

Bounded Search Tree Algorithms for Parameterized Cograph Deletion: Efficient Branching Rules by Exploiting Structures of Special Graph Classes*

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Abstract. Many *fixed-parameter tractable* algorithms using a bounded search tree have been repeatedly improved, often by describing a larger number of branching rules involving an increasingly complex case analysis. We introduce a novel and general search strategy that branches on the forbidden subgraphs of a graph class relaxation. By using the class of P_4 -sparse graphs as the relaxed graph class, we obtain efficient bounded-search tree algorithms for several parameterized deletion problems. We give the first non-trivial bounded search tree algorithms for the cograph edge-deletion problem and the trivially perfect edge-deletion problems. For the cograph vertex deletion problem, a refined analysis of the runtime of our simple bounded search algorithm gives a faster exponential factor than those algorithms designed with the help of complicated case distinctions and non-trivial running time analysis [21] and computer-aided branching rules [11].

Keywords: Fixed-parameter tractability; edge-deletion; graph modification; cographs; trivially perfect graphs; quasi-threshold graphs; bounded search tree.

1 Introduction

A graph is a *cograph* [23] if it has no induced subgraph isomorphic to a P_4 , an induced path on four vertices. The name originates from *complement reducible graphs* as cographs are also characterized as being those graphs G which are either disconnected or else its complement \bar{G} is disconnected [23]. They are a well-studied class of graphs and many NP-complete problems on graphs have been shown to have polynomial time solutions when the input is a cograph [6].

A *graph modification problem* is a general term for a problem that takes a graph as input and asks how the graph can be modified to arrive at a new graph with a desired property. Usually, graph modifications are edge additions

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or deletions, vertex additions or deletions, or combinations of these. Our work on the following problems originally stems from studying social networks from which edge removals are made to reveal underlying structures in the network.

Many parameterized graph modification problems are tackled by a bounded search tree method where the size of the search tree usually dominates the computation time. This paper presents a framework for designing branching rules for bounded search tree algorithms. Our strategy exploits the structure of specialized graph classes in order to design efficient algorithms for (hard) problems on general graphs. We illustrate this method by giving algorithms that solve four different graph modification problems.

The *cograph edge-deletion problem* is the problem of determining when a graph $G = (V, E)$ has a set S of at most k edges which can be removed in order to make $G_2 = (V, E \setminus S)$ a cograph. This problem is known to be NP-complete [18] and also known to be *fixed parameter tractable* [4]. Similarly, the *trivially perfect edge-deletion problem* asks whether k edges can be removed to turn a graph into a trivially perfect graph (a graph which is P_4 and C_4 -free.) We first show how to solve these problems in linear time on a relaxed graph class, the P_4 -sparse graphs, and design algorithms to solve these problems on general graphs by branching towards P_4 -sparse graphs. Furthermore, we give improved algorithms for the vertex-deletion version of these problems.

We note that since the class of cographs is self-complementary, an algorithm solving k -edge-deletion problem also serves as a solution to the problem of k -edge-addition to cographs. Similarly, the k -edge-deletion problem to trivially perfect graphs serves as a solution to the k -edge-addition problem to co-trivially perfect graphs.

This paper is structured as follows: Section 2 summarizes previous results related to graph modification problems and gives some background on the class of P_4 -sparse graphs; Section 3 gives edge-deletion algorithms to cographs and to trivially perfect graphs; Section 4 designs *vertex deletion* algorithms for cographs and *trivially perfect* graph and their improvements using HITTING SET; Section 5 summarizes and discusses these results and suggests a number of directions for future work.

2 Previous Results and Background

2.1 Previous Fixed-Parameter Tractability Results

While cographs can be recognized in linear time [6], it is also known that it is NP-complete to decide whether a graph is a cograph with k extra edges [18]. Graph modification problems have been studied extensively: Yannakakis shows that vertex-deletion problems to many types of structures is NP-hard [26]. Elmallah and Colbourn give hardness results for many edge-deletion problems [18].

Recently, much research has been devoted to finding fixed-parameter tractable algorithms for graph modification problems: Guo [12] studied edge deletion to split graphs, chain graphs, threshold graphs and co-trivially perfect graphs;

Kaplan et al. [15] studied edge-addition problems to chordal graphs, strongly chordal graphs and proper interval graphs; Cai [4] showed fixed-parameter tractability for the edge deletion, edge addition, and edge editing problem to any class of graphs defined by a finite set of forbidden induced subgraphs. The constructive proof implies that k -edge-deletion problems to a class of graphs defined by a finite number of forbidden subgraphs is $O(M^k p(m+n))$ where p is some polynomial and M is the maximum over the number of edges in each of the forbidden induced subgraphs defining that graph class in question. For k -edge-deletions to P_4 -free graphs in particular, Cai's result implies an algorithm running in $O(3^k(m+n))$ time. This algorithm would work by finding a $P_4: a-b-c-d$ in a graph and branching on the 3 possible ways of removing an edge in order to destroy the P_4 (that is, removing either the edge $\{a, b\}$ or $\{b, c\}$ or $\{c, d\}$).

Nikolopoulos and Palios study the edge-deletion to cograph problem for a graph $G - xy$ where G is a cograph and xy is some edge of G [22]. Lokshtanov et al. study cograph edge-deletion sets to determine whether they are *minimal*, but not a minimum edge-deletion set [17]. To the best of our knowledge, ours is the first study that specifically addresses the edge-deletion problem to cographs. We present a bounded search tree algorithm that solves k -edge-deletion to cographs in $O(2.562^k(m+n))$ time by performing a search until we arrive at a P_4 -sparse graph and then optimally solving the remainder of the problem using the structure of P_4 -sparse graphs.

Graph modification problems can also be regarded as a type of graph recognition problem. Following the notation of Cai [4], for any class of graphs \mathcal{C} , we call $\mathcal{C} + ke$ the set of all graphs which can be composed by adding k extra edges to a graph from class \mathcal{C} . Similarly, $\mathcal{C} - ke$ is the set of graphs which are formed from a graph from class \mathcal{C} with k edge removals. Replacing 'edges' by 'vertices' in these definitions gives analogous classes for $\mathcal{C} + kv$ and $\mathcal{C} - kv$. A k -edge-deletion problem to a class of graphs \mathcal{C} can thusly be restated as a recognition problem for the class of $\mathcal{C} + ke$ graphs. Our results on cographs here can be restated as recognition algorithms for the classes: Cograph+ ke , Cograph- ke , Cograph+ kv , Trivially Perfect+ ke .

2.2 Background Information: P_4 -sparse graphs

One generalization to the class of cographs is formed by allowing P_4 s to exist in a graph but in restricted amounts. Hoáng [13] introduced P_4 -sparse graphs to be those for which every induced subgraph on five vertices induces at most one P_4 . This immediately implies a forbidden induced subgraph characterization which restricts any subgraph of five vertices inducing two or more P_4 s. We include these graphs in Figure 1.

A special graph structure called a *spider* [14] commonly occurs in graph classes of bounded cliquewidth. We define two types of spiders here:

Definition 1. A graph $G = (V, E)$ is a thin spider if V can be partitioned into K , S and R such that:

- i) K is a clique, S is a stable set, and $|K| = |S| \geq 2$.

