in [n]$. For $1 \leq m < n - k \cdot l^*$ a sequence $(m_0, m_1, ..., m_k)$ is called a candidate of type (n, l^*, k, m) if it is without repetitions, $m_0 = m$ and for each even $0 \leq i < k$, $\{m_i, m_{i+1}, m_{i+2}\} = \{l, l + l^*, l + 2l^*\}$ for some $l > 0$. Note that for given (n, l^*, k, m), there are at most 4 candidates of type (n, l^*, k, m) (and at most 2 if $k > 2$).*

Claim 2.9. *Let $n \in \mathbb{N}$, $k \in \mathbb{N}$ be even, and $\bar{q} \in \mathfrak{P}$ be proper for l^*. For $1 \leq m < n - k \cdot l^*$ let $E^n_{\bar{q},m}$ be the following event (on the probability space $M^n_{\bar{q}}$): "No candidate of of type (n, l^*, k, m) is a chain of triangles." Then $M^n_{\bar{q}}$ satisfies with probability 1: $M^n_{\bar{q}} \models \neg\psi_k$ iff $M^n_{\bar{q}} \models \bigwedge_{1\leq m<n-k\cdot l^*} E^n_{\bar{q},m}$*

Proof. The "only if" direction is immediate. For the "if" direction note that by 2.6(2), with probability 1, only a candidate can be a chain of triangles, and the claim follows immediately. □

The following claim shows that by adding enough zeros at the end of $\bar{q}$ we can make sure that ψ_k holds in $M^n_{\bar{q}}$ with probability close to 1. Note that we do not make a "strong" use of the properness of $\bar{q}$, i.e we do not use item (2) of Definition 2.5.

Claim 2.10. *Let $\bar{q} \in \mathfrak{P}^{fin}$ be proper for l^*, $k \in \mathbb{N}$ be even, and $\zeta > 0$ be some rational. Then there exists $\bar{q}' \in \mathfrak{P}^{fin}$ such that $\bar{q} \lhd^{prop} \bar{q}'$ and $Pr[M^{n_{\bar{q}'}}_{\bar{q}'} \models \psi_k] \geq 1 - \zeta$.*

Proof. For $n > n_{\bar{q}}$ denote by $\bar{q}^n$ the member of $\mathfrak{P}$ with $n_{\bar{q}^n} = n$ and $(q^n)_l$ is q_l if $l < n_{\bar{q}}$ and 0 otherwise. Note that $\bar{q} \lhd^{prop} \bar{q}^n$, hence if we show that for n large enough we have $Pr[M^n_{\bar{q}^n} \models \psi_k] \geq 1 - \zeta$ then we will be done by putting $\bar{q}' = \bar{q}^n$. Note that (recalling Definition 2.3(5)) $M^n_{\bar{q}} = M^n_{\bar{q}^n}$ so below we may confuse between them. Now set $n^* = \max\{n_{\bar{q}}, k \cdot l^*\}$. For any $n > n^*$ and $1 \leq m \leq n - n^*$ consider the sequence $s(m) = (m, m+l^*, m+2l^*, ..., m+k\cdot l^*)$ (note that $s(m)$ is a candidate of type (n, l^*, k, m)). Denote by E_m the event that $s(m)$ is a chain of triangles (in $M^n_{\bar{q}}$). We then have:

$$Pr[M^n_{\bar{q}} \models E_m] \geq (q_{l^*})^k \cdot (q_{2l^*})^{k/2} \cdot (\prod_{l=1}^{n_{\bar{q}}-1} (1 - p_l))^{2(k+1)}.$$

Denote the expression on the right by $p^*_{\bar{q}}$ and note that it is positive and depends only on k and $\bar{q}$ (but not on n). Now assume that $n > 6\cdot n^*$ and that $1 \leq m < m' \leq n - n^*$ are such that $m' - m > 2\cdot n^*$. Then the distance between the sequences $s(m)$ and $s(m')$ is larger than $n_{\bar{q}}$ and hence the events E_m and $E_{m'}$ are independent. We conclude that $Pr[M^n_{\bar{q}} \not\models \psi_k] \leq (1 - p^*_{\bar{q}})^{n/(2\cdot n^*+1)} \to_{n\to\infty} 0$ and hence by choosing n large enough we are done. □

The following claim shows that under our assumptions we can always find a long initial segment $\bar{q}$ of some member of $Gen_1(\bar{p})$ such that ψ_k holds in $M^n_{\bar{q}}$ with probability close to 0. This is where we make use of our assumptions on $\bar{p}$ and the properness of $\bar{q}$.

Claim 2.11. *Let $\bar{p} \in \mathfrak{P}^{inf}$, $\epsilon > 0$ and assume that for an unbounded set of $n \in \mathbb{N}$ we have $\prod_{l=1}^n (1 - p_l) \leq n^{-\epsilon}$. Let $k \in \mathbb{N}$ be even such that $k \cdot \epsilon > 2$. Let $\bar{q} \in Gen^r_1(\bar{p})$ be proper for l^*, and $\zeta > 0$ be some rational. Then there exists $r' > r$ and $\bar{q}' \in Gen^{r'}_1(\bar{p})$ such that $\bar{q} \lhd^{prop} \bar{q}'$ and $Pr[M^{n_{\bar{q}'}}_{\bar{q}'} \models \neg\psi_k] \geq 1 - \zeta$.*

Proof. First recalling Definition 2.5 let $l^{**} = 3l^* + 2$, and for $l \geq n_{\bar{q}}$ define $r(l) := \lceil (l - n_{\bar{q}} + 1)/l^{**} \rceil$. Now for each $n > n_{\bar{q}} + l^{**}$ denote by $\bar{q}_n$ the member of $\mathfrak{P}$ defined by:

$$(q_n)_l = \begin{cases} q_l & 0 < l < n_{\bar{q}} \\ 0 & n_{\bar{q}} \leq l < n \text{ and } l \not\equiv 1 \mod l^{**} \\ p_{r+r(l)} & n_{\bar{q}} \leq l < n \text{ and } l \equiv 1 \mod l^{**}. \end{cases}$$

Note that $n_{\bar{q}_n} = n$, $\bar{q}_n \in Gen_1^{r'}(\bar{p})$ where $r' = r + r(n-1) > r$ and $\bar{q} \lhd^{prop} \bar{q}_n$. Hence if we show that for some n large enough we have $Pr[M^n_{\bar{q}_n} \models \neg\psi_k] \geq 1 - \zeta$ then we will be done by putting $\bar{q}' = \bar{q}_n$. As before let $n^* := \max\{kl^*, n_{\bar{q}} + l^*\}$. Now fix some $n > n^*$ and for $1 \leq m < n - k \cdot l^*$ let $s(m)$ be some candidate of type (n, l^*, k, m). Denote by $E = E(s(m))$ the event that $s(m)$ is a chain of triangles in $M^n_{\bar{q}_n}$. We then have:

$$Pr[M^n_{\bar{q}_n} \models E] \leq (q_{l^*})^k \cdot (q_{2l^*})^{k/2} \cdot (\prod_{n^*+1}^{\lfloor (n-n^*)/2 \rfloor} (1-(q_i)_l))^k.$$

Now denote:

$$p^*_{\bar{q}} := (q_{l^*})^k \cdot (q_{2l^*})^{k/2} \cdot (\prod_{l=1}^{n^*} (1-(q_i)_l))^{-k}$$

and note that it is positive and does not depend on n. Together we get:

$$Pr[M^n_{\bar{q}_n} \models E] \leq p^* \cdot (\prod_{l=1}^{\lfloor (n-n^*)/2 \rfloor} (1-(q_i)_l))^k \leq p^*_{\bar{q}} \cdot (\prod_{l=1}^{\lfloor (n-n^*)/(2l^{**}) \rfloor} (1-p_l))^k.$$

For each $1 \leq m < n - k \cdot l^*$ the number of candidates of type (n, l^*, k, m) is at most 4, hence the total number of candidates is no more then $4n$. We get that the expected number (in the probability space $M^n_{\bar{q}_n}$) of candidates which are a chain of triangles is at most $p^*_{\bar{q}} \cdot (\prod_{l=1}^{\lfloor (n-n^*)/(2l^{**}) \rfloor} (1-p_l))^k \cdot 4n$. Let E^* be the following event: "No candidate is a chain of triangles". Then using Claim 2.9 and Markov's inequality we get:

$$Pr[M^n_{\bar{q}} \models \psi_k] = Pr[M^n_{\bar{q}} \not\models E^*] \leq p^*_{\bar{q}} \cdot (\prod_{l=1}^{\lfloor (n-n^*)/(2l^{**}) \rfloor} (1-p_l))^k \cdot 4n.$$

Finally by our assumptions, for an unbounded n we have $\prod_{l=1}^{\lfloor (n-n^*)/(2l^{**}) \rfloor} (1 - p_l) \leq (\lfloor (n-n^*)/(2l^{**}) \rfloor)^{-\epsilon}$, and note that for n large enough we have $(\lfloor (n-n^*)/(2l^{**}) \rfloor)^{-\epsilon} \leq n^{-\epsilon/2}$. Hence for unbounded $n \in \mathbb{N}$ we have $Pr[M^n_{\bar{q}} \models \psi_k] \leq p^*_{\bar{q}} \cdot 4 \cdot n^{1-\epsilon \cdot k/2}$, and as $\epsilon \cdot k > 2$ this tends to 0 as n tends to ∞, so we are done. $\square$

We are now ready to prove Lemma 2.2. First as $(*)$ of 1.5 does not hold we have some $\epsilon > 0$ such that for an unbounded set of $n \in \mathbb{N}$, we have $\prod_{l=1}^{n} (1-p_l) \leq n^{-\epsilon}$. Let $k \in \mathbb{N}$ be even such that $k \cdot \epsilon > 2$. Now for each $i \in \mathbb{N}$ we will construct a pair $(\bar{q}_i, r_i)$ such that the following holds:

(1) For $i \in \mathbb{N}$, $\bar{q}_i \in Gen_1^{r_i}(\bar{p})$ and put $n_i := n_{\bar{q}_i}$.
(2) For $i \in \mathbb{N}$, $\bar{q}_i \lhd^{prop} \bar{q}_{i+1}$.
(3) For each odd $i > 0$, $Pr[M^{n_i}_{\bar{q}_i} \models \psi_k] \geq 1 - \frac{1}{i}$ and $r_i = r_{i-1}$.
(4) For each even $i > 0$, $Pr[M^{n_i}_{\bar{q}_i} \models \neg\psi_k] \geq 1 - \frac{1}{i}$ and $r_i > r_{i-1}$.

Clearly if we construct such $\langle (\bar{q}_i, r_i) : i \in \mathbb{N} \rangle$ then by taking $\bar{q} = \cup_{i \in \mathbb{N}} \bar{q}_i$ (recall observation 2.4), we have $\bar{q} \in Gen_1(\bar{p})$ and both ψ_k and $\neg\psi_k$ holds infinitely often in $M^n_{\bar{q}}$, thus finishing the proof. We turn to the construction of $\langle (\bar{q}_i, r_i) : i \in \mathbb{N} \rangle$, and naturally we use induction on $i \in \mathbb{N}$.

Case 1: $i = 0$. Let $l_1 < l_2$ be the first and second indexes such that $p_{l_i} > 0$. Put $r_0 := l_2$. If $l_2 \le 2l_1$ define $\bar{q}_0$ by:

$$(q_0)_l = \begin{cases} p_l & l \le l_1 \\ 0 & l_1 \le l \le 2l_1 \\ p_{l_2} & l = 2l_1. \end{cases}$$

Otherwise if $l_2 > 2l_1$ define $\bar{q}_0$ by:

$$(q_0)_l = \begin{cases} 0 & l < \lceil l_2/2 \rceil \\ p_{l_1} & l = \lceil l_2/2 \rceil \\ 0 & \lceil l_2/2 \rceil < l < 2\lceil l_2/2 \rceil \\ p_{l_2} & l = 2\lceil l_2/2 \rceil. \end{cases}$$

clearly $\bar{q}_0 \in Gen_1^{r_0}(\bar{p})$ as desired, and note that $\bar{q}_0$ is proper (for either l_1 or $\lceil l_2/2 \rceil$).

Case 2: $i > 0$ is odd. First set $r_i = r_{i-1}$. Next we use Claim 2.10 where we set: $\bar{q}_{i-1}$ for $\bar{q}$, $\frac{1}{i}$ for ζ and $\bar{q}_i$ is the one promised by the claim. Note that indeed $\bar{q}_{i-1} \lhd^{prop} \bar{q}_i$, $\bar{q}_i \in gen^{r_i}(\bar{p})$ and $Pr[M_{\bar{q}_i}^{n_i} \models \psi_k] \ge 1 - \frac{1}{i}$.

Case 3: $i > 0$ is even. We use Claim 2.11 where we set: $\bar{q}_{i-1}$ for $\bar{q}$, $\frac{1}{i}$ for ζ and $(r_i, \bar{q}_i)$ are $(r', \bar{q}')$ promised by the claim. Note that indeed $\bar{q}_{i-1} \lhd^{prop} \bar{q}_i$, $\bar{q}_i \in Gen_1^{r_i}(\bar{p})$ and $Pr[M_{\bar{q}_i}^{n_i} \models \psi_k] \ge 1 - \frac{1}{i}$. This completes the proof of Lemma 2.2.

3. Decreasing coordinates

In this section we prove Theorem 1.5 for $j \in \{2, 3\}$. As before, the "if" direction is an immediate conclusion of Theorem 1.2. Moreover as $Gen_3(\bar{p}) \subseteq Gen_2(\bar{p})$ it remains to prove that if $(*)$ of 1.5 fails then the 0-1 strongly fails for some $\bar{q} \in Gen_3(\bar{p})$. We divide the proof into two cases according to the behavior of $\sum_{l=1}^{n} p_i$, which is an approximation of the expected number of neighbors of a given node in $M_{\bar{p}}^n$. Define:

$$(**) \qquad \lim_{n\to\infty} \log(\sum_{i=1}^{n} p_i)/\log n = 0.$$

Assume that $(**)$ above fails. Then for some $\epsilon > 0$, the set $\{n \in \mathbb{N} : \sum_{i=1}^{n} p_i \ge n^{\epsilon}\}$ is unbounded, hence we finish by Lemma 3.1. On the other hand if $(**)$ holds then $\sum_{i=1}^{n} p_i$ increases slower then any positive power of n, formally for all $\delta > 0$ for some $n_\delta \in \mathbb{N}$ we have $n > n_\delta$ implies $\sum_{i=1}^{n} p_i \le n^{\delta}$. As we assume that $(*)$ of Theorem 1.5 fails we have for some $\epsilon > 0$ the set $\{n \in \mathbb{N} : \prod_{i=1}^{n}(1 - p_i) \le n^{-\epsilon}\}$ is unbounded. Together (with $-\epsilon/6$ as δ) we have that the assumptions of Lemma 3.2 hold, hence we finish the proof.

Lemma 3.1. *Let $\bar{p} \in \mathfrak{P}^{inf}$ be such that $p_l < 1$ for $l > 0$. Assume that for some $\epsilon > 0$ we have for an unbounded set of $n \in \mathbb{N}$: $\sum_{l \le n} p_l \ge n^{\epsilon}$. Then for some $\bar{q} \in Gen_3(\bar{p})$ and $\psi = \psi_{isolated} := \exists x \forall y \neg x \sim y$, both ψ and $\neg\psi$ holds infinitely often in $M_{\bar{q}}^n$.*

Proof. We construct a series, $(\bar{q}_1, \bar{q}_2, ...)$ such that for $i > 0$: $\bar{q}_i \in \mathfrak{P}^{fin}$, $\bar{q}_i \lhd \bar{q}_{i+1}$ and $\cup_{i>0} \bar{q}_i \in Gen_3(\bar{p})$. For $i \ge 1$ denote $n_i := n_{\bar{q}_i}$. We will show that:

$*_{even}$ For even $i > 1$: $Pr[M_{\bar{q}_i}^{n_i} \models \psi] \ge 1 - \frac{1}{i}$.

$*_{odd}$ For odd $i > 1$: $Pr[M_{\bar{q}_i}^{n_i} \models \neg\psi] \ge 1 - \frac{1}{i}$.

Taking $\bar{q} = \cup_{i>0}\bar{q}_i$ will then complete the proof. We construct $\bar{q}_i$ by induction on $i > 0$:

Case 1 $i = 1$: Let $n_1 = 2$ and $(q_1)_1 = p_1$.

