# GPGCD, an Iterative Method for Calculating Approximate GCD, for Multiple Univariate Polynomials 

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#### Abstract

We present an extension of our GPGCD method, an iterative method for calculating approximate greatest common divisor (GCD) of univariate polynomials, to multiple polynomial inputs. For a given pair of polynomials and a degree, our algorithm finds a pair of polynomials which has a GCD of the given degree and whose coefficients are perturbed from those in the original inputs, making the perturbations as small as possible, along with the GCD. In our GPGCD method, the problem of approximate GCD is transferred to a constrained minimization problem, then solved with the so-called modified Newton method, which is a generalization of the gradient-projection method, by searching the solution iteratively. In this paper, we extend our method to accept more than two polynomials with the real coefficients as an input.


## 1 Introduction

For algebraic computations on polynomials and matrices, approximate algebraic algorithms are attracting broad range of attentions recently. These algorithms take inputs with some "noise" such as polynomials with floating-point number coefficients with rounding errors, or more practical errors such as measurement errors, then, with minimal changes on the inputs, seek a meaningful answer that reflect desired property of the input, such as a common factor of a given degree. By this characteristic, approximate algebraic algorithms are expected to be applicable to more wide range of problems, especially those to which exact algebraic algorithms were not applicable.

As an approximate algebraic algorithm, we consider calculating the approximate greatest common divisor (GCD) of univariate polynomials, such that, for a given pair of polynomials and a degree $d$, finding a pair of polynomials which has a GCD of degree $d$ and whose coefficients are perturbations from those in the original inputs, with making the perturbations as small as possible, along with the GCD. This problem has been extensively studied with various approaches including the Euclidean method on the polynomial remainder sequence (PRS) ([1], 2], 3]), the singular value decomposition (SVD) of the Sylvester matrix ([4],
[5]), the QR factorization of the Sylvester matrix or its displacements ( 6], [7], [8), Padé approximation [9, optimization strategies ([10], [11], 12], [13], [14]). Furthermore, stable methods for ill-conditioned problems have been discussed ([6], [15], 16]).

Among methods in the above, we focus our attention on optimization strategies. Already proposed algorithms utilize iterative methods including the LevenbergMarquardt method [10, the Gauss-Newton method 14 and the structured total least norm (STLN) method ([11, [12]). Among them, STLN-based methods have shown good performance calculating approximate GCD with sufficiently small perturbations efficiently.

In this paper, we discuss an extension of the GPGCD method, proposed by the present author ([17, [21]), an iterative method with transferring the original approximate GCD problem into a constrained optimization problem, then solving it by the so-called modified Newton method [18, which is a generalization of the gradient-projection method [19]. In the previous papers ([17], [21]), we have shown that our method calculates approximate GCD with perturbations as small as those calculated by the STLN-based methods and with significantly better efficiency than theirs. While our previous methods accept two polynomials with the real or the complex coefficients as inputs and outputs, respectively, we extend it to handle more than two polynomial inputs with the real coefficients in this paper.

The rest part of the paper is organized as follows. In Section 2 we transform the approximate GCD problem into a constrained minimization problem for the case with the complex coefficients. In Section 3, we show details for calculating the approximate GCD, with discussing issues in minimizations. In Section 4, we demonstrate performance of our algorithm with experiments.

## 2 Formulation of the Approximate GCD Problem

Let $P_{1}(x), \ldots, P_{n}(x)$ be real univariate polynomials of degree $d_{1}, \ldots, d_{n}$, respectively, given as

$$
P_{i}(x)=p_{d_{i}}^{(i)} x^{d_{i}}+\cdots p_{1}^{(i)} x+p_{0}^{(i)}
$$

for $i=1, \ldots, n$, with $\min \left\{d_{1}, \ldots, d_{n}\right\}>0$. We permit $P_{i}$ and $P_{j}$ be relatively prime for any $i \neq j$ in general. For a given integer $d$ satisfying $\min \left\{d_{1}, \ldots, d_{n}\right\}>$ $d>0$, let us calculate a deformation of $P_{1}(x), \ldots, P_{n}(x)$ in the form of

$$
\tilde{P}_{i}(x)=P_{i}(x)+\Delta P_{i}(x)=H(x) \cdot \bar{P}_{i}(x)
$$

where $\Delta P_{i}(x)$ is a real polynomial whose degrees do not exceed $d_{i}$, respectively, $H(x)$ is a polynomial of degree $d$, and $\bar{P}_{i}(x)$ and $\bar{P}_{j}(x)$ are pairwise relatively prime for any $i \neq j$. In this situation, $H(x)$ is an approximate GCD of $P_{1}(x), \ldots, P_{n}(x)$. For a given $d$, we try to minimize $\left\|\Delta P_{1}(x)\right\|_{2}^{2}+\cdots+\left\|\Delta P_{n}(x)\right\|_{2}^{2}$, the norm of the deformations.

