SEIDEL MINOR, PERMUTATION GRAPHS AND COMBINATORIAL PROPERTIES (EXTENDED ABSTRACT)

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ABSTRACT. A permutation graph is an intersection graph of segments lying between two parallel lines. A Seidel complementation of a finite graph at one of it vertex ν consists to complement the edges between the neighborhood and the non-neighborhood of ν . Two graphs are Seidel complement equivalent if one can be obtained from the other by a successive application of Seidel complementation.

In this paper we introduce the new concept of Seidel complementation and Seidel minor, we then show that this operation preserves cographs and the structure of modular decomposition.

The main contribution of this paper is to provide a new and succinct characterization of permutation graphs *i.e.* A graph is a permutation graph if and only if it does not contain the following graphs: C_5 , C_7 , XF_6^2 , XF_5^{2n+3} , C_{2n} , $n \ge 6$ and their complement as Seidel minor. In addition we provide a O(n + m)-time algorithm to output one of the forbidden Seidel minor if the graph is not a permutation graph.

Keywords: Graphs, Permutation graphs, Seidel complementation, Seidel minor, Modular decomposition, Cographs, Local complementation, Well Quasi Order.

1. INTRODUCTION

The aim of this paper is to present a new local operator on graphs, called Seidel complementation, and to show how this local operator leads to a new and compact characterization, by Seidel minors, of permutation graphs – intersection graph of segments between parallel lines – this characterization is in the same spirit as the famous Kuratowski's characterization of planar graph [22] which is: a graph is planar if and only if it does not contain K_5 neither $K_{3,3}$ as topological minor.

The Seidel complementation of a graph at a given vertex ν consists to complement the edges between the neighborhood of ν and its non-neighborhood. A schema of Seidel complementation is presented figure 1.

The main result of this paper is: A graph is a permutation graph if and only if it does not contain the following graphs C_5 , C_7 , XF_6^2 , XF_5^{2n+3} , C_{2n} , $n \ge 6$ and their complement as Seidel minors. This results consistutes, in a sense, an improvement compared to the characterization of permutation graphs by forbidden induced subgraphs which counts no less than 18 finite graphs, and 14 infinite families. As an algorithmic consequence of our result, we provide in linear time, when the graph is not a permutation graph, one of the obstruction by Seidel minor.

In order to proove the main result we study what are the relationship between Seidel complementation and modular decomposition. We prove that primality of a graph is invariant

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under Seidel complemention. We then provide a complete characterization of Seidel equivalent cographs, and we provide a linear time algorithm to decide if two cographs are Seidel equivalent.

Actually a similar approach, using local operators, has been used to characterize circle graphs, intersection of chords in a circle. This operator is the local complementation – *i.e.* complement the graph induced by the neighborhood of a vertex – Local complementation and vertex minor were used by Bouchet [6, 5, 7] study the structure of Circle graphs. He finally obtained a very elegant characterization of circle graphs [8]: A graph is a circle graph if and only if it does not contain W_5, W_7 and BW_3 as vertex minor. Recently Geelen and Oum [15] gave a characterization of the same flavor, using pivot minor. Vertex minor encountered a new celebrity with the fundamental work of Oum and Seymour [25, 26, 27] on the study of rank-width and later by Courcelle and Oum [11]. Another aspect of vertex minors and local complementation have been studied by Arratia *et al.* [2, 3, 4] in their serie on Interlace polynomials of graphs.

The Seidel complementation comes from a modification of another well known graph transformation introduced by Seidel in its seminal paper [29]. This operator is called after his name, the Seidel switch. Seidel switch have been intensively studied since its introduction, Colbourn *et al.* [10] proved that to decide if two graphs are Seidel switch equivalent is ISO-Complete, this results was independently proved in [20]. The Seidel switch has also applications in graph coloring [19]. Other interesting applications of Seidel switch concerns structural graph properties [16, 18] recently Montgolfier *et al.* [23, 24] used it to characterize graph completely decomposable *w.r.t.* Bi-join decomposition. Seidel switch is not only relevant to the study of graphs, Ehrenfeucht *et al.* [13] showed the interest of this operation for the study of 2structures and recently Bui-Xuan *et al.* extended this results to broader structures called Homogeneous relations [9].

All the above mentionned local operators have applications in various of area of computer science. For instance local complementation constitutes, in Bio-Informatic, an elementary tool to sort signed permutations by reversals [28, 31]. Recently these local operators were employed in quantum computing. In their papers Van den Nest *et al.* [33] and Hein *et al.* [17] consider local complementation on graph states. More recently Severini [30] used, this time, Seidel switch on two-colorable graph states.