Case 2 even $i > 1$: As $(\bar{q}_{i-1}, n_{i-1})$ are given, let us define $\bar{q}_i$ were $n_i > n_{i-1}$ is to be determined later: $(q_i)_l = (q_{i-1})_l$ for $l < n_{i-1}$ and $(q_i)_l = 0$ for $n_{i-1} \leq l < n_i$. For $x \in [n_i]$ let E_x be the event: "x is an isolated point". Denote $p' := (\prod_{0<l<n_{i-1}}(1-(q_{i-1})_l)^2$ and note that $p' > 0$ and does not depend on n_i. Now for $x \in [n_i]$, $Pr[M^{n_i}_{\bar{q}_i} \models E_x] \geq p'$, furthermore if $x, x' \in [n_i]$ and $|x - x'| > n_{i-1}$ then E_x and $E_{x'}$ are independent in $M^{n_i}_{\bar{q}_i}$. We conclude that $Pr[M^{n_i}_{\bar{q}_i} \models \neg\psi] \leq (1-p)^{\lfloor n_i/(n_{i-1}+1)\rfloor}$ which approaches 0 as $n_i \to \infty$. So by choosing n_i large enough we have $*_{even}$.

Case 3 odd $i > 1$: As in case 2 let us define $\bar{q}_i$ were $n_i > n_{i-1}$ is to be determined later: $(q_i)_l = (q_{i-1})_l$ for $l < n_{i-1}$ and $(q_i)_l = p_l$ for $n_{i-1} \leq l < n_i$. Let $n' = \max\{n < n_i/2 : n = 2^m \text{ for some } m \in \mathbb{N}\}$, so $n_i/4 \leq n' < n_i/2$. Denote $a = \sum_{0<l\leq n'}(q_i)_l$ and $a' = \sum_{0<l\leq\lfloor n/4\rfloor}(q_i)_l$. Again let E_x be the event: "x is isolated". Now as $n' < n_i/2$, $Pr[M^{n_i}_{\bar{q}_i} \models E_x] \leq \prod_{0<l\leq n'}(1-(q_i)_l)$. By a repeated use of: $(1-x)(1-y) \leq (1-\frac{x+y}{2})^2$ we get $Pr[M^{n_i}_{\bar{q}_i} \models E_x] \leq (1-\frac{a}{n'})^{n'}$ which for n' large enough is smaller then $2 \cdot e^{-a}$, and as $a' \leq a$, we get $Pr[M^{n_i}_{\bar{q}_i} \models E_x] \leq 2 \cdot e^{-a'}$. By the definition of a' and $\bar{q}_i$ we have $a' = \sum_{l=1}^{\lfloor n_1/4\rfloor} p_l - \sum_{l<n_{i-1}}(p_l - (q_{i-1})_l)$. By our assumption for an unbounded set of $n_i \in \mathbb{N}$ we have $a' \geq (\lfloor n_i/4\rfloor)^\epsilon - \sum_{l<n_{i-1}}(p_l - (q_{i-1})_l)$. But as the sum on the right is independent of n_i we have (again for n_i large enough): $a' \geq (n_i/5)^\epsilon$. Consider the expected number of isolated points in the probability space $M^{n_i}_{\bar{q}_i}$, denote this number by $X(n_i)$. By all the above we have:

$$X(n_i) \leq n_i \cdot 2 \cdot e^{-a} \leq n_i \cdot 2 \cdot e^{-a'} \leq 2n_i \cdot e^{-(n_i/5)^\epsilon}.$$

The last expression approaches 0 as $n_i \to \infty$. So by choosing n_i large enough (while keeping $a' \geq (n_i/5)^\epsilon$ we have $*_{odd}$.

Finally notice that indeed $\cup_{i>0}\bar{q}_i \in Gen_3(\bar{p})$, as the only change we made in the inductive process is decreasing p_l to 0 for $n_{i-1} < l \leq n_i$ and i is even. □

Lemma 3.2. *Let $\bar{p} \in \mathfrak{P}^{inf}$ be such that $p_l < 1$ for $l > 0$. Assume that for some $\epsilon > 0$ we have for an unbounded set of $n \in \mathbb{N}$:*

(α) $\sum_{l\leq n} p_l \leq n^{\epsilon/6}$.

(β) $\prod_{l\leq n}(1-p_l) \leq n^{-\epsilon}$.

Let $k = \lceil\frac{6}{\epsilon}\rceil + 1$ and $\psi = \psi_k$ be the sentence "saying" there exists a connected component which is a path of length k, formally:

$$\psi_k := \exists x_1 ... \exists x_k \bigwedge_{1\leq i\neq j\leq k} x_i \neq x_j \wedge \bigwedge_{1\leq i<k} x_i \sim x_{i+1} \wedge \forall y (\bigwedge_{1\leq i\leq k} x_i \neq y) \to (\bigwedge_{1\leq i\leq k} \neg x_i \sim y).$$

Then for some $\bar{q} \in Gen_3(\bar{p})$, both ψ and $\neg\psi$ holds infinitely often in $M^n_{\bar{q}}$.

Proof. The proof follows the same line as the proof of 3.1. We construct an increasing series, $(\bar{q}_1, \bar{q}_2, ...)$, and demand $*_{even}$ and $*_{odd}$ as in 3.1. Taking $\bar{q} = \cup_{i>0}\bar{q}_i$ will then complete the proof. We construct $\bar{q}_i$ by induction on $i > 0$:

Case 1 $i = 1$: Let $l(*) := \min\{l > 0 : p_l > 0\}$ and define $n_1 = l(*) + 1$ and $(q_1)_l = p_l$ for $l < n_1$.

Case 2 even $i > 1$: As before, for $n_i > n_{i-1}$ define: $(q_i)_l = (q_{i-1})_l$ for $l < n_{i-1}$ and $(q_i)_l = 0$ for $n_{i-1} \leq l < n_i$. For $1 \leq x < n_i - k \cdot l(*)$ let E^x be the event: "$(x, x+l(*), ..., x+l(*)(k-1))$ exemplifies ψ." Formally E^x holds in $M^{n_i}_{\bar{q}_i}$ iff $\{(x, x+$

$l(*), ..., x+l(*)(k-1))\}$ is isolated and for $0 \leq j < k-1$, $\{x+jl(*), x+(j+1)l(*)\}$ is an edge of $M^{n_i}_{\bar{q}_i}$. The remainder of this case is similar to case 2 of Lemma 3.1 so we will not go into details. Note that $Pr[M^{n_i}_{\bar{q}_i} \models E^x] > 0$ and does not depend on n_i, and if $|x-x'|$ is large enough (again not depending on n_i) then E^x and $E^{x'}$ are independent in $M^{n_i}_{\bar{q}_i}$. We conclude that by choosing n_i large enough we have $*_{even}$.

Case 3 odd $i > 1$: In this case we make use of the fact that almost always, no $x \in [n]$ have to many neighbors. Formally:

Claim 3.3. *Let $\bar{q} \in \mathfrak{P}^{inf}$ be such that $q_l < 1$ for $l > 0$. Let $\delta > 0$ and assume that for an unbounded set of $n \in \mathbb{N}$ we have, $\sum_{l=1}^{n} q_l \leq n^{\delta}$. Let E^n_δ be the event: "No $x \in [n]$ have more than $8n^{2\delta}$ neighbors". Then we have:*

$$\limsup_{n\to\infty} Pr[E^n_\delta \textit{ holds in } M^n_{\bar{q}}] = 1.$$

Proof. First note that the size of the set $\{l > 0 : q_l > n^{-\delta}\}$ is at most $n^{2\delta}$. Hence by ignoring at most $2n^{2\delta}$ neighbors of each $x \in [n]$, and changing the number of neighbors in the definition of E^n_δ to $6n^{2\delta}$ we may assume that for all $l > 0$, $q_l \leq n^{-\delta}$. The idea is that the number of neighbors of each $x \in [n]$ can be approximated (or in our case only bounded from above) by a Poisson random variable with parameter close to $\sum_{i=l}^{n} q_l$. Formally, for each $l > 0$ let B_l be a Bernoulli random variable with $Pr[B_l = 1] = q_l$. For $n \in \mathbb{N}$ let X^n be the random variable defined by $X^n := \sum_{l=1}^{n} B_l$. For $l > 0$ let Po_l be a Poisson random variable with parameter $\lambda_l := -\log(1-q_l)$ that is for $i = 0, 1, 2, ...$ $Pr[Po_l = i] = e^{-\lambda_l}\frac{(\lambda_l)^i}{i!}$. Note that $Pr[B_l = 0] = Pr[Po_l = 0]$. Now define $Po^n := \sum_{i=1}^{n} Po_l$. By the last sentence we have $Po^n \geq_{st} X^n$ (Po^n is stochastically larger than X^n) that is, for $i = 0, 1, 2, ...$ $Pr[Po^n \geq i] \geq Pr[X^n \geq i]$. Now Po^n (as the sum of Poisson random variables) is a Poisson random variable with parameter $\lambda^n := \sum_{l=1}^{n} \lambda_l$. Let $n \in \mathbb{N}$ be such that $\sum_{l=1}^{n} q_l \leq n^{\delta}$, and define $n' = n'(n) := min\{n' \geq n : n' = 2^m \text{ for some } m \in \mathbb{N}\}$, so $n \leq n' < 2n$. For $0 < l \leq n'$ let q'_l be q_l if $l \leq n$ and 0 otherwise, so we have: $\prod_{l=1}^{n} 1 - q_l = \prod_{l=1}^{n'} 1 - q'_l$ and $\sum_{l=1}^{n} q_l = \sum_{l=1}^{n'} q'_l$. Note that if $0 \leq p, q \leq 1/4$ then $(1-p)(1-q) \geq (1-\frac{p+q}{2})^2 \cdot \frac{1}{2}$. By a repeated use of the last inequality we get that $\prod_{i=l}^{n'}(1-q'_l) \geq (1-\frac{\sum_{i=l}^{n'} q'_l}{n'})^{n'} \cdot \frac{1}{n'}$. We can now evaluate λ^n:

$$\begin{aligned}
\lambda^n &= \sum_{l=1}^{n} \lambda_l = \sum_{l=1}^{n} -\log(1-q_l) = -\log(\prod_{l=1}^{n}(1-q_l)) = -\log(\prod_{l=1}^{n'}(1-q'_l)) \\
&\leq -\log[(1-\frac{\sum_{l=1}^{n'} q'_l}{n'})^{n'} \cdot \frac{1}{n'}] = -\log[(1-\frac{\sum_{l=1}^{n} q_l}{n'})^{n'} \cdot \frac{1}{n'}] \\
&\approx -\log[e^{-\sum_{l=1}^{n} q_l} \cdot \frac{1}{n'}] \leq -\log[e^{-n^{\delta}} \cdot \frac{1}{2n}] \leq -\log[e^{-n^{2\delta}}] = n^{2\delta}.
\end{aligned}$$

Hence by choosing $n \in \mathbb{N}$ large enough while keeping $\sum_{l=1}^{n} q_l \leq n^{\delta}$ (which is possible by our assumption) we have $\lambda^n \leq n^{2\delta}$. We now use the Chernoff bound for Poisson random variable: If Po is a Poisson random variable with parameter λ and $i > 0$ we have $Pr[Po \geq i] \leq e^{\lambda(i/\lambda - 1)} \cdot (\frac{\lambda}{i})^i$. Applying this bound to Po^n (for n as above) we get:

$$Pr[Po^n \geq 3n^{2\delta}] \leq e^{\lambda^n(3n^{2\delta}/\lambda^n - 1)} \cdot (\frac{\lambda^n}{3n^{2\delta}})^{3n^{2\delta}} \leq e^{3n^{2\delta}} \cdot (\frac{\lambda^n}{3n^{2\delta}})^{3n^{2\delta}} \leq (\frac{e}{3})^{3n^{2\delta}}.$$

Now for $x \in [n]$ let X^n_x be the number of neighbors of x in $M^n_{\bar{q}}$ (so X^n_x is a random variable on the probability space $M^n_{\bar{q}}$). By the definition of $M^n_{\bar{q}}$ we have $X^n_x \leq_{st} 2 \cdot X^n \leq_{st} 2 \cdot Po^n$. So for unbounded $n \in \mathbb{N}$ we have for all $x \in [n]$, $Pr[X^n_x \geq 6n^{2\delta}] \leq (\frac{e}{3})^{3n^{2\delta}}$. Hence by the Markov inequality for unbounded $n \in \mathbb{N}$ we have,

$$Pr[E^n \text{ does not hold in } M^n_{\bar{q}}] = Pr[\text{for some } x \in [n], X^n_x \geq 3n^{2\delta}] \leq n \cdot (\frac{e}{3})^{6n^{2\delta}}.$$

But the last expression approaches 0 as n approaches ∞, Hence we are done proving the claim. $\square$

We return to **Case 3** of the proof of 3.2, and it remains to construct $\bar{q}_i$. As before for $n_i > n_{i-1}$ define: $(q_i)_l = (q_{i-1})_l$ for $l < n_{i-1}$ and $(q_i)_l = p_l$ for $n_{i-1} \leq l < n_i$. By the claim above and (α) is our assumptions, for n_i large enough we have $Pr[E^{n_i}_{\epsilon/6}$ holds in $M^{n_i}_{\bar{q}_i}] \geq 1/2i$, so assume in the rest of the proof that n_i is indeed large enough, and assume that $E^{n_i}_{\epsilon/6}$ holds in $M^{n_i}_{\bar{q}_i}$, and all the probabilities on the space $M^{n_i}_{\bar{q}_i}$ will be conditioned to $E^{n_i}_{\epsilon/6}$ (even if not explicitly said so). A k-tuple $\bar{x} = (x_1, ..., x_k)$ of members of $[n_i]$ is called a k-path (in $M^{n_i}_{\bar{q}_i}$) if it is without repetitions and for $0 < j < k$ we have $M^{n_i}_{\bar{q}_i} \models x_j \sim x_{j+1}$. A k-path is isolated if in addition no member of $\{x_1, ..., x_k\}$ is connected to a member of $[n_i] \setminus \{x_1, ..., x_k\}$. Now (recall we assume $E^{n_i}_{\epsilon/6}$) with probability 1: the number of k-paths in $M^{n_i}_{\bar{q}_i}$ is at most $8^k \cdot n^{1+k\epsilon/3}$. For each $(x_1, ..., x_k)$ without repetitions we have:

$$Pr[(x_1, ..., x_k) \text{ is isolated in } M^{n_i}_{\bar{q}_i}] = \prod_{j=1}^{k} \prod_{y \neq x_j} (1 - (q_i)_{|x_j - y|}) \leq (\prod_{l=1}^{\lfloor n_i/2 \rfloor} (1 - (q_i)_l))^k.$$

By assumption (β) we have for unbounded set of $n_i \in \mathbb{N}$:

$$\prod_{l=1}^{\lfloor n_i/2 \rfloor} (1 - (q_i)_l) \leq \prod_{l=n_i-1}^{\lfloor n_i/2 \rfloor} (1 - p_l) \leq \prod_{l<n_i} (1 - q_l) \cdot (\lfloor n_i/2 \rfloor)^{-\epsilon} \leq (n_i)^{-\epsilon/2}.$$

Together letting $Y(n_i)$ be the expected number of isolated k tuples in $M^{n_i}_{\bar{q}_i}$ we have:

$$Y(n_i) \leq 8^k \cdot (n_i)^{1+k\epsilon/3} \cdot (n_i)^{-k\epsilon/2} = 8^k \cdot (n_i)^{1-k\epsilon/6} \to_{n_i \to \infty} 0.$$

So by choosing n_i large enough and using Markov's inequality, we have $*_{odd}$, and we are done. $\square$

4. Allowing some probabilities to equal 1

In this section we analyze the hereditary 0-1 law for $\bar{p}$ where some of the p_i-s may equal 1. For $\bar{p} \in \mathfrak{P}^{inf}$ let $U^*(\bar{p}) := \{l > 0 : p_l = 1\}$. The situation $U^*(\bar{p}) \neq \emptyset$ was discussed briefly in the end of section 4 of [1], an example was given there of some $\bar{p}$ consisting of only ones and zeros with $|U^*(\bar{p})| = \infty$ such that the 0-1 law fails for $M^n_{\bar{p}}$. We follow the lines of that example and prove that if $|U^*(\bar{p})| = \infty$ and $j \in \{1, 2, 3\}$, then the j-hereditary 0-1 law for L fails for $\bar{p}$. This is done in 4.1. The case $0 < |U^*(\bar{p})| < \infty$ is also studied and a full characterization of the j-hereditary 0-1 law for L is given in 4.6 for $j \in \{2, 3\}$, and for $j = 1$, $1 < |U^*(\bar{p})|$. The case $j = 1$ and $1 = |U^*(\bar{p})|$ is discussed in section 5.