For a real univariate polynomial $P(x)$ represented as $P(x)=p_{n} x^{n}+\cdots+p_{0} x^{0}$, let $C_{k}(P)$ be a real $(n+k, k+1)$ matrix defined as

$$
C_{k}(P)=\left(\begin{array}{ccc}
p_{n} & & \\
\vdots & \ddots & \\
p_{0} & & p_{n} \\
& \ddots & \vdots \\
& & p_{0}
\end{array}\right),
$$

and let $\boldsymbol{p}$ be the coefficient vector of $P(x)$ defined as

$$
\begin{equation*}
\boldsymbol{p}=\left(p_{n}, \ldots, p_{0}\right) \tag{1}
\end{equation*}
$$

In this paper, for a generalized Sylvester matrix, we use a formulation by Rupprecht [20, Sect. 3]. Then, a generalized Sylvester matrix for $P_{1}, \ldots, P_{n}$ becomes as

$$
N\left(P_{1}, \ldots, P_{n}\right)=\left(\begin{array}{ccccc}
C_{d_{1}-1}\left(P_{2}\right) & C_{d_{2}-1}\left(P_{1}\right) & \mathbf{0} & \ldots & \mathbf{0}  \tag{2}\\
C_{d_{1}-1}\left(P_{3}\right) & \mathbf{0} & C_{d_{3}-1}\left(P_{1}\right) & \cdots & \mathbf{0} \\
\vdots & \vdots & & \ddots & \vdots \\
C_{d_{1}-1}\left(P_{n}\right) & \mathbf{0} & \ldots & \mathbf{0} & C_{d_{n}-1}\left(P_{1}\right)
\end{array}\right)
$$

and the $k$-th subresultant matrix (with $\min \left\{d_{1}, \ldots, d_{n}\right\}>k \geq 0$ ) is also defined similarly as

$$
\begin{align*}
& N_{k}\left(P_{1}, \ldots, P_{n}\right) \\
& \quad=\left(\begin{array}{ccccc}
C_{d_{1}-1-k}\left(P_{2}\right) & C_{d_{2}-1-k}\left(P_{1}\right) & \mathbf{0} & \ldots & \mathbf{0} \\
C_{d_{1}-1-k}\left(P_{3}\right) & \mathbf{0} & C_{d_{3}-1-k}\left(P_{1}\right) & \cdots & \mathbf{0} \\
\vdots & \vdots & & \ddots & \vdots \\
C_{d_{1}-1-k}\left(P_{n}\right) & \mathbf{0} & \ldots & \mathbf{0} & C_{d_{n}-1-k}\left(P_{1}\right)
\end{array}\right) \tag{3}
\end{align*}
$$

with

$$
\begin{equation*}
r_{k}=d_{1}+d_{2}+\cdots+d_{n}-(n-1) k+(n-2) d_{1} \tag{4}
\end{equation*}
$$

rows and

$$
\begin{equation*}
c_{k}=d_{1}+d_{2}+\cdots+d_{n}-n \cdot k \tag{5}
\end{equation*}
$$

columns.
Calculation of GCD is based on the following fact.
Proposition 1 (Rupprecht [20, Proposition 3.1]). $N_{k}\left(P_{1}, \ldots, P_{n}\right)$ has full rank if and only if $\operatorname{deg}\left(\operatorname{gcd}\left(P_{1}, \ldots, P_{n}\right)\right) \leq k$.

Thus, for a given degree $d$, if $N_{d-1}\left(\tilde{P}_{1}, \ldots, \tilde{P}_{n}\right)$ is rank-deficient, then there exist real univariate polynomials $U_{1}(x), \ldots, U_{n}(x)$ of degree at most $d_{1}-d, \ldots, d_{n}-$ $d$, respectively, satisfying

$$
\begin{equation*}
U_{1} \tilde{P}_{i}+U_{i} \tilde{P}_{1}=0 \tag{6}
\end{equation*}
$$

for $i=2, \ldots, n$. In such a case, if $U_{i}$ and $U_{j}$ are pairwise relatively prime for any $i \neq j$, then $H=\frac{\tilde{P}_{1}}{U_{1}}=-\frac{\tilde{P}_{2}}{U_{2}}=\cdots=-\frac{\tilde{P}_{n}}{U_{n}}$ becomes the expected GCD. Therefore, for given polynomials $P_{1}, \ldots, P_{n}$ and a degree $d$, our problem is to find perturbations $\Delta P_{1}, \ldots, \Delta P_{n}$ along with cofactors $U_{1}, \ldots, U_{n}$ satisfying (6) with making $\left\|\Delta P_{1}(x)\right\|_{2}^{2}+\cdots+\left\|\Delta P_{n}(x)\right\|_{2}^{2}$ as small as possible.

By representing $\tilde{P}_{i}(x)$ and $U_{i}(x)$ as

$$
\begin{align*}
\tilde{P}_{i}(x) & =\tilde{p}_{d_{i}}^{(i)} x^{d_{i}}+\cdots+\tilde{p}_{1}^{(i)} x+\tilde{p}_{0}^{(i)} \\
U_{i}(x) & =u_{d_{i}-d}^{(i)} x^{d_{i}-d}+\cdots+u_{1}^{(i)} x+u_{0}^{(i)} \tag{7}
\end{align*}
$$

we express the objective function and the constraint as follows. For the objective function, $\left\|\Delta P_{1}(x)\right\|_{2}^{2}+\cdots+\left\|\Delta P_{n}(x)\right\|_{2}^{2}$ becomes as

$$
\begin{equation*}
\left\|\Delta P_{1}(x)\right\|_{2}^{2}+\cdots+\left\|\Delta P_{n}(x)\right\|_{2}^{2}=\sum_{i=1}^{n}\left\{\sum_{j=0}^{d_{i}}\left(\tilde{p}_{j}^{(i)}-p_{j}^{(i)}\right)^{2}\right\} \tag{8}
\end{equation*}
$$