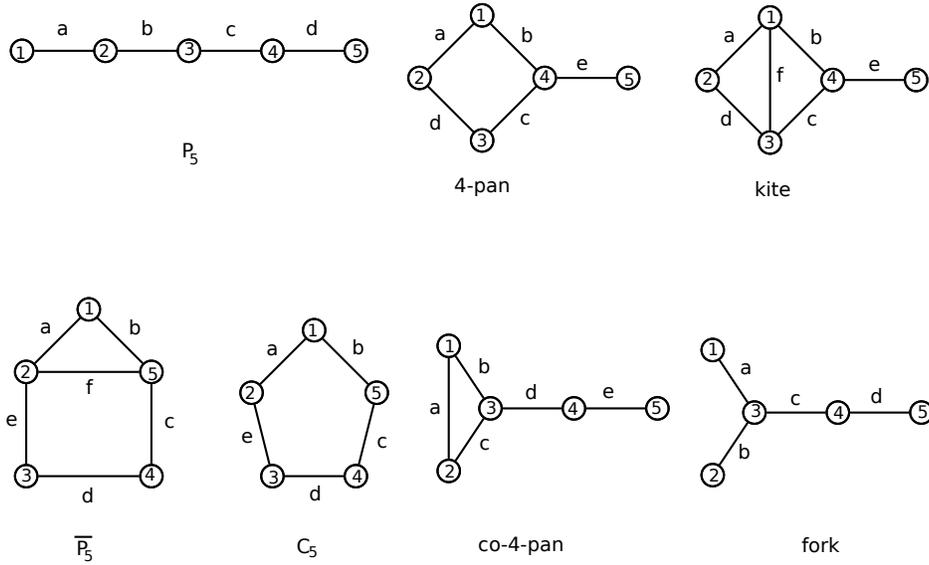


Fig. 1. The forbidden induced subgraphs for P_4 -sparse graphs

- ii) every vertex in R is adjacent to every vertex of K and to no vertex in S .
- iii) each vertex in S has a unique neighbour in K , that is: there exists a bijection $f : S \rightarrow K$ such that every vertex $k \in K$ is adjacent to $f(k) \in S$ and to no other vertex in S .

A graph G is called a *thick spider* if \overline{G} is a thin spider. Note that the vertex sets K and S swap roles under graph complementation, that condition (i) and (ii) hold for thick spiders, and that statement (iii) changes to saying that every vertex in S has a unique *non-neighbour* in K . The sets K , S and R are called the *body*, *feet* and *head* of the spider, respectively. The edges with one endpoint in S are called *thin legs* or *thick legs* for thin spiders or thick spiders, respectively. Examples of spiders are given in Figure 2.

Hoàng [13] defined a graph G to be P_4 -sparse if every induced subgraph with exactly five vertices contains at most one P_4 . The following decomposition theorem for P_4 -sparse graphs was proven in [14]:

Lemma 1. [14] *Let G be a P_4 -sparse graph. Then exactly one of the following is true:*

- i) G is disconnected
- ii) \overline{G} is disconnected
- iii) G is a spider

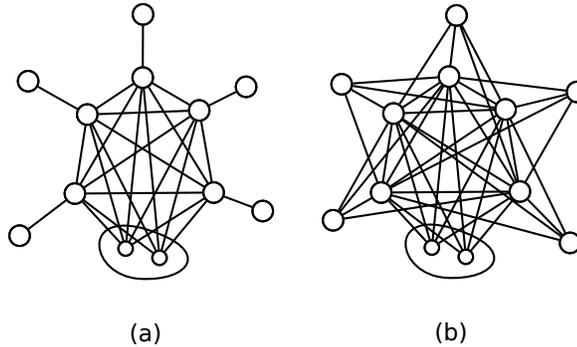


Fig. 2. (a) A thin spider and (b) thick spider with $|K| = |S| = 5$ and $|R| = 2$

We conclude our discussion of P_4 -sparse graphs with an observation that will be useful to us:

Lemma 2. *Let G be a spider with body K and feet S . Then every edge $\{k_1, k_2\}$ with $k_1, k_2 \in K$ is in one P_4 in $K \cup S$.*

Proof. A P_4 can not contain 3 vertices of K . If G is a thin spider, let each k_i be adjacent to each s_i . The edge $\{k_1, k_2\}$ is only in the $P_4 \{s_1, k_1, k_2, s_2\}$. If G is a thick spider, let each k_i be adjacent to every foot s_j where $i \neq j$. The edge $\{k_1, k_2\}$ is only in the $P_4 \{s_1, k_2, k_1, s_2\}$. \square

3 Edge-Deletion Algorithms

In this section, we give algorithms for two edge-deletion problems.

Problem 1. COGRAPH DELETION (G, k) :

Given graph $G = (V, E)$, does there exist a set S of at most k edges such that $(V, E \setminus S)$ is a cograph?

A graph is *trivially perfect* if it has no induced subgraphs isomorphic to a P_4 or a C_4 [25]. Every trivially perfect graph is a cograph and every cograph is a P_4 -sparse graph.

Problem 2. TRIVIALY PERFECT DELETION (G, k) :

Given graph $G = (V, E)$, does there exist a set S of at most k edges such that $(V, E \setminus S)$ is a trivially perfect graph?

The idea of the algorithms in this section is to focus on the forbidden subgraphs of P_4 -sparse graphs so that efficient branching rules can be designed systematically. This depends critically on whether these problems can be solved polynomially on P_4 -sparse graphs. We first show how to solve the cograph deletion problem on P_4 -sparse graphs in linear time.

3.1 Computing Cograph Edge-Deletion Sets on P_4 -sparse Graphs in Linear Time

We show that a linear time divide-and-conquer algorithm can be designed to find the minimum cograph deletion set for P_4 -sparse graphs.

Definition 2. Let G be a graph and \overline{G} be the complement of G . The vertex sets corresponding to the maximal connected components of \overline{G} are called the co-components of G . If \overline{G} is connected, then we say that G is co-connected.

Proposition 1. Let G be a P_4 -sparse graph and $M(G)$ be the size of a minimum edge-deletion set required to turn G into a P_4 -free graph. Then:

- i) if G is disconnected with components V_1, \dots, V_t , then $M(G) = \sum_{i=1}^t M(V_i)$
- ii) if \overline{G} is disconnected with co-components V_1, \dots, V_t , then $M(G) = \sum_{i=1}^t M(V_i)$
- iii) if G is a spider with head R , body K and feet S , then

$$M(R \cup K \cup S) = M(R) + M(K \cup S)$$

Proof. (i) This follows from the fact that a P_4 is connected and so any P_4 is in only one connected component, even after some edge deletions.

(ii) It is easy to verify that an edge joining two vertices in separate co-components can not be in a P_4 (or else in the graph complement this would imply a P_4 contains vertices in separate connected components as a P_4 is self-complementary.) After any edge-deletions within a co-component are made, the vertex sets of separate co-components are still completely joined, and so any new P_4 s will not include any two vertices in separate co-components.

(iii) Call a *leg edge* any edge joining a vertex $s \in S$ with a vertex $k \in K$, a *head edge* any edge joining some $r_1 \in R$ with some $r_2 \in R$, a *body edge* any edge joining two vertices in K , and call a *neck edge* any edge joining some $r \in R$ with some $k \in K$.

The structural definition of a spider says that every vertex in K is adjacent to every vertex in $K \cup R$, even after the removal of any leg edges and head edges. Thus a P_4 can never contain an edge $\{r, k\}$ with $r \in R$ and $k \in K$ even after leg and head edge removals. We will show that there is an optimal solution without body edges.

