# Approximation Algorithms for Non-Single-minded Profit-Maximization Problems with Limited Supply\*

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#### Abstract

We consider *profit-maximization* problems for *combinatorial auctions* with *non-single minded valuation functions* and *limited supply*. There are *n* customers and *m* items, each of which is available is in some limited supply or capacity. Each customer *j* has a value  $v_j(S)$  for each subset *S* of items specifying the maximum amount she is willing to pay for that set (with  $v_j(\emptyset) = 0$ ). A feasible solution to the profitmaximization problem consists of item prices and an allocation  $(S_1, \ldots, S_n)$  of items to customers such that (i) the price of the set  $S_j$  assigned to *j* is at most  $v_j(S_j)$ , and (ii) the number of customers who are allotted an item is at most its capacity. The goal is find a feasible solution that maximizes the total profit earned by selling items to customers.

We obtain fairly general results that relate the approximability of the profit-maximization problem to that of the corresponding *social-welfare-maximization* (SWM) problem, which is the problem of finding an allocation  $(S_1, \ldots, S_n)$  satisfying the capacity constraints that has maximum total value  $\sum_j v_j(S_j)$ . For *subadditive valuations* (and hence *submodular, XOS valuations*), we obtain a solution with profit OPT<sub>SWM</sub>/ $O(\log c_{\max})$ , where OPT<sub>SWM</sub> is the optimum social welfare and  $c_{\max}$  is the maximum itemsupply; thus, this yields an  $O(\log c_{\max})$ -approximation for the profit-maximization problem. Furthermore, given *any* class of valuation functions, if the SWM problem for this valuation class has an LP-relaxation (of a certain form) and an algorithm "verifying" an *integrality gap* of  $\alpha$  for this LP, then we obtain a solution with profit OPT<sub>SWM</sub>/ $O(\alpha \log c_{\max})$ , thus obtaining an  $O(\alpha \log c_{\max})$ -approximation.

The latter result immediately yields an  $O(\sqrt{m} \log c_{\max})$ -approximation for the profit maximization problem for combinatorial auctions with *arbitrary valuations*. As another application of this result, we consider the non-single-minded tollbooth problem on trees (where items are edges of a tree, and customers desire paths of the tree). We devise an O(1)-approximation algorithm for the corresponding SWM problem satisfying the desired integrality-gap requirement, and thereby obtain an  $O(\log c_{\max})$ approximation for the non-single-minded tollbooth problem on trees. For the special case, when the tree is a path, we also obtain an incomparable  $O(\log m)$ -approximation (via a different approach) for subadditive valuations, and arbitrary valuations with unlimited supply. Our approach for the latter problem also gives an  $\frac{e}{e-1}$ -approximation algorithm for the multi-product pricing problem in the Max-Buy model, with limited supply, improving on the previously known approximation factor of 2.

# **1** Introduction

Profit (or revenue) maximization is a classic and fundamental economic goal, and the design of computationallyefficient item-pricing schemes for various profit-maximization problems has received much recent attention [1, 20, 4, 2, 5, 3]. We study the algorithmic problem of *item-pricing for profit-maximization* for *general* 

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(multi unit) combinatorial auctions (CAs) with limited supply. There are n customers and m items. Each item is available is in some limited supply or capacity, and each customer j has a value  $v_i(S)$  for each subset S of items specifying the maximum amount she is willing to pay for that set (with  $v_i(\emptyset) = 0$ ). Given a pricing of the items, a *feasible allocation* is an assignment of a (possibly empty) subset  $S_j$  to each customer j satisfying (i) the *budget constraints*, which require that the price of  $S_i$  (i.e., the total price of the items in  $S_i$ ) is at most  $v_i(S_i)$ , and (ii) the *capacity constraints*, which stipulate that the number of customers who are allocated an item be at most the supply of that item. The objective is to determine item prices that maximize the total profit or revenue earned by selling items to customers. Guruswami et al. [20] introduced the envyfree version of the problem, where there is the additional constraint that the set assigned to a customer must maximize her utility (defined as value-price). Item pricing has an appealing simplicity and enforces a basic notion of fairness wherein the seller does not discriminate between customers who get the same item(s). Our focus on item pricing is in keeping with the vast majority of work on algorithms for profit-maximization (e.g., the above references; in fact, with unlimited supply and unit-demand valuations, our problem reduces to the Max-Buy model in [1]). Various current trading practices are described by item pricing, and thus it becomes pertinent to understand what guarantees are obtainable via such schemes. Profit-maximization problems are typically NP-hard, so we will be interested in designing approximation algorithms for these problems. Throughout, a  $\rho$ -approximation algorithm for a maximization problem, where  $\rho \geq 1$ , denotes a polytime algorithm that returns a solution of value at least (optimum value)/ $\rho$ .

The framework of combinatorial auctions is an extremely rich framework that encapsulates a variety of applications. In fact, recognizing the generality of the envy-free profit-maximization problem for CAs, Guruswami et al. proceeded to study various more-tractable special cases of the problem. In particular, they introduced the following two structured problems in the single-minded (SM) setting, where each customer desires a single fixed set: (a) the *tollbooth problem* where the items are edges of a graph and the customersets correspond to paths in this graph, which can be interpreted as the problem of pricing transportation links or network connections. (b) a further special case called the *highway problem* where the graph is a path, which can also be motivated from a scheduling perspective (the path corresponds to a time-horizon). The non-SM versions of even such structured problems can be used to capture various interesting scenarios. For instance, in a computer network, users may consider different possibilities for connecting to the network, and the price they are willing to pay may depend on where they connect. The goal is to determine how to price the bandwidth along the network links so as to maximize the profit obtained. For an application of the non-single minded highway problem, consider customers who are interested in executing their jobs on a machine(s) (or using a service, such as a hotel room). A customer is willing to pay for this service, but the amount paid depends on when her job is scheduled. We want to price the time units and schedule the jobs so as to maximize the profit.