For the constraint, (6) becomes as

$$
\begin{equation*}
N_{d-1}\left(\tilde{P}_{1}, \ldots, \tilde{P}_{n}\right) \cdot{ }^{t}\left(\boldsymbol{u}_{1}, \ldots, \boldsymbol{u}_{n}\right)=\mathbf{0} \tag{9}
\end{equation*}
$$

where $\boldsymbol{u}_{i}$ is the coefficient vector of $U_{i}(x)$ defined as in (11). Furthermore, we add another constraint for the coefficient of $U_{i}(x)$ as

$$
\begin{equation*}
\left\|U_{1}\right\|_{2}^{2}+\cdots+\left\|U_{n}\right\|_{2}^{2}=1 \tag{10}
\end{equation*}
$$

which can be represented together with (9) as

$$
\left(\begin{array}{cccc}
\boldsymbol{u}_{1} & \cdots & \boldsymbol{u}_{n} & -1  \tag{11}\\
N_{d-1}\left(\tilde{P}_{1}, \ldots, \tilde{P}_{n}\right) & \mathbf{0}
\end{array}\right) \cdot{ }^{t}\left(\boldsymbol{u}_{1}, \ldots, \boldsymbol{u}_{n}, 1\right)=\mathbf{0}
$$

where (10) has been put on the top of (9). Note that, in (11), we have total of

$$
\begin{equation*}
\bar{d}=d_{1}+\cdots+d_{n}-(n-1)(d-1)+(n-2) d_{1}+1 \tag{12}
\end{equation*}
$$

equations in the coefficients of polynomials in (7) as a constraint, with the $j$-th row of which is expressed as $g_{j}=0$.

Now, we substitute the variables

$$
\begin{equation*}
\left(\tilde{p}_{d_{1}}^{(1)}, \ldots, \tilde{p}_{0}^{(1)}, \ldots, \tilde{p}_{d_{n}}^{(n)}, \ldots, \tilde{p}_{0}^{(n)}, u_{d_{1}-d}^{(1)}, \ldots, u_{0}^{(1)}, \ldots, u_{d_{n}-d}^{(n)}, \ldots, u_{0}^{(n)}\right) \tag{13}
\end{equation*}
$$

as $\boldsymbol{x}=\left(x_{1}, \ldots, x_{2\left(d_{1}+\cdots+d_{n}\right)+(2-d) n}\right)$, then (8) and (11) become as

$$
\begin{align*}
& f(\boldsymbol{x})=\left(x_{1}-p_{d_{1}}^{(1)}\right)^{2}+\cdots+\left(x_{d_{1}}-p_{0}^{(1)}\right)^{2}+\cdots \\
& \cdots+\left(x_{d_{1}+\cdots+d_{n-1}+n}-p_{d_{n}}^{(n)}\right)^{2}+\cdots+\left(x_{d_{1}+\cdots+d_{n-1}+d_{n}+n}-p_{0}^{(n)}\right)^{2}  \tag{14}\\
& \quad \boldsymbol{g}(\boldsymbol{x})={ }^{t}\left(g_{1}(\boldsymbol{x}), \ldots, g_{\bar{d}}(\boldsymbol{x})\right)=\mathbf{0} \tag{15}
\end{align*}
$$

respectively, where $\bar{d}$ in (15) is defined as in (12). Therefore, the problem of finding an approximate GCD can be formulated as a constrained minimization problem of finding a minimizer of the objective function $f(\boldsymbol{x})$ in (14), subject to $\boldsymbol{g}(\boldsymbol{x})=\mathbf{0}$ in (15).

## 3 The Algorithm for Approximate GCD

We calculate an approximate GCD by solving the constrained minimization problem (14), (15) with the gradient projection method by Rosen [19] (whose initials become the name of our GPGCD method) or the modified Newton method by Tanabe [18 (for review, see the author's previous paper [17]). Our preceding experiments ([17, Sect. 5.1], [21, Sect. 4]) have shown that the modified Newton method was more efficient than the original gradient projection method while the both methods have shown almost the same convergence property, thus we adopt the modified Newton method in this paper.

In applying the modified Newton method to the approximate GCD problem, we discuss issues in the construction of the algorithm in detail, such as

- Representation of the Jacobian matrix $J_{\boldsymbol{g}}(\boldsymbol{x})$ and certifying that $J_{\boldsymbol{g}}(\boldsymbol{x})$ has full rank (Sect. 3.1),
- Setting the initial values (Sect. 3.2),
- Regarding the minimization problem as the minimum distance problem (Sect. 3.3),
- Calculating the actual GCD and correcting the coefficients of $\tilde{P}_{i}$ (Sect. (3.4), as follows.