The paper is organized as follows, in section 2 we formally introduce the Seidel complement operation and its associated minor: the Seidel minor. Then we present some structural properties of Seidel complementation and we briefly recall the notions and notations used in the sequel of the paper. In section 3 we present what are the connections between Seidel complementation and Modular decomposition, namely we prove that Seidel complementation preserves the structure of modular decomposition of a graph. Then we show that cographs are closed under this relation. In addition we provide a linear time algorithm to decide whether two cographs are Seidel complement equivalent or not.

The section 4 is devoted to the proof of the main theorem, namely to prove that a graph is a permutation graph if and only if it does not contain any of the forbidden Seidel minors. We also prove that permutation graphs are not WQO w.r.t. Seidel minor. We then derive from the main theorem a linear time algorithm to output one of the forbidden Seidel minor if the graph is not a permutation graph.

SEIDEL MINOR

2. Definitions and notations

In the sequel of the paper the graphs used are undirected, finite, loopless and simple. Here are some notations used in the papers. The graph induced by a subset of vertices X is noted G[X]. By $N(\nu)$, ν a vertex, we mean the neighborhood of ν , the set of non-neighbors of ν is represented by $\overline{N(\nu)}$. Sometimes we need to use a refinement of the neighborhood on a subset of vertices X, noted $N_X(\nu)$ it is simply $N(\nu) \cap X$.

Definition 2.1 (Seidel complement). Le G = (V, E) be a finite undirected graph. And let v be a vertex of V. The Seidel complement at v on G, noted G * v is defined as follows: Inverse all the edges between G[N(v)] and $G[\overline{N}(v)]$.

 $(G * v) = (V, E_1 \cup E_2 \cup E_3)$. Where $E_1 = \{e = xy | x \text{ and } y \in \{v\} \cup N(v) \text{ and } e \in E\}$, $E_2 = \{e = xy | x \text{ and } y \in \overline{N}(v) \text{ and } e \in E\}$ and $E_3 = \{e = xy | x \in N(v), y \in \overline{N}(v) \text{ and } e \notin E\}$.

From the previous definition it is straightforward to notice that G * v * v = G. As a remark it is clear that G * v * v = G. However contrary to the Seidel switch for Seidel complementation G * v * u is not (always) isomorph to G * u * v.

Proposition 2.2. Let G be a graph, if vw is an edge of G. Then G * v * w * v = G * w * v * * w. This operation is noted G * vw.

Definition 2.3 (Seidel Minor). Let G = (V, E) and H = (V', E') be two graphs. H is a Seidel Minor of G (noted $H \leq_S G$) if H can be obtained from G by a sequence of the following operations:

- Perform a Seidel complemention at a vertex ν of G,
- Delete a vertex ν of G.

Definition 2.4 (Seidel Equivalent Graphs). Let G = (V, E) and H = (V, F) be two (finite) graphs. G and H are said to be Seidel equivalent if and only if there exists a word ω defined on V^{*} such that $G * \omega \cong H$.

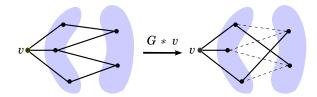


FIGURE 1. An illustration of the Seidel complement concept

3. MODULAR DECOMPOSITION AND COGRAPHS

In this section we investigate the relations between Seidel complementation and modular decomposition. This study is relevant in order to prove the main result. Actually a permutation graph is uniquely representable if and only if it is prime w.r.t. to modular decomposition. And one of the results of this section is to prove that if the graph considered is prime w.r.t. modular decomposition this property is preserved by Seidel complementation. From the point of view of permutation graphs it means that if the graph is uniquely representable so are their Seidel complement equivalent graphs.

Let us now briefly introduce the definition of module. A module in a graph is subset of vertices M such that any vertex outside M is either completely connected to M or is completely disjoint from M. Modular decomposition is a decomposition of graph introduced by Gallai [14]. The modular decomposition of a graph G is the decomposition of G into its modules. Without going too deep into the details, there exists for each graph a unique modular decomposition tree, and it is possible to compute it in linear time (*cf.* [32]).

In the sequel of this section we show that if G is prime, *i.e.* not decomposable, *w.r.t.* modular decomposition, then applying a Seidel complementation at any vertex of the graph preserves this property. Actually, it is interesting to notice that this phenomenon occurs for local complementation w.r.t. Split decomposition (*cf.* [12]) which is a generalization of modular decomposition.

Then we prove that the family of cographs is closed under Seidel minor. We show that any cographs that are Seidel complement equivalent are at distance at most 1 –the size of the sequence of Seidel complementation– and then from this properties we design a linear time algorithm to decide if two cographs are Seidel complement equivalent.

3.1. Modular decomposition.

Theorem 3.1. Le G = (V, E) be graph, and let v be an arbitrary vertex of G. G is prime w.r.t. modular decomposition if and only if G * v is prime w.r.t. to modular decomposition.