**Our results.** We obtain fairly general *polytime approximation guarantees* for profit-maximization problems involving *combinatorial auctions* with *limited supply* and *non-single-minded* valuations. We obtain results for both (a) certain structured valuation classes, namely *subadditive valuations* (where  $v(A) + v(B) \ge$  $v(A \cup B)$  for any two sets A, B) and hence, *submodular* valuations, which have been intensely studied recently [14, 16, 3, 28, 10]; and (b) *arbitrary valuations*. Our results relate the approximability of the profitmaximization problem to that of the corresponding *social-welfare-maximization* (SWM) problem, which is the problem of finding an allocation  $(S_1, \ldots, S_n)$  satisfying the capacity constraints that has maximum total value  $\sum_j v_j(S_j)$ . Our main theorem, stated informally below and proved in Section 3, shows that any LP-based approximation algorithm that provides an integrality-gap bound for the SWM problem with a given class of valuations, can be leveraged to obtain a corresponding approximation guarantee for the profitmaximization problem with that class of valuations. Let  $c_{\max} \leq n$  denote the maximum item supply, and OPT<sub>SWM</sub> denote the optimum value of the SWM problem, which is clearly an upper bound on the maximum profit achievable. **Theorem 1.1 (Informal statement).** (*i*) For the class of subadditive (and hence submodular) valuations, one can obtain a solution with profit  $OPT_{SWM}/O(\log c_{max})$ .

(ii) Given any class of valuations for which the corresponding SWM problem admits a packing-type LP relaxation with an integrality gap of  $\alpha$  as "verified" by an  $\alpha$ -approximation algorithm, one can obtain a solution with profit OPT<sub>SWM</sub>/ $O(\alpha \log c_{max})$ .

(Part (ii) above does not imply part (i), because for part (ii) we require an integrality-gap guarantee which, roughly speaking, means that we require an algorithm that returns a "good" solution for *every* profile of n valuations.)

A key notable aspect of our theorem is its versatility. One can simply "plug in" various known (or easily derivable) results about the SWM problem to obtain approximation algorithms for various limitedsupply profit-maximization problems. For example, as *corollaries* of part (ii) of our theorem, we obtain an  $O(\sqrt{m} \log c_{\max})$ -approximation for profit-maximization for combinatorial auctions with *arbitrary valuations*, and an  $O(\log c_{\max})$ -approximation for the *non-single-minded tollbooth problem* on trees (see Section 3.1). The first result follows from the various known  $O(\sqrt{m})$ -approximation algorithms for the SWM problem for CAs with arbitrary valuations that also bound the integrality gap [26, 23]. For the second result, we devise a suitable O(1)-approximation for the SWM problem corresponding to non-single-minded tollbooth on trees, by adapting the randomized-rounding approach of Chakrabarty et al. [9].

Notice that with bundle-pricing, which is often used in the context of mechanism design for CAs, the profit-maximization problem becomes equivalent to the SWM problem. Thus, our results provide worstcase bounds on how item-pricing (which may be viewed as a fairness constraint on the seller) diminishes the revenue of the seller versus bundle-pricing. It is also worth remarking that our algorithms for an arbitrary valuation class (i.e., part (ii) above) can be modified in a simple way to return prices and an allocation  $(S_1, \ldots, S_n)$  with the following  $\epsilon$ -"one-sided envy-freeness" property while diminishing the profit by a  $(1 - \epsilon)$ -factor: for every non-empty  $S_j$ , the utility that j obtains from  $S_j$  is at least  $\epsilon$  times the maximum utility j may obtain from any set (see Remark 3.7).

The only previous guarantees for limited-supply CAs with a general valuation-class are those obtained via a reduction in [2], showing that an  $\alpha$ -approximation for the SWM problem and an algorithm for the unlimited-supply SM problem that returns profit at least  $OPT_{SWM}/\beta$  yield an  $\alpha\beta$ -approximation. A simple "grouping-by-density" approach gives  $\beta = O(\log m + \log n)$ ; using the best known bound on  $\beta$  [5] yields an  $O(\alpha(\log m + \log c_{\max}))$  guarantee, which is significantly weaker than our guarantees. (E.g., we obtain an  $O(\alpha)$ -approximation for constant  $c_{\max}$ .) The  $O(\log c_{\max})$ -factor we incur is unavoidable if one compares the profit against the optimal social welfare: a well-known example with one item,  $n = c_{\max}$  players shows a gap of  $H_{c_{\max}} := 1 + \frac{1}{2} + \cdots + \frac{1}{c_{\max}}$  between the optima of the SWM- and profit-maximization problems. Almost all results for profit-maximization for CAs with non-SM valuations also compare against the optimum social welfare, so they also incur this factor. Also, it is easy to see that with  $c_{\max} = 1$ , the profit-maximization problem reduces to the SWM problem. Thus, we obtain an  $m^{\frac{1}{2}}$ -, or n-, inapproximability for CAs with even SM valuations (see, e.g., [19]), and *APX*-hardness for CAs with subadditive, submodular valuations, and the tollbooth problem on trees.

The proof of Theorem 1.1 is based on considering a natural LP-relaxation (P) for the SWM problem and its dual. A crucial observation is that an optimal primal solution combined with the optimal values of the dual variables corresponding to the primal supply constraints can be seen as furnishing a "feasible" solution with a *fractional* allocation to (even) the (envy-free) profit-maximization problem. [12] utilized this observation to design an approximation algorithm for the *single-minded envy-free* profit-maximization problem. But even with unit capacities and *one* non-single-minded customer, there is an  $\Omega(m)$ -factor gap between the optimum (integer or fractional) social-welfare and the optimum profit achievable by an envyfree pricing (see, e.g., [3]). Our approach is similar to the one in [12], but as suggested by the above fact, we need new ingredients to exploit the greater flexibility afforded by the profit-maximization problem (vs. the envy-free problem) and turn the above observation into an approximation algorithm *even for non-single-minded valuations*. As in [12], we argue that there must be an optimal dual solution with suitable, possibly lowered, item-capacities yielding profit (with the fractional allocation) comparable to OPT<sub>SWM</sub>. A suitable rounding of the optimal primal solution with these capacities then yields a good allocation, which combined with the prices obtained yields the desired approximation bounds. Here, for part (ii) of Theorem 1.1, we leverage a decomposition technique of [8]. Thus, our work shows that (in contrast to the envy-free setting) for profit-maximization problems, one can obtain a great deal of mileage from the LP-relaxation of the SWM problem and exploit LP-based techniques to obtain guarantees even for various non-single-minded valuation classes.