Theorem 4.1. *Let $\bar{p} \in \mathfrak{P}^{inf}$ be such that $U^*(\bar{p})$ is infinite, and j be in $\{1, 2, 3\}$. Then $M^n_{\bar{p}}$ does not satisfy the j-hereditary weak convergence law for L.*

Proof. We start with the case $j = 1$. The idea here is similar to that of section 2. We show that some $\bar{q} \in Gen_1(\bar{p})$ has a structure (similar to the "proper" structure defined in 2.5) that allows us to identify the sections "close" to 1 or n in $M^n_{\bar{q}}$. It is then easy to see that if $\bar{q}$ has infinitely many ones and infinitely many "long" sections of consecutive zeros, then the sentence saying: "there exists an edge connecting vertexes close to the the edges", will exemplify the failure of the 0-1 law for $M^n_{\bar{q}}$. This is formulated below. Consider the following demands on $\bar{q} \in \mathfrak{P}^{inf}$:

(1) Let $l^* < l^{**}$ be the first two members of $U^*(\bar{q})$, then l^* is odd and $l^{**} = 2 \cdot l^*$.
(2) If l_1, l_2, l_3 all belong to $\{l > 0 : q_l > 0\}$ and $l_1 + l_2 = l_3$ then $l_1 = l_2 = l^*$.
(3) The set $\{n \in \mathbb{N} : n - 2l^* < l < n \Rightarrow q_l = 0\}$ is infinite.
(4) The set $U^*(\bar{q})$ is infinite.

We first claim that some $\bar{q} \in Gen_1(\bar{p})$ satisfies the demands (1)-(4) above. This is straight forward. We inductively add enough zeros before each nonzero member of $\bar{p}$ guaranteing that it is larger than the sum of any two (not necessarily different) nonzero members preceding it. We continue until we reach l^*, then by adding zeros either before l^* or before l^{**} we can guarantee that l^* is odd and that $l^{**} = 2 \cdot l^*$, and hence (1) holds. We then continue the same process from l^{**}, adding at least $2l^*$ zero's at each step. This guaranties (2) and (3). (4) follows immediately form our assumption that $U^*(\bar{p})$ is infinite. Assume that $\bar{q}$ satisfies (1)-(4) and $n \in \mathbb{N}$. With probability 1 we have:

$$\{x, y, z\} \text{ is a triangle in } M^n_{\bar{q}} \text{ iff } \{x, y, z\} = \{l, l + l^*, l + l^{**}\} \text{ for some } 0 < l \leq n.$$

To see this use (1) for the "if" direction and (2) for the "only if" direction. We conclude that letting $\psi_{ext}(x)$ be the L sentence saying that x belongs to exactly one triangle, for each $n \in \mathbb{N}$ and $m \in [n]$ with probability 1 we have:

$$M^n_{\bar{q}} \models \psi_{ext}[m] \text{ iff } m \in [1, l^*] \cup (n - l^*, n].$$

We are now ready to prove the failure of the weak convergence law in $M^n_{\bar{q}}$, but in the first stage let us only show the failure of the convergence law. This will be useful for other cases (see Remark 4.2 below). Define

$$\psi := (\exists x \exists y)\psi_{ext}(x) \wedge \psi_{ext}(y) \wedge x \sim y.$$

Recall that l^* is the *first* member of $U^*(\bar{p})$, hence for some $p > 0$ (not depending on n) for any $x, y \in [1, l^*]$ we have $Pr[M^n_{\bar{q}} \models \neg x \sim y] \geq p$ and similarly for any $x, y \in (n - l^*, n]$. We conclude that:

$$Pr[(\exists x \exists y)(x, y \in [1, l^*] \text{ or } x, y \in (n - l^*, n]) \text{ and } x \sim y] \leq 1 - p^{2\binom{l^*}{2}} < 1.$$

By all the above, for each l such that $q_l = 1$ we have $Pr[M^{l+1}_{\bar{q}} \models \psi] = 1$, as the pair $(1, l+1)$ exemplifies ψ in $M^{l+1}_{\bar{q}}$ with probability 1. On the other hand if n is such that $n - 2l^* < l < n \Rightarrow q_l = 0$ then $Pr[M^n_{\bar{q}} \models \psi] \leq 1 - p^{2\binom{l^*}{2}}$. Hence by (3) and (4) above, ψ exemplifies the failure of the convergence law for $M^n_{\bar{q}}$ as required.

We return to the proof of the failure of the weak convergence law. Define:

$$\begin{aligned}\psi' \quad = \quad & \exists x_0 ... \exists x_{2l^*-1}[\bigwedge_{0 \leq i < i' < 2l^*} x_i \neq x_{i'} \wedge \forall y((\bigwedge_{0 \leq i < 2l^*} y \neq x_i) \to \neg\psi_{ext}(y)) \\ & \wedge \bigwedge_{0 \leq i < 2l^*} \psi_{ext}(x_i) \wedge \bigwedge_{0 \leq i < l^*} x_{2i} \sim x_{2i+1}].\end{aligned}$$

We will show that both ψ' and $\neg\psi'$ holds infinitely often in $M^n_{\bar{q}}$. First let $n \in \mathbb{N}$ be such that $q_{n-l^*} = 1$. Then by choosing for each $0 \leq i < l^*$, $x_{2i} := i+1$ and $x_{2i+1} := n - l^* + 1 + i$, we will get that the sequence $(x_0, ..., x_{2l^*-1})$ exemplifies ψ' in $M^n_{\bar{q}}$ (with probability 1). As by assumption (4) above the set $\{n \in \mathbb{N} : q_{n-l^*} = 1\}$ is unbounded we have $\limsup_{n\to\infty}[M^n_{\bar{q}} \models \psi'] = 1$. For the other direction let $n \in \mathbb{N}$ be such that for each $n - 2l^* < l < n$, $q_l = 0$. Then $M^n_{\bar{q}}$ satisfies (again with probability 1) for each $x, y \in [1, l^*] \cup (n - l^*, n]$ such that $x \sim y$: $x \in [1, l^*]$ iff $y \in [1, l^*]$. Now assume that $(x_0, ..., x_{2l^*-1})$ exemplifies ψ' in $M^n_{\bar{q}}$. Then for each $0 \leq i < l^*$, $x_{2i} \in [1, l^*]$ iff $x_{2i+1} \in [1, l^*]$. We conclude that the set $[1, l^*]$ is of even size, thus contradicting (1). So we have $Pr[M^n_{\bar{q}} \models \psi'] = 0$. But by assumption (3) above the set of natural numbers, n, for which we have $n - 2l^* < l < n$ implies $q_l = 0$ is unbounded, and hence we have $\limsup_{n\to\infty}[M^n_{\bar{q}} \models \neg\psi'] = 1$ as desired.

We turn to the proof of the case $j \in \{2, 3\}$, and as $Gen_3(\bar{p}) \subseteq Gen_2(\bar{p})$ it is enough to prove that for some $\bar{q} \in Gen_3(\bar{p})$ the 0-1 law for L strongly fails in $M^n_{\bar{q}}$. Motivated by the example mentioned above appearing in the end of section 4 of [1], we let ψ be the sentence in L implying that each edge of the graph is contained in a cycle of length 4. Once again we use an inductive construction of $(\bar{q}_1, \bar{q}_2, \bar{q}_3, ...)$ in $\mathfrak{P}^{fin}$ such that $\bar{q} = \bigcup_{i>0} \bar{q}_i \in Gen_3(\bar{p})$ and both ψ and $\neg\psi$ hold infinitely often in $M^n_{\bar{q}}$. For $i = 1$ let $n_{\bar{q}_1} = n_1 := \min\{l : p_l = 1\} + 1$ and define $(q_1)_l = 0$ if $0 < l < n_1 - 1$ and $(q_1)_{n_1-1} = 1$. For even $i > 1$ let $n_{\bar{q}_i} = n_i := \min\{l > 4n_{i-1} : p_l = 1\} + 1$ and define $(q_i)_l = (q_{i-1})_l$ if $0 < l < n_{i-1}$, $(q_i)_l = 0$ if $n_{i-1} \leq l < n_i - 1$ and $(q_1)_{n_1-1} = 1$. For odd $i > i$ recall $n_1 = \min\{l : p_l = 1\} + 1$ and let $n_{\bar{q}_i} = n_i := n_{i-1} + n_1$. Now define $(q_i)_l = (q_{i-1})_l$ if $0 < l < n_{i-1}$ and $(q_i)_l = 0$ if $n_{i-1} \leq l < n_i$. Clearly we have for even $i > 1$, $Pr[M^{n_i+1}_{\bar{q}_{n_i+1}} \models \psi] = 0$ and for odd $i > 1$ $Pr[M^{n_i}_{\bar{q}_{n_i}} \models \psi] = 1$. Note that indeed $\bigcup_{i>0} \bar{q}_i \in Gen_3(\bar{p})$, hence we are done. $\square$

Remark 4.2. *In the proof of the failure of the convergence law in the case $j = 1$ the assumption $|U^*(\bar{p})| = \infty$ is not needed, our proof works under the weaker assumption $|U^*(\bar{p})| \geq 2$ and for some $p > 0$, $\{l > 0 : p_l > p\}$ is infinite. See below more on the case $j = 1$ and $1 < |U^*(\bar{p})| < \infty$.*

Lemma 4.3. *Let $\bar{q} \in \mathfrak{P}^{inf}$ and assume:*

(1) *Let $l^* < l^{**}$ be the first two members of $U^*(\bar{q})$ (in particular assume $|U^*(\bar{q})| \geq 2$) then $l^{**} = 2 \cdot l^*$.*
(2) *If l_1, l_2, l_3 all belong to $\{l > 0 : q_l > 0\}$ and $l_1 + l_2 = l_3$ then $\{l_1, l_2, l_3\} = \{l, l + l^*, l + l^{**}\}$ for some $l \geq 0$.*
(3) *Let l^{***} be the first member of $\{l > 0 : 0 < q_l < 1\}$ (in particular assume $|\{l > 0 : 0 < q_l < 1\}| \geq 1$) then the set $\{n \in \mathbb{N} : n \leq l \leq n + l^{**} + l^{***} \Rightarrow q_l = 0\}$ is infinite.*

Then the 0-1 law for L fails for $M^n_{\bar{q}}$.

Proof. The proof is similar to the case $j = 1$ in the proof of Theorem 4.1, hence we will not go into detail. Below n is some large enough natural number (say larger than $3 \cdot l^{**} \cdot l^{***}$) such that (3) above holds, and if we say that some property holds in $M^n_{\bar{q}}$ we mean it holds there with probability 1. Let $\psi^1_{ext}(x)$ be the formula in L implying that x belongs to at most two distinct triangles. Then for all $m \in [n]$:

$$M^n_{\bar{q}} \models \psi^1_{ext}[m] \text{ iff } m \in [1, l^{**}] \cup (n - l^{**}, n].$$

Similarly for any natural $t < n/3l^{**}$ define (using induction on t):

$$\psi^t_{ext}(x) := (\exists y \exists z) x \sim y \wedge x \sim z \wedge y \sim z \wedge (\psi^{t-1}_{ext}(y) \vee \psi^{t-1}_{ext}(z))$$

we then have for all $m \in [n]$:

$$M^n_{\bar{q}} \models \psi^t_{ext}[m] \text{ iff } m \in [1, tl^{**}] \cup (n - tl^{**}, n].$$

Now for $1 \leq t < n/3l^{**}$ let $m^*(t)$ be the minimal number of edges in $M^n_{\bar{q}}|_{[1,t\cdot l^{**}]\cup(n-t\cdot l^{**},n]}$ i.e only edges with probability one and within one of the intervals are counted, formally

$$m^*(t) := 2 \cdot |\{(m, m') : m < m' \in [1, t \cdot l^{**}] \text{ and } q_{m'-m} = 1\}|.$$

Let $1 \leq t^* < n/3l^{**}$ be such that $l^{***} < l^{**} \cdot t^*$ (it exists as n is large enough). Note that $m^*(t^*)$ depends only on $\bar{q}$ and not on n hence we can define

$$\psi := \text{"There exists exactly } m^*(t^*) \text{ couples } \{x, y\} \text{ s.t. } \psi^{t^*}_{ext}(x) \wedge \psi^{t^*}_{ext}(y) \wedge x \sim y.\text{"}$$

We then have $Pr[m^n_{\bar{q}} \models \psi] \leq (1 - q_{l^{***}})^2 < 1$ as we have $m^*(t^*)$ edges on $[1, t^*l^{**}] \cup (n - t^*l^{**}, n]$ that exist with probability 1, and at least two additional edges (namely $\{1, l^{***} + 1\}$ and $\{n - l^{***}, n\}$) that exist with probability $q_{l^{***}}$ each. On the other hand if we define:

$$p' := \prod\{1 - q_{m'-m} : m < m' \in [1, t^* \cdot l^{**}] \text{ and } q_{m'-m} < 1\}$$

and note that p' does not depend on n, then (recalling assumption (3) above) we have $Pr[m^n_{\bar{q}} \models \psi] \geq (p')^2 > 0$ thus completing the proof. $\square$

Lemma 4.4. *Let $\bar{q} \in \mathfrak{P}^{inf}$ be such that for some $l_1 < l_2 \in \mathbb{N} \setminus \{0\}$ we have: $0 < p_{l_1} < 1$, $p_{l_2} = 1$ and $p_l = 0$ for all $l \notin \{l_1, l_2\}$. Then the 0-1 law for L fails for $M^n_{\bar{q}}$.*

Proof. Let ψ be the sentence in L "saying" that some vertex has exactly one neighbor and this neighbor has at least three neighbors. Formally:

$$\psi := (\exists x)(\exists! y) x \sim y \wedge (\forall z) x \sim z \rightarrow (\exists u_1 \exists u_2 \exists u_3) \bigwedge_{0<i<j\leq 3} u_i \neq u_j \wedge \bigwedge_{0<i\leq 3} z \sim u_i.$$

We first show that for some $p > 0$ and $n_0 \in \mathbb{N}$, for all $n > n_0$ we have $Pr[M^n_{\bar{q}} \models \psi] > p$. To see this simply take $n_0 = l_1 + l_2 + 1$ and $p = (1 - p_{l_1})(p_{l_1})$. Now for $n > n_0$ in $M^n_{\bar{q}}$, with probability $1 - p_{l_1}$ the node $1 \in [n]$ has exactly one neighbor (namely $1 + l_2 \in [n]$) and with probability at least p_{l_1}, $1 + l_2$ is connected to $1 + l_1 + l_2$, and hence has three neighbors (1, $1 + 2l_2$ and $1 + l_1 + l_2$). This yields the desired result. On the other hand for some $p' > 0$ we have for all $n \in \mathbb{N}$, $Pr[M^n_{\bar{q}} \models \neg\psi] > p'$. To see this note that for all n, only members of $[1, l_2] \cup (n - l_2, n]$ can possibly exemplify ψ, as all members of $(l_2, n - l_2]$ have at least two neighbors with probability one. For each $x \in [1, l_2] \cup (n - l_2, n]$, with probability at least $(1 - p_1)^2$, x dose not exemplify ψ (since the unique neighbor of x has less then three neighbors). As the size of $[1, l_2] \cup (n - l_2, n]$ is $2 \cdot l_2$ we get $Pr[M^n_{\bar{q}} \models \neg\psi] > (1 - p_1)^{2l_2} := p' > 0$. Together we are done. $\square$

Lemma 4.5. *Let $\bar{p} \in \mathfrak{P}^{inf}$ be such that $|U^*(\bar{p})| < \infty$ and $p_i \in \{0, 1\}$ for $i > 0$. Then $M^n_{\bar{p}}$ satisfy the 0-1 law for L.*

Proof. Let S^n be the (not random) structure in vocabulary $\{Suc\}$, with universe $[n]$ and Suc is the successor relation on $[n]$. It is straightforward to see that any sentence $\psi \in L$ has a sentence $\psi^S \in \{Suc\}$ such that

$$Pr[M^n_{\bar{p}} \models \psi] = \begin{cases} 1 & S^n \models \psi^S \\ 0 & S^n \not\models \psi^S. \end{cases}$$

Also by a special case of Gaifman's result from [3] we have: for each $k \in \mathbb{N}$ there exists some $n_k \in \mathbb{N}$ such that if $n, n' > n_k$ then S^n and $S^{n'}$ have the same first order theory of quantifier depth k. Together we are done. $\square$

Conclusion 4.6. *Let $\bar{p} \in \mathfrak{P}^{inf}$ be such that $0 < |U^*(\bar{p})| < \infty$.*

(1) *The 2-hereditary 0-1 law holds for $\bar{p}$ iff $|\{l > 0 : p_l > 0\}| > 1$.*
(2) *The 3-hereditary 0-1 law holds for $\bar{p}$ iff $\{l > 0 : 0 < p_l < 1\} \neq \emptyset$.*
(3) *If furthermore $1 < |U^*(\bar{p})|$ then the 1-hereditary 0-1 law holds for $\bar{p}$ iff $\{l > 0 : 0 < p_l < 1\} \neq \emptyset$.*

Proof. For (1) note that if indeed $|\{i > 0 : p_l > 0\}| > 1$ then some $\bar{q} \in Gen_2(\bar{p})$ is as in the assumption of Lemma 4.4, otherwise any $\bar{q} \in Gen_2(\bar{p})$ has at most 1 nonzero member hence $M^n_{\bar{q}}$ satisfy the 0-1 law by either 4.5 or 1.2.