Proof. Let us proceed by contradiction. Let us assume that G is prime and G * v has a module M. We have to consider two cases: (1) $v \in M$ and (2) $v \notin M$

 $\mathbf{v} \in \mathbf{M}$: Since \mathcal{M} is not trivial we have $|\mathcal{M}| \ge 2$ and $|\overline{\mathcal{M}}| \ge 1$.

We can identify four representant vertices of G. Let A be a vertex of $\overline{N(\nu)} \cap M$, let B be a vertex of $\overline{N(\nu)} \cap \overline{M}$, let C be a vertex of $N(\nu) \cap \overline{M}$ and let D be a vertex of $N(\nu) \cap M$. Since M is a module we have the following edges: CA and CD and the following non edges: BA and BD (*cf.* figure 2(a)).

By definition of Seidel complementation at a vertex, it is equivalent to swap the edges and non-edges between the neighborhood and the non-neighborhood of ν . We obtain the result depicted in figure 2(b). Now we can clearly see that $\overline{M} \cup \{\nu\}$ is a module in G, and since $|\overline{M}| \ge 1$ we obtain a non trivial module. Thus a contradiction.

 $\mathbf{v} \notin \mathbf{M}$: Let us consider the case where \mathbf{v} does not belong to \mathbf{M} . We can assume, *w.l.o.g.*, that $\mathbf{M} \subseteq \mathbf{N}(\mathbf{v})$. We can partition $\mathbf{N}(\mathbf{v})$ into A_1, A_2 such that $\mathbf{N}_{\mathbf{M}}(A_1) = \mathbf{M}$ and $\mathbf{N}_{\mathbf{M}}(A_2) = \emptyset$. And similarly we can partition $\overline{\mathbf{N}(\mathbf{x})}$ into B_1, B_2 such that $\mathbf{N}_{\mathbf{M}}(B_1) = \mathbf{M}$ and $\mathbf{N}_{\mathbf{M}}(B_2) = \emptyset$. (cf. figure 2(c)-(d))

Since we have proceeded to a Seidel complement on ν , the original configuration in G is such that $N_M(B_1) = \emptyset$ and $N_M(B_2) = M$. This is the only change *w.r.t.* M. So M is also a module in G. Contradiction.

3.2. Cographs. Cographs are the graphs which are completely decomposable w.r.t. modular decomposition. There exist several characterizations of cographs, one of them is given by a list fordidden induced subgraphs, *i.e.* cographs are the graphs without P_4 –a path on four vertices– as induced subgraph. Another fundamental properties of cograph is the fact that its modular decomposition tree –called its co-tree– has only serie (1) and parallel (0) nodes as internal nodes. An example of cograph and its associated co-tree is given in figure 3(a). A co-tree is a rooted tree, where the leafs represent the vertices of the graphs. And the internal nodes of the co-tree encode the adjacency of the vertices of the graph. Two vertices are

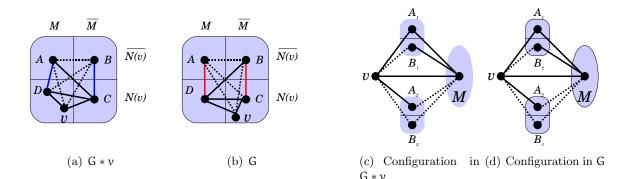


FIGURE 2. Details of theorem 3.1. The figures (a)-(b) correspond to the case where ν belongs to M. And the figures (c)-(d) correspond to the other case.

adjacent iff their Least Common Ancestor¹ (LCA) is a serie node (1). Conversely two vertices are disconnected iff their LCA is a parallel node (0). The following theorem shows that the class of cographs is closed under Seidel complemention.

Theorem 3.2. Let G = (V, E) a cograph, and v a vertex of G, then G * v is also a cograph.

Proof. Let T be the co-tree of G. The Seidel complementation at a vertex ν is obtained as follows: Let T' be the tree obtained by $T * \nu$. $P(\nu)$ becomes the new root of T'. and now the parent of ν in T' is the former root, to know R(T). In other words by performing a Seidel complementation we have reversed the path from $P(\nu)$ to R(T).

It is easy to see that G[N(v)] and G[N(v)] are not modified. Now to see that the adjacency between G[N(v)] and $G[\overline{N(v)}]$ is reversed, it is sufficient to remark that for two vertices, one belonging to the neighborhood of v and the other one belonging to the non neighborhood of v. If these two vertices are adjacent in G it means that their LCA is a serie node. We can notice that this node lies on the path from v to the root of T. After proceeding to a Seidel complementation their LCA is modified and it is now a parallel node. Consequently reversing the adjacency between the neighborhood and the non-neighborhood.

A schema of the Seidel complement of the co-tree is given in figure 3(b).