Consider an edge-deletion set E' such that $G - E'$ is a cograph, and let $E'' \subset E'$ be the set of body edges and neck edges in E' . Consider the P_4 s in $G - E' + E''$ (the P_4 s created when adding E'' back to G .) In $G - E' + E''$, K and R are completely joined and K is a clique and so no P_4 crosses the neck. So any P_4 s in $G - E' + E''$ are strictly in $K \cup S$ or strictly in R . Since E' is a cograph deletion set, the induced graph on R in $G - E' + E''$ is P_4 -free. In $K \cup S$, the body edges added back may be in a P_4 with two leg edges, and if so, this P_4 will be unique by Lemma 2. Adding the body edges from E'' can not create a P_4 involving a body edge not in E'' , so we just concentrate on the unique P_4 that each of these added body edges may have created. By deleting

one of these leg edges for each body edge that creates a P_4 , we create a new deletion set $E' - E'' + E'''$ where E''' is a set of leg edges and $|E'''| \leq |E''|$, so this new edge deletion set is a solution no larger than E' which does not use body or neck edges. \square

We note that parts (i) and (ii) of Proposition 1 apply to any graph G , and not just P_4 -sparse graphs.

Lemma 3. *Let G be a thin spider with body $K = \{k_1, \dots, k_{|K|}\}$ and legs $S = \{s_1, \dots, s_{|K|}\}$, and $\{s_i, k_j\}$ is an edge if and only if $i = j$. Then a minimum cograph edge-deletion set for $K \cup S$ is $\{\{s_i, k_i\}, i = 1..|K| - 1\}$.*

Proof. Since K is a clique and S is stable, every P_4 in $K \cup S$ has its endpoints in S . Furthermore, every pair of vertices in S are in a unique P_4 . Deleting any $|S| - 1$ thin legs will clearly destroy all of the P_4 s, so this edge-deletion set is indeed a cograph edge-deletion set. To see that it is of minimum size, assume there is a deletion set of size $|K| - 2$ or less in which two legs are not part of the deletion set. Let these two legs be $\{s_1, k_1\}$ and $\{s_2, k_2\}$ and call them “permanent” in this case. Since $\{s_1, k_1, k_2, s_2\}$ is a P_4 and the edges $\{s_1, k_1\}$ and $\{s_2, k_2\}$ are not in the deletion-set, it must be that $\{k_1, k_2\}$ is in the deletion set. There at most $|K| - 3$ other edges in the deletion set. Now $\{s_1, k_1, k_j, k_2\}$ induces a P_4 for every $j = 3 \dots |K|$. This means that the permanent edge $\{s_1, k_1\}$ is still in $|K| - 2$ P_4 s and every pair of these P_4 s have distinct edges aside from $\{s_1, k_1\}$. Thus it is impossible to destroy all of these remaining P_4 s with only $|K| - 3$ additional deletions or less. \square

Lemma 4. *Let G be a thick spider with body $K = \{k_1, \dots, k_{|K|}\}$ and feet $S = \{s_1, \dots, s_{|K|}\}$, and $\{s_i, k_j\}$ is an edge if and only if $i \neq j$. Then a minimum cograph edge-deletion set for $K \cup S$ is $\{\{k_i, s_j\}, i < j\}$.*

Proof. Every edge in $K \cup S$ is in exactly one P_4 : an edge $\{k_i, k_j\}$ is only in the $P_4 \{s_j, k_i, k_j, s_i\}$ and any edge $\{s_i, k_j\}$ is only in the $P_4 \{s_i, k_j, k_i, s_j\}$ so the number of P_4 s in $K \cup S$ is $\binom{|S|}{2}$, and since no two of these P_4 s share an edge, at least $\binom{|S|}{2}$ deletions are required. Consider the edge set $T = \{\{k_i, s_j\}, i < j\}$. When deleting T from $K \cup S$, K is still a clique and S is still a stable set, and so if there is any P_4 in $(K \cup S) \setminus T$, its endpoints must still be in S . But after deletion of T , we have that the neighbourhood of s_i is $N(s_i) = \{k_{i+1}, \dots, k_{|K|}\}$ which means that $N(s_i) \subset N(s_j)$ for all $i > j$, and so no two vertices in S can be the endpoints of a P_4 . So T indeed destroys all the P_4 s in $K \cup S$ and since $|T| = \binom{|S|}{2}$, this is a minimum set. \square

Theorem 1. *Algorithm 1 correctly solves the cograph edge-deletion problem for P_4 -sparse graphs and can be implemented in $O(m + n)$ time.*

Proof. The correctness of Algorithm 1 follows from Lemma 3, Lemma 4 and Proposition 1.

Algorithm 1 can be implemented in linear time, as the spider structure of P_4 -sparse graphs can be identified in linear time [14]. Identifying the connected or

co-components can also be done in linear time, as these types of vertex partitions are special cases of the more general notion of a *homogeneous set* or *module*, and there are a number of modular decomposition algorithms running in linear time [19], [7]. \square

Our algorithm to find cograph edge-deletion sets in P_4 -sparse graphs is presented in Algorithm 1.

Algorithm SPIDER(G):

Input: A P_4 -Sparse Graph $G = (V, E)$

Output: A set $S \subset E$ such that $(V, E \setminus S)$ is a P_4 -free graph

if G (or \overline{G}) *is disconnected* **then**

 Let V_1, \dots, V_t be the components or co-components of G ;
 $S \leftarrow S \cup_{i=1}^t \text{SPIDER}(V_i)$;

end

G is a spider with $K = \{k_1, \dots, k_{|K|}\}$ and $S = \{s_1, \dots, s_{|K|}\}$;

if G *is a thin spider* **then**

 Notation: k_i adjacent to s_j if and only if $i = j$;
 Add edge $\{k_i, s_i\}$ to solution set S for every $i = 1, \dots, |K| - 1$;

end

if G *is a thick spider* **then**

 Notation: k_i adjacent to s_j if and only if $i \neq j$;
 Add edge $\{k_i, s_j\}$ to solution set S for every pair $i < j$;

end

Return $S \cup \text{SPIDER}(R)$;

Algorithm 1: Cograph edge-deletion algorithm for P_4 -sparse graphs

3.2 A Bounded Search Tree Algorithm for Cograph Edge-Deletion

The bounded search tree algorithm (Algorithm 2) finds 5-vertex subsets that induce at least 2 P_4 s, branches on the possible ways of destroying the P_4 s, and then finally arrives at a P_4 -sparse graph and calls Algorithm 1. This algorithm either terminates with a call to the subroutine (in the case that a spider structure is encountered) or detects a cograph structure early, or else its integer parameter k has been reduced to 0 or less in which case the number of allowed edge-deletions has been exhausted without reaching a cograph.

Refer to Figure 1 for the possible subgraphs the general search algorithm may encounter. We refer to specific edges as they are labeled in Figure 1 for each subgraph. The pseudocode description of the general search algorithm branches on one of the deletion sets given in the table below.

Let H be one of the forbidden subgraphs from Figure 1. The possible edge-deletion sets to destroy the P_4 s in H are:

Subgraph	Minimal Edge Deletion Sets
C_5	{a,c}, {a,d}, {b,d}, {b,e}, {c,e}
P_5	{a,d}, {b}, {c}
\overline{P}_5	{a,b}, {e,c}, {d,e}, {c,d}, {a,d,f}, {a,c,f}, {b,d,f}, {b,e,f}
4-pan	{a,d}, {a,c}, {b,c}, {b,d}, {e}
co-4-pan	{b,c}, {d}, {e}
fork	{a,b}, {c}, {d}
kite	{a,d}, {a,c,f}, {b,d,f}, {b,c}, {e}

Algorithm COGRAPHDELETION(G, k)

Input: A Graph $G = (V, E)$ and a positive integer k

Output: A set S of edges of G with $|S| \leq k$ where $(V, E \setminus S)$ is a cograph if it exists, otherwise NO

Initialize $S = \emptyset$;

if G is a cograph **then**

 Return S ;

end

if $k \leq 0$ **then**

 Return NO;

end

Apply a P_4 -sparse recognition algorithm;

if G is P_4 -sparse **then**

$S \leftarrow S \cup \text{SPIDER}(G)$;

 If $|S| \leq k$, return S ; Otherwise, return NO;

end

else

 A forbidden graph H from Figure 1 exists;

foreach minimal edge-deletion set E' for H **do**

$S \leftarrow S \cup E'$;

 COGRAPHDELETION($G - E', k - |E'|$);

end

end

Algorithm 2: Bounded search tree algorithm computing a cograph edge-deletion set

It is routine to verify that any edge-deletion set from each of the 7 induced subgraph cases must contain one of the deletion set cases given in the table. Since every P_4 in the graph must be destroyed with an edge deletion, encountering any of these 7 configurations necessitates the need to apply one of the corresponding deletions.