### 3.1 Representation and the rank of the Jacobian Matrix

By the definition of the constraint (15), we have the Jacobian matrix $J_{\boldsymbol{g}}(\boldsymbol{x})$ (with the original notation of variables (13) for $\boldsymbol{x}$ ) as

$$
J_{\boldsymbol{g}}(\boldsymbol{x})=\left(\begin{array}{cccccccc}
\mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & & & \\
C_{d_{1}}\left(U_{2}\right) & C_{d_{2}}\left(U_{1}\right) & \mathbf{0} & \cdots & \mathbf{0} & & & \\
C_{d_{1}}\left(U_{3}\right) & \mathbf{0} & C_{d_{3}}\left(U_{1}\right) & & \mathbf{0} & & & \\
\vdots & \vdots & & \ddots & \vdots & & & \\
C_{d_{1}}\left(U_{n}\right) & \mathbf{0} & \cdots & \mathbf{0} & C_{d_{n}}\left(U_{1}\right) & & \\
& 2 \cdot{ }^{t} \boldsymbol{u}_{1} & 2 \cdot{ }^{t} \boldsymbol{u}_{2} & 2 \cdot{ }^{t} \boldsymbol{u}_{3} & \cdots & 2 \cdot{ }^{t} \boldsymbol{u}_{n} \\
& C_{d_{1}-d}\left(P_{2}\right) & C_{d_{2}-d}\left(P_{1}\right) & \mathbf{0} & \cdots & \mathbf{0} \\
& C_{d_{1}-d}\left(P_{3}\right) & \mathbf{0} & C_{d_{3}-d}\left(P_{1}\right) & & \mathbf{0} \\
& \vdots & \vdots & & \ddots & \vdots \\
& C_{d_{1}-d}\left(P_{n}\right) & \mathbf{0} & \cdots & \mathbf{0} & C_{d_{n}-d}\left(P_{1}\right)
\end{array}\right),
$$

which can easily be constructed in every iteration. Note that the number of rows in $J_{\boldsymbol{g}}(\boldsymbol{x})$ is equal to $\bar{d}$ in (12), which is equal to the number of constraints, while the number of columns is equal to $2\left(d_{1}+\cdots+d_{n}\right)+(2-d) n$, which is equal to the number of variables (see (131)).

In executing iterations, we need to keep that $J_{\boldsymbol{g}}(\boldsymbol{x})$ has full rank: otherwise, we are unable to decide proper search direction. For this requirement, we have the following observations.

Proposition 2. Assume that we have $\operatorname{deg} d<\min \left\{d_{1}, \ldots, d_{n}\right\}-1$ and $\operatorname{deg} U_{i} \geq$ 1 for $i=1, \ldots, n$. Let $\boldsymbol{x}^{*} \in V_{\boldsymbol{g}}$ be any feasible point satisfying (15). Then, if the corresponding polynomials do not have a GCD whose degree exceeds d, then $J_{\boldsymbol{g}}\left(\boldsymbol{x}^{*}\right)$ has full rank.

Proof. Let

$$
\boldsymbol{x}^{*}=\left(\tilde{p}_{d_{1}}^{(1)}, \ldots, \tilde{p}_{0}^{(1)}, \ldots, \tilde{p}_{d_{n}}^{(n)}, \ldots, \tilde{p}_{0}^{(n)}, u_{d_{1}-d}^{(1)}, \ldots, u_{0}^{(1)}, \ldots, u_{d_{n}-d}^{(n)}, \ldots, u_{0}^{(n)}\right)
$$

as in (13), with its polynomial representation expressed as in (7) (note that this assumption permits the polynomials $\tilde{P}_{i}(x)$ to be relatively prime in general). To verify our claim, we show that we have $\operatorname{rank}\left(J_{\boldsymbol{g}}\left(\boldsymbol{x}^{*}\right)\right)=\bar{d}=d_{1}+\cdots+d_{n}-(n-$ $1)(d-1)+(n-2) d_{1}+1$ (see (12) ). Let us divide $J_{\boldsymbol{g}}\left(\boldsymbol{x}^{*}\right)$ into two column blocks such that $J_{\boldsymbol{g}}\left(\boldsymbol{x}^{*}\right)=\left(J_{\mathrm{L}} \mid J_{\mathrm{R}}\right)$, where $J_{\mathrm{L}}$ and $J_{\mathrm{R}}$ are expressed as

$$
\begin{aligned}
J_{\mathrm{L}} & =\left(\begin{array}{ccccc}
\mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\
C_{d_{1}}\left(U_{2}\right) & C_{d_{2}}\left(U_{1}\right) & \mathbf{0} & \cdots & \mathbf{0} \\
C_{d_{1}}\left(U_{3}\right) & \mathbf{0} & C_{d_{3}}\left(U_{1}\right) & & \mathbf{0} \\
\vdots & \vdots & & \ddots & \vdots \\
C_{d_{1}}\left(U_{n}\right) & \mathbf{0} & \ldots & \mathbf{0} & C_{d_{n}}\left(U_{1}\right)
\end{array}\right), \\
J_{\mathrm{R}} & =\left(\begin{array}{ccccc}
2 \cdot{ }^{t} \boldsymbol{u}_{1} & 2 \cdot{ }^{t} \boldsymbol{u}_{2} & 2 \cdot{ }^{t} \boldsymbol{u}_{3} & \cdots & 2 \cdot{ }^{t} \boldsymbol{u}_{n} \\
C_{d_{1}-d}\left(P_{2}\right) & C_{d_{2}-d}\left(P_{1}\right) & \mathbf{0} & \cdots & \mathbf{0} \\
C_{d_{1}-d}\left(P_{3}\right) & \mathbf{0} & C_{d_{3}-d}\left(P_{1}\right) & \mathbf{0} \\
\vdots & \vdots & & \ddots & \vdots \\
C_{d_{1}-d}\left(P_{n}\right) & \mathbf{0} & \cdots & \mathbf{0} & C_{d_{n}-d}\left(P_{1}\right)
\end{array}\right)
\end{aligned}
$$

respectively. Then, we have the following lemma.
Lemma 1. We have $\operatorname{rank}\left(J_{\mathrm{L}}\right)=\bar{d}=d_{1}+\cdots+d_{n}-(n-1)(d-1)+(n-2) d_{1}$.
Proof. Let $\bar{J}$ be a submatrix of $J_{\mathrm{L}}$ by eliminating the top row. Since the number of rows in $J_{\mathrm{L}}$ is equal to $\bar{d}=d_{1}+\cdots+d_{n}-(n-1)(d-1)+(n-2) d_{1}$, we show that $\bar{J}$ has full rank.