Remark 3.3 (Exchange property). Actually a Seidel complemention on a cograph, or more precisely on its co-tree is equivalent to exchange the root of the co-tree with the vertex ν used to proceed to the Seidel complement, *i.e.* the vertex ν is attached to the former root of the co-tree and the new root is the former parent of the vertex ν .

Except this transformation the others parts of the co-tree remain unchanged, *i.e.* the number and the types of internal nodes are preserved, and no internal nodes are merged.

The following theorem show that if two cographs are Seidel complement equivalent, then they are at distance at most 1.

Theorem 3.4. Let G = (V, E) and H = (V, F) be two cographs. G and H are Seidel complement equivalent if and only if H is at distance at most one of G, i.e. there exists one vertex v of V such that $G \cong H * v$.

¹The LCA of two leafs x and y is first node in common on the paths from the leafs to the root.

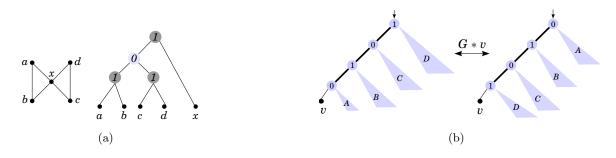


FIGURE 3. (a) An example of cograph on 5 vertices and its respective co-tree. (b)A schema of a Seidel complement at a vertex ν on a co-tree.

Proof. If G and H are isomorphic then it is done. Let us now consider the case: G and H are not isomorph but are Seidel complement equivalent (*cf.* 2.4). This proof relies on theorem 3.4 and uses as a key tool the observation made in remark 3.3.

Let us consider that H is at distance 2 of G, *i.e.* there exist ν and u two vertices of H such that $G \cong H * \nu * u$. Using the remark 3.3 once we have done the Seidel complement using ν we have exchanged $P(\nu)$ and R(T(H)). In other words $P(\nu)$ is now the root of $T(H * \nu)$ and now ν is connected to the node which used to be the root of T(H). When we proceed to the second Seidel complement, using the vertex u on $H * \nu$, once again we exchange P(u) and the root of $T(H * \nu)$, and we connect now u to the previous root *i.e.* to $R(T(H * \nu))$.

Let us now consider what is the situation when we proceed directly to a Seidel complement using u. After the operation using u the co-tree obtained T(H * u) has for root P(u) and now u is connected to the former root of T(H).

If we now look to the two trees obtained, T(H * v * u) and T(H * u) we can easily see that the two trees are isomorph. And we can remark that in a sense u and v have been "switched".

Consequently we deduce that we can always reduce a sequence of Seidel complementation of length k ($k \ge 2$) by applying this procedure. And proceeding greedily, it yields that two co-graphs are Seidel complement equivalent if and only if they are at distance at most 1. Better insights appear clearly in figure 4.

Corollary 3.5. The number of cographs that are Seidel complement equivalent to a given cograph G on n vertices is at most O(n).

Proof. It is a direct consequence of theorem 3.4 since all the graphs are at distance at most one. It means that the number of different graphs, up to isomorphism, is no more than O(n), *i.e.* from G each vertex v can give a different graph by proceeding to a Seidel complement G*v. \Box

Corollary 3.6. To decide if two cographs G and H are Seidel complement equivalent can be computed in linear tine O(n).

Proof. Let us consider the co-trees T(G) and T(H). We modify T(G) and T(H) as follows: Let T'(G) be the co-tree of G on which we add a dummy vertex attached to the root of T(G). We proceed in a similar manner for T'(H).

G and H are Seidel complement equivalent if and only if T'(G) and T'(H) are isomorph. \Rightarrow This direction is easy, since according to the previous remark, and theorem 3.4. That if G and H are Seidel complement equivalent then T'(G) and T'(H) are isomorph.

 $\Leftarrow \text{ Let us assume now that } \mathsf{T}'(\mathsf{G}) \text{ and } \mathsf{T}'(\mathsf{H}) \text{ are isomorph and let } \phi: V(\mathsf{T}'(\mathsf{G})) \mapsto V(\mathsf{T}'(\mathsf{H}))$

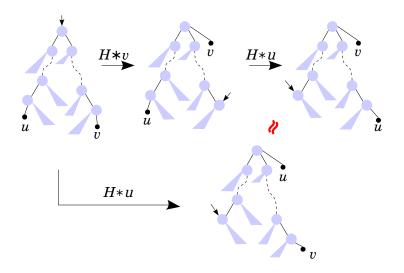


FIGURE 4. Schemas of co-trees and their transformations. The roots of the trees are indicated with the ingoing arc.

be the mapping function. The isomorphism considered here is the labelled isomorphism, *i.e.* labels of the internal nodes, 0 or 1, are preserved.