In Section 4, we consider an alternate approach for the non-SM highway problem that (does not use  $OPT_{SWM}$  as an upper bound and) achieves an (incomparable)  $O(\log m)$ -approximation factor. We decompose the instance via an exponential-size *configuration LP*, which is solved approximately using the ellipsoid method and rounded via randomized rounding. Here, we use LP duality to handle dependencies arising from the non-SM setting.

**Theorem 1.2.** There is an  $O(\log m)$ -approximation algorithm for the non-single-minded highway problem with (i) subadditive valuations with limited supply; and (ii) arbitrary valuations with unlimited supply.

It is worth noting that the non-SM highway problem with subadditive valuations can be used to capture some *multi-product pricing* problems in the so-called *Max-Buy* model (a customer buys the most expensive product she can afford), with or without a *price ladder*, considered by Aggarwal et al. [1]. (Indeed, the case without a ladder (resp., with a ladder) can be modeled by a set of disjoint intervals (resp., a Laminar set of intervals sharing the right end-point), where customers' valuations are defined on each of these intervals). In fact, our algorithm in Theorem 1.2 is based on combining ideas from the PTAS for the version with a price ladder in [1], and the  $\frac{e}{e-1}$ -approximation algorithm for the one without a ladder. We observe that our method gives the following result<sup>1</sup> for the multi-product pricing problem, which improves on the 2-approximation result in [6].

**Corollary 1.3.** There is an  $\frac{e}{e-1}$ -approximation algorithm for the multi-product pricing problem in the Max-Buy model, with limited supply.

**Related work.** There has been a great deal of recent work on approximation algorithms for various kinds of pricing problems; see, e.g., [1, 3, 12, 10, 20, 6, 18, 15], and the references therein. However, to our knowledge, the only approximation results for profit-maximization for non-single-minded CAs with (general) limited supply (i.e., not necessarily unit- or unlimited- supply) are: (1) those gleaned from the reduction in [2] coupled with the guarantee in [5]; and (2) the 2-approximation algorithm of [6] for *unit-demand* valuations (where each customer wants at most one item). (For this very structured subclass of submodular valuations, this 2-approximation result is better than the guarantee we obtain using Theorem 1.1; hopwever, our method .) We briefly survey the work in three special cases that have been studied: unit supply, unlimited supply, and single-minded (SM) valuations.

As remarked earlier, with unit capacities, the profit-maximization problem reduces to the SWM problem, which is a relatively well-studied problem. The approximation guarantees known for the SWM problem for CAs with unit capacities are (i) 2 for subadditive valuations [16]; (ii)  $\frac{e}{e-1}$  [14, 28] for submodular valuations; and (iii)  $\Theta(\sqrt{m})$  for arbitrary valuations [26, 23]. Recently, Balcan et al. [3] and Chakraborty et al. [10] considered the unit-capacity problem with subadditive valuations in the *online* setting where customers arrive online and select their *utility-maximizing* set from the unallotted items given the current prices. The guarantees they obtain in this constrained setting are naturally worse than the guarantees known in the

<sup>&</sup>lt;sup>1</sup>This result was recently rediscovered in [17].

offline setting. The unlimited-supply setting with arbitrary valuations has been less studied; [3] gave an  $O(\log m + \log n)$ -approximation algorithm by extending an algorithm in [20] for SM valuations.

The single-minded profit-maximization problem has received much attention. The work that is most relevant to ours is Cheung and Swamy [12], who obtain an approximation guarantee of the same flavor as in part (ii) of Theorem 1.1. They obtain an envy-free solution of profit  $OPT_{SWM}/O(\alpha \log c_{max})$  using an LP-based  $\alpha$ -approximation for the SWM problem (in the SM setting, this is equivalent to the integrality-gap requirement we have); we make use of portions of their analysis in proving our results. For the unlimited-supply SM problem, [20] gave an  $O(\log m + \log n)$ -approximation guarantee, which was improved by Briest and Krysta [5]. A variety of approximation results based on dynamic programming have been obtained [20, 21, 4, 5, 19] that yield exact algorithms or approximation schemes for various restricted instances, or pseudopolynomial or quasipolynomial time algorithms. On the hardness side, a reduction from the set-packing problem shows that achieving an approximation factor better than  $m^{\frac{1}{2}}$ , or n, is *NP*-hard even when  $c_{max} = 1$ , even for the (SM) tollbooth problem on grid graphs [19], and [13, 5, 20, 11] prove various hardness results for unlimited-supply instances.

Finally, we note that the singe-minded version of the highway problem admits a PTAS in the unlimited supply case [18] and a quasi-PTAS for the limited supply case [15] (with an  $\epsilon$ -approximate notion of envy-freeness), while the non-single minded version is APX-hard (since it includes the multi-product pricing problem which was proved to be APX-hard in the Max-Buying setting in [1]).

# **2** Problem definition and preliminaries

**Profit-maximization problems for combinatorial auctions.** The general setup of *profit-maximization* problems for (multi unit) *combinatorial auctions* (CAs) is as follows. There are *n* customers and *m* items. Let  $[n] := \{1, \ldots, n\}$  and  $[m] := \{1, \ldots, m\}$ . Each item *e* is available in some limited supply or capacity  $c_e$ . Each customer *j* has a *valuation function*  $v_j : 2^{[m]} \mapsto \mathbb{R}_+$ , where  $v_j(S)$  specifies the maximum amount that customer *j* is willing to pay for the set *S*; equivalently this is *j*'s value for receiving the set *S* of items. We assume that  $v_j(\emptyset) = 0$ ; we often assume for convenience that  $v_j(S) \leq v_j(T)$  for  $S \subseteq T$ , but this monotonicity requirement is not crucial for our results. The objective is to find non-negative prices  $p_e \geq 0$  for the items, and an allocation  $(S_1, \ldots, S_n)$  of items to customers (where  $S_j$  could be empty) so as to maximize the total profit  $\sum_{j \in [n]} \sum_{e \in S_j} p_e = \sum_{e \in [m]} p_e |\{j : e \in S_j\}|$  while satisfying the following two constraints.