The runtime of the algorithm is dominated by the size of the search tree. The spider structure can be identified in linear time. When k is the parameter measuring the number of edge deletions left to make, the size $T(k)$ of the search tree produced by this process is found from each branch rule separately:

1. C_5 : five branches, each reducing the parameter by 2 gives $T(k) = 5T(k - 2)$ and so $T(k) \leq 2.237^k$
2. P_5 : $T(k) = 2T(k - 1) + T(k - 2)$ giving $T(k) \leq 2.415^k$
3. \overline{P}_5 : $T(k) = 4T(k - 2) + 4T(k - 3)$ giving $T(k) \leq 2.383^k$
4. 4-pan: $T(k) = T(k - 1) + 4T(k - 2)$ giving $T(k) \leq 2.562^k$
5. co-4-pan: $T(k) = 2T(k - 1) + T(k - 2)$ giving $T(k) \leq 2.415^k$
6. fork: $T(k) = 2T(k - 1) + T(k - 2)$ giving $T(k) \leq 2.415^k$
7. kite: $T(k) = T(k - 1) + 2T(k - 2) + 2T(k - 3)$ giving $T(k) \leq 2.270^k$

The size of the search tree is thus upper-bounded by the worst case of deleting P_4 s in a 4-pan: $T(k) \leq 2.562^k$.

Theorem 2. *Algorithm 2 correctly solves the cograph k -edge-deletion problem in $O(2.562^k(n + m))$ time.*

Proof. Jamison and Olariu [14] give a linear time recognition algorithm for P_4 -sparse graphs. In the case that the graph being tested is not P_4 -sparse, the algorithm terminates upon finding a 5-set of vertices isomorphic to one of the forbidden subgraphs shown in Figure 1. In $O(m + n)$ time on a general graph, we can find one of the subgraphs in Figure 1 or else assert that our graph is P_4 -sparse. □

3.3 A Bounded Search Tree Algorithm for Edge-Deletion to Trivially Perfect Graphs

In [12], Guo studied the edge-deletion problem for *complements* of trivially perfect graphs. We know of no prior study of the specific problem of deleting edges to a trivially perfect graph. A naïve solution would find a subgraph isomorphic to either a P_4 or a C_4 and then branch on the possible ways of deleting an edge from that subgraph, resulting in a worst-case search tree of size $O(4^k)$. A minor observation that deleting any one edge from a C_4 always results in the other forbidden subgraph, P_4 , allows us to branch on the 6 possible ways of deleting any 2 edges from a C_4 . This results in a worst-case search tree of size $O(3^k)$ due to the 3 edges in a P_4 .

We use our strategy of branching towards a relaxation class of trivially perfect graphs. The 6 possible ways of deleting 2 edges from C_4 yield a branching factor of $\sqrt{6}^k \leq 2.45^k$, and since removing two edges from any C_4 is necessary to arrive at a (P_4, C_4) -free graph, our algorithm will begin by performing this branching step before running a P_4 -sparse recognition algorithm. Then we proceed as in the previous section, finding any P_4 -sparse forbidden subgraph and branching on the ways of deleting P_4 s and C_4 s in it. Once no P_4 -sparse obstruction exists,

we solve the problem optimally on the resulting specialized structure (a C_4 -free P_4 -sparse graph.) The branching rules become simpler in that only 5 of the 7 graphs in Figure 1 need consideration. In particular, the 4 -pan that caused the bottleneck of Algorithm 2, is no longer considered and this changes the runtime of the process from $O(2.562^k)$ to $O(2.450^k)$.

One main difference in this algorithm from Algorithm 2 is that C_4 s are found and destroyed first, and after any of the P_4 -sparse deletions are made, the process restarts with looking for C_4 s to destroy again. Once the C_4 s are destroyed and the resulting graph is P_4 -sparse, we proceed with removing edges with edge-deletion algorithm for thin or thick spiders (Algorithm 1).

Algorithm TRIVIALPERFECTEDGEDELETION(G, k)

Input: A Graph $G = (V, E)$ and a positive integer k

Output: A set S of edges of G with $|S| \leq k$ where $(V, E \setminus S)$ is trivially perfect if it exists, otherwise NO

```

Initialize  $S = \emptyset$ ;
if  $G$  is a trivially perfect then
    | Return  $S$ ;
end
if  $k \leq 0$  then
    | Return NO;
end
while There exists  $H$  isomorphic to  $C_4$  do
    | Create 6 branches corresponding to the possible ways of removing any
    | 2 edges in  $H$ 
end
Apply a  $P_4$ -sparse recognition algorithm;
if  $G$  is  $P_4$ -sparse then
    |  $S \leftarrow S \cup \text{SPIDER}(G)$ ;
    | If  $|S| \leq k$ , return  $S$ ; Otherwise, return NO.
end
else
    | A forbidden graph  $H$  from Figure 1 exists;
    | foreach minimal vertex-deletion set  $S'$  for  $H$  do
    | | Add the vertices  $S'$  to the solution set  $S$ ;
    | | TRIVIALPERFECTEDGEDELETION( $G - S', k - |S'|$ );
    | end
end

```

Algorithm 3: Bounded search tree algorithm finding a trivially perfect edge-deletion set

The correctness of decomposing the edge-deletion problem into separate problems on $K \cup S$ and R depends on a proposition similar to Proposition 1.

Proposition 2. *Let G be a C_4 -free graph and $M(G)$ be the size of a minimum edge-deletion set required to turn G into a (P_4, C_4) -free graph. Then:*

- i) if G is disconnected with components V_1, \dots, V_t , then $M(G) = \sum_{i=1}^t M(V_i)$
- ii) if \overline{G} is disconnected, G is a complete join between a clique and a smaller C_4 -free graph, H , and $M(G) = M(H)$.
- iii) if G is a spider with head R , body K and feet S , then

$$M(R \cup K \cup S) = M(R) + M(K \cup S)$$

Proof. Case i): If G has more than one connected component, any edge deletions made in one component cannot create a P_4 or a C_4 in a different connected component.

Case ii): \overline{G} is disconnected. Let H be a set of at least 2 vertices inducing a connected component in \overline{G} . Then H induces a C_4 -free graph in G since any induced subgraph of a C_4 -free graph is C_4 -free. Since H is connected in \overline{G} , there must be two non-adjacent vertices u, v of H in G . Let x and y be any two vertices not in H . If x and y are not adjacent, then $\{u, x, y, v\}$ induces a C_4 in G , which is impossible. So any vertices outside of H must induce a clique in G . Furthermore, since H is a connected component in \overline{G} , every vertex in H is adjacent to every vertex of the clique $G \setminus H$. It follows, then, that no P_4 in G includes a vertex of $G \setminus H$, and after any edge deletions in H , no P_4 or C_4 can include a vertex of $G \setminus H$. Hence $M(G) = M(H)$.