Using the result of theorem 3.4 we know that cographs are at distance at most 1. It is thus sufficient to find the actual vertex to transform one co-tree into another.

Let us call the dummy vertices added to turn T(G) (resp. T(H)) into T'(G) (resp. T'(H)) du_G and du_H. Now if since we want to transform T(H) into T(G) it suffices to pick a vertex f in T(H) such that it is the image by φ of du_G *i.e.* $f = \varphi(du_G)$. Once we have obtained this vertex in T(G) it is sufficient to proceed to a Seidel complement on f, H * f, so now P(f)is the root of T(H * f) as requested since f was an image of du_G and f is now attached to the former root R(H). Consequently we have shown that when T'(G) and T'(H) are isomorph we can find a vertex permitting to transform T(H) into T(G) and hence proving that they are Seidel complement equivalent.

This procedure can be achieved in linear time, since decide if two given trees are isomorph is well known to be linear [1], and the find the actual vertex and perform the Seidel complementation is done in constant time. \Box

Proposition 3.7. The Seidel complementation of a cograph on its co-tree can be performed in O(1)-time.

Proof. It suffices to consider the co-tree of G. As proven in previous lemmas to perform a Seidel complementation at a vertex ν is equivalent to exchange a vertex -i.e. a leaf - with the root of the tree. We need to store, in a lookup table, for each vertex its parent node in the tree and the root of the tree. Updating the structure is done in constant time.

4. Permutation graphs

In this section we show that the class of permutation graphs is closed under Seidel minor, and we prove the main theorem that states that a graph is a permutation graph if and only if it does not contain none of the following graph: C_5 , C_7 , XF_6^2 , XF_5^{2n+3} , C_{2n} , $n \ge 6$ and their complement as Seidel minor.

We also show that a Seidel complementation at a vertex on the permutation diagram can be achieved in constant time. Where as this operation can be quadratic on the graph itself.

Definition 4.1 (Permutation graph). A graph G = (V, E) is a permutation graph if there exist two permutations σ_1, σ_2 on $V = \{1, ..., n\}$. And two vertices u, v of V are adjacent iff $\sigma_1(u) < \sigma_1(v)$ and $\sigma_2(v) < \sigma_2(u)$.

An example of permutation graph is presented in figure 5

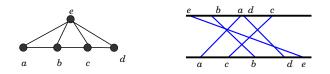


FIGURE 5. A permutation graph and its representation

Theorem 4.2 (Gallai'67 [14]). A permutation graph is uniquely representable iff it is prime w.r.t. modular decomposition

Theorem 4.3 ([14]²). A graph is a permutation graph if and only if it does not contain one of the finite graphs as induced subgraphs T_2 , X_2 , X_3 , X_{30} , X_{31} , X_{32} , X_{33} , X_{34} , X_{36} and their complement and does not contain the graphs given by the infinite families: XF_1^{2n+3} , XF_5^{2n+3} , XF_6^{2n+2} , XF_1^{n+1} , XF_3^n , XF_4^n , the Holes, and their complement.

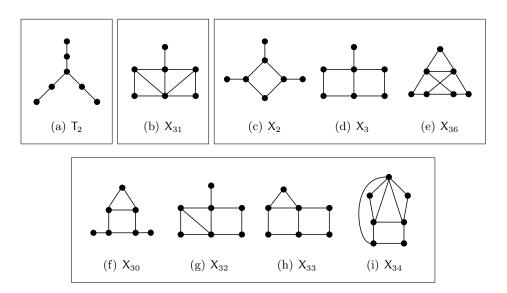


FIGURE 6. Finite forbidden induced subgraphs for permutation graphs.

Theorem 4.4. Let G = (V, E) a permutation graph, and v a vertex of G, then G * v is also a permutation graph.

²http://wwwteo.informatik.uni-rostock.de/isgci/classes/AUTO_3080.html

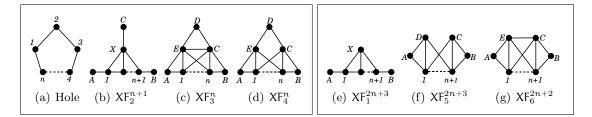


FIGURE 7. Forbidden infinite families for permutation graphs. The families in the left box (a)-(d) contains asteroidal triples. The families in the right box (e)-(g) do not contain asteroidal triple, the key point is the parity of the dashed path.

Proof. Let G = (V, E) be a permutation graph and v a vertex of G. Let us prove that G * v remains a permutation graph. To do so we present now the transformation on D(G), the permutation diagram of G, which corresponds to the Seidel complementation at v. In order to give a better insight this transformation is depicted in figure 8(a). Let σ_1 be $A \cdot v \cdot B$ and σ_2 be $C \cdot v \cdot D$. Where A is a word on $V \setminus \{v\}$ and B is a word on $V \setminus \{v\}$ and similarly C is a word on $V \setminus \{v\}$ and D is a word on $V \setminus (C \cup \{v\})$.