- Budget constraints. Each customer j can afford to buy her assigned set:  $p(S_j) := \sum_{e \in S_j} p_e \le v_j(S_j)$ .
- Capacity constraints. Each element e is assigned to at most  $c_e$  customers:  $|\{j \in [n] : e \in S_j\}| \le c_e$ .

Since the valuations may be arbitrary set functions, an explicit description of the input may require exponential (in m) space. Hence, we assume that the valuations are specified via an oracle. As is standard in the literature on combinatorial auctions and profit-maximization problems (see, e.g., [24, 16, 3, 10]), we assume that a valuation v is specified by a *demand oracle*, which means that given item prices  $\{p_e\}$ , the demand-oracle returns a set S that maximizes the *utility* v(S) - p(S). We use  $c_{\max} := \max_e c_e$  to denote the maximum item supply.

An LP relaxation. We consider a natural linear programming (LP) relaxation (P) of the SWM problem for combinatorial auctions, and its dual (D). Throughout, we use j to index customers, e to index items, and S to index sets of items. We use the terms supply and capacity, and customer and player interchangeably.

$$\max \qquad \sum_{j,S} v_j(S) x_{j,S} \tag{P}$$

s.t. 
$$\sum_{S} x_{j,S} \le 1 \quad \forall j$$
 (1)

$$\sum_{j} \sum_{S:e \in S}^{S} x_{j,S} \le c_e \quad \forall e \tag{2}$$
$$x_{j,S} \ge 0 \quad \forall j,S$$

$$\begin{array}{ll} \min & \sum_{e} c_{e} y_{e} + \sum_{j} z_{j} \\ \text{s.t.} & \sum_{e \in S} y_{e} + z_{j} \geq v_{j}(S) \quad \forall j, S \\ & y_{e}, z_{j} \geq 0 \qquad \forall e, j. \end{array}$$

(D)

In the primal LP, we have a variable  $x_{j,S}$  for each customer j and set S that indicates if j receives set S, and we relax the integrality constraints on these variables to obtain the LP relaxation. The dual (D) has variables  $z_j$  and  $y_e$  for each customer j and element e respectively, which correspond to the primal constraints (1) and (2) respectively. Although (D) has an exponential number of constraints, it can be solved efficiently given demand oracles for the valuations as these oracles yield the desired separation oracle for (D). This in turn implies that (P) can be solved efficiently. We say that an algorithm  $\mathcal{A}$  for the SWM problem is an *LP-based*  $\alpha$ -approximation algorithm for a class  $\mathcal{V}$  of valuations if for every instance involving valuation functions  $(v_1, \ldots, v_n)$ , where each  $v_j \in \mathcal{V}$ ,  $\mathcal{A}$  returns an integer solution of value at least LP-optimum/ $\alpha$ . For example, the algorithm in [16] is an LP-based 2-approximation algorithm for the class of subadditive valuations.

**Definition 2.1.** We say that an algorithm  $\mathcal{A}$  for the SWM problem "verifies" an integrality gap of (at most)  $\alpha$  for an LP-relaxation of the SWM problem (e.g., (P)), if for every profile of (monotonic) valuation functions  $(v_1, \ldots, v_n)$ ,  $\mathcal{A}$  returns an integer solution of value at least (LP-optimum)/ $\alpha$ .

As emphasized above, an integrality-gap-verifying algorithm above must "work" for every valuationprofile. (Note that for the SWM problem, one can always assume that the valuation is monotonic, since we can always move from a set to its subset (as items may be left unallotted).) In particular, an LP-based  $\alpha$ -approximation algorithm for a given *structured class* of valuations (e.g., submodular or subadditive valuations) does not verify the integrality gap for the LP-relaxation. This is the precise reason why our guarantee for subadditive valuations (part (i) of Theorem 1.1) does not follow from part (ii) of Theorem 1.1. In certain cases however, one may be able to encapsulate the combinatorial structure of the SWM problem with a structured valuation class by formulating a stronger LP-relaxation for the SWM problem, and thereby prove that an approximation algorithm for the structured valuation class is in fact an integrality-gap-verifying approximation algorithm with respect to this stronger LP-relaxation. For example, in Section 3.1 we consider the setting where items are edges of a tree and customers desire paths of the tree. This leads to the structured valuation where, for a set of edges T,  $v(T) = \max\{v(P) : P \text{ is a path in } T\}$  (with  $v(P) \ge 0$  being the value for path P). We design an O(1)-approximation algorithm for such valuations, and formulate a stronger LP for the corresponding SWM problem for which our algorithm verifies a constant integrality gap. (For the SWM problem with subadditive valuations, it is not known how to exploit the underlying structure and formulate an efficiently-solvable LP-relaxation with O(1) integrality gap.)

For a given instance  $\mathcal{I} = (m, n, \{v_j\}_{j \in [n]}, \{c(e)\}_{e \in [m]})$ , our algorithms will consider different capacity vectors  $k \leq c$ . We use  $(P_k)$  and  $(D_k)$  to denote respectively (P) and (D) with capacity-vector  $k = (k_e)$ , and OPT(k) to denote their common optimal value. Let OPT := OPT(c) denote the optimum value of (P) (and (D)) with the original capacities. We will utilize the following facts that follow from complementary slackness, and a rounding result that follows from the work of Carr and Vempala [8], and was made explicit in [24].

**Claim 2.2.** Let  $k = (k_e)$  be any capacity-vector, and let  $x^*$  and  $(y^*, z^*)$  be optimal solutions to  $(P_k)$  and  $(D_k)$  respectively.

(i) If  $x_{j,S}^* > 0$ , then  $\sum_{e \in S} y_e^* \le v_j(S)$ ; (ii) If  $x_{j,S}^* > 0$ , and  $v_j$  is subadditive, then  $\sum_{e \in T} y_e^* \le v_j(T)$  for any  $T \subseteq S$ ; (iii) If  $y_e^* > 0$ , then  $\sum_{j,S:e \in S} x_{j,S}^* = k_e$ .