For $i=2, \ldots, n$, let us divide column blocks $C_{d_{1}}\left(U_{i}\right)$ and $C_{d_{i}}\left(U_{1}\right)$ as

$$
\begin{align*}
C_{d_{1}}\left(U_{i}\right) & =(\overbrace{C_{d_{1}\left(U_{i}\right)_{\mathrm{L}}}^{d+1}}^{\overbrace{C_{d_{1}}\left(U_{i}\right)_{\mathrm{R}}}^{d_{1}-d})}, \\
C_{d_{1}}\left(U_{i}\right)_{\mathrm{L}} & \left.=\binom{C_{d}\left(U_{i}\right)}{\mathbf{0}}\right\}_{d_{1}-d}, \quad C_{d_{1}}\left(U_{i}\right)_{\mathrm{R}}=\binom{\mathbf{0}}{C_{d_{1}-d-1}\left(U_{i}\right)} \begin{array}{c}
\}^{d+1} \\
d_{1}+d_{i}-2 d
\end{array},  \tag{16}\\
C_{d_{i}}\left(U_{1}\right) & =(\overbrace{C_{d_{i}}\left(U_{1}\right)_{\mathrm{L}}}^{d+1} \overbrace{C_{d_{i}}\left(U_{1}\right)_{\mathrm{R}}}^{d_{1}-d}), \\
C_{d_{i}}\left(U_{1}\right)_{\mathrm{L}} & \left.=\binom{C_{d}\left(U_{1}\right)}{\mathbf{0}}\right\}_{d_{1}-d}, \quad C_{d_{i}}\left(U_{1}\right)_{\mathrm{R}}=\binom{\mathbf{0}}{C_{d_{i}-d-1}\left(U_{1}\right)} \begin{array}{c}
\} d+1 \\
\} d_{1}+d_{i}-2 d
\end{array}, \tag{17}
\end{align*}
$$

respectively, thus $\bar{J}$ is expressed as

Then, by exchanges of columns, we can transform $\bar{J}$ to $\hat{J}=\left(\hat{J}_{\mathrm{L}} \hat{J}_{\mathrm{R}}\right)$, where

$$
\begin{array}{r}
\hat{J}_{\mathrm{L}}=\left(\begin{array}{ccccc}
C_{d_{1}}\left(U_{2}\right)_{\mathrm{L}} C_{d_{2}}\left(U_{1}\right)_{\mathrm{L}} & \mathbf{0} & \cdots & \mathbf{0} \\
C_{d_{1}}\left(U_{3}\right)_{\mathrm{L}} & \mathbf{0} & C_{d_{3}}\left(U_{1}\right)_{\mathrm{L}} & & \mathbf{0} \\
\vdots & \vdots & & \ddots & \vdots \\
C_{d_{1}}\left(U_{n}\right)_{\mathrm{L}} & \mathbf{0} & \cdots & \mathbf{0} & C_{d_{n}}\left(U_{1}\right)_{\mathrm{L}}
\end{array}\right), \\
\hat{J}_{\mathrm{R}}=\left(\begin{array}{ccccc}
C_{d_{1}}\left(U_{2}\right)_{\mathrm{R}} & C_{d_{2}}\left(U_{1}\right)_{\mathrm{R}} & \mathbf{0} & \cdots & \mathbf{0} \\
C_{d_{1}}\left(U_{3}\right)_{\mathrm{R}} & \mathbf{0} & C_{d_{3}}\left(U_{1}\right)_{\mathrm{R}} & & \mathbf{0} \\
\vdots & \vdots & & \ddots & \vdots \\
C_{d_{1}}\left(U_{n}\right)_{\mathrm{R}} & \mathbf{0} & \cdots & \mathbf{0} & C_{d_{n}}\left(U_{1}\right)_{\mathrm{R}}
\end{array}\right) .
\end{array}
$$

We see that nonempty rows in $\hat{J}_{\mathrm{R}}$ consist of $N\left(U_{1}, \ldots, U_{n}\right)$, a generalized Sylvester matrix for $U_{1}, \ldots, U_{n}$ (see (2)). By the assumption, $U_{1}, \ldots, U_{n}$ are pairwise relatively prime, thus, by Prop. $1, \operatorname{rank}\left(\hat{J}_{\mathrm{R}}\right)$ is equal to the number of nonempty rows in $\hat{J}_{\mathrm{R}}$, which is equal to $d_{2}+\cdots+d_{n}+(n-1)\left(d_{1}-2 d\right)$ (see (16) and (17)).