For (2) note that if $\{i > 0 : 0 < p_l < 1\} \neq \emptyset$ then some $\bar{q} \in Gen_3(\bar{p})$ is as in the assumption of Lemma 4.4, otherwise any $\bar{q} \in Gen_3(\bar{p})$ is as in the assumption of Lemma 4.5 and we are done.

Similarly for (3) note that if $1 < |U^*(\bar{p})|$ and $\{l > 0 : 0 < p_l < 1\} \neq \emptyset$ then some $\bar{q} \in Gen_1(\bar{p})$ satisfies assumptions (1)-(3) of Lemma 4.3, otherwise any $\bar{q} \in Gen_1(\bar{p})$ is as in the assumption of Lemma 4.5 and we are done. $\square$

5. When exactly one probability equals 1

In this section we assume:

Assumption 5.1. *$\bar{p}$ is a fixed member of $\mathfrak{P}^{inf}$ such that $|U^*(\bar{p})| = 1$ hence denote $U^*(\bar{p}) = \{l^*\}$, and assume*

$$(*)' \qquad \lim_{n\to\infty} \log\Big(\prod_{l \in [n]\setminus\{l^*\}} (1 - p_l)\Big)/\log(n) = 0.$$

We try to determine when the 1-hereditary 0-1 law holds. The assumption of $(*)'$ is justified as the proof in section 2 works also in this case and in fact in any case that $U^*(\bar{p})$ is finite. To see this replace in section 2 products of the form $\prod_{l<n}(1 - p_l)$ by $\prod_{l<n, l \notin U^*(\bar{p})}(1 - p_l)$, sentences of the form "x has valency m" by "x has valency $m + 2|U^*(\bar{p})|$", and similar simple changes. So if $(*)'$ fails then the 1-hereditary weak convergence law fails, and we are done. It seems that our ability to "identify" the l^*-boundary (i.e. the set $[1, l^*] \cup (n - l^*, n])$ in $M^n_{\bar{p}}$ is closely related to the holding of the 0-1 law. In Conclusion 5.6 we use this idea and give a necessary condition on $\bar{p}$ for the 1-hereditary weak convergence law. The proof uses methods similar to those of the previous sections. Finding a sufficient condition for the 1-hereditary 0-1 law seems to be harder. It turns out that the analysis of this case is, in a way, similar to the analysis when we add the successor relation to our vocabulary. This is because the edges of the form $\{l, l + l^*\}$ appear with probability 1 similarly to the successor relation. There are, however, some obvious differences. Let L^+ be the vocabulary $\{\sim, S\}$, and let $(M^+)^n_{\bar{p}}$ be the random L^+

structure with universe $[n]$, $\sim$ is the same as in $M^n_{\bar{p}}$, and $S^{(M^+)^n_{\bar{p}}}$ is the successor relation on $[n]$. Now if for some $l^{**} > 0$, $0 < p_{l^{**}} < 1$ then $(M^+)^n_{\bar{p}}$ does not satisfy the 0-1 law for L^+. This is because the elements 1 and $l^{**}+1$ are definable in L^+ and hence some L^+ sentence holds in $(M^+)^n_{\bar{p}}$ iff $\{1, l^{**}+1\}$ is an edge of $(M^+)^n_{\bar{p}}$ which holds with probability $p_{l^{**}}$. In our case, as in L we can not distinguish edges of the form $\{l, l+l^*\}$ from the rest of the edged, the 0-1 law may hold even if such l^* exists. In Lemma 5.10 below we show that if, in fact, we can not "identify the edges" in $M^n_{\bar{p}}$ then the 0-1 law, holds in $M^n_{\bar{p}}$. This is translated in Theorem 5.14 to a sufficient condition on $\bar{p}$ for the 0-1 law holding in $M^n_{\bar{p}}$, but not necessarily for the 1-hereditary 0-1 law. The proof uses "local" properties of graphs. It seems that some form of "1-hereditary" version of 5.14 is possible. In any case we could not find a necessary and sufficient condition for the 1-hereditary 0-1 law, and the analysis of this case is not complete.

We first find a necessary condition on $\bar{p}$ for the 1-hereditary weak convergence law. Let us start with a definition of a structure on a sequence $\bar{q} \in \mathfrak{P}$ that enables us to "identify" the l^*-boundary in $M^n_{\bar{q}}$.

Definition 5.2. (1) *A sequence $\bar{q} \in \mathfrak{P}$ is called nice if:*
   (a) *$U^*(\bar{q}) = \{l^*\}$.*
   (b) *If $l_1, l_2, l_3 \in \{l < n_{\bar{q}} : q_l > 0\}$ then $l_1 + l_2 \neq l_3$.*
   (c) *If $l_1, l_2, l_3, l_4 \in \{l < n_{\bar{q}} : q_l > 0\}$ then $l_1 + l_2 + l_3 \neq l_4$.*
   (d) *If $l_1, l_2, l_3, l_4 \in \{l < n_{\bar{p}} : q_l > 0\}$, $l_1 + l_2 = l_3 + l_4$ and $l_1 + l_2 < n_{\bar{q}}$ then $\{l_1, l_2\} = \{l_3, l_4\}$.*

(2) *Let ϕ^1 be the following L-formula:*

$$\phi^1(y_1, z_1, y_2, z_2) := y_1 \sim z_1 \wedge z_1 \sim z_2 \wedge z_2 \sim y_2 \wedge y_2 \sim y_1 \wedge y_1 \neq z_2 \wedge z_1 \neq y_2.$$

(3) *For $k \geq 0$ define by induction on k the L-formula $\phi^1_k(y_1, z_1, y_2, z_2)$ by:*
   - *$\phi^1_0(y_1, z_1, y_2, z_2) := y_1 = y_2 \wedge z_1 = z_2 \wedge y_1 \neq z_1$.*
   - *$\phi^1_1(y_1, z_1, y_2, z_2) := \phi^1(y_1, z_1, y_2, z_2)$.*
   - *$\phi^1_{k+1}(y_1, z_1, y_2, z_2) :=$*
     $(\exists y \exists z)[(\phi^1_k(y_1, z_1, y, z) \wedge \phi^1(y, z, y_2, z_2)) \vee (\phi^1_k(y_2, z_2, y, z)\phi^1(y_1, z_1, y, z))]$.

(4) *For $k_1, k_2, \in \mathbb{N}$ let $\phi^2_{k_1,k_2}$ be the following L-formula:*

$$\phi^2_{k_1,k_2}(y, z) := (\exists x_1 \exists x_2 \exists x_3 \exists x_4)[\phi^1_{k_1}(y, z, x_1, x_2) \wedge \phi^1_{k_2}(x_2, x_1, x_3, x_4) \wedge \neg x_3 \sim x_4].$$

(5) *For $k_1, k_2, \in \mathbb{N}$ let $\phi^3_{k_1,k_2}$ be the following L formula:*

$$\phi^3_{k_1,k_2}(x) := (\exists! y)[x \sim y \wedge \neg \phi^2_{k_1,k_2}(x, y)].$$

Observation 5.3. *Let $\bar{q} \in \mathfrak{P}$ be nice and $n \in \mathbb{N}$ be such that $n < n_{\bar{q}}$. Then the following holds in $M^n_{\bar{q}}$ with probability 1:*

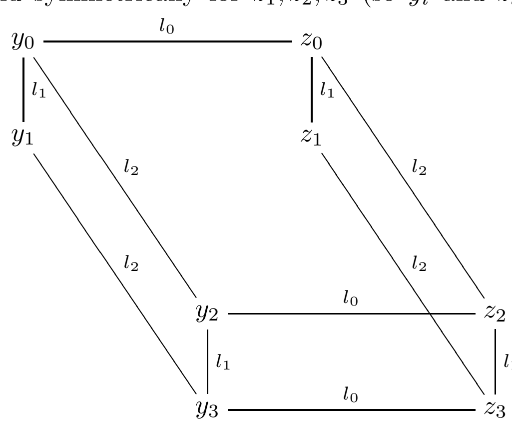
(1) *For $y_1, z_1, y_2, z_2 \in [n]$, if $M^n_{\bar{q}} \models \phi^1[y_1, z_1, y_2, z_2]$ then $y_1 - z_1 = y_2 - z_2$. (Use (d) in the definition of nice).*
(2) *For $k \in \mathbb{N}$ and $y_1, z_1, y_2, z_2 \in [n]$, if $M^n_{\bar{q}} \models \phi^1_k[y_1, z_1, y_2, z_2]$ then $y_1 - z_1 = y_2 - z_2$. (Use (1) above and induction on k).*
(3) *For $k_1, k_2 \in \mathbb{N}$ and $y, z \in [n]$, if $M^n_{\bar{q}} \models \phi^2_{k_1,k_2}[y, z]$ then $|y - z| \neq l^*$. (Use (2) above and the definition of $\phi^2_{k_1,k_2}(y, z)$).*
(4) *For $k_1, k_2 \in \mathbb{N}$ and $x \in [n]$, if $M^n_{\bar{q}} \models \phi^3_{k_1,k_2}[x]$ then $x \in [1, l^*] \cup (n - l^*, n]$. (Use (3) above).*

The following claim shows that if $\bar{q}$ is nice (and have a certain structure) then, with probability close to 1, $\phi^3_{3,0}[y]$ holds in $M^n_{\bar{q}}$ for all $y \in [1, l^*] \cup (n - l^*, n]$. This, together with (4) in the observation above gives us a "definition" of the l^*-boundary in $M^n_{\bar{q}}$.

Claim 5.4. *Let $\bar{q} \in \mathfrak{P}^{fin}$ be nice and denote $n = n_{\bar{q}}$. Assume that for all $l > 0$, $q_l > 0$ implies $l < \lfloor n/3 \rfloor$. Assume further that for some $\epsilon > 0$, $0 < q_l < 1 \Rightarrow \epsilon < q_l < 1 - \epsilon$. Let $y_0 \in [1, l^*] \cup (n - l^*, n]$. Denote $m := |\{0 < l < n_{\bar{p}} : 0 < q_l < 1\}|$. Then:*

$$Pr[M^n_{\bar{q}} \models \neg\phi^3_{3,0}[y_0]] \leq \Big(\sum_{\{y \in [n] : |y_0 - y| \neq l^*\}} q_{|y_0 - y|}\Big)(1 - \epsilon^{11})^{m/2-1}.$$

Proof. We deal with the case $y_0 \in [1, l^*]$, the case $y_0 \in (n - l^*, n]$ is symmetric. Let $z_0 \in [n]$ be such that $l_0 := z_0 - y_0 \in \{0 < l < n : 0 < q_l < 1\}$ (so $l_0 \neq l^*$ and $l_0 < \lfloor n/3 \rfloor$), and assume that $M^n_{\bar{q}} \models y_0 \sim z_0$. For any $l_1, l_2 < \lfloor n/3 \rfloor$ denote (see diagram below): $y_1 := y_0 + l_1$, $y_2 := y_0 + l_2$, $y_3 := y_2 + l_1 = y_1 + l_2 = y_0 + l_1 + l_2$ and symmetrically for z_1, z_2, z_3 (so y_i and z_i for $i \in \{0, 1, 2, 3\}$ all belong to $[n]$).



The following holds in $M^n_{\bar{q}}$ with probability 1: If for some $l_1, l_2 < \lfloor n/3 \rfloor$ such that (l_0, l_1, l_2) is without repetitions, we have:

$(*)_1$ (y_0, y_1, y_3, y_2), (z_0, z_1, z_3, z_2) and (y_2, y_3, z_3, z_2) are all circles in $M^n_{\bar{q}}$.
$(*)_2$ $\{y_1, z_1\}$ is *not* an edge of $M^n_{\bar{q}}$.

Then $M^n_{\bar{q}} \models \phi^2_{0,3}[y_0, z_0]$. Why? As (y_1, y_0, z_0, z_1), in the place of (x_1, x_2, x_3, x_4), exemplifies $M^n_{\bar{p}} \models \phi^2_{0,3}[y_0, z_0]$. Let us fix $z_0 = y_0 + l_0$ and assume that $M^n_{\bar{q}} \models y_0 \sim z_0$. (Formally we condition the probability space $M^n_{\bar{q}}$ to the event $y_0 \sim z_0$.) Denote

$$L^{y_0,z_0} := \{(l_1, l_2) : q_{l_1}, q_{l_2} > 0, l_0 \neq l_1, l_0 \neq l_2, l_1 \neq l_2\}.$$

For $(l_1, l_2) \in L^{y_0,z_0}$, the probability that $(*)_1$ and $(*)_2$ holds, is $(1 - q_{l_0})(q_{l_0})^2(q_{l_1})^4(q_{l_2})^4$. Denote the event that $(*)_1$ and $(*)_2$ holds by $E^{y_0,z_0}(l_1, l_2)$. Note that if $(l_1, l_2), (l'_1, l'_2) \in L^{y_0,z_0}$ are such that (l_1, l_2, l'_1, l'_2) is without repetitions and $l_1 + l_2 \neq l'_1 + l'_2$ then the events $E^{y_0,z_0}(l_1, l_2)$ and $E^{y_0,z_0}(l'_1, l'_2)$ are independent. Now recall that $m := |\{l > 0 : \epsilon < q_l < 1 - \epsilon\}|$. Hence we have some $L' \subseteq L^{y_0,z_0}$ such that: $|L'| = \lfloor m/2 - 1 \rfloor$, and if $(l_1, l_2), (l'_1, l'_2) \in L'$ then the events $E^{y_0,z_0}(l_1, l_2)$ and $E^{y_0,z_0}(l'_1, l'_2)$ are independent. We conclude that

$$Pr[M^n_{\bar{q}} \models \neg\phi^2_{0,3}[y_0, z_0] | M^n_{\bar{q}} \models y_0 \sim z_0] \leq$$
$$(1 - (1 - q_{l_0})(q_{l_0})^2(q_{l_1})^4(q_{l_2})^4)^{m/2-1} \leq (1 - \epsilon^{11})^{m/2-1}.$$

This is a common bound for all $z_0 = y_0 + l_0$, and the same bound holds for all $z_0 = y_0 - l_0$ (whenever it belongs to $[n]$). We conclude that the expected number of $z_0 \in [n]$ such that: $|z_0 - y_0| \neq l^*$, $M^n_{\bar{q}} \models y_0 \sim z_0$ and $M^n_{\bar{q}} \models \neg\phi^2_{0,3}[y_0, z_0]$ is at most $(\sum_{\{y\in[n]:|y_0-y|\neq l^*\}} q_{|y_0-y|})(1-\epsilon^{11})^{m/2-1}$. Now by (3) in Observation 5.3, $M^n_{\bar{q}} \models \phi^2_{0,3}[y_0, y_0 + l^*]$. By Markov's inequality and the definition of $\phi^3_{0,3}(x)$ we are done. $\square$

We now prove two lemmas which allow us to construct a sequence $\bar{q}$ such that for $\varphi := \exists x\phi^3_{0,3}(x)$ both φ and $\neg\varphi$ will hold infinitely often in $M^n_{\bar{q}}$.