*Proof.* Parts (i) and (iii) follow directly from the complementary slackness (CS) conditions: part (i) follows from the CS condition for  $x_{j,S}^*$ , since  $z_i^* \ge 0$ ; part (iii) uses the CS condition for  $y_e^*$  and the corresponding primal constraint (2). For part (ii), again, by the CS conditions we have  $\sum_{e \in S} y_e^* + z_j^* = v_j(S)$ . Also, dual feasibility implies that  $\sum_{e \in S \setminus T} y_e^* + z_j^* \ge v_j(S \setminus T)$ . Subtracting this from the first equation and using subadditivity yields  $\sum_{e \in T} y_e^* \le v_j(S) - v_j(S \setminus T) \le v_j(T)$ .

*Remark* 2.3. As mentioned above, we will sometimes consider a different LP-relaxation when considering the SWM problem with a structured class of valuations. Roughly speaking, the only properties we require of this LP are that it should: (a) include a constraint similar to (2) that encodes the supply constraints; and (b) be a *packing LP*, i.e., have the form  $Ax \le b$ ,  $x \ge 0$  where A is a nonnegative matrix. Given this, parts (i) and (iii) of Claim 2.2 continue to hold with  $y_e$  denoting (as before) the dual variable corresponding to the supply constraint for item e, since the dual is then a covering LP.

**Lemma 2.4** ([8, 24]). Given a fractional solution x to the LP-relaxation of an SWM problem that is a packing LP (e.g.,  $(P_k)$ ), and a polytime integrality-gap-verifying  $\alpha$ -approximation algorithm  $\mathcal{A}$  for this LP, one can express  $\frac{x}{\alpha}$  as a convex combination of integer solutions to the LP in polytime. In particular, one can round x to a random integer solution  $\hat{x}$  satisfying the following "rounding property":  $\frac{x_{j,S}}{\alpha} \leq \Pr[\hat{x}_{j,S} = 1] \leq x_{j,S} \ \forall j, S$ .

# **3** The main algorithm and its applications

Claim 2.2 leads to the simple, but important observation that if  $k \le c$  and the optimal primal solution  $x^*$  is integral, then by using  $\{y_e^*\}$  as the prices, one obtains a feasible solution to the profit-maximization problem with profit  $\sum_e k_e y_e^*$ . There are two main obstacles encountered in leveraging this observation and turning it into an approximation algorithm. First,  $(P_k)$  will of course not in general have an integral optimal solution. Second, it is not clear what capacity-vector  $k \le c$  to use: for instance,  $\sum_e c_e y_e^*$  could be much smaller than OPT (it is easy to construct such examples), and in general,  $\sum_e k_e y_e^*$  could be quite small for a given capacity-vector  $k \le c$ . We overcome these difficulties by taking an approach similar to the one in [12].

We tackle the second difficulty by utilizing a key lemma proved by Cheung and Swamy [12], which is stated in a slightly more general form in Lemma 3.2 so that it can be readily applied to various profitmaximization problems. This lemma implies that one can efficiently compute a capacity-vector  $k \le c$  and an optimal dual solution  $(y^*, z^*)$  to  $(D_k)$  such that  $\sum_e k_e y_e^*$  is  $(OPT - OPT(1))/O(\log c_{\max})$ , where 1 denotes the all-one's vector (Corollary 3.4). To handle the first difficulty, notice that part (i) of Claim 2.2 implies that one can still use  $\{y_e^*\}$  as the prices, provided we obtain an allocation (i.e., integer solution)  $\hat{x}$ that only assigns a set S to customer j (i.e.,  $\hat{x}_{j,S} = 1$ ) if  $x_{j,S}^* > 0$ . (In contrast, in the envy-free setting, if we use  $\{y_e^*\}$  as the prices then *every* customer j with  $z_j^* > 0$ , and hence  $\sum_S x_{j,S}^* = 1$ , must be assigned a set S with  $x_{j,S}^* > 0$ ; this may be impossible with non-single-minded valuations, whereas this is easy to accomplish with single-minded valuations (as there is only one set per customer).) Furthermore, for subadditive valuations, part (ii) of Claim 2.2 shows that it suffices to obtain an allocation where  $\hat{x}_{j,T} = 1$ implies that there is some set  $S \supseteq T$  with  $x_{j,S}^* > 0$ . This is precisely what our algorithms do. We show that one can round  $x^*$  into an integer solution  $\hat{x}$  satisfying the above *structural* properties, and in addition ensure that the profit obtained,  $\sum_{j,T} \hat{x}_{j,T} (\sum_{e \in T} y_e^*)$ , is "close" to  $\sum_e k_e y_e^*$  (Lemma 3.5). So if  $\sum_e k_e y_e^*$ is OPT/ $O(\log c_{\max})$  then applying this rounding procedure to the optimal primal solution to (P\_k) yields a "good" solution. On the other hand, Corollary 3.4 implies that if this is not the case, then OPT(1) must be large compared to OPT, and then we observe that an  $\alpha$ -approximation to the SWM problem trivially yields a solution with profit OPT(1)/ $\alpha$  (Lemma 3.1). (As mentioned earlier, in the envy-free setting and unit capacities, there can be an  $\Omega(m)$ -gap between the optimum profit and the optimum social welfare.) Thus, in either case we obtain the desired approximation.

The algorithm is described precisely in Algorithm 1. If we use an LP-relaxation different from (P) for the SWM problem with a given valuation class that satisfies the properties stated in Remark 2.3, then the only (obvious) change to Algorithm 1 is that we now use this LP and its dual (with the appropriate capacity-vector) instead of (P) and (D) above.

#### Algorithm 1 Non-single-minded profit-maximization

**Input:** a profit-maximization instance  $\mathcal{I} = (m, n, \{v_j\}, \{c_e\})$  with a demand oracle for each valuation  $v_j$ 1. Define  $k^1, k^2, \ldots, k^{\ell}$  as the following capacity-vectors. Let  $k_e^1 = 1 \quad \forall e$ . For j > 1, let  $k_e^j = \min\{\lceil (1+\epsilon)k_e^{j-1}\rceil, c_e\}$ ; let  $\ell$  be the smallest index such that  $k^{\ell} = c$ .