On the other hand, in $\hat{J}_{L}$, column blocks $C_{d_{2}}\left(U_{1}\right)_{\mathrm{L}}, C_{d_{3}}\left(U_{1}\right)_{\mathrm{L}}, \ldots, C_{d_{n}}\left(U_{1}\right)_{\mathrm{L}}$ are lower triangular matrices with $d+1$ diagonal elements, which shows that $\operatorname{rank}\left(\hat{J}_{L}\right)$ is equal to the sum of the number of columns in $C_{d_{2}}\left(U_{1}\right)_{\mathrm{L}}, C_{d_{3}}\left(U_{1}\right)_{\mathrm{L}}, \ldots$, $C_{d_{n}}\left(U_{1}\right)_{\mathrm{L}}$, which is equal to $(n-1)(d+1)$.

Furthermore, we see that the row position of diagonal elements in $C_{d_{2}}\left(U_{1}\right)_{\mathrm{L}}$, $C_{d_{3}}\left(U_{1}\right)_{\mathrm{L}}, \ldots, C_{d_{n}}\left(U_{1}\right)_{\mathrm{L}}$ correspond to the position of the empty rows in $\hat{J}_{\mathrm{R}}$, thus the columns in $C_{d_{2}}\left(U_{1}\right)_{\mathrm{L}}, C_{d_{3}}\left(U_{1}\right)_{\mathrm{L}}, \ldots, C_{d_{n}}\left(U_{1}\right)_{\mathrm{L}}$ are linearly independent along with the columns in $\hat{J}_{R}$. Therefore, we have

$$
\operatorname{rank}(\bar{J})=\operatorname{rank}\left(\hat{J}_{L}\right)+\operatorname{rank}\left(\hat{J}_{R}\right)=d_{1}+\cdots+d_{n}-(n-1)(d-1)+(n-2) d_{1},
$$

which proves the lemma.
Proof of Proposition 2 (continued). By the assumptions, we have at least one nonzero coordinate in the top row in $J_{\mathrm{R}}$, while we have no nonzero coordinate in the top row in $J_{\mathrm{L}}$, thus we have $\operatorname{rank}\left(J_{\boldsymbol{g}}(\boldsymbol{x})\right)=d_{1}+\cdots+d_{n}-$ $(n-1)(d-1)+(n-2) d_{1}+1$, which proves the proposition.

Proposition 2 says that, under certain conditions, so long as the search direction in the minimization problem satisfies that corresponding polynomials have a GCD of degree not exceeding $d$, then $J_{\boldsymbol{g}}(\boldsymbol{x})$ has full rank, thus we can safely calculate the next search direction for approximate GCD.

### 3.2 Setting the Initial Values

At the beginning of iterations, we give the initial value $\boldsymbol{x}_{0}$ by using the singular value decomposition (SVD) [22] of $N_{d-1}\left(P_{1}, \ldots, P_{n}\right)$ (see (3)) as $N_{d-1}=$ $U \Sigma^{t} V, U=\left(\boldsymbol{w}_{1}, \ldots, \boldsymbol{w}_{c_{d-1}}\right), \Sigma=\operatorname{diag}\left(\sigma_{1}, \ldots, \sigma_{c_{d-1}}\right), V=\left(\boldsymbol{v}_{1}, \ldots, \boldsymbol{v}_{c_{d-1}}\right)$, where $\boldsymbol{w}_{j} \in \mathbf{R}^{r_{d-1}}, \boldsymbol{v}_{j} \in \mathbf{R}^{c_{d-1}}$ with $r_{k}$ and $c_{k}$ as in (4) and (5), respectively, and $\Sigma=\operatorname{diag}\left(\sigma_{1}, \ldots, \sigma_{c_{d-1}}\right)$ denotes the diagonal matrix with the $j$-th diagonal element of which is $\sigma_{j}$. Note that $U$ and $V$ are orthogonal matrices. Then, by a property of the SVD [22, Theorem 3.3], the smallest singular value $\sigma_{c_{d-1}}$ gives the minimum distance of the image of the unit sphere $\mathrm{S}^{c_{d-1}-1}$, given as $\mathrm{S}^{c_{d-1}-1}=\left\{\boldsymbol{x} \in \mathbf{R}^{c_{d-1}} \mid\|\boldsymbol{x}\|_{2}=1\right\}$, by $N_{d-1}\left(P_{1}, \ldots, P_{n}\right)$, represented as $N_{d-1} \cdot \mathrm{~S}^{c_{d-1}-1}=\left\{N_{d-1} \boldsymbol{x} \mid \boldsymbol{x} \in \mathbf{R}^{c_{d-1}},\|\boldsymbol{x}\|_{2}=1\right\}$, from the origin, along with $\sigma_{c_{d-1}} \boldsymbol{w}_{c_{d-1}}$ as its coordinates. Thus, we have $N_{d-1} \cdot \boldsymbol{v}_{c_{d-1}}=\sigma_{c_{d-1}} \boldsymbol{w}_{c_{d-1}}$. For $\boldsymbol{v}_{c_{d-1}}={ }^{t}\left(\bar{u}_{d_{1}-d}^{(1)}, \ldots, \bar{u}_{0}^{(1)}, \ldots, \bar{u}_{d_{n}-d}^{(n)}, \ldots, \bar{u}_{0}^{(n)}\right)$, let $\bar{U}_{i}(x)=\bar{u}_{d_{i}-d}^{(i)} x^{x_{i}-d}+\cdots+$ $\bar{u}_{0}^{(i)} x^{0}$ for $i=1, \ldots, n$. Then, $\bar{U}_{1}(x), \ldots, \bar{U}_{n}(x)$ give the least norm of $U_{1} P_{i}+U_{i} P_{1}$ satisfying $\left\|U_{1}\right\|_{2}^{2}+\cdots+\left\|U_{n}\right\|_{2}^{2}=1$ by putting $U_{i}(x)=\bar{U}_{i}(x)$ in (7).