Lemma 5.5. *Assume$\bar{p}$ satisfy $\sum_{l>0} p_l = \infty$, and let $\bar{q} \in Gen^r_1(\bar{p})$ be nice. Let $\zeta > 0$ be some rational number. Then there exists some $r' > r$ and $\bar{q}' \in Gen^{r'}_1(\bar{p})$ such that: $\bar{q}'$ is nice, $\bar{q} \lhd \bar{q}'$ and $Pr[M^{n_{\bar{q}'}}_{\bar{q}'} \models \varphi] \leq \zeta$.*

Proof. Define $p^1 := (\prod_{l\in[n_{\bar{q}}]\setminus\{l^*\}}(1-p_l))^2$, and choose $r' > r$ large enough such that $\sum_{r<l\leq r'} p_l \geq 2l^* \cdot p^1/\zeta$. Now define $\bar{q}' \in Gen^{r'}_1(\bar{p})$ in the following way:

$$q'_l = \begin{cases} q_l & 0 < l < n_{\bar{q}} \\ 0 & n_{\bar{q}} \leq l < (r'-r)\cdot n_{\bar{q}} \\ p_{r+i} & l = (r'-r+i)\cdot n_{\bar{q}} \text{ for some } 0 < i \leq (r'-r) \\ 0 & (r'-r)\cdot n_{\bar{q}} \leq l < 2(r'-r)\cdot n_{\bar{q}} \text{ and } l \not\equiv 0 \pmod{n_{\bar{q}}}. \end{cases}$$

Note that indeed $\bar{q}'$ is nice and $\bar{q} \lhd \bar{q}'$. Denote $n := n_{\bar{q}'} = 2(r'-r)\cdot n_{\bar{q}}$. Note further that every member of $M^n_{\bar{q}'}$ have at most one neighbor of distance more more than $n/2$, and all the rest of its neighbors are of distance at most $n_{\bar{q}}$. We now bound from above the probability of $M^n_{\bar{q}'} \models \exists x\phi^3_{0,3}(x)$. Let x be in $[1, l^*]$. For each $0 < i \leq (r'-r)$ denote $y_i := x + (r'-r+i)\cdot n_{\bar{q}}$ (hence $y_i \in [n/2, n]$) and let E_i be the following event: "$M^n_{\bar{q}'} \models y_i \sim z$ iff $z \in \{x, y_i + l^*, y_i - l^*\}$". By the definition of $\bar{q}'$, each y_i can only be connected to either x of to members of $[y - n_{\bar{q}}, y + n_{\bar{q}}]$, hence we have

$$Pr[E_i] = q'_{(r'-r+i)\cdot n_{\bar{q}}} \cdot p^1 = p_{r+i} \cdot p^1.$$

As $i \neq j \Rightarrow n/2 > |y_i - y_j| > n_{\bar{q}}$ we have that the E_i-is are independent events. Now if E_i holds then by the definition of $\phi^2_{0,3}$ we have $M^n_{\bar{q}'} \models \neg\phi^2_{0,3}[x, y_i]$, and as $M^n_{\bar{q}'} \models \neg\phi^2_{0,3}[x, x+l^*]$ this implies $M^n_{\bar{q}'} \models \neg\phi^3_{0,3}[x]$. Let the random variable X denote the number of $0 < i \leq (r'-r)$ such that E_i holds in $M^n_{\bar{q}'}$. Then by Chebyshev's inequality we have:

$$Pr[M^n_{\bar{q}'} \models \phi^3_{0,3}[x]] \leq Pr[X=0] \leq \frac{Var(X)}{Exp(X)^2} \leq \frac{1}{Exp(X)} \leq \frac{p^1}{\sum\limits_{0<i\leq(r'-r)} p_{r+i}} \leq \frac{\zeta}{2l^*}.$$

This is true for each $x \in [1, l^*]$ and the symmetric argument gives the same bound for each $x \in (n-l^*, n]$. Finally note that if $x, x+l^*$ both belong to $[n]$ then $M^n_{\bar{q}'} \models \neg\phi^2_{0,3}[x, x+l^*]$ (see 5.3(4)). Hence if $x \in (l^*, n-l^*]$ then $M^n_{\bar{q}'} \models \neg\phi^3_{0,3}[x]$. We conclude that:

$$Pr[M^n_{\bar{q}'} \models \exists x\phi^3_{0,3}(x)] = Pr[M^n_{\bar{q}'} \models \phi] \leq \zeta$$

as desired. $\square$

Lemma 5.6. *Assume $\bar{p}$ satisfy $0 < p_l < 1 \Rightarrow \epsilon < p_l < 1-\epsilon$ for some $\epsilon > 0$, and $\sum_{n=1}^{\infty} p_n = \infty$. Let $\bar{q} \in Gen^r_1(\bar{p})$ be nice, and $\zeta > 0$ be some rational number.*

Then there exists some $r' > r$ and $\bar{q}' \in Gen_1^{r'}(\bar{p})$ such that: $\bar{q}'$ is nice, $\bar{q} \lhd \bar{q}'$ and $Pr[M_{\bar{q}'}^{n_{\bar{q}'}} \models \varphi] \geq 1 - \zeta$.

Proof. This is a direct consequence of Claim 5.4. For each $r' > r$ denote $m(r') := |\{0 < l \leq r' : 0 < p_l < 1\}|$. Trivially we can choose $r' > r$ such that $m(r')(1 - \epsilon^{11})^{m(r')/2-1} \leq \zeta$. As $\bar{q}$ is nice there exists some nice $\bar{q}' \in Gen_1^{r'}(\bar{p})$ such that $\bar{q} \lhd \bar{q}'$. Note that

$$\sum_{\{y \in [n] : |1-y| \neq l^*\}} q'_{|1-y|} \leq \sum_{\{0 < l < n_{\bar{q}'} : l \neq l^*\}} q'_l \leq m(r')$$

and hence by 5.4 we have:

$$Pr[M_{\bar{q}'}^n \models \neg\phi] \leq Pr[M_{\bar{q}'}^n \models \neg\phi_{2,0}^3[1]] \leq m(r')(1 - \epsilon^{11})^{m(r')/2-1} \leq \zeta$$

as desired. □

From the last two lemmas we conclude:

Conclusion 5.7. *Assume that $\bar{p}$ satisfy $0 < p_l < 1 \Rightarrow \epsilon < p_l < 1 - \epsilon$ for some $\epsilon > 0$, and $\sum_{n=1}^{\infty} p_n = \infty$. Then $\bar{p}$ does not satisfy the 1-hereditary weak convergence law for L.*

The proof is by inductive construction of $\bar{q} \in Gen_1(\bar{p})$ such that for $\varphi := \exists x \phi_{0,3}^3(x)$ both φ and $\neg\varphi$ hold infinitely often in $M_{\bar{q}}^n$, using Lemmas 5.5, 5.6 as done on previous proofs.

From Conclusion 5.7 we have a necessary condition on $\bar{p}$ for the 1-hereditary weak convergence law. We now find a sufficient condition on $\bar{p}$ for the (not necessarily 1-hereditary) 0-1 law. Let us start with definitions of distance in graphs and of local properties in graphs.

Definition 5.8. *Let G be a graph on vertex set $[n]$.*

(1) *For $x, y \in [n]$ let $dist^G(x, y) := \min\{k \in \mathbb{N} : G$ has a path of length k from x to $y\}$. Note that for each $k \in \mathbb{N}$ there exists some L-formula $\theta_k(x, y)$ such that for all G and $x, y \in [n]$:*

$$G \models \theta_k[x, y] \quad \text{iff} \quad dist^G(x, y) \leq k.$$

(2) *For $x \in [n]$ and $r \in \mathbb{N}$ let $B^G(r, x) := \{y \in [n] : dist^G(x, y) \leq r\}$ be the ball with radius r and center x in G.*

(3) *An L-formula $\phi(x)$ is called r-local if every quantifier in ϕ is restricted to the set $B^G(r, x)$. Formally each appearance of the form $\forall y...$ in ϕ is of the form $(\forall y)\theta_r(x, y) \to ...$, and similarly for $\exists y$ and other variables. Note that for any G, $x \in [n]$, $r \in \mathbb{N}$ and an r-local formula $\phi(x)$ we have:*

$$G \models \phi[x] \quad \text{iff} \quad G|_{B(r,x)} \models \phi[x].$$

(4) *An L-sentence is called* local *if it has the form*

$$\exists x_1 ... \exists x_m \bigwedge_{1 \leq i \leq m} \phi(x_i) \bigwedge_{1 \leq i < j \leq m} \neg\theta_{2r}(x_i, x_j)$$

where $\phi = \phi(x)$ is an r-local formula for some $r \in \mathbb{N}$.

(5) *For $l, r \in \mathbb{N}$ and an L-formula $\phi(x)$ we say that the l-boundary of G is r-indistinguishable by $\phi(x)$ if for all $z \in [1, l] \cup (n - l, n]$ there exists some $y \in [n]$ such that $B^G(r, y) \cap ([1, l] \cup (n - l, n]) = \emptyset$ and $G \models \phi[z] \leftrightarrow \phi[y]$*

We can now use the following famous result from [3]:

Theorem 5.9 (**Gaifman's Theorem**). *Every L-sentence is logically equivalent to a boolean combination of local L-sentences.*

We will use Gaifman's theorem to prove:

Lemma 5.10. *Assume that for all $k \in \mathbb{N}$ and k-local L-formula $\varphi(z)$ we have:*

$$\lim_{n\to\infty} Pr[\textit{The } l^*\textit{-boundary of } M^n_{\bar{p}} \textit{ is } k\textit{-indistinguishable by } \varphi(z)] = 1.$$

Then the 0-1 law for L holds in $M^n_{\bar{p}}$.

Proof. By Gaifman's theorem it is enough if we prove that the 0-1 law holds in $M^n_{\bar{p}}$ for local L-sentences. Let

$$\psi := \exists x_1 ... \exists x_m \bigwedge_{1\le i\le m} \phi(x_i) \bigwedge_{1\le i<j\le m} \neg\theta_{2r}(x_i, x_j)$$

be some local L-sentence, where $\phi(x)$ is an r-local formula.

Define $\mathfrak{H}$ to be the set of all 4-tuples (l, U, u_0, H) such that: $l \in \mathbb{N}$, $U \subseteq [l]$, $u_0 \in U$ and H is a graph with vertex set U. We say that some $(l, U, u_0, H) \in \mathfrak{H}$ is r-proper for $\bar{p}$ (but as $\bar{p}$ is fixed we usually omit it) if it satisfies:

($*_1$) For all $u \in U$, $dist^H(u_0, u) \le r$.
($*_2$) For all $u \in U$, if $dist^H(u_0, u) < r$ then $u + l^*, u - l^* \in U$.
($*_3$) $Pr[M^l_{\bar{p}}|_U = H] > 0$.

We say that a member of $\mathfrak{H}$ is proper if it is r-proper for some $r \in \mathbb{N}$.

Let H be a graph on vertex set $U \subseteq [l]$ and G be a graph on vertex set $[n]$. We say that $f : U \to [n]$ is a strong embedding of H in G if:

- f in one-to one.
- For all $u, v \in U$, $H \models u \sim v$ iff $G \models f(u) \sim f(v)$.
- For all $u, v \in U$, $f(u) - f(v) = u - v$.
- If $i \in Im(f)$, $j \in [n] \setminus Im(f)$ and $|i - j| \ne l^*$ then $G \models \neg i \sim j$.

We make two observations which follow directly from the definitions:

(1) If $(l, U, u_0, H) \in \mathfrak{H}$ is r-proper and $f : U \to [n]$ is a strong embedding of H in G then $Im(f) = B^G(r, f(u_0))$. Furthermore for any r-local formula $\phi(x)$ and $u \in U$ we have, $G \models \phi[f(u)]$ iff $H \models \phi[u]$.
(2) Let G be a graph on vertex set $[n]$ such that $Pr[M^n_{\bar{p}} = G] > 0$, and $x \in [n]$ be such that $B^G(r-1, x)$ is disjoint to $[1, l^*] \cup (n - l^*, n]$. Denote by m and M the minimal and maximal elements of $B^G(r, x)$ respectively. Denote by U the set $\{i - m + 1 : i \in B^G(r, x)\}$ and by H the graph on U defined by $H \models u \sim v$ iff $G \models (u + m - 1) \sim (v + m - 1)$. Then the 4-tuple $(M - m + 1, U, x - m + 1, H)$ is an r-proper member of $\mathfrak{H}$. Furthermore for any r-local formula $\phi(x)$ and $u \in U$ we have, $G \models \phi[u - m + 1]$ iff $H \models \phi[u]$.

We now show that for any proper member of $\mathfrak{H}$ there are many disjoint strong embeddings into $M^n_{\bar{p}}$. Formally:

Claim 5.11. *Let $(l, U, u_0, H) \in \mathfrak{H}$ be proper, and $c > 1$ be some fixed real. Let E^n_c be the following event on $M^n_{\bar{p}}$: "For any interval $I \subseteq [n]$ of length at least n/c there exists some $f : U \to I$ a strong embedding of H in $M^n_{\bar{p}}$". Then*

$$\lim_{n\to\infty} Pr[E^n_c \textit{ holds in } M^n_{\bar{p}}] = 1.$$

We skip the proof of this claim an almost identical lemma is proved in [1] (see Lemma at page 8 there).

We can now finish the proof of Lemma 5.10. Recall that $\phi(x)$ is am r-local formula. We consider two possibilities. First assume that for some r-proper $(l, U, u_0, H) \in \mathfrak{H}$ we have $H \models \phi[u_0]$. Let $\zeta > 0$ be some real. Then by the claim above, for n large enough, with probability at least $1 - \zeta$ there exists $f_1, ..., f_m$ strong embeddings of H into $M^n_{\bar{p}}$ such that $\langle Im(f_i) : 1 \leq i \leq m \rangle$ are pairwise disjoint. By observation (1) above we have:

- For $1 \leq i < j \leq m$, $B^{M^n_{\bar{p}}}(r, f_i(u_0)) \cap B^{M^n_{\bar{p}}}(r, f_j(u_0)) = \emptyset$.
- For $1 \leq i \leq m$, $M^n_{\bar{p}} \models \phi[f_i(u_0)]$.

Hence $f_1(u_0), ..., f_m(u_0)$ exemplifies ψ in $M^n_{\bar{p}}$, so $Pr[M^n_{\bar{p}} \models \psi] \geq 1 - \zeta$ and as ζ was arbitrary we have $\lim_{n\to\infty} Pr[M^n_{\bar{p}} \models \psi] = 1$ and we are done.

Otherwise assume that for all r-proper $(l, U, u_0, H) \in \mathfrak{H}$ we have $H \models \neg\phi[u_0]$. We will show that $\lim_{n\to\infty} Pr[M^n_{\bar{p}} \models \psi] = 0$ which will finish the proof. Towards contradiction assume that for some $\epsilon > 0$ for unboundedly many $n \in \mathbb{N}$ we have $Pr[M^n_{\bar{p}} \models \psi] \geq \epsilon$. Define the L-formula:

$$\varphi(z) := (\exists x)(\theta_{r-1}(x, z) \wedge \phi(x)).$$

Note that $\varphi(z)$ is equivalent to a k-local formula for $k = 2r - 1$. Hence by the assumption of our lemma for some (large enough $n \in \mathbb{N}$) we have with probability at least $\epsilon/2$: $M^n_{\bar{p}} \models \psi$ and the l^*-boundary of $M^n_{\bar{p}}$ is k-indistinguishable by $\varphi(z)$. In particular for some $n \in \mathbb{N}$ and G a graph on vertex set $[n]$ we have:

(α) $Pr[M^n_{\bar{p}} = G] > 0$.
(β) $G \models \psi$.
(γ) The l^*-boundary of G is k-indistinguishable by $\varphi(z)$.

By (β) for some $x_0 \in [n]$ we have $G \models \phi[x_0]$. If x_0 is such that $B^G(r - 1, x_0)$ is disjoint to $[1, l^*] \cup (n - l^*, n]$ then by (α) and observation (2) above we have some r-proper $(l, U, u_0, H) \in \mathfrak{H}$ such that $H \models \phi[u_0]$ in contradiction to our assumption. Hence assume that $B^G(r-1, x_0)$ is not disjoint to $[1, l^*] \cup (n-l^*, n]$ and let $z_0 \in [n]$ belong to their intersection. So by the definition of $\varphi(z)$ we have $G \models \varphi[z_0]$ and by (γ) we have some $y_0 \in [n]$ such that $B^G(k, y_0) \cap ([1, l^*] \cup (n - l^*, n]) = \emptyset$ and $G \models \varphi[y_0]$. Again by the definition of $\varphi(z)$, and recalling that $k = 2r - 1$ we have some $x_1 \in [n]$ such that $B^G(r - 1, x_1) \cap ([1, l^*] \cup (n - l^*, n]) = \emptyset$ and $G \models \phi[x_1]$. So again by (α) and observation (2) we get a contradiction. □

Remark 5.12. *Lemma 5.10 above gives a sufficient condition for the 0-1 law. If we are only interested in the convergence law, then a weaker condition is sufficient, all we need is that the probability of any local property holding in the l^*-boundary converges. Formally:*

Assume that for all $r \in \mathbb{N}$ and r-local L-formula, $\phi(x)$, and for all $1 \leq l \leq l^*$ we have: Both $\langle Pr[M^n_{\bar{p}} \models \phi[l] : n \in \mathbb{N}\rangle$ and $\langle Pr[M^n_{\bar{p}} \models \phi[n - l + 1] : n \in \mathbb{N}\rangle$ converge to a limit. Then $M^n_{\bar{p}}$ satisfies the convergence law.

The proof is similar to the proof of Lemma 5.10. A similar proof on the convergence law in graphs with the successor relation is Theorem 2(i) in [1].