Case iii): Notice that no C_4 can include a vertex s from S in a spider even after removals of leg edges and head edges since the neighbourhood of s induces a clique. Since K is a clique, and every $k \in K$ is adjacent to every $r \in R$, there can not exist a C_4 in $K \cup R$ unless the C_4 is completely contained in R . So no C_4 contains an edge from R to K . Therefore, any edge $e = \{r, k\}$ with $r \in R$ and $k \in K$ is not in any C_4 in G , and for any subset of leg edges and head edges E' the edge $e = \{r, k\}$ is not in any C_4 in $G - E'$. Combining this with Proposition 1 for P_4 s establishes the decomposition. \square

Proposition 2 shows us that since all the C_4 s are destroyed in the branching stage of TRIVIALYPERFECTEDGEDELETION(G, k), once we arrive at a C_4 -free spider, we are free to delete leg edges without creating a new C_4 .

The runtime of Algorithm 3 is dominated by the branching rules once again. Encountering a C_4 results in 6 branches which delete 2 edges each. The resulting recurrence is $T(k) = 6T(k - 2)$ and so $T(k) \leq 2.450^k$. Having deleted all the C_4 s, we no longer include the \overline{P}_5 or the 4 -pan cases in our analysis. The runtime analysis for the rest remain unchanged: $C_5 : 2.237^k$, $P_5 : 2.415^k$, co-4-pan: 2.415^k , fork: 2.415^k , kite: 2.270^k . The search tree is thus bounded by the C_4 case of size $O(2.450^k)$. Finding a C_4 directly is a problem that is currently best-achieved using matrix multiplication [16], so this entire process *as described* runs in $O(2.450^k n^\alpha)$ where $O(n^\alpha)$ is the time required for matrix multiplication ($\alpha \leq 2.376$ [5]).

We can, in fact, modify the algorithm to run linearly in n and m by observing that a graph is P_4 -free and C_4 -free if and only if it is a chordal cograph. By first running a certifying chordal recognition algorithm [24], we can either deduce

that there is no C_4 or else find a C_4 or a C_5 or a larger induced cycle (and thus a P_5) and branch on these subgraphs according to the rules we gave, and if the graph is chordal then we apply a P_4 -sparse recognition algorithm to find one of the other forbidden induced subgraph, branch on it, and then re-apply the chordal recognition process.

Theorem 3. *Finding a trivially perfect k -edge-deletion set can be solved in $O(2.450^k(n + m))$ time.*

4 Vertex-Deletion Algorithms

This section shows how our general method can be used to solve vertex-deletion version of our prior two problems:

Problem 3. COGRAPH VERTEX-DELETION (G, k) :

Given graph $G = (V, E)$, does there exist a set S of at most k vertices such that $G - S$ is a cograph?

Problem 4. TRIVIALY PERFECT VERTEX-DELETION (G, k) :

Given graph $G = (V, E)$, does there exist a set S of at most k vertices such that $G - S$ is a trivially perfect graph?

4.1 Vertex-Deletion to Cographs

Since removing a vertex set S from a graph $G = (V, E)$ is equivalent to taking the induced subgraph on the vertex set $V \setminus S$, these problems are also often named *maximum induced subgraph* problems. In our case of asking if there is a vertex set of size at most k that can be removed to leave behind a cograph, this is equivalent to asking if there is an induced cograph subgraph of size at least $|V| - k$. Removing a vertex from G can never create a new induced subgraph in G , and so deleting vertices to destroy induced subgraphs is commonly modeled as a HITTING SET problem. In this case in which each P_4 maps to a 4-set in a HITTING SET instance, we have the restricted problem of a 4-HITTING SET. Algorithms for such vertex-deletion problems should always be compared against the state-of-the-art algorithms of d -HITTING SET if not anything else. d -HITTING SET is a well-studied NP-complete problem which admits fixed-parameter tractable algorithms. The first improved analysis of d -HITTING SET by Niedermeier and Rossmanith [21] give a search tree of size $O(3.30^k)$, and a more detailed and involved analysis by Fernau [8] improves the bound to $O(3.148^k)$. This is the best known bound for 4-HITTING SET to date.

The simple spider structure of P_4 sparse graphs allows us to describe a simple algorithm for the vertex-deletion problem to cographs. The runtime of this simple algorithm matches that of [11] and of [21]. The algorithm in [11] used branching rules that were designed by breaking the P_4 s in every subgraph of size t . Testing various values of t deduced that rules based on subgraphs of size 7 yielded the optimal runtime of an algorithm of this sort, with runtime $O(3.30^k)$. Their

algorithm builds branching rules from 447 graphs of size 7, while our algorithm only involves seven graphs on 5 vertices (Figure 1.)

In the following subsection, we use the analysis technique of [21] to show that the runtime of our bounded search tree algorithm is $O(3.115^k)$, hence improving on Fernau's $O(3.148^k)$. Our runtime could be improved further if we were to use the methods of Fernau [8], but such an analysis is extensive and would sidetrack from the focus of this paper.

We describe the subroutine SPIDER VERTEX-DELETION here. The algorithm works in the same way as Algorithm 1, taking as input a P_4 -sparse graph and returning the optimal number of vertices to remove in order to break all P_4 s in the graph. For thin spiders, every pair of feet is the end-pair of a P_4 , and removing any $|S| - 1$ vertices from S will destroy all the P_4 s in the body and legs. Removing less than $|S| - 1$ will leave at least two thin legs and hence a P_4 , so $|S| - 1$ is necessary.

Since a set of 4 vertices induces a P_4 in a graph G if and only if they induce a P_4 in \overline{G} , deleting any $|K| - 1$ vertices from K in a thick spider will destroy all the P_4 s in $K \cup S$. In either the thin or thick spider case, the subroutine is then applied to head R . This concludes the description of SPIDER VERTEX-DELETION. The correctness of the algorithm is asserted by the following proposition:

Proposition 3. *Let G be a spider with head R , body K and feet S . Let $M(G)$ be the minimum number of vertices required to remove from G in order to turn G into a cograph. Then $M(G) = M(R) + M(S \cup K)$.*

Proof. Deleting vertices from a graph can never create a new P_4 . We know from Proposition 1 that no P_4 includes vertices from both K and R . Deleting any vertices from $K \cup S$ will not destroy P_4 s in R , and vertex deletions from R will not destroy any P_4 s in $K \cup S$. Hence $M(G) = M(R) + M(S \cup K)$. \square

Corollary 1. *The algorithm SPIDER VERTEX-DELETION described above correctly solves the cograph vertex deletion problem for spiders in linear time.*

For a general graph, we proceed as in the cograph edge-deletion algorithm. We find P_4 -sparse obstructions and branch on the possible ways of deleting

vertices to destroy all P_4 s, repeating until the remaining graph is P_4 -sparse. The pseudocode description is given in 4.