The following transformation $\sigma_1 * \nu = B \cdot \nu$. A and $\sigma_2 * \nu = D \cdot \nu \cdot C$ corresponds to a Seidel complementation at ν . We have to prove that the graphs induced by the neighborhood $G[N(\nu)]$ and $G[\overline{N(\nu)}]$ are unchanged. Let us begin with the non-neighborhood of ν . It is easy to notice on figure 8(a) that the non neighborhood of ν is contained in the two vertical rectangles, one on the left of ν and the other one on their right, (A, C) and (B, D). by proceedind to the transformation described above, and by keeping the order of the words, it is easy to notice that first of all, theses vertices remains disconnected of ν and since the order of vertices in the words are preserved then this subgraphs remain unchanged. In a similar manner for the subgraph induced by the neighborhood of ν , now the vertices of their neighborhood are contained in the gray crosses (A, D) and (B, C) and for the same reason as for non-neighborhood, the order, the subgraphs remains unchanged and it is still connected to ν .

Now let us consider the less obvious part which is to inverse the adjacency between G[N(v)]and $G[\overline{N(v)}]$. Let w be a neighbor of v and let u be a non-neighbor of v. Let us assume, w.l.o.g., that w and u are connected. Let us consider the case where u belongs to the (A, C)rectangle and $w \in (A, D)$, if $uv \in E$ it means that $\sigma_1(w) < \sigma_1(u)$ and $\sigma_2(u) < \sigma_2(v)$, after proceeding to a Seidel complement at v we obtain $\sigma_1 * v$ and $\sigma_2 * v$ but now according to the transformation we have $\sigma_1 * v(w) < \sigma_1 * v(u)$ and $\sigma_2 * v(w) < \sigma_1 * v(u)$. And according to the definition 4.1 now u and w are no longer connected. The proof is similar for the other cases. Consequently Seidel complementation preserves permutation graphs. \Box

Corollary 4.5. The Seidel complementation at a vertex v of a permutation graph can be achieved in O(1)-time.

Proof. It is sufficient to consider the permutation representation of G as two doubly linked list. Then the Seidel complementation consists to apply the pattern described in the proof of theorem (4.4). It consist *w.l.o.g.* on σ_1 to exchange A and B: $A \cdot v \cdot B$ becomes $B \cdot v \cdot A$. So it suffices to change the successor of v in the list as the first element of A and the predecessor

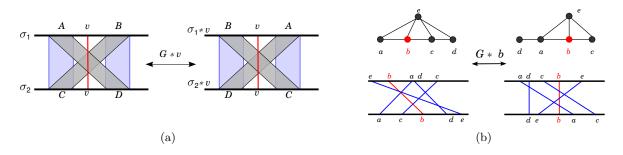


FIGURE 8. (a) Schematic view of Seidel complementation on permutation diagram and (b) An example of Permutation graph and a Seidel complementation at a vertex b

of ν as the last element of B. Then update the first and last element of the new list. All this operations can obviously be done in constant time.

One can remark that to perform a Seidel complementation at a vertex on graph can require in the worst case $O(n^2)$ -time. It suffices to consider a the graph consituted of a Star $K_{1,n}$ and a stable S_n . its size is 2n + 1 with n + 1 connected components. Applying a Seidel complementation on the vertex of degree n results in a connected graph with $O(n^2)$ edges.

4.1. Finite Families. In this section we show that it is possible to reduce the list of fordidden induced subgraphs by using Seidel Complementation. Actually a lot of forbidden subgraphs are Seidel equivalent. The graphs that are Seidel complement equivalent are in the same box in figure 6. Finally, the list of forbidden graphs is reduced from 18 induced subgraphs to only 6 finite Seidel minor. The fordidden Seidel minors are C_5 , C_7 , XF_6^2 and their complement.

Proposition 4.6. The graphs X_3 , X_2 , X_{36} (cf. figure 6(c)-(e)) are Seidel complement equivalent.

Proposition 4.7. The graphs X_{30} , X_{32} , X_{33} and X_{34} (cf. figure 6(f)-(i)) are Seidel complement equivalent.

Proposition 4.8. The graph XF_4^0 is a Seidel minor of T_2 and X_{31} .

Proposition 4.9. The graph C_6 is a Seidel minor of XF_4^0 .

Proof. Applying a Seidel complementation on the degree 2 vertex of the C_4 in XF_4^0 we obtain C_6 .