We now use 5.10 to get a sufficient condition on $\bar{p}$ for the 0-1 law holding in $M^n_{\bar{p}}$. Our proof relays on the assumption that $M^n_{\bar{p}}$ contains few circles, and only those that are "unavoidable". We start with a definition of such circles:

Definition 5.13. *Let $n \in \mathbb{N}$.*

(1) *For a sequence $\bar{x} = (x_0, x_1, ..., x_k) \subseteq [n]$ and $0 \leq i < k$ denote $l_i^{\bar{x}} := x_{i+1} - x_i$.*
(2) *A sequence $(x_0, x_1, ..., x_k) \subseteq [n]$ is called possible for $\bar{p}$ (but as $\bar{p}$ is fixed we omit it and similarly below) if for each $0 \leq i < k$, $p_{|l_i^{\bar{x}}|} > 0$.*
(3) *A sequence $(x_0, x_1, ..., x_k)$ is called a circle of length k if $x_0 = x_k$ and $\langle \{x_i, x_{i+1}\} : 0 \leq i < k \rangle$ is without repetitions.*
(4) *A circle of length k, is called simple if $(x_0, x_1, ..., x_{k-1})$ is without repetitions.*
(5) *For $\bar{x} = (x_0, x_1, ..., x_k) \subseteq [n]$, a pair $(S \cup A)$ is called a symmetric partition of $\bar{x}$ if:*
   - *$S \cup A = \{0, ..., k-1\}$.*
   - *If $i \neq j$ belong to A then $l_i^{\bar{x}} + l_j^{\bar{x}} \neq 0$.*
   - *The sequence $\langle l_i^{\bar{x}} : i \in S \rangle$ can be partitioned into two sequences of length $r = |S|/2$: $\langle l_i : 0 \leq i < r \rangle$ and $\langle l'_i : 0 \leq i < r \rangle$ such that $l_i + l'_i = 0$ for each $0 \leq i < r$.*

(6) *For $\bar{x} = (x_0, x_1, ..., x_k) \subseteq [n]$ let $(Sym(\bar{x}), Asym(\bar{x}))$ be some symmetric partition of $\bar{x}$ (say the first in some prefixed order). Denote $Sym^+(\bar{x}) := \{i \in Sym(\bar{x}) : l_i^{\bar{x}} > 0\}$.*
(7) *We say that $\bar{p}$ has no unavoidable circles if for all $k \in \mathbb{N}$ there exists some $m_k \in \mathbb{N}$ such that if $\bar{x}$ is a possible circle of length k then for each $i \in Asym(\bar{x})$, $|l_i^{\bar{x}}| \leq m_k$.*

Theorem 5.14. *Assume that $\bar{p}$ has no unavoidable circles, $\sum_{l=1}^{\infty} p_l = \infty$ and $\sum_{l=1}^{\infty} (p_l)^2 < \infty$. Then $M_{\bar{p}}^n$ satisfies the 0-1 law for L.*

Proof. Let $\phi(x)$ be some r-local formula, and j^* be in $\{1, 2, ..., l^*\} \cup \{-1, -2, ..., -l^*\}$. For $n \in \mathbb{N}$ let $z_n^* = z^*(n, j^*)$ equal j^* if $j^* > 0$ and $n - j^* + 1$ if $j^* < 0$ (so z_n^* belongs to $[1, l^*] \cup (n - l^*, n]$). We will show that with probability approaching 1 as $n \to \infty$ there exists some $y^* \in [n]$ such that $B^{M_{\bar{p}}^n}(r, y^*) \cap ([1, l^*] \cup (n - l^*, n]) = \emptyset$ and $M_{\bar{p}}^n \models \phi[z_n^*] \leftrightarrow \phi[y^*]$. This will complete the proof by Lemma 5.10. For simplicity of notation assume $j^* = 1$ hence $z_n^* = 1$ (the proof of the other cases is similar). We use the notations of the proof of 5.10. In particular recall the definition of the set $\mathfrak{H}$ and of an r-proper member of $\mathfrak{H}$. Now if for two r-proper members of $\mathfrak{H}$, (l^1, x^1, U^1, H^1) and (l^2, x^2, U^2, H^2) we have $H^1 \models \phi[x^1]$ and $H^2 \models \neg\phi[x^2]$ then by Claim 5.11 we are done. Otherwise all r-proper members of $\mathfrak{H}$ give the same value to $\phi[x]$ and without loss of generality assume that if $(l, x, U, H) \in \mathfrak{H}$ is a r-proper then $H \models \phi[x]$ (the dual case is identical). If $\lim_{n\to\infty} Pr[M_{\bar{p}}^n \models \phi[1]] = 1$ then again we are done by 5.11. Hence we may assume that:

$\odot$ For some $\epsilon > 0$, for an unbounded set of $n \in \mathbb{N}$, $Pr[M_{\bar{p}}^n \models \neg\phi[1]] \geq \epsilon$.

In the construction below we use the following notations: 2 denotes the set $\{0, 1\}$. ${}^k2$ denotes the set of sequences of length k of members of 2, and if η belongs to ${}^k2$ we write $|\eta| = k$. ${}^{\leq k}2$ denotes $\bigcup_{0 \leq i \leq k} {}^k2$ and similarly ${}^{<k}2$. $\langle\rangle$ denotes the empty sequence, and for $\eta, \eta' \in {}^{\leq k}2$, $\eta\hat{\ }\eta'$ denotes the concatenation of η and η'. Finally for $\eta \in {}^k2$ and $k' < k$, $\eta|_{k'}$ is the initial segment of length k' of η.

Call $\bar{y}$ a saturated tree of depth k in $[n]$ if:

- $\bar{y} = \langle y_\eta \in [n] : \eta \in {}^{\leq k}2 \rangle$.
- $\bar{y}$ is without repetitions.

- $\{y_{\langle 0\rangle}, y_{\langle 1\rangle}\} = \{y_{\langle\rangle} + l^*, y_{\langle\rangle} - l^*\}$.
- If $0 < l < k$ and $\eta \in {}^{l}2$ then $\{y_\eta + l^*, y_\eta - l^*\} \subseteq \{y_{\eta\hat{}\langle 0\rangle}, y_{\eta\hat{}\langle 1\rangle}, y_{\eta|_{l-1}}\}$.

Let G be a graph with set of vertexes $[n]$, and $i \in [n]$. We say that $\bar{y}$ is a circle free saturated tree of depth k for i in G if:

(i) $\bar{y}$ is a saturated tree of depth k in $[n]$.
(ii) $G \models i \sim y_{\langle\rangle}$ but $|i - y_{\langle\rangle}| \neq l^*$.
(iii) For each $\eta \in {}^{<k}2$, $G \models y_\eta \sim y_{\eta\hat{}\langle 0\rangle}$ and $G \models y_\eta \sim y_{\eta\hat{}\langle 1\rangle}$.
(iv) None of the edges described in (ii),(iii) belongs to a circle of length $\leq 6k$ in G.
(v) Recalling that $\bar{p}$ have no unavoidable circles let m_{2k} be the one from definition 5.13(7). For all $\eta \in {}^{\leq k}2$ and $y \in [n]$ if $G \models y_\eta \sim y$ and $y \notin \{y_{\eta\hat{}\langle 0\rangle}, y_{\eta\hat{}\langle 1\rangle}, y_{\eta|_{l-1}}, i\}$ then $|y - y_\eta| > m_{2k}$.

For $I \subseteq [n]$ we say that $\langle \bar{y}^i : i \in I\rangle$ is a circle free saturated forest of depth k for I in G if:

(a) For each $i \in I$, $\bar{y}^i$ is a circle free saturated tree of depth k for i in G.
(b) As sets $\langle \bar{y}^i : i \in I\rangle$ are pairwise disjoint.
(c) If $i_1, i_2 \in I$ and $\bar{x}$ is a path of length $k' \leq k$ in G from $y^{i_1}_{\langle\rangle}$ to i_2, then for some $j < k'$, $(x_j, x_{j+1}) = (y^{i_1}_{\langle\rangle}, i_1)$.

Claim 5.15. *For $n \in \mathbb{N}$ and G a graph on $[n]$ denote by $I^*_k(G)$ the set $([1, l^*] \cup (n - l^*, n]) \cap B^G(1, k)$. Let $E^{n,k}$ be the event: "There exists a circle free saturated forest of depth k for $I^*_k(G)$". Then for each $k \in \mathbb{N}$:*

$$\lim_{n\to\infty} Pr[E^{n,k} \textit{ holds in } M^n_{\bar{p}}] = 1.$$

Proof. Let $k \in \mathbb{N}$ be fixed. The proof proceeds in six steps:

Step 1. We observe that only a bounded number of circles starts in each vertex of $M^n_{\bar{p}}$. Formally For $n, m \in \mathbb{N}$ and $i \in [n]$ let $E^1_{n,m,i}$ be the event: "More than m different circles of length at most $12k$ include i". Then for all $\zeta > 0$ for some $m = m(\zeta)$ (m depends also on $\bar{p}$ and k but as those are fixed we omit them from the notation and similarly below) we have:

$\circledast_1$ For all $n \in \mathbb{N}$ and $i \in [n]$, $Pr_{M^n_{\bar{p}}}[E^1_{n,m,i}] \leq \zeta$.

To see this note that if $\bar{x} = (x_0, ..., x_{k'})$ is a possible circle in $[n]$, then

$$Pr[\bar{x} \text{ is a weak circle in } M^n_{\bar{p}}] := p(\bar{x}) = \prod_{i\in Asym(\bar{x})} p_{|l^{\bar{x}}_i|} \cdot \prod_{i\in Sym^+(\bar{x})} (p_{l^{\bar{x}}_i})^2.$$

Now as $\bar{p}$ has no unavoidable, circles let m_{12k} be as in 5.13(7). Then the expected number of circles of length $\leq 12k$ starting in $i = x_0$ is

$$\sum_{\substack{k'\leq 12k, \bar{x}=(x_0,\ldots,x_{k'})\\ \text{is a possible circle}}} p(\bar{x}) \leq (m_{12k})^{12k} \cdot \sum_{0<l_1,\ldots,l_{6k}<n} \prod_{i=1}^{6k} (p_{l_i})^2 \leq (m_1 2k)^{12k} \cdot (\sum_{0<l<n} (p_l)^2)^{6k}.$$

But as $\sum_{0<l<n} (p_l)^2$ is bounded by $\sum_{l=1}^{\infty} (p_l)^2 := c^* < \infty$, if we take $m = (m_{12k})^{12k} \cdot (c^*)^{6k}/\zeta$ then we have $\circledast_1$ as desired.

Step 2. We show that there exists a positive lower bound on the probability that a circle passes through a given edge of $M^n_{\bar{p}}$. Formally: Let $n \in \mathbb{N}$ and $i, j \in [n]$ be such that $p_{|i-j|} > 0$. Denote By $E^2_{n,i,j}$ the event: "There does not exists a circle of length $\leq 6k$ containing the edge $\{i,j\}$". Then there exists some $q_2 > 0$ such that:

$\circledast_2$ For any $n \in \mathbb{N}$ and $i,j \in [n]$ such that $p_{|i-j|} > 0$, $Pr_{M^n_{\bar{p}}}[E^2_{n,i,j} | i \sim j] \geq q_2$.

To see this call a path $\bar{x} = (x_0, ..., x_{k'})$ good for $i,j \in [n]$ if $x_0 = j$, $x_{k'} = i$, $\bar{x}$ does not contain the edge $\{i,j\}$ and does not contain the same edge more than once. Let $E'^2_{n,i,j}$ be the event: "There does not exists a path good for i,j of length $< 6k$". Note that for $i,j \in [n]$ and G a graph on $[n]$ such that $G \models i \sim j$ we have: $(i,j,x_2,...,x_{k'})$ is a circle in G iff $(j,x_2,...,k_{k'})$ is a path in G good for i,j. Hence for such G we have: $E^2_{n,i,j}$ holds in G iff $E'^2_{n,i,j}$ holds in G. Since the events $i \sim j$ and $E'^2_{n,i,j}$ are independent in $M^n_{\bar{p}}$ we conclude:

$$Pr_{M^n_{\bar{p}}}[E^2_{n,i,j} | i \sim j] = Pr_{M^n_{\bar{p}}}[E'^2_{n,i,j} | i \sim j] = Pr_{M^n_{\bar{p}}}[E'^2_{n,i,j}].$$

Next recalling Definition 5.13(7) let m_k be as there. Since $\sum_{l>0}(p_l)^2 < \infty$, $(p_l)^2$ converges to 0 as l approaches infinity, and hence so does p_l. Hence for some $m^0 \in \mathbb{N}$ we have $l > m^0$ implies $p_l < 1/2$. Let $m^*_k := \max\{m_{6k}, m^0\}$. We now define for a possible path $\bar{x} = (x_0, ...x_{k'})$, $Large(\bar{x}) = \{0 \leq r < k' : |l^{\bar{x}}_r| > m^*_k\}$. Note that as $\bar{p}$ have no unavoidable circles we have for any possible circle $\bar{x}$ of length $\leq 6k$, $Large(\bar{x}) \subseteq Sym(\bar{x})$, and $|Large(\bar{x})|$ is even. We now make the following claim: For each $0 \leq k^* \leq \lfloor k/2 \rfloor$ let $E'^{2,k^*}_{n,i,j}$ be the event: "There does not exists a path, $\bar{x}$, good for i,j of length $< 6k$ with $|Large(\bar{x})| = 2k^*$". Then there exists a positive probability q_{2,k^*} such that for any $n \in \mathbb{N}$ and $i,j \in [n]$ we have:

$$Pr_{M^n_{\bar{p}}}[E'^{2,k^*}_{n,i,j}] \geq q_{2,k^*}.$$

Then by taking $q_2 = \prod_{0 \leq k^* \leq \lfloor k/2 \rfloor} q_{2,k^*}$ we will have $\circledast_2$. Let us prove the claim. For $k^* = 0$ we have (recalling that no circle consists only of edges of length l^*):

$$\begin{aligned} Pr_{M^n_{\bar{p}}}[E'^{2,0}_{n,i,j}] &= \prod_{\substack{k' \leq 6k,\ \bar{x}=(i=x_0,j=x_1,\dots,x_{k'}) \\ \text{is a possible circle, } |Large(\bar{x})|=0}} (1 - \prod_{r=1}^{k'-1} p_{|l^{\bar{x}}_r|}) \\ &\geq (1 - \max\{p_l : 0 < l \leq m^*_k, l \neq l^*\})^{6k \cdot (m^*_k)^{6k-1}}. \end{aligned}$$

But as the last expression is positive and depends only on $\bar{p}$ and k we are done. For $k^* > 0$ we have:

$$\begin{aligned} Pr_{M^n_{\bar{p}}}[E'^{2,k^*}_{n,i,j}] &= \prod_{\substack{k' \leq 6k,\ \bar{x}=(i=x_0,j=x_1,\dots,x_{k'}) \\ \text{is a possible circle, } |Large(\bar{x})|=k^*}} (1 - \prod_{m=1}^{k'-1} p_{|l^{\bar{x}}_m|}) \\ &= \prod_{\substack{k' \leq 6k,\ \bar{x}=(i=x_0,j=x_1,\dots,x_{k'}) \\ \text{is a possible circle,} \\ |Large(\bar{x})|=k^*\ ,0 \notin Large(\bar{x})}} (1 - \prod_{m=1}^{k'-1} p_{|l^{\bar{x}}_m|}) \cdot \prod_{\substack{k' \leq 6k,\ \bar{x}=(i=x_0,j=x_1,\dots,x_{k'}) \\ \text{is a possible circle,} \\ |Large(\bar{x})|=k^*\ ,0 \in Large(\bar{x})}} (1 - \prod_{m=1}^{k'-1} p_{|l^{\bar{x}}_m|}). \end{aligned}$$

But the product on the left of the last line is at least

$$[\prod_{l_1,\dots,l_{k^*} > m^*_k} (1 - \prod_{m=1}^{k^*} (p_{l_m})^2)]^{(m^*_k)^{(6k-2k^*)} \cdot (6k)^{2k^*}},$$

and as $\sum_{l>m^*_k}(p_l)^2 \leq c^* < \infty$ we have $\sum_{l_1,...,l_{k^*}>m^*_k}\prod_{m=1}^{k^*}(p_{l_m})^2 \leq (c^*)^{k^*} < \infty$ and hence $\prod_{l_1,...,l_{k^*}>m^*_k}(1-\prod_{m=1}^{k^*}(p_{l_m})^2) > 0$ and we have a bound as desired. Similarly the product on the right is at least

$$[\prod_{l_1,...,l_{k^*-1}>m^*_k}(1-\prod_{m=1}^{k^*-1}(p_{l_m})^2)\cdot 1/2]^{(m^*_k)^{(6k-2k^*-1)}\cdot(6k)^{2k^*}},$$

and again we have a bound as desired.