- 2. For each vector  $k = k^j$ ,  $j = 1, ..., \ell$ , compute an optimal solution  $(y^{(k)}, z^{(k)})$  to  $(D_k)$ . Select  $u \in$  $\{k^1, \ldots, k^\ell\}$  that maximizes  $\sum_e u_e y_e^{(u)}$ .
- 3. Compute an optimal solution  $\overline{x^{(u)}}$  to  $(\mathbf{P}_u)$ . Use Round $(u, x^{(u)})$  to convert  $x^{(u)}$  to a feasible allocation.
- 4. Use an LP-based  $\alpha$ -approximation algorithm for the SWM problem (with the given valuation class) to compute an  $\alpha$ -approximate solution to the SWM problem with unit capacities, and a pricing scheme for this allocation that yields profit equal to the social-welfare value of the allocation.
- 5. Return the better of the following two solutions: (1) allocation computed in step 3 with  $\{y_e^{(u)}\}\$  as the prices; (2) allocation and pricing scheme computed in step 4.

### Round( $\mu = (\mu_e), x^*$ )

**Subadditive valuations:** First, independently for each player j, assign j at most one set S by choosing set S with probability  $x_{j,S}^*$ . If an item e gets allotted to more than  $\mu_e$  customers this way, then arbitrarily select  $\mu_e$  customers from among these customers and assign e to these customers. Given item prices, this algorithm can be derandomized via the method of conditional expectations.

**General valuation class:** Given an integrality-gap-verifying  $\alpha$ -approximation algorithm (for (P<sub>µ</sub>)), use Lemma 2.4 to decompose  $\frac{x^*}{\alpha}$  into a convex combination  $\sum_{r=1}^{\ell} \lambda_r \hat{x}^r$  of integer solutions to  $(\mathbf{P}_{\mu})$ . (Here  $\sum_r \lambda_r = 1$  and  $\lambda_r \ge 0$  for each r.) Return  $\hat{x}^{(r)}$  with probability  $\lambda_r$ . Given item prices, this algorithm can be derandomized by choosing the solution in  $\{\hat{x}^{(1)}, \dots, \hat{x}^{(r)}\}\$  achieving maximum profit.

**Analysis.** The analysis of Algorithm 1 for both subadditive valuations and a general valuation class proceeds very similarly with the only point of difference being in the analysis of the rounding procedure (Lemma 3.5). First, observe that if we have an allocation  $(S_1, \ldots, S_n)$  that is feasible with unit capacities, then since the sets  $S_j$  are disjoint we can charge each customer her valuation for the assigned set by pricing one of her items at this value, and hence, obtain profit equal to the social-welfare value  $\sum_{i} v_j(S_j)$ of the allocation.

**Lemma 3.1.** Given an LP-based  $\alpha$ -approximation algorithm for the SWM problem with a given valuation class, one can compute a solution that achieves profit at least  $OPT(1)/\alpha$ .

**Lemma 3.2** ([12] paraphrased). Let  $(C_k)$  denote the LP: min  $k^T y + b^T z$  s.t.  $(y, z) \in \mathcal{P} \subseteq \mathbb{R}^{m+n}_+$ , where  $k, y \in \mathbb{R}^m_+, b, z \in \mathbb{R}^n_+, \mathcal{P} \neq \emptyset$ . Let  $(y^{(k)}, z^{(k)})$  be an optimal solution to  $(C_k)$  that maximizes  $k^T y$  among all optimal solutions, and opt(k) denote the optimal value. Let  $k^1, \ldots, k^\ell$ , and u be as defined in steps 1 and 2 respectively of Algorithm 1. Then,  $\sum_{e} u_e y_e^{(u)} \ge (\operatorname{opt}(c) - \operatorname{opt}(1))/(2(1+\epsilon)H_{c_{\max}}).$ 

*Proof.* We mimic the proof in [12]. First, note that opt(k) is well-defined for all  $k \ge 0$ .

For j > 1, define  $d^j = k^j - k^{j-1}$ . Note that  $0 \le d_e^j \le k_e^j$  for all e. Let  $e^*$  be an item with  $c_{e^*} = c_{\max}$ .

**Claim 3.3.** For any j > 1, we have  $d_{e^*}^j / k_{e^*}^j = \max_e (d_e^j / k_e^j)$  and  $d_{e^*}^j / k_{e^*}^{j-1} = \max_e (d_e^j / k_e^{j-1})$ .

*Proof.* It is easy to argue the following by induction on j: (i) if  $k_e^{j-1} < c_e$  and  $k_{e'}^{j-1} < c_{e'}$ , then  $k_e^{j-1} = k_{e'}^{j-1}$ ; and (ii) if  $c_e \leq c_{e'}$ , then  $k_e^j \leq k_{e'}^j$  and  $d_e^j \leq d_{e'}^j$ . It is clear that  $d_e^j > 0$  iff  $k_e^{j-1} < c_e$ , and that  $k_{e*}^{j-1} < c_{e*}$  for all j > 1. Combining these facts, for any e with  $d_e^j > 0$ , we have  $k_e^{j-1} < c_e$  and so  $k_e^{j-1} = k_{e*}^{j-1}$ , and since  $c_e \leq c_{e*}$ , we have  $k_e^j \leq k_{e*}^j$  and  $d_e^j \leq d_{e*}^j$ . Thus,  $\frac{d_{e*}^j}{k_{e*}^j} \geq \frac{d_e^j}{k_e^j}$  and  $\frac{d_{e*}^j}{k_{e*}^{j-1}} \geq \frac{d_e^j}{k_e^{j-1}}$ .

Let  $P = \sum_{e} u_e y_e^{(u)} = \max_{k=k^1,\dots,k^\ell} \sum_{e} k_e y_e^{(k)}$ . Then, for every j > 1, we have

$$P \cdot \frac{d_{e^*}^j}{k_{e^*}^{j-1}} \ge \operatorname{opt}(k^j) - \operatorname{opt}(k^{j-1}).$$
(3)

This follows because, considering  $(y, z) = (y^{(k^{j-1})}, z^{(k^{j-1})})$ , which is a feasible solution for  $(C_{k^{j-1}})$ , and an optimal solution for  $(C_{k^{j-1}})$ , the RHS is at most  $\sum_e d_e^j y_e \leq \max_e \frac{d_e^j}{k_e^{j-1}} \cdot \sum_e k_e^{j-1} y_e \leq \frac{d_{e^*}^j}{k_{e^*}^{j-1}}P$ , where the last inequality follows again from Claim 3.3 and since  $\sum_e k_e^{j-1} y_e \leq P$ . Since  $k_{e^*}^j \leq 2(1+\epsilon)k_{e^*}^{j-1}$ , we can upper bound the coefficient of P in the above inequality by  $2(1+\epsilon)k_e^{j-1}$ .