Algorithm COGRAPHVERTEXDELETION(G, k)

Input: A Graph $G = (V, E)$ and a positive integer k

Output: A set S of vertices of G with $|S| \leq k$ where $(V \setminus S, E)$ is a cograph if it exists, otherwise NO

Initialize $S = \emptyset$;

if G is a cograph **then**

 Return S ;

end

if $k \leq 0$ **then**

 Return NO;

end

Apply a P_4 -sparse recognition algorithm;

if G is P_4 -sparse **then**

$S \leftarrow S \cup \text{SPIDER VERTEX-DELETION}(G)$;

 If $|S| \leq k$, return S ; Otherwise, return NO.

end

else

 A forbidden graph H from Figure 1 exists;

foreach minimal vertex-deletion set S' for H **do**

 Add the vertices S' to the solution set S ;

 COGRAPHVERTEXDELETION($G - S', k - |S'|$);

end

end

Algorithm 4: Bounded search tree algorithm finding a cograph vertex-deletion set

The branching rules for the vertex deletions are given in a table as before:

Subgraph	Minimal Vertex Deletion Sets
C_5	$\{1,2\}, \{1,3\}, \{1,4\}, \{1,5\}, \{2,3\}, \{2,4\}, \{2,5\}, \{3,4\}, \{3,5\}, \{4,5\}$
P_5	$\{1,5\}, \{2\}, \{3\}, \{4\}$
\overline{P}_5	$\{1\}, \{3\}, \{4\}, \{2,5\}$
4-pan	$\{2\}, \{4\}, \{5\}, \{1,3\}$
co-4-pan	$\{3\}, \{4\}, \{5\}, \{1,2\}$
fork	$\{3\}, \{4\}, \{5\}, \{1,2\}$
kite	$\{2\}, \{4\}, \{5\}, \{1,3\}$

The runtime of the algorithm is dominated by the branching steps. The runtime $T(k)$ for the C_5 case depends on 10 branches, while each of the other cases have equivalent runtime analysis.

1. C_5 : ten branches, each reducing the parameter by 2 gives $T(k) = 10T(k-2)$ and so $T(k) \leq 3.163^k$

2. All others: $T(k) = 3T(k-1) + T(k-2)$ giving $T(k) \leq 3.303^k$

The runtime of this vertex-deletion algorithm is bounded by $O(3.303^k(m+n))$, matching the runtime of the cograph vertex deletion algorithm generated by automated branching rule design [11].

Theorem 4. *Algorithm 4 solves the vertex-deletion problem for cographs in $O(3.303^k(m+n))$ time.*

4.2 Improvement using Hitting-Set

The 4-HITTING SET algorithm of [21] involves an analysis which counts when a branch choice can be made on a 3-set. Without counting these cases, an algorithm for 4-HITTING SET which only makes choices on 4-sets will have a search tree size of 4^k . By keeping track of when 3-sets are created in the search process and by branching on 3-sets whenever they are available, the authors of [21] are able to improve the upper-bound to the size of the search tree to $O(3.30^k)$.

We show here that using a similar technique to that in [21] can improve our search tree size. For an instance (G, k) of cograph vertex-deletion, we will use an implicit instance of 4-HITTING SET where each set of 4 vertices inducing a P_4 in G corresponds to a 4-set. A cograph deletion set for G will correspond to the hitting set of the set of 4-sets.

We adapt the notion of dominance from HITTING SET to that of P_4 s: a vertex v P_4 -dominates u if v exists in every P_4 that u is in.

Following the HITTING SET method, we observe that if v P_4 -dominates u , then any hitting set that contains u could be replaced with a hitting set containing v . Using this observation, we *mark* u in the graph to signify that it will not be in our solution. When we encounter P_4 s involving u , the P_4 only needs to be considered as a 3-set to hit. Marking u in G is equivalent to removing u in the implicit 4-HITTING SET instance.

Our vertex-deletion algorithm given in the previous subsection applies a P_4 -sparse recognition algorithm to find one of the 7 forbidden configurations from Figure 1. We illustrate how to proceed to the branching step when encountering a P_5 : $\{v_1, v_2, v_3, v_4, v_5\}$ with v_1 and v_5 as the endpoints:

If we put v_2 in S , remove v_2 from the graph G and reduce the parameter k by 1. Any set in the hitting set instance H containing v_2 is removed. Otherwise (if v_2 is not in S) we mark v_2 in G and remove v_2 from H , possibly creating some 3-sets. If v_3 is put in S , reduce k by 1 and remove v_3 from the graph, as before. Otherwise (if v_3 is also not in S) then mark v_3 in the graph and remove v_3 from H . If v_4 is in S , reduce k by 1 and remove any set containing v_4 . Otherwise (if none of v_2, v_3, v_4 are in S) we add v_1 and v_5 to S , remove them from G and reduce the parameter k by 2.

If we first ensure that P_4 -dominated vertices have been removed from consideration, some vertices in the P_5 (or analogous forbidden subgraph) may be marked. We do not need to build branches on cases asking if a vertex v is in S if v is already marked. If we encounter a P_4 in one of our forbidden subgraphs

consisting of four marked vertices, we can terminate that branch of the search tree and backtrack.

When encountering any of the P_4 -sparse forbidden subgraphs: P_5, \overline{P}_5 , kite, fork, 4-pan, co-4-pan, we have in each case 3 vertices whose removal will break both P_4 s in the obstruction, or else two vertices which must be removed together. Call those first 3 vertices the *breaking vertices*. Our process is summarized in the following algorithm:

Algorithm COGRAPHVERTEXDELETIONHITTINGSET(G, k)

Input: A Graph $G = (V, E)$ and a positive integer k

1. If any 3-set has been created, branch on that 3-set. Repeat until there are no more 3-sets;
2. If any vertex is P_4 -dominated, mark it in G and remove it from the hitting set. Go to step 1.;
3. Find one of P_5, \overline{P}_5 , kite, fork, 4-pan, co-4-pan.;
4. If there is a P_4 all of whose vertices are marked, STOP and backtrack.;
5. Branch on the (up to 3) unmarked breaking vertices using the cases as described above. Go to step 1.;
6. If all three breaking vertices are marked, include the other two vertices in S and Go to 1.;
7. If no such subgraph can be found, our graph is an *extended P_4 -sparse graph* (See below.) Solve the remainder optimally without search.

Algorithm 5: Using Hitting-Set for Cograph Vertex-Deletion

Let v and v' be breaking vertices encountered after steps 1 and 2 cannot be applied any further. If v is not in any other P_4 besides the two P_4 s in the obstruction graph found in step 3, then v is P_4 -dominated by v' , but this cannot happen if steps 1 and 2 are done to exhaustion. So we have that v must be in another P_4 not involving v' . When branching on v , we consider $v \in S$, in which case v is removed from G , or $v \notin S$ in which case we remove v from H , creating at least one 3-set since we established that v must be in another P_4 not containing v' .

Step 3 can be performed with a linear-time algorithm recognizing $(P_5, \overline{P}_5, \text{kite, fork, 4-pan, co-4-pan})$ -free graphs. These are called *extended P_4 -sparse graphs* by Giakoumakis and Vanherpe [10]. This ensures we do not encounter a C_5 at this stage of the process. They showed:

Theorem 5. [10] *If C is a C_5 in an extended P_4 -sparse graph, then C is a prime module.*

Let C be a C_5 in our graph after reaching step 7 of our hitting-set process. Observe that every 4-set of C induces a P_4 , so no vertex of C is contained in a nontrivial module or else we will have one of the forbidden subgraphs of extended P_4 -sparse graphs which we have already destroyed. Further, C can not be a module in some P_4 or else that P_4 extends to one of the forbidden graphs already destroyed (see Figure 3.) It must be that C is a set of 5 vertices inducing

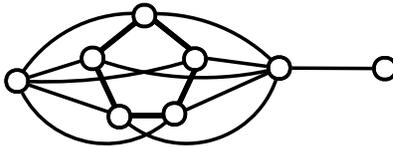


Fig. 3. An impossible configuration for a C_5 in an extended P_4 -sparse graph

a 5-cycle and not overlapping with any other P_4 . Since C does not intersect with any other existing P_4 s left in G , we are free to choose any two vertices of C to add to S and delete from G .

After deleting every C_5 from the extended P_4 -sparse graph, we have a conventional P_4 -sparse graph and we proceed with vertex deletions for spiders using SPIDER VERTEX-DELETION described in the previous subsection.