Step 3. Denote

$$E^3_{n,i,j} := E^2_{n,i,j} \wedge \bigwedge_{r=1,...,k}(E^2_{n,j+(r-1)l^*,j+rl^*} \wedge E^2_{n,j,j-(r-1)l^*,j-rl^*})$$

and let $q_3 = q_2^{(2l^*+1)}$. We then have:

$\circledast_3$ For any $n \in \mathbb{N}$ and $i,j \in [n]$ such that $p_{|i-j|} > 0$ and $j+kl^*, j-kl^* \in [n]$, $Pr_{M^n_{\bar{p}}}[E^3_{n,i,j}|i \sim j] \geq q_3$.

This follows immediately from $\circledast_2$, and the fact that if i,i',j,j' all belong to $[n]$ then the probability $Pr_{M^n_{\bar{p}}}[E^2_{n,i,j}|E^2_{n,i',j'}]$ is no smaller then the probability $Pr_{M^n_{\bar{p}}}[E^2_{n,i,j}]$.

Step 4. For $i,j \in [n]$ such that $j+kl^*, j-kl^* \in [n]$ denote by $E^4_{n,i,j}$ the event: "$E^3_{n,i,j}$ holds and for $x \in \{j+rl^* : r \in \{-k,-k+1,...,k\}\}$ and $y \in [n] \setminus \{i\}$ we have $x \sim y \Rightarrow (|x-y| = l^* \vee |x-y| > m_{2k})$". Then for some $q_4 > 0$ we have:

$\circledast_4$ For any $n \in \mathbb{N}$ and $i,j \in [n]$ such that $p_{|i-j|} > 0$ and $j+kl^*, j-kl^* \in [n]$, $Pr_{M^n_{\bar{p}}}[E^4_{n,i,j}|i \sim j] \geq q_4$.

To see this simply take $q_4 = q_3 \cdot (\prod_{l\in\{1,...,m_{2k}\}\setminus\{l^*\}}(1-p_l))^{2k+1}$, and use $\circledast_3$.

Step 5. For $n \in \mathbb{N}$, $S \subseteq [n]$, and $i \in [n]$ let $E^5_{n,S,i}$ be the event: "For some $j \in [n] \setminus S$ we have $i \sim j$, $|i-j| \neq l^*$ and $E^4_{n,i,j}$". Then for each $\delta > 0$ and $s \in \mathbb{N}$, for $n \in \mathbb{N}$ large enough (depending on δ and s) we have:

$\circledast_5$ For all $i \in [n]$ and $S \subseteq [n]$ with $|S| \leq s$, $Pr_{M^n_{\bar{p}}}[E^5_{n,S,i}] \geq 1-\delta$.

First let $\delta > 0$ and $s \in \mathbb{N}$ be fixed. Second for $n \in \mathbb{N}$, $S \subseteq [n]$ and $i \in [n]$ denote by $J^{n,S}_i$ the set of all possible candidates for j, namely $J^{n,S}_i := \{j \in (kl^*, n-kl^*] \setminus S : |i-j| \neq l^*\}$. For $j \in J^{n,\emptyset}_i$ let $U_j := \{j+rl^* : r \in \{-k,-k+1,...,k\}\}$. For $m \in \mathbb{N}$ and G a graph on $[n]$ call $j \in J^{n,S}_i$ a candidate of type (n,m,S,i) in G, if each $j' \in U(j)$, belongs to at most m different circles of length at most $6k$ in G. Denote the set of all candidates of type (n,m,S,i) in G by $J^{n,S}_i(G)$. Now let $X^{n,m}_i$ be the random variable on $M^n_{\bar{p}}$ defined by:

$$X^{n,m}_i(M^n_{\bar{p}}) = \sum\{p_{|i-j|} : j \in J^{n,S}_i(M^n_{\bar{p}})\}.$$

Denote $R^{n,S}_i := \sum\{p_{|i-j|} : j \in J^{n,S}_i\}$. Trivially for all n,m,S,i as above, $X^{n,m}_i \leq R^{n,S}_i$. On the other hand, by $\circledast_1$ and the definition of a candidate, for all $\zeta > 0$ we can find $m = m(\zeta) \in \mathbb{N}$ such that for all n,S,i as above and $j \in J^{n,S}_i$, the probability that j is a candidate of type (n,m,S,i) in $M^n_{\bar{p}}$ is at least $1-\zeta$. Then for such m we have: $Exp(X^{n,m}_i) \geq R^{n,S}_i(1-\zeta)$. Hence we have $Pr_{M^n_{\bar{p}}}[X^{n,m}_i \leq R^{n,S}_i/2] \leq 2\zeta$. Recall that $\delta > 0$ was fixed, and let $m^* = m(\delta/4)$. Then for all n,S,i as above we have with probability at least $1-\delta/2$, $X^{n,m^*}_i(M^n_{\bar{p}}) \geq R^{n,S}_i/2$. Now denote $m^{**} := (2l^*+1)(m^*+2m_{2k})6k(m^*+1)$, and fix $n \in \mathbb{N}$ such that $\sum_{0<l<n}p_l > 2\cdot((m^{**}/(q_4\cdot\delta)\cdot 2m_{2k}(2l^*+1)+(s+2kl^*+2))$. Let $i \in [n]$ and $S \subseteq [n]$ be such that

$|S| \leq s$. We relatives our probability space $M^n_{\bar{p}}$ to the event $X^{n,m^*}_i(M^n_{\bar{p}}) \geq R^{n,S}_i/2$, and all probabilities until the end of Step 5 will be conditioned to this event. If we show that under this assumption we have, $Pr_{M^n_{\bar{p}}}[E^5_{n,S,i}] \geq 1 - \delta/2$ then we will have $\circledast_5$.

Let G be a graph on $[n]$ such that, $X^{n,m^*}_i(G) \geq R^{n,S}_i/2$. For $j \in J^{n,S}_i$ let $C_j(G)$ denote the set of all the pairs of vertexes which are relevant for the event $E^4_{n,i,j}$. Namely $C_j(G)$ will contain: $\{i,j\}$, all the edges $\{u,v\}$ such that : $u \in U(j)$, $v \neq i$ and $|u-v| < m_{2k}$, and all the edges that belong to a circle of length $\leq 6k$ containing some member of $U(j)$. We make some observations:

(1) $X^{n,m^*}_i(G) \geq (m^{**}/(q_4 \cdot \delta)) \cdot 2m_{2k}(2l^* + 1)$.
(2) There exists $J^1(G) \subseteq J^{n,S}_i$ such that:
  (a) The sets $U(j)$ for $j \in J^1(G)$ are pairwise disjoint. Moreover if $j_1, j_2 \in J^1(G)$, $u_l \in U(j_l)$ for $l \in \{1,2\}$ and $j_1 \neq j_2$ then $|u_1 - u_2| > m_{2k}$.
  (b) Each $j \in J^1(G)$ is a candidate of type (n, m^*, S, i) in G.
  (c) The sum $\sum\{p_{|i-j|} : j \in J^1(G)\}$ is at least $m^{**}/(q_4 \cdot \delta)$.

  [To see this use (1) and construct J^1 by adding the candidate with the largest $p_{|i-j|}$ that satisfies (a). Note that each new candidate excludes at most $m_{2k}(2l^* + 1)$ others.]
(3) Let j belong to $J^1(G)$. Then the set $\{j' \in J^1(G) : C_j(G) \cap C_{j'}(G) \neq \emptyset\}$ has size at most m^{**}. [To see this use (2)(b) above, the fact that two circles of length $\leq 6k$ that intersect in an edge give a circle of length $\leq 12k$ and similar trivial facts.]
(4) From (3) we conclude that there exists $J^2(G) \subseteq j^1(G)$ and $\langle j_1, ... j_r \rangle$ an enumeration of $J^2(G)$ such that:
  (a) For any $1 \leq r' \leq r$ the sets $C(j_{r'})$ and $\cup_{1 \leq r'' < r'} C(j_{r''})$ are disjoint.
  (b) The sum $\sum\{p_{|i-j|} : j \in J^2(G)\}$ is greater or equal $1/(q_4 \cdot \delta)$.

Now for each $j \in J^{n,S}_i$ let E^*_j be the event: "$i \sim j$ and $E^4_{n,i,j}$". By $\circledast_4$ we have for each $j \in J^{n,S}_i$, $Pr_{M^n_{\bar{p}}}[E^*_j] \geq q_4 \cdot p_{|i-j|}$. Recall that we condition the probability space $M^n_{\bar{p}}$ to the event $X^{n,m^*}_i(M^n_{\bar{p}}) \geq R^{n,S}_i/2$, and let $\langle j_1, ... j_r \rangle$ be the enumeration of $J^2(M^n_{\bar{p}})$ from (4) above. (Formally speaking r and each $j_{r'}$ is a function of $M^n_{\bar{p}}$). We then have for $1 \leq r' < r'' \leq r$, $Pr_{M^n_{\bar{p}}}[E^*_{j_{r'}} | E^*_{j_{r''}}] \geq Pr_{M^n_{\bar{p}}}[E^*_{j_{r'}}]$, and $Pr_{M^n_{\bar{p}}}[E^*_{j_{r'}} | \neg E^*_{j_{r''}}] \geq Pr_{M^n_{\bar{p}}}[E^*_{j_{r'}}]$. To see this use (2)(a) and (4)(a) above and the definition of $C_j(G)$.

Let the random variables X and X' be defined as follows. X is the number of $j \in J^2(M^n_{\bar{p}})$ such that E^*_j holds in $M^n_{\bar{p}}$. In other words X is the sum of r random variables $\langle Y_1, ..., Y_r \rangle$, where for each $1 \leq r' \leq r$, $Y_{r'}$ equals 1 if $E^*_{j_{r'}}$ holds, and 0 otherwise. X' is the sum of r independent random variables $\langle Y'_1, ..., Y'_r \rangle$, where for each $1 \leq r' \leq r$ $Y'_{r'}$ equals 1 with probability $q_4 \cdot p_{|i-j_{r'}|}$ and 0 with probability $1 - q_4 \cdot p_{|i-j_{r'}|}$. Then by the last paragraph for any $0 \leq t \leq r$,

$$Pr_{M^n_{\bar{p}}}[X \geq t] \geq Pr[X' \geq t].$$

But $Exp(X') = Exp(X) = q_4 \cdot \sum_{1 \leq r' \leq r} p_{|i-j_{r'}|}$ and by (4)(b) above this is grater or equal $1/\delta$. Hence by Chebyshev's inequality we have:

$$Pr_{M^n_{\bar{p}}}[\neg E^5_{n,S,i}] \leq Pr_{M^n_{\bar{p}}}[X = 0] \leq Pr[X' = 0] \leq \frac{Var(X')}{Exp(X')^2} \leq \frac{1}{Exp(X')} \leq \delta$$

as desired.

Step 6. We turn to the construction of the circle free saturated forest. Let $\epsilon > 0$, and we will prove that for $n \in \mathbb{N}$ large enough we have $Pr[E^{n,k}$ holds in $M^n_{\bar{p}}] \geq 1-\epsilon$. Let $\delta = \epsilon/(l^* 2^{k+2})$ and $s = 2l^*((k+2^k)(2l^*k+1))$. Let $n \in \mathbb{N}$ be large enough such that $\circledast_5$ holds for n, k, δ and s. We now choose (formally we show that with probability at least $1-\epsilon$ such a choice exists) by induction on $(i,\eta) \in I^*_k(M^n_{\bar{p}}) \times {}^{\leq k}2$ (ordered by the lexicographic order) $y^i_\eta \in [n]$ such that:

(1) $\langle y^i_\eta \in [n] : (i,\eta) \in I^*_k(M^n_{\bar{p}}) \times {}^{\leq k}2 \rangle$ is without repetitions.
(2) If $\eta = \langle\rangle$ then $M^n_{\bar{p}} \models i \sim y^i_\eta$, but $|i - y^i_\eta| \neq l^*$.
(3) If $\eta \neq \langle\rangle$ then $M^n_{\bar{p}} \models y^i_\eta \sim y^i_{\eta|_{|\eta|-1}}$.
(4) If $\eta = \langle\rangle$ then $M^n_{\bar{p}}$ satisfies E^4_{n,i,y^i_η} else, denoting $\rho := \eta|_{|\eta|-1}$, $M^n_{\bar{p}}$ satisfies $E^4_{n,y^i_\rho,y^i_\eta}$.

Before we describe the choice of y^i_η, we need to define sets $S^i_\eta \subseteq [n]$. For a graph G on $[n]$ and $i \in I^*_k(G)$ let $S^*_i(G)$ be the set of vertexes in the first (in some pre fixed order) path of length $\leq k$ from 1 to i in G. Now let $S^*(G) = \bigcup_{i \in I^*_k(G)} S^*_i(G)$. For $(i,\eta) \in I^*_k(M^n_{\bar{p}}) \times {}^{\leq k}2$ and $\langle y^{i'}_{\eta'} \in [n] : (i',\eta') <_{lex} (i,\eta)\rangle$ define:

$$S^i_\eta(G) = S^*(G) \cup \{[y^{i'}_{\eta'} - kl^*, y^{i'}_{\eta'} + kl^*] : (i'\eta') <_{lex} (i,\eta)\}.$$

Note that indeed $|S^*(G)| \leq s$ for all G. In the construction below when we write S^i_η we mean $S^i_\eta(M^n_{\bar{p}})$ where $\langle y^{i'}_{\eta'} \in [n] : (i',\eta') <_{lex} (i,\eta)\rangle$ were already chosen. Now the choice of y^i_η is as follows:

- If $\eta = \langle\rangle$ by $\circledast_5$ with probability at least $1-\delta$, $E^5_{n,S^i_\eta,i}$ holds in $M^n_{\bar{p}}$ hence we can choose y^i_η that satisfies (1)-(4).
- If $\eta = \langle 0\rangle$ (resp. $\eta = \langle 1\rangle$) choose $y^i_\eta = y^i_{\langle\rangle} - l^*$ (resp. $y^i_\eta = y^i_{\langle\rangle} + l^*$). By the induction hypothesis and the definition of $E^4_{n,i,j}$ this satisfies (1)-(4) above.
- If $|\eta| > 1$, $|y^i_{\eta|_{|\eta|-1}} - y^i_{\eta|_{|\eta|-2}}| \neq l^*$ and $\eta(|\eta|) = 0$ (resp. $\eta(|\eta|) = 1$) then choose $y^i_\eta = y^i_{\eta|_{|\eta|-1}} - l^*$ (resp. $y^i_\eta = y^i_{\eta|_{|\eta|-1}} + l^*$). Again by the induction hypothesis and the definition of $E^4_{n,i,j}$ this satisfies (1)-(4).
- If $|\eta| > 1$, $y^i_{\eta|_{|\eta|-1}} - y^i_{\eta|_{|\eta|-2}} = l^*$ (resp. $y^i_{\eta|_{|\eta|-1}} - y^i_{\eta|_{|\eta|-2}} = -l^*$) and $\eta(|\eta|) = 0$, then choose $y^i_\eta = y^i_{\eta|_{|\eta|-1}} - l^*$ (resp. $y^i_\eta = y^i_{\eta|_{|\eta|-1}} + l^*$).
- If $|\eta| > 1$, $|y^i_{\eta|_{|\eta|-1}} - y^i_{\eta|_{|\eta|-2}}| = l^*$ and $\eta(|\eta|) = 1$. Then by $\circledast_5$ with probability at least $1-\delta$, $E^5_{n,S^i_\eta,y^i_{\eta|_{|\eta|-1}}}$ holds in $M^n_{\bar{p}}$, and hence we can choose y^i_η that satisfies (1)-(4).

At each step of the construction above the probability of "failure" is at most δ, hence with probability at least $1-(l^*2^{k+2})\delta = 1-\epsilon$ we compleat the construction. It remains to show that indeed $\langle y^i_\eta : i \in I^n, \eta \in {}^{\leq k}2\rangle$ is a circle free saturated forest of depth k for I^*_k in $M^n_{\bar{p}}$. This is straight forward from the definitions. First each $\langle y^i_\eta : \eta \in {}^{\leq k}2\rangle$ is a saturated tree of depth k in $[n]$ by its construction. Second (ii) and (iii) in the definition of a saturated tree holds by (2) and (3) above (respectively). Third note that by (4) each edge (y,y') of our construction satisfies $E^2_{n,y,y'}$ and $E^4_{n,y,y'}$ hence (iv) and (v) (respectively) in the definition of a saturated tree follows. Lastly we need to show that (c) in the definition of a saturated forest holds. To see this note that if $i_1, i_2 \in i^*_k(M^n_{\bar{p}})$ then by the definition of $S^i_\eta(M^n_{\bar{p}})$ there exists a path of length $\leq 2k$ from i_1 to i_2 with all its vertexes in $S^i_\eta(M^n_{\bar{p}})$.