Since  $k_{e^*}^j \leq 2(1+\epsilon)k_{e^*}^{j-1}$ , we can upper bound the coefficient of P in the above inequality by  $2(1+\epsilon)\sum_{t=k_{e^*}^{j-1}+1}^{k_{e^*}^j} 1/t$ . Thus, adding (3) for all j > 1 gives  $P \cdot 2(1+\epsilon)(H_{c_{\max}}-1) \geq \operatorname{opt}(c) - \operatorname{opt}(1)$ .

**Corollary 3.4.** The capacity-vector u computed in step 2 of Algorithm 1 satisfies the inequality  $\sum_{e} u_e y_e^{(u)} \ge (OPT(c) - OPT(1))/(2(1 + \epsilon)H_{c_{\max}}).$ 

We now analyze the rounding procedure for general and subadditive valuations. Together with Lemma 3.1 and Corollary 3.4, this yields Theorem 3.6.

**Lemma 3.5.** Let  $\hat{x}$  be the (random) integer solution returned by procedure Round in step 3 of Algorithm 1. Then  $\hat{x}$  combined with the pricing scheme  $y^{(u)}$  is a feasible solution to the profit-maximization problem with probability 1, which achieves expected profit at least (i)  $\left(1 - \frac{1}{e}\right) \sum_{e} u_e y_e^{(u)}$  for subadditive valuations; and (ii)  $\sum_{e} u_e y_e^{(u)} / \alpha$  for a general valuation class.

*Proof.* Feasibility is immediate from Claim 2.2 since if a player j is assigned a set S then (i) for a general class of valuations,  $x_{j,S}^{(u)} > 0$ , and (ii) for subadditive valuations, there is some set  $T \supseteq S$  such that  $x_{j,T}^{(u)} > 0$ . The bound on the profit with a general valuation class follows from part (iii) of Claim 2.2 since each item e is assigned to an expected number of  $(\sum_{j,S:e\in S} x_{j,S}^{(u)})/\alpha$  players. Note that we only need the rounding property in Lemma 2.4 (and not how it is obtained). To lower-bound the profit achieved with subadditive valuations, we show that the expected number of players who are allotted an item f is at least  $(1 - \frac{1}{e}) \sum_{j,S:f\in S} x_{j,S}^{(u)}$  (here, e is the base of the natural logarithm). Notice that this implies the claim since the expected profit is then at least  $(1 - \frac{1}{e}) \sum_{f} y_{f}^{(u)} (\sum_{j,S:f\in S} x_{j,S}^{(u)}) = (1 - \frac{1}{e}) \sum_{f} u_{f} y_{f}^{(u)}$  (where the last equality follows from part (iii) of Claim 2.2).

Let  $X = (X_{j,S})$  be the random, possibly infeasible solution computed after the first rounding step. Now fix an item f. To avoid clutter, we use  $x_j$  and  $X_j$  below as shorthand for  $\sum_{S:f \in S} x_{j,S}^{(u)}$  and  $\sum_{S:f \in S} X_{j,S}$ respectively. Also, let  $g_j = \frac{x_j}{u_f}$ . The expected number of players who are allotted item f after the subsequent "cleanup" step is

$$\begin{split} & \mathrm{E}[\min\{u_f, \sum_j X_j\}] = x_1 \Big( 1 + \mathrm{E}\left[\min\{u_f - 1, \sum_{j \ge 2} X_j\}\right] \Big) + (1 - x_1) \mathrm{E}\left[\min\{u_f, \sum_{j \ge 2} X_j\}\right] \\ & \ge x_1 + \mathrm{E}\left[\min\{u_f, \sum_{j \ge 2} X_j\}\right] \Big(x_1 \cdot \frac{u_f - 1}{u_f} + 1 - x_1\Big) = x_1 + \Big(1 - \frac{x_1}{u_f}\Big) \mathrm{E}\left[\min\{u_f, \sum_{j \ge 2} X_j\}\right] \\ & \ge u_f \Big(g_1 + (1 - g_1)g_2 + \dots + (1 - g_1)\dots(1 - g_{n-1})g_n\Big) = u_f \Big(1 - \prod_{j=1}^n (1 - g_j)\Big) \end{split}$$

Thus,  $\operatorname{E}\left[\min\{u_f, \sum_j X_j\}\right] \ge u_f\left(1 - \left(1 - \frac{\sum_j g_j}{n}\right)^n\right) \ge u_f\left(1 - \left(1 - \frac{1}{n}\right)^n\right) \sum_j g_j \ge \left(1 - \frac{1}{e}\right) \sum_j x_j$ . Here the penultimate inequality follows from the fact that  $1 - \left(1 - \frac{a}{n}\right)^n$  is a concave function of a, and hence is at least  $\left(1 - \left(1 - \frac{1}{n}\right)^n\right)a$  when  $a \in [0, 1]$  (note that  $\sum_j g_j \le 1$ ).

**Theorem 3.6.** Algorithm 1 runs in time poly (input size,  $\frac{1}{\epsilon}$ ) and achieves an

- (i)  $O(\log c_{\max})$ -approximation for subadditive valuations, using the 2-approximation algorithm for the SWM problem with subadditive valuations in [16];
- (ii)  $O(\alpha \log c_{\max})$ -approximation for a general valuation class given an integrality-gap-verifying  $\alpha$ -approximation algorithm for the SWM problem.