Let $T(k)$ be the number of leaves in a search tree of our vertex deletion problem, and let $B(k)$ be the number of leaves in a search tree for this problem whose root branched on a 3-set. Step 4 of Algorithm 5, can (at worst) branch on each of the breaking vertices. If the first vertex is put in S , we reduce k by 1 and so we have a $T(k - 1)$ branch. If we assume the first vertex is not in S and select the second vertex to be in S , the parameter decreases by 1. Since this first vertex is not P_4 -dominated (or else it would have been marked), it must be in another P_4 and so marking it will create at least one 3-set in H , giving a $B(k - 1)$ branch. Along the same lines, if we choose the third breaking vertex, we arrive at another $B(k - 1)$ branch. In the final case of deleting the two non-breaking vertices, we create a $T(k - 2)$ branch. Together this puts an upper bound on $T(k)$ of $T(k - 1) + 2B(k - 1) + T(k - 2)$.

Similarly, when branching on a 3-set, $B(k) \leq T(k - 1) + 2B(k - 1)$. Together, these two recurrences give a simultaneous system from which one can show $B(k) \leq 3.115^k$ and $T(k) \leq 1.115B(k)$ with a straightforward induction proof.

Theorem 6. *Algorithm 5 solves the cograph vertex-deletion problem in $O(3.115^k)$ time.*

The method of analyzing the search tree size created upon the existence of a 3-set shows that our search tree size is smaller than the $O(3.30^k)$ for 4-HITTING SET found by Niedermeier and Rossmanith [21]. Fernau [8] refines this analysis process by keeping track of the *the number* of $(d - 1)$ -sets in d -HITTING SET, arriving at $O(3.148^k)$ for 4-hitting set. Specifically, Fernau's analysis involves expressions $T^i(k)$ for $i = 0, 1, 2, 3$ where i is the number of 3-sets in an instance of 4-hitting set (in our case, $B(k)$ is $T^1(k)$.) We are confident that a similar refinement in the analysis of our vertex-deletion algorithm would reveal further gains, but our presented algorithm is already shown to have a smaller search space.

4.3 Vertex-Deletion for Trivially Perfect Graphs

Given a graph G , our task now is to find the largest induced trivially perfect subgraph in G . Equivalently, given a value k , we want know whether we can delete at most k vertices in order to turn the graph P_4 -free and C_4 -free.

In the edge-deletion version of this problem from the previous section, we deleted at least 2 edges from all C_4 s in the branching process since 2 edges is necessary, and this was algorithmically appealing as it decreased the parameter by 2. The vertex-deletion problem does not share this luxury: there are 4 vertices in a C_4 and only a single vertex removal is required to turn it into a (P_4, C_4) -free graph. This will result in a more complicated procedure to delete all remaining C_4 s in the P_4 -sparse graph that remains after the search process.

We will proceed directly to finding the P_4 -sparse obstructions and branching on them to turn each one into a (P_4, C_4) -free graph. This yields a worst-case runtime of $O(3.303^k)$, as summarized by the following table for each obstruction graph H :

Subgraph	Minimal Vertex Deletion Sets
C_5	$\{1,2\}, \{1,3\}, \{1,4\}, \{1,5\}, \{2,3\}, \{2,4\}, \{2,5\}, \{3,4\}, \{3,5\}, \{4,5\}$
P_5	$\{1,5\}, \{2\}, \{3\}, \{4\}$
\overline{P}_5	$\{1,2\}, \{1,3\}, \{1,4\}, \{1,5\}, \{2,3\}, \{2,4\}, \{2,5\}, \{3,4\}, \{3,5\}, \{4,5\}$
4-pan	$\{2\}, \{4\}, \{1,3\}, \{1,5\}, \{3,5\}$
co-4-pan	$\{3\}, \{4\}, \{5\}, \{1,2\}$
fork	$\{3\}, \{4\}, \{5\}, \{1,2\}$
kite	$\{2\}, \{4\}, \{5\}, \{1,3\}$

The runtime for each of these cases is summarized below:

1. C_5 : five branches, each reducing the parameter by 2 gives $T(k) = 10T(k-2)$ and so $T(k) \leq 3.167^k$
2. P_5 : $T(k) = 3T(k-1) + T(k-2)$ giving $T(k) \leq 3.303^k$
3. \overline{P}_5 : $T(k) = 10T(k-2)$ giving $T(k) \leq 3.167^k$
4. 4-pan: $T(k) = 2T(k-1) + 3T(k-2)$ giving $T(k) \leq 3^k$
5. co-4-pan: $T(k) = 3T(k-1) + T(k-2)$ giving $T(k) \leq 3.303^k$
6. fork: $T(k) = 3T(k-1) + T(k-2)$ giving $T(k) \leq 3.303^k$
7. kite: $T(k) = 3T(k-1) + T(k-2)$ giving $T(k) \leq 3.303^k$

After all the forbidden configurations of P_4 -sparse graphs have been destroyed, we are left with a P_4 -sparse graph from which we must delete vertices to destroy the remainder of the P_4 s and C_4 s. While C_4 s do not exist in a thin or thick spider, C_4 s will exist across co-components. Namely, if A_1 and A_2 are two non-clique co-components, then any two nonadjacent vertices x_1 and y_1 in A_1 and any two nonadjacent vertices x_2 and y_2 in A_2 will induce a 4-cycle. Since each connected component and co-component of a P_4 -sparse graph must be a spider, every induced 4-cycle must be the type that crosses non-clique co-components.

In order for this P_4 -sparse graph to be C_4 free, all but one of the co-components must be a clique. The only P_4 s that will be left to delete will be

those strictly in the non-clique co-component. To determine the optimal way at arriving at this point, let us introduce some notation: for a P_4 -sparse graph G , let A_1, A_2, \dots, A_t be the co-components of G . Let $\omega_i = \omega(A_i)$ be the size of a maximum clique in A_i , and η_i be the size of a minimum cograph vertex-deletion set, as found by the algorithm SPIDER VERTEX-DELETION.

We seek to find i such that deleting all co-components $A_j, j \neq i$ into cliques, plus SPIDER VERTEX-DELETION(A_i) is a minimum. That is, we want to find i that minimizes

$$\eta_i + \sum_{j \neq i} |A_j| - \omega_j.$$

For a particular G , $\sum |A_i| = n$ is fixed, as is $\sum \omega_i$. We see that the expression above is minimized when i is chosen such that $|A_i| - \omega_i - \eta_i$ is a maximum.

Our algorithm is as follows:

Algorithm TRIVIALPERFECTVERTEXDELETION(G, k)

Input: A Graph $G = (V, E)$ and a positive integer k

Output: YES if there exists a set S of at most k vertices so that $G - S$ is trivially perfect, NO otherwise.

while G is not P_4 -sparse **do**

 Let H be a P_4 -sparse obstruction subgraph;
 Branch on the possible ways of deleting the P_4 s and C_4 s from H ;
 Let k_1 be the number of vertex deletions made in this stage;

end

G is P_4 -sparse. Let A_1, \dots, A_t be the co-components of G ;

Fix i to be the lowest index maximizing $|A_i| - \omega_i - \eta_i$;

for $j \neq i$ **do**

 Fix a maximum clique of A_j ;
 Delete any vertex of A_j which is not in this maximum clique;

end

Let k_2 be the number of vertex deletions made in the for-loop;

Let k_3 be the number of deletions performed in SPIDER

VERTEX-DELETION(A_i);

if $k_1 + k_2 + k_3 \leq k$ **then**

 return YES;

end

return NO;

Algorithm 6: Trivially Perfect Vertex-Deletion Algorithm

Since maximum cliques can be found in linear time on P_4 -sparse graphs, it is clear that this algorithm runs in polynomial time for any fixed k . The runtime is dominated by the exponential factor from the search tree, which was shown to be $O(3.303^k)$.