Therefore, we admit the coefficients of $P_{1}, \ldots, P_{n}, \bar{U}_{1}, \ldots, \bar{U}_{n}$ as the initial values of the iterations as

$$
\boldsymbol{x}_{0}=\left(p_{d_{1}}^{(1)}, \ldots, p_{0}^{(1)}, \ldots, p_{d_{n}}^{(n)}, \ldots, p_{0}^{(n)}, \bar{u}_{d_{1}-d}^{(1)}, \ldots, \bar{u}_{0}^{(1)}, \ldots, \bar{u}_{d_{n}-d}^{(n)}, \ldots, \bar{u}_{0}^{(n)}\right)
$$

### 3.3 Regarding the Minimization Problem as the Minimum Distance (Least Squares) Problem

Since we have the object function $f$ as in (14), we have

$$
\begin{aligned}
\nabla f(\boldsymbol{x})=2 \cdot{ }^{t}( & x_{1}-p_{d_{1}}^{(1)}, \ldots, x_{d_{1}}-p_{0}^{(1)}, \ldots \\
& \left.x_{d_{1}+\cdots+d_{n-1}+n}-p_{d_{n}}^{(n)}, \ldots, x_{d_{1}+\cdots+d_{n-1}+d_{n}+n}-p_{0}^{(n)}, 0, \ldots, 0\right)
\end{aligned}
$$

However, we can regard our problem as finding a point $\boldsymbol{x} \in V_{\boldsymbol{g}}$ which has the minimum distance to the initial point $\boldsymbol{x}_{0}$ with respect to the $\left(x_{1}, \ldots, x_{d_{1}+\cdots+d_{n-1}+d_{n}+n}\right)$ coordinates which correspond to the coefficients in $P_{i}(x)$. Therefore, as in the case for two polynomials (see the author's previous papers ([17, [21)), we change the objective function as $\bar{f}(\boldsymbol{x})=\frac{1}{2} f(\boldsymbol{x})$, then solve the minimization problem of $\bar{f}(\boldsymbol{x})$, subject to $\boldsymbol{g}(\boldsymbol{x})=\mathbf{0}$.

### 3.4 Calculating the Actual GCD and Correcting the Deformed Polynomials

After successful end of the iterations, we obtain the coefficients of $\tilde{P}_{i}(x)$ and $U_{i}(x)$ satisfying (6) with $U_{i}(x)$ are relatively prime. Then, we need to compute
the actual GCD $H(x)$ of $\tilde{P}_{i}(x)$. Although $H$ can be calculated as the quotient of $\tilde{P}_{i}$ divided by $U_{i}$, naive polynomial division may cause numerical errors in the coefficient. Thus, we calculate the coefficients of $H$ by the so-called least squares division [14], followed by correcting the coefficients in $\tilde{P}_{i}$ by using the calculated $H$, as follows.

For polynomials $\tilde{P}_{i}$, and $U_{i}$ represented as in (7) and $H$ represented as

$$
H(x)=h_{d} x^{d}+\cdots+h_{0} x^{0}
$$

solve the equations $H U_{i}=\tilde{P}_{i}$ with respect to $H$ as solving the least squares problems of a linear system

$$
\begin{equation*}
C_{d}\left(U_{i}\right)^{t}\left(h_{d} \ldots, h_{0}\right)={ }^{t}\left(p_{d_{i}}^{(i)}, \cdots, p_{0}^{(i)}\right) \tag{18}
\end{equation*}
$$

Let $H_{i}(x) \in \mathbf{R}[x]$ be a candidate for the GCD whose coefficients are calculated as the least squares solutions of (18). Then, for $i=2, \ldots, n$, calculate the norms of the residues as

$$
r_{i}=\sum_{j=1}^{n}\left\|P_{j}-H_{i} U_{j}\right\|_{2}^{2}
$$

and set the GCD $H(x)$ be $H_{i}(x)$ giving the minimum value of $r_{i}$ so that the perturbed polynomials make the minimum amount of perturbations in total.

Finally, for the chosen $H(x)$, correct the coefficients of $\tilde{P}_{i}(x)$ as $\tilde{P}_{i}(x)=$ $H(x) \cdot U_{i}(x)$ for $i=1, \ldots, n$.

## 4 Experiments

We have implemented our GPGCD method on the computer algebra system Maple and compared its performance with a method based on the structured total least norm (STLN) method [11] for randomly generated polynomials with approximate GCD. The tests have been carried out on Intel Core2 Duo Mobile Processor T7400 (in Apple MacBook "Mid-2007" model) at 2.16 GHz with RAM 2GB, under MacOS X 10.5.

In the tests, we have generated random polynomials with GCD then added noise, as follows. First, we have generated a monic polynomial $P_{0}(x)$ of degree $m$ with the GCD of degree $d$. The GCD and the prime parts of degree $m-d$ are generated as monic polynomials and with random coefficients $c \in[-10,10]$ of floating-point numbers. For noise, we have generated a polynomial $P_{\mathrm{N}}(x)$ of degree $m-1$ with random coefficients as the same as for $P_{0}(x)$. Then, we have defined a test polynomial $P(x)$ as $P(x)=P_{0}(x)+\frac{e_{P}}{\left\|P_{\mathrm{N}}(x)\right\|_{2}} P_{\mathrm{N}}(x)$, scaling the noise such that the 2 -norm of the noise for $P$ is equal to $e_{P}$. In the present test, we set $e_{P}=0.1$.