Now if $\bar{x}$ is a path of length $\leq k$ from $y^{i_1}_{\langle\rangle}$ to i_2 and $(y^{i_1}_{\langle\rangle}, i_1)$ is not an edge of $\bar{x}$, then necessarily $\{y^{i_1}_{\langle\rangle}, i_1\}$ is included in some circle of length $\leq 3k+2$. A contradiction to the choice of $y^{i_1}_{\langle\rangle}$. This completes the proof of the claim. $\square$

By $\odot$ and the claim above we conclude that, for some large enough $n \in \mathbb{N}$, there exists a graph $G = ([n], \sim)$ such that:

(1) $G \models \neg\phi[1]$.
(2) $Pr[M^n_{\bar{p}} = G] > 0$.
(3) There exists $\langle \bar{y}^i : i \in I^*_r(G)\rangle$, a circle free saturated forest of depth r for $I^*_r(G)$ in G.

Denote $B = B^G(1, r)$, $I = I^*_r(G)$, and we will prove that for some r-proper $(l, u_0, U, H) \in \mathfrak{H}$ we have $(B, 1) \cong (H, u_0)$ (i.e. there exists a graph isomorphism from $G|_B$ to H mapping 1 to u_0). As ϕ is r-local we will then have $H \models \neg\phi[u_0]$ which is a contradiction of our assumption and we will be done. We turn to the construction of (l, u_0, U, H). For $i \in I$ let $r(i) = r - dist^G(1, i)$. Denote

$$Y := \{y^i_\eta : i \in I, \eta \in {}^{<r(i)}2\}.$$

Note that by (ii)-(iii) in the definition of a saturated tree we have $Y \subseteq B$. We first define a one-to-one function $f : B \to \mathbb{Z}$ in three steps:

Step 1. For each $i \in I$ define

$$B_i := \{x \in B : \text{ there exists a path of length } \leq r(i) \text{ from } x \text{ to } i \text{ disjoint to } Y\}$$

and $B^0 := I \cup \bigcup_{i\in I} B_i$. Now define for all $x \in B^0$, $f(x) = x$. Note that:

- $\bullet_1$ $f|_{B^0}$ is one-to-one (trivially).
- $\bullet_2$ If $x \in B^0$ and $dist^G(1, x) < r$ then $x + l^* \in [n] \Rightarrow x + l^* \in B^0$ and $x - l^* \in [n] \Rightarrow x - l^* \in B^0$ (use the definition of a saturated tree).

Step 2. We define $f|_Y$. We start by defining $f(y)$ for $y \in \bar{y}^1$, so let $\eta \in {}^{\leq r}2$ and denote $y = y^1_\eta$. We define $f(y)$ using induction on η were ${}^{\leq r}2$ is ordered by the lexicographic order. First if $\eta = \langle\rangle$ then define $f(y) = 1 - l^*$. If $\eta \neq \langle\rangle$ let $\rho : \eta|_{|\eta|-1}$, and consider $u := f(y^1_\rho)$. Denote $F = F_\eta := \{f(y^1_{\eta'}) : \eta' <_{lex} \eta\}$. Now if $u - l^* \notin F$ define $f(y) = u - l^*$. If $u - l^* \in F$ but $u + l^* \notin F$ define $f(y) = u + l^*$. Finally, if $u - l^*, u + l^* \in F$, choose some $l = l_\eta$ such that $p_l > 0$ and $u - l < \min F - rl^* - n$, and define $f(y) = u - l$. Note that by our assumptions $\{l : p_l > 0\}$ is infinite so we can always choose l as desired. Note further that we chose $f(y)$ such that $f|_{\bar{y}^1}$ is one-to-one. Now for each $i \in I \cap [1, l^*]$ and $\eta \in {}^{<r(i)}2$, define $f(y^i_\eta) = f(y^1_\eta) + (f(i) - 1)$ (recall that $f(i) = i$ was defined in Step 1, and that $k(i) \leq k(1)$ so $f(y^i_\eta)$ is well defined). For $i \in I \cap (n - l^*, n]$ preform a similar construction in "reversed directions". Formally define $f(y^i_{\langle\rangle}) = i + l^*$, and the induction step is similar to the case $i = 1$ above only now choose l such that $u + l > \max F + rl^* + n$, and define $f(y) = u + l$. Note that:

- $\bullet_3$ $f|_Y$ is one-to-one.
- $\bullet_4$ $f(Y) \cap f(B^0) = \emptyset$. In fact:
- $\bullet_4^+$ $f(Y) \cap [n] = \emptyset$.
- $\bullet_5$ If $i \in I \cap [1, l^*]$ then $i - l^* \in f(Y)$ (namely $i - l^* = f(y^i_{\langle\rangle})$).
- $\bullet_5'$ If $i \in I \cap (n - l^*, n]$ then $i + l^* \in f(Y)$ (namely $i + l^* = f(y^i_{\langle\rangle})$).

- $\bullet_6$ If $y \in Y \setminus \{y^i_{\langle\rangle} : i \in I\}$ and $dist^G(1, y) < r$ then $f(y) + l^*, f(y) - l^* \in f(Y)$. (Why? As if $dist^G(1, y^i_\eta) < r$ then $|\eta| < r(i)$, and the construction of **Step 2**).

Step 3. For each $i \in I$ and $\eta \in {}^{<r(i)}2$, define

$$B^i_\eta := \{x \in B : \text{ there exists a path of length } \leq r(i) \text{ from } x \text{ to } y^i_\eta \text{ disjoint to } Y \setminus \{y^i_\eta\}\}$$

and $B^1 := \bigcup_{i \in I, \eta \in {}^{<r(i)}2} B^i_\eta$.

We now make a few observations:

(α) If $i_1, i_2 \in I$ then, in G there exists a path of length at most $2r$ from i_1 to i_2 disjoint to Y. Why? By the definition of I and (c) in the definition of a saturated forest.

(β) B^0 and B^1 are disjoint and cover B. Why? Trivially they cover B, and by (α) and (iv) in the definition of a saturated tree they are disjoint.

(γ) $\langle B^i_\eta : i \in I, \eta \in {}^{<r(i)}2\rangle$ is a partition of B^1. Why? Again trivially they cover B^1, and by (iv) in the definition of a saturated tree they are disjoint.

(δ) If $\{x, y\}$ is an edge of $G|_B$ then either $x, y \in B^0$, $\{x, y\} = \{i, y^i_{\langle\rangle}\}$ for some $i \in I$, $\{x, y\} \subseteq Y$ or $\{x, y\} \subseteq B^i_\eta$ for some $i \in I$ and $\eta \in {}^{<r(i)}2$. (Use the properties of a saturated forest.)

We now define $f|_{B^1}$. Let $\langle (B_j, y_j) : j < j^*\rangle$ be some enumeration of $\langle (B^i_\eta, y^i_\eta) : i \in I, \eta \in {}^{<r(i)}2\rangle$. We define $f|_{B_j}$ by induction on $j < j^*$ so assume that $f|_{(\cup_{j'<j} B_{j'})}$ is already defined, and denote: $F = F_j := f(B^0) \cup f(Y) \cup f(\cup_{j'<j} B_{j'})$. Our construction of $f|_{B_j}$ will satisfy:

- $f|_{B_j}$ is one-to-one.
- $f(B_j)$ is disjoint to F_j.
- If $y \in B_j$ then either $f(y) = y$ or $f(y) \notin [n]$.

Let $\langle z^j_s : s < s(j)\rangle$ be some enumeration of the set $\{z \in B_j : G \models y_j \sim z\}$. For each $s < s(j)$ choose $l(j, s)$ such that $p_{l(j,s)} > 0$ and:

$\otimes$ If $k \leq 4r$, $(m_1, ..., m_k)$ are integers with absolute value not larger than $4r$ and not all equal 0, and $(s_1, ...s_k)$ is a sequence of natural numbers smaller than $j(s)$ without repetitions. Then $|\sum_{1 \leq i \leq m}(m_i \cdot l(j, s_i))| > n + \max\{|x| : x \in F_j\}$.

Again as $\{l : p_l > 0\}$ is infinite we can always choose such $l(j, s)$. We now define $f|_{B_j}$. For each $y \in B_j$ let $\bar{x} = (x_0, ...x_k)$ be a path in G from y to y_j, disjoint to $Y \setminus \{y_j\}$, such that k is minimal. So we have $x_0 = y$, $x_k = y_j$, $k \leq r$ and $\bar{x}$ is without repetitions. Note that by the definition of B_j such a path exists. For each $0 \leq t < k$ define

$$l_t = l_t(\bar{x}) \begin{cases} l(j, s) & l^{\bar{x}}_t = |y_j - z^j_s| \text{ for some } s < s(j) \\ -l(j, s) & l^{\bar{x}}_t = -|y_j - z^j_s| \text{ for some } s < s(j) \\ l^{\bar{x}}_t & \text{otherwise.} \end{cases}$$

Now define $f(y) = f(y_j) + \sum_{0 \leq t < k} l_t$. We have to show that $f(y)$ is well defined. Assume that both $\bar{x}_1 = (x_0, ...x_{k_1})$ and $\bar{x}_2 = (x'_0, ...x'_{k_1})$ are paths as above. Then $k_1 = k_2$ and $\bar{x} = (x_0, ..., x_{k_1}, x'_{k_2-1}, ..., x'_0)$ is a circle of length $k_1 + k_2 \leq 2r$. By (v) in the definition of a saturated tree we know that for each $s < s(j)$, $|y_j - z^j_s| > m_{2r}$. Hence as $\bar{p}$ is without unavoidable circles we have for each $s < s(j)$ and $0 \leq t < k_1 + k_2$, if $|l^{\bar{x}}_t| = |y_j - z^j_s|$ then $t \in Sym(\bar{x})$. (see definition 5.13(6,7)).

Now put for $w \in \{1,2\}$ and $s < s(j)$, $m^+_w(s) := |\{0 \leq t < k_w : l^{\bar{x}_w}_t = y_j - z^j_s\}|$ and similarly $m^-_w(s) := |\{0 \leq t < k_w : -l^{\bar{x}_w}_t = y_j - z^j_s\}|$. By the definition of $\bar{x}$ we have, $m^+_1(s) - m^-_1(s) = m^+_2(s) - m^-_2(s)$. But from the definition of $l_t(\bar{x})$ we have for $w \in \{1,2\}$,

$$\sum_{0 \leq t < k_w} l_t(\bar{x}_w) = \sum_{0 \leq t < k_w} l^{\bar{x}_w}_t + \sum_{s < s(j)} (m^+_w(s) - m^-_w(s))(l(j,s) - (y_j - z^j_s)).$$

Now as $\sum_{0 \leq t < k_1} l^{\bar{x}_1}_t = \sum_{0 \leq t < k_2} l^{\bar{x}_2}_t$ we get $\sum_{0 \leq t < k_1} l_t(x_1) = \sum_{0 \leq t < k_2} l_t(x_2)$ as desired.

We now show that $f|_{B_j}$ is one-to-one. Let $y^1 \neq y^2$ be in B_j. So for $w \in \{1,2\}$ we have a path $\bar{x}_w = (x^w_0, ... x^w_{k_w})$ from y^w to y_j. as before, for $s < s(j)$ denote $m^+_w(s) := |\{0 \leq t < k_w : l^{\bar{x}_w}_t = y_j - z^j_s\}|$ and similarly $m^-_w(s)$. By the definition of f_{B_j} we have

$$f(y^1) - f(y^2) = y^1 - y^2 + \sum_{s < s(j)} [(m^+_1(s) - m^-_1(s)) - (m^+_2(s) - m^-_2(s))] \cdot l(j,s).$$

Now if for each $s < s(j)$, $m^+_1(s) - m^-_1(s) = m^+_2(s) - m^-_2(s)$ then we are done as $y^1 \neq y^2$. Otherwise note that for each $s < s(j)$, $|m^+_1(s) - m^-_1(s) = m^+_2(s) - m^-_2(s)| \leq 4r$. Note further that $|\{s < s(j) : m^+_1(s) - m^-_1(s) = m^+_2(s) - m^-_2(s) \neq 0\}| \leq 4r$. Hence by $\otimes$, and as $|y^1 - y^2| \leq n$ we are done.

Next let $y \in B_j$ and $\bar{x} = (x_0, ..., x_k)$ be a path in G from y to y_j. For each $s < s(j)$ define $m^+(s)$ and $m^-(s)$ as above, hence we have $f(y) = y_j + \sum_{s<s(j)}(m^+(s) - m^-(s))l(j,s)$. Consider two cases. First if $(m^+(s) - m^-(s)) = 0$ for each $s < s(j)$ then $f(y) = y$. Hence $f(y) \notin f(B^0) = B^0$ (by (β) above), $f(y) \notin f(Y)$ (as $f(Y) \cap [n] = \emptyset$) and $f(y) \notin f(\cup_{j'<j} B_{j'})$ (by (γ) and the induction hypothesis). So $f(y) \notin F_j$. Second assume that for some $s < s(j)$, $(m^+(s) - m^-(s)) \neq 0$. Then by the $\otimes$ we have $f(y) \notin [n]$ and furthermore $f(y) \notin F_j$. In both cases the demands for $f|_{B_j}$ are met and we are done. After finishing the construction for all $j < j^*$ we have $f|_{B^1}$ such that:

- $\bullet_7$ $f|_{B^1}$ is one-to-one.
- $\bullet_8$ $f(B^1)$ is disjoint to $f(B^0) \cup f(Y)$.
- $\bullet_9$ If $y \in B^1$ and $dist^G(1,y) < r$ then $f(y) + l^*, f(y) - l^* \in f(B^1)$. In fact $f(y + l^*) = f(y) + l^*$ and $f(y - l^*) = f(y) - l^*$. (By the construction of Step 3.)

Putting $\bullet_1 - \bullet_9$ together we have constructed $f : B \to \mathbb{Z}$ that is one-to-one and satisfies:

- $(\circ)$ If $y \in B$ and $dist^G(1,y) < r$ then $f(y) + l^*, f(y) - l^* \in f(B)$. Furthermore:
- $(\circ\circ)$ $\{y, f^{-1}(f(y) - l^*)\}$ and $\{y, f^{-1}(f(y) + l^*)\}$ are edges of G.

For $(\circ\circ)$ use: $\bullet_2$ with the definition of $f|_{B^0}$, $\bullet_5 + \bullet'_5$ with the fact that $G \models i \sim y^i_{\langle\rangle}$, $\bullet_6$ with the construction of Step 2 and $\bullet_9$.

We turn to the definition of (l, u_0, U, H) and the isomorphism $h : B \to H$. Let $l_{min} = \min\{f(b) : b \in B\}$ and $l_{max} = \max\{f(b) : b \in B\}$. Define:

- $l = l_{min} + l_{max} + 1$.
- $u_0 = l_{min} + 2$.
- $U = \{z + l_{min} + 1 : z \in Im(f)\}$.
- For $b \in B$, $h(b) = f(b) + l_{min} + 1$.
- For $u, v \in U$, $H \models u \sim v$ iff $G \models h^{-1}(u) \sim h^{-1}(v)$.

As f was one-to-one so is h, and trivially it is onto U and maps 1 to u_0. Also by the definition of H, h is a graph isomorphism. So it remains to show that (l, u_0, U, H) is r-proper. First $(*)_1$ in the definition of proper is immediate from the definition of H. Second for $(*)_2$ in the definition of proper let $u \in U$ be such that $dist^H(u_0, u) < r$. Denote $y := h^{-1}(u)$ then by the definition of H we have $dist^G(1, y) < r$, hence by $(\circ)$, $f(y) + l^*, f(y) - l^* \in f(B)$ and hence by the definition of h and U, $u + l^*, u - l^* \in U$ as desired. Lastly to see $(*)_3$ let $u, u' \in U$ and denote $y = h^{-1}(u)$ and $y' = h^{-1}(u')$. Assume $|u - u'| = l^*$ then by $(\circ\circ)$ we have $G \models y \sim y'$ and by the definition of H, $H \models u \sim u'$. Now assume that $H \models u \sim u'$ then $G \models y \sim y'$. Using observation (δ) above and rereading 1-3 we see that $|u - u'|$ is either l^*, $|y - y'|$, l_η for some $\eta \in {}^{<r}2$ (see Step 2) or $l(j, s)$ for some $j < j^*, s < s(j)$ (see step 3). In all cases we have $P_{|u-u'|} > 0$. Together we have $(*)_3$ as desired. This completes the proof of Theorem 5.14. $\square$

References

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