Theorem 7. *Algorithm 6 is a fixed-parameter tractable algorithm which solves the vertex-deletion problem for trivially perfect graphs in $O(3.303^k)$.*

Of course, a hitting set-style improvement similar to the previous section could be applied here.

5 Conclusions and Future Work

We presented a general method for solving a variety of graph modification problems by limiting a search to a superclass of graphs which are close enough to the target that an optimal polynomial-time subroutine can be used. The algorithms presented here depend on the fact that deleting edges or vertices to a cograph or a trivially-perfect graph can be solved in linear time when the input is a P_4 -sparse graph. For general input, we gave the first non-trivial algorithm for the cograph edge-deletion problem (running in $O(2.562^k)$) and trivially-perfect edge-deletion problem (running in $O(2.450^k)$ time.) We gave simple algorithms to find minimum vertex-deletion sets to cographs and trivially perfect graphs whose runtime of $O(3.303^k)$ matched the existing literature. We also illustrated how a careful branching strategy and refined analysis technique improved the runtime to $O(3.115^k)$ for the cograph vertex deletion problem, and noted that these vertex-deletion problems can likely yield to further improvements with a more detailed analysis.

Our general method of branching toward a superclass of the target class benefits from few branching rules. This paper concentrated on only 7 graph configurations of P_4 -sparse graphs, allowing to explicitly define the small number branching rules when compared to the automated method of [11].

This paper leaves open many opportunities for future work. Firstly, the bound of $O(2.562^k)$ for cograph edge-deletion is due to the basic branching rules when encountering the 4-pan subgraph. Any improvement on this branching step would reduce the bottleneck of the cograph edge-deletion algorithm. Alternately, one could remove the 4-pan from the list of subgraphs to branch on provided this is accompanied with an efficient algorithm to optimally solve cograph edge-deletion on a $(C_5, P_5, \overline{P}_5, \text{kite}, \text{fork}, \text{co-4-pan})$ -free graphs (which are a restricted version of *semi P_4 -sparse graphs* [9]). Such an algorithm would improve cograph edge-deletion from $O(2.562^k)$ to $O(2.415^k)$.

The literature on graph classes is extensive [2], and many of these classes admit polynomial time solutions to many NP-complete problems. We expect that new and fast fixed-parameter tractable algorithms will soon develop through the use of superclasses as we have used in this paper. It would be interesting to see if the linear time algorithms for treewidth and minimum fill-in on permutation graphs [20] and on distance hereditary graphs [3], or polynomial time algorithms on weakly chordal graphs [1] could be used to design simple FPT algorithms for those problems.

References

1. V. Bouchitt and I. Todinca. Treewidth and minimum fill-in of weakly triangulated graphs. In Christoph Meinel and Sophie Tison, editors, *STACS 99*, volume 1563 of

- Lecture Notes in Computer Science*, pages 197–206. Springer Berlin / Heidelberg, 1999.
2. A. Brandstädt, V.B. Le, and J.P. Spinrad. *Graph classes: a survey*. Society for Industrial and Applied Mathematics, Philadelphia, PA, USA, 1999.
 3. H. J. Broersma, E. Dahlhaus, and T. Kloks. A linear time algorithm for minimum fill-in and treewidth for distance hereditary graphs. *Discrete Applied Mathematics*, 99(1-3):367 – 400, 2000.
 4. L. Cai. Fixed-parameter tractability of graph modification problems for hereditary properties. *Inf. Process. Lett.*, 58(4):171–176, 1996.
 5. D. Coppersmith and S. Winograd. Matrix multiplication via arithmetic progressions. *J. Symb. Comput.*, 9(3):251–280, 1990.
 6. D.G. Corneil, Y. Perl, and L.K. Stewart. A linear recognition algorithm for cographs. *SIAM J. Comput.*, 14:926–934, 1985.
 7. A. Cournier and M. Habib. A new linear algorithm for modular decomposition. In Sophie Tison, editor, *CAAP*, volume 787 of *Lecture Notes in Computer Science*, pages 68–84. Springer, 1994.
 8. H. Fernau. Parameterized algorithms for d -hitting set: The weighted case. *Theor. Comput. Sci.*, 411(16-18):1698–1713, 2010.
 9. J.-L. Fouquet and V. Giakoumakis. On semi- P_4 -sparse graphs. *Discrete Mathematics*, 165-166:277–300, 1997.
 10. V. Giakoumakis and J.-M. Vanherpe. On extended P_4 -reducible and extended P_4 -sparse graphs. *Theor. Comput. Sci.*, 180(1-2):269–286, 1997.
 11. J. Gramm, J. Guo, F. Hüffner, and R. Niedermeier. Automated generation of search tree algorithms for hard graph modification problems. *Algorithmica*, 39(4):321–347, 2004.
 12. J. Guo. Problem kernels for NP-complete edge deletion problems: Split and related graphs. In *ISAAC*, pages 915–926, 2007.
 13. C.T. Hoàng. Perfect graphs, (Ph.D. thesis). *School of Computer Science, McGill University Montreal*, 1985.
 14. B. Jamison and S. Olariu. Recognizing P_4 -sparse graphs in linear time. *SIAM J. Comput.*, 21(2):381–406, 1992.
 15. H. Kaplan, R. Shamir, and R.E. Tarjan. Tractability of parameterized completion problems on chordal, strongly chordal, and proper interval graphs. *SIAM J. Comput.*, 28(5):1906–1922, 1999.
 16. D. Kratsch and J. Spinrad. Between $O(nm)$ and $O(n^\alpha)$. *SIAM J. Comput.*, 36(2):310–325, 2006.
 17. D. Lokshtanov, F. Mancini, and C. Papadopoulos. Characterizing and computing minimal cograph completions. *Discrete Appl. Math.*, 158(7):755–764, 2010.
 18. E.S. El Mallah and C.J. Colbourn. Edge deletion problems: properties defined by weakly connected forbidden subgraphs. *Proc. Eighteenth Southeastern Conference on Combinatorics, Graph Theory, and Computing*, Congressus Numerantium 61:275–285, 1988.
 19. R.M. McConnell and J. Spinrad. Modular decomposition and transitive orientation. *Discrete Mathematics*, 201(1-3):189–241, 1999.
 20. D. Meister. Computing treewidth and minimum fill-in for permutation graphs in linear time. In Dieter Kratsch, editor, *Graph-Theoretic Concepts in Computer Science*, volume 3787 of *Lecture Notes in Computer Science*, pages 91–102. Springer Berlin / Heidelberg, 2005.
 21. R. Niedermeier and P. Rossmanith. An efficient fixed-parameter algorithm for 3-hitting set. *J. Discrete Algorithms*, 1(1):89–102, 2003.

22. S.D. Nikolopoulos and L. Palios. Adding an edge in a cograph. In D. Kratsch, editor, *WG*, volume 3787 of *Lecture Notes in Computer Science*, pages 214–226. Springer, 2005.
23. D. Seinsche. On a property of the class of n -colorable graphs. *J. Combin. Theory (B)*, 16(2):191–193, 1974.
24. R.E. Tarjan and M. Yannakakis. Addendum: Simple linear-time algorithms to test chordality of graphs, test acyclicity of hypergraphs, and selectively reduce acyclic hypergraphs. *SIAM J. Comput.*, 14(1):254–255, 1985.
25. J.-H. Yan, J.-J. Chen, and G.J. Chang. Quasi-threshold graphs. *Discrete Applied Mathematics*, 69(3):247–255, 1996.
26. M. Yannakakis. The effect of a connectivity requirement on the complexity of maximum subgraph problems. *J. ACM*, 26(4):618–630, 1979.