4.2. Infinite Families. We show in this section that actually forbidden infinite families under the relation on induced subgraphs are redundant when the Seidel minor operation is considered. Consequently the following propositions allows us to reduce from 14 infinite families with the induced subgraph relation to only 4 infinite families under Seidel minor. The forbidden families are XF_5^{2n+3} and C_{2n} , $n \ge 6$ and their complement.

Proposition 4.10. The Hole is a Seidel minor of XF_3^n , XF_4^n and $XF_2n + 1$.

Proposition 4.11. XF_5^{2n+1} is a Seidel minor of XF_6^{2n+2} .

Proposition 4.12. XF_5^{2n+1} is a Seidel minor of XF_1^{2n+3} .

Proposition 4.13. XF_5^{2n+1} is a Seidel minor of C_{2n+3} .

4.3. Main Theorem.

Definition 4.14 (Seidel Complement Stable). A graph G = (V, E) is said to be Seidel complement stable if: $\forall v \in V : G \cong G * v$

Few smalls graphs are Seidel complement stable, for instance, P_4 , C_5 , and more trivially K_n the clique on ν vertices and S_n the stable on n vertices.

Lemma 4.15. The graph XF_5^n is Seidel complement stable.

Due to lack of space the proof is postponed in the appendix p. 14.

Lemma 4.16. The Seidel stable class of the hole C_n is consituted of C_n , XF_4^{n-6} .

The previous lemma means that the only graphs that are Seidel complement equivalent to C_n are C_n and XF_4^{n-6} . Due to lack of space the proof is omitted, but in few words, it relies on the "regular" structure of XF_5^n and the lemma 4.15.

Theorem 4.17 (Main Theorem). A graph is a permutation graph if and only if it does not contain as finite graphs C_5 , C_7 and XF_6^2 and their complement and as infinite families XF_5^{2n+3} and C_{2n} , $n \ge 6$ and their complement as Seidel minor.

Sketch of Proof. This theorem relies on Gallai's theorem 4.3. If G is not a permutation graph then it contains one of the graph listed in theorem 4.3 as an induced subgraph. Thanks to the previous proposition we are able to reduce each of this induced subgraphs into a smaller set of graphs which are now forbidden Seidel minor. It remains to prove that this list is minimal. Concerning infinite family lemma 4.15 prove that it is not possible to get rid of this family since it is Seidel stable. Concerning Even Holes (since Odd holes are dismissed because they contain XF_5^{2n-1} as Seidel minors) the lemma 4.16 says that it is not possible to get rid of them. The same kind of argument holds for the finite graphs.

Theorem 4.18. If G is not a permutation graph a Seidel minor certificate can be given in O(n + m) time.

Proof. If G is not a permutation graph, Kratsch *et al.* [21] gave a linear time algorithm to output one of the graph used in theorem 4.3 as induced. Consequently, once one of this graph is found, one can use the above propositions to turn it into a graph of the list given by theorem 4.17. And it is clear that is done in linear time in the worst case.

Corollary 4.19 (Permutation graphs not WQO). The class of permutation graphs is not WQO under Seidel minor relation.

Proof. XF_5^{2n+3} constitutes an obstruction for permutation graphs. But since for even values XF_5^{2n} this graph is a permutation graph. Furthermore it is easy to check that for k and l two positives integers such that k < l. XF_5^{2k} is not an induced subgraph of XF_5^{2l} . Consequently the family XF_5^{2n} is an infinite family of finite permutation graphs. Since XF_5^n is Seidel stable by lemma 4.15, these graphs are not comparable each other with the Seidel minor relation. It is thus an infinite antichain for Seidel minor relation and consequently permutation graphs are not Wqo under Seidel minor relation.

5. Conclusion and Perspectives

We have shown that the new paradigm of Seidel minor allows to provide a nice and compact characterization of permutation graphs. In addition we provided a linear time algorithm to output a certificate that the graph is not a permutation graph.

A lot of questions remain open. Concerning the distance, *i.e.* the size of the sequence of Seidel complementation, between two Seidel complement equivalent graphs, a natural question : is there a polynomial upper bound for this distance ?

What is the status of the problem to decide if given two graphs are Seidel complement equivalent. Is that harder, easier or equivalent to the ISO problem ?

Another natural question lies on the fact that theorem 4.17 is obtained using Gallai's result on forbidden induced subgraphs. Is that possible to give a direct proof of theorem 4.17 whitout using Gallai's result.

Another direction concerns graph decomposition Oum and Seymour [27] have shown that Local Complementation preserves rank-width. Is there a graph decomposition that is preserved by Seidel complementation?

Finally it could be interesting to generalize the Seidel complement operator to directed graphs, and possibly to hypergraphs.

We hope that this Seidel minor will be relevant in the future as a tool to study graph decomposition and to provide similar characterizations, as the one presented for permutations graphs, to other graph classes.