In this test, we have compared our implementation against a method based on the structured total least norm (STLN) method [11], using their implementation (see Acknowledgments). In their STLN-based method, we have used the procedure R_con_mulpoly which calculates the approximate GCD of several polynomials in $\mathbf{R}[x]$. The tests have been carried out on Maple 13 with Digits=15

Table 1. Test results for $(m, d, n): n$ input polynomials of degree $m$ with the degree of approximate GCD $d$. See Section 4 for details.

| Ex. | $(m, d, n)$ | \#Fail |  | Error |  | \#Iterations |  | Time (sec.) |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  |  | STLN | GPGCD | STLN | GPGCD | STLN | GPGCD | STLN | GPGCD |
| 1 | $(10,5,3)$ | 0 | 0 | $2.31 e-3$ | $2.38 e-3$ | 5.50 | 11.2 | 1.17 | 0.45 |
| 2 | $(10,5,5)$ | 0 | 0 | $5.27 e-3$ | $5.22 e-3$ | 4.70 | 13.5 | 3.10 | 1.53 |
| 3 | $(10,5,10)$ | 0 | 0 | $5.48 e-3$ | $5.62 e-3$ | 4.40 | 17.9 | 12.49 | 8.59 |
| 4 | $(20,10,3)$ | 0 | 0 | $5.17 e-3$ | $5.40 e-3$ | 4.50 | 12.0 | 3.35 | 1.52 |
| 5 | $(20,10,5)$ | 0 | 0 | $5.89 e-3$ | $5.85 e-3$ | 4.40 | 12.7 | 10.37 | 4.97 |
| 6 | $(20,10,10)$ | 0 | 1 | $6.31 e-3$ | $6.20 e-3$ | 4.00 | 25.6 | 44.62 | 43.16 |
| 7 | $(40,20,3)$ | 0 | 0 | $5.32 e-3$ | $5.39 e-3$ | 4.90 | 12.8 | 13.60 | 5.83 |
| 8 | $(40,20,5)$ | 0 | 0 | $6.01 e-3$ | $5.97 e-3$ | 4.30 | 12.1 | 41.46 | 17.92 |
| 9 | $(40,20,10)$ | 0 | 0 | $6.41 e-3$ | $6.25 e-3$ | 4.10 | 8.90 | 200.88 | 60.21 |

executing hardware floating-point arithmetic. For every example, we have generated 10 random test cases as in the above. In executing the GPGCD method, we set $u=100$ and a threshold of the 2 -norm of the "update" vector in each iteration $\varepsilon=1.0 \times 10^{-8}$; in R_con_mulpoly, we set the tolerance $e=1.0 \times 10^{-8}$.

Table 1 shows the results of the test. In each test, we have given several polynomials of the same degree as the input. The second column with ( $m, d, n$ ) denotes the degree of input polynomials, degree of GCD, and the number of input polynomials, respectively. The columns with "STLN" are the data for the STLNbased method, while those with "GPGCD" are the data for the GPGCD method. "\#Fail" is the number of "failed" cases such as: in the STLN-based method, the number of iterations exceeds 50 times (which is the built-in threshold in the program), while, in the GPGCD method, the number of iterations exceeds 100 times. All the other data are the average over results for the "not failed" cases: "Error" is the sum of perturbation $\sum_{i=1}^{n}\left\|\tilde{P}_{i}-P_{i}\right\|_{2}^{2}$, where " $a e-b$ " denotes $a \times 10^{-b}$; "\#Iterations" is the number of iterations; "Time" is computing time in seconds.

We see that, in the most of tests, both methods calculate approximate GCD with almost the same amount of perturbations. In the most of tests, the GPGCD method runs faster than STLN-based method. However, running time of the GPGCD method increases as much as that of the STLN-based method in some cases with relatively large number of iterations (such as Ex. 6). There is a case in which the GPGCD method does not converge (Ex. 6). Factors leading to such phenomena is under investigation.

## 5 Concluding Remarks

Based on our previous research ([17, [21), we have extended our GPGCD method for more than two input polynomials with the real coefficients. We have shown that, at least theoretically, our algorithm properly calculates an approximate GCD under certain conditions for multiple polynomial inputs.

Our experiments have shown that, in the case that the number of iterations is relatively small, the GPGCD method calculates an approximate GCD efficiently with almost the same amount of perturbations as the STLN-based method. However, computing time of the GPGCD method increases as the number of iterations becomes larger; it suggests that we need to reduce the computing time of each iteration in the GPGCD method for the cases with relatively large number of iterations. It is desirable to have more detailed experiments for analyzing stability, performance for input polynomials of larger degree, etc.

For the future research, generalizing this result to polynomials with the complex coefficients will be among our next problems. It is also an interesting problem how the choice of $P_{1}$ affects the performance of the algorithm. Furthermore, one can also use arbitrary linear combination to transform $\operatorname{gcd}\left(P_{1}, P_{2}, \ldots, P_{n}\right)$ to $\operatorname{gcd}\left(P_{1}, a_{2} P_{2}+\cdots+a_{n} P_{n}\right)$. This will reduce the size of the generalized Sylvester matrix and will be another approach for calculating approximate GCD.

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