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APPENDIX A. ADDITIONAL PROOFS

Proof of lemma 4.15. XF_5^n is a path of length n dominated by two non-adjacent vertices C and D. In addition to that, a vertex A is connected to D and 1, and a vertex B is connected to C and n + 1. This graph is represented in figure 7(f).

The degree sequence for this graph for $n \ge 1$ is 2; 2; $4 \times n$; n + 2; n + 2. Except for n = 3 the degree sequence allows to "identify" the vertices. A and B are the vertices of degree 2, C and D are the vertices of degree n + 1 and the vertices of the path [1, n + 1] are the vertices of degree 4.

Now let us formulate two easy observations. Since the graph presents of lot of symmetries, *i.e.* A is equivalent to B; C is equivalent to D. It suffices to check that the graph obtained after a Seidel complement on the following vertices will preserve the desired properties. So the set of vertices to consider is $\{A, D, 1, \ldots, \lceil n+1 \rceil\}$.

Now two easy observations: G denotes XF_5^n . $G \cong G * D$. Since D is connected to $\{A, 1, \ldots, n+1\}$. After the Seidel complement it means that C is now connected to only B and A. And it also means that B is connected to C and since it was only connected to n + 1 in the original graph B is now connected to $\{A, 1, \ldots, n\}$. So now the path is consituted of the vertices $\{A, 1, \ldots, n\}$, B and D dominate this path and C and n + 1 consitute the extremities. The function φ is given by this permutation.

$$\sigma = \begin{pmatrix} A & B & C & D & 1 & 2 & \dots & n+1 \\ 1 & D & A & C & 2 & 3 & \dots & B \end{pmatrix}$$

Let us show now that $G \cong G * A$. by definition the subgraph induced by $\{B, C, 2, ..., n + 1\}$ remains unchanged. The vertex 1 is now connected to $\{3, ..., n + 1, B\}$, and is still connected to A and D. Concerning D, it now only connected to B and C in G[N(A)]. So the bijection φ is given by the following permutation:

$$\sigma = \begin{pmatrix} A & B & C & D & 1 & 2 & \dots & n+1 \\ A & n & C & n+1 & D & B & \dots & n-1 \end{pmatrix}$$

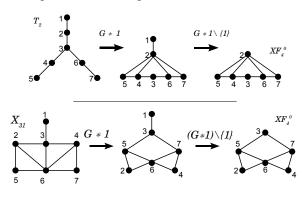
It is easy to see that $G \cong G * 1$. The path is $3, 4, \ldots, n + 1, B, D, 1, C$. The vertex A is connected to $\{3, 4, \ldots, n+1, B, D, 1\}$ and the vertex 2 is connected to $\{4, \ldots, n+1, B, D, 1, C\}$. Let us consider the case for the vertex 2. Actually $G \cong G * 2$ The path is $\{4, 5, \ldots, n + 1, B, D, 2, C, A\}$. and the vertex 1 is connected to $\{4, 5, \ldots, n+1, B, D, 2, C\}$. And the vertex 3 is connected to $\{5, \ldots, n+1, B, D, 2, C, A\}$.

Concerning the vertices on the path let us consider the case of their vertex k such that $k \in [3, n-1]$. It is clear that the graph $G[\{C, D, k-1, k, k+1\}]$ remains unchanged as for the graph $G[V \setminus \{C, D, k-1, k, k+1\}]$. Now let us briefly check that the degree sequence is relevant with the desired goal. The vertex C is now connected to A, k-1, k and k+1. So it is 4. A similar thing happens for D. It is now connected to B, k-1, k and k+1. Concerning k-1 and k+1. k-1 is connected to every vertex except k-2 and k+1 so its degree is n+2. And k+1 is connected to every vertex except k-1 and k+2. Concerning A and B their degree are now equal to 4 (because of k-1 and k+1). And concerning the vertices k-2 and k+2 their degree equal to 2 because they are no longer connected to C, D, $k \pm 1$

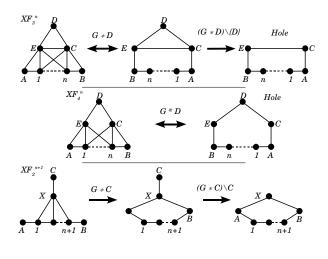
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but are now connected to $k \pm 1$ (*i.e.* k - 1 and k + 1 swap roles). Now the extremities of the path are k - 2 and k + 2. The path is of the form: $k - 2, \ldots, 1, A, C, k, D, B, n + 1, n, \ldots, k + 2$ Consequently the graph XF_5^n is Seidel complement stable.

Proof of proposition 4.8. $XF_4^0 <_S T_2$ and $XF_4^0 <_S X_{31}$



Proof of proposition 4.10. The Hole is a Seidel minor of XF_3^n , XF_4^n and $XF_2n + 1$.



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