*Proof.* By Lemmas 3.1, 3.5, and Corollary 3.4, for subadditive valuations, the profit obtained is at least  $\max\left\{\frac{\operatorname{OPT}(\mathbf{1})}{2}, (1-\frac{1}{e})(\operatorname{OPT}(c) - \operatorname{OPT}(\mathbf{1}))/(2(1+\epsilon)H_{c_{\max}})\right\} \ge \operatorname{OPT}(c)/(4(1+\epsilon)H_{c_{\max}}).$  Similarly for a general valuation class, we obtain profit at least  $\frac{1}{\alpha} \cdot \max\left\{\operatorname{OPT}(\mathbf{1}), (\operatorname{OPT}(c) - \operatorname{OPT}(\mathbf{1}))/(2(1+\epsilon)H_{c_{\max}})\right\} \ge \operatorname{OPT}(c)/(4\alpha(1+\epsilon)H_{c_{\max}}).$ 

Remark 3.7. Note that if the allocation  $(S_1, \ldots, S_n)$  returned by Algorithm 1 is obtained via Round, then  $S_j$  is always a subset of a *utility-maximizing set* of j, and with a general valuation class, if  $S_j \neq \emptyset$ , it is a utility-maximizing set (under the computed prices). (For submodular valuations, this implies that  $v_j(S_j) - v_j(S_j \setminus \{e\}) \ge (\text{price of } e)$  for all  $e \in S_j$ .) If  $(S_1, \ldots, S_n)$  is obtained in step 4, then we may assume that  $v_j(S_j) = \max_{T \subseteq S_j} v_j(T)$  (since we have a demand oracle for  $v_j$ ); with a general valuation class, this solution can be modified to yield an approximate "one-sided envy-freeness" property. We compute  $(S_1, \ldots, S_n)$  by rounding  $x^{(1)}$  as described in Lemma 2.4. Now choose prices  $\{p'_e\}$  (arbitrarily) such that  $p' \ge y^{(1)}$  and  $p'(S_j) = \max\{y^{(1)}(S_j), (1 - \epsilon)v_j(S_j)\}$  for every j. Since any non-empty  $S_j$  is a utility-maximizing set under  $y^{(1)}$ , it follows that (a) p' is a valid item-pricing yielding profit at least  $(1 - \epsilon) \sum_j v_j(S_j)$ ; (b) if  $S_j \neq \emptyset$ , then the utility j derives from  $S_j$  under p' is at least  $\epsilon(\max$  utility of j under p').

These properties prevent a kind of "cheating" that may occur in profit-maximization problems. To elaborate, although monotonicity of the valuation is an innocuous assumption for the SWM problem, with profit-maximization this can lead to the following artifact: a customer j desires a set A but is allotted  $B \supseteq A$  (with  $v_j(B) = v_j(A)$ ) and items in A have 0 price and items in  $B \setminus A$  have positive prices, so that j ends up paying for items she never wanted! The above properties ensure that (we may assume that) the solution computed by our algorithm does not have this artifact. In fact, if j desires one of k sets  $A_1, \ldots, A_k$ , then our algorithm will assign j a set  $S_j \in \{\emptyset, A_1, \ldots, A_k\}$ . We could also prevent this artifact by dropping monotonicity of the valuations.

#### 3.1 Applications

Arbitrary valuation functions. The integrality gap of (P) is known to be  $\Theta(\sqrt{m})$ , and there are efficient (deterministic) algorithms that verify this integrality gap [26, 23]. So Theorem 3.6 immediately yields an  $O(\sqrt{m} \log c_{\max})$ -approximation algorithm for the profit-maximization problem for combinatorial auctions with arbitrary valuations.

Non-single-minded tollbooth problem on trees. In this profit-maximization problem, items are *edges* of a tree and customers desire paths of the tree. More precisely, let  $\mathcal{P}$  denote the set of all paths in the tree (including  $\emptyset$ ). Each customer j has a value  $v_j(S) \ge 0$  for path  $S \in \mathcal{P}$ , and may be assigned any (one) path of the tree. Notice that this leads to the *structured* valuation function  $v_j : 2^{[m]} \mapsto \mathbb{R}_+$  where  $v_j(T) = \max\{v_j(S) : S \text{ is a path in } T\}$ . Note that  $v_j$  need *not* be subadditive. We use Algorithm 1 to obtain an  $O(\log c_{\max})$ -approximation guarantee by formulating an LP-relaxation of the SWM problem that is tailored to this setting and designing an O(1)-integrality-gap-verifying algorithm for this LP.

The "new" LP is almost identical to (P), except that we now only have variables  $x_{j,S}$  for  $S \in \mathcal{P}$ . Correspondingly, in the dual (D), we only have a constraint for (j, S) when  $S \in \mathcal{P}$ . Clearly, this new LP satisfies the properties stated in Remark 2.3, so parts (i) and (iii) of Claim 2.2 hold for this new LP, and so does Lemma 2.4. Thus, we only need to design an O(1)-integrality-gap-verifying algorithm for this new LP to apply Theorem 3.6. Let  $\{v_j : \mathcal{P} \mapsto \mathbb{R}_+\}_{j \in [n]}$  be any instance and  $x^*$  be an optimal solution to this new LP for this instance. We design a randomized algorithm that returns a (random) integer solution  $\hat{x}$  of expected objective value  $\Omega(\sum_{j,S \in \mathcal{P}} v_j(S)x_{j,S}^*)$ . This algorithm can be derandomized using the work of [27]; this yields an O(1)-integrality-gap-verifying algorithm is a generalization of the one proposed by [9] for unsplittable flow on a line. Root the tree at an arbitrary node. Define the *depth* of an edge (a, b) to be the minimum of the distances of a and b to the root. Define the depth of an edge-set T to be the minimum depth of any edge in T. Let  $\alpha = 0.01$ .

- 1. Independently, for every customer j, choose at most one set (i.e., path) S, by picking S with probability  $\alpha x_{i,S}^*$ . Let  $S_j$  be the set assigned to j. (If j is unassigned, then  $S_j = \emptyset$ .)
- 2. Let  $W = \emptyset$ . Consider the sets  $\{S_j\}$  in non-decreasing order of their depth (breaking ties arbitrarily). For each set  $T = S_j$ , if T can be added to  $\{S_i : i \in W\}$  without violating any capacities, add j to W; otherwise discard T.

Let  $\hat{x}$  be the (random) integer solution computed. Using a similar argument as in [9], we prove in Appendix A that if we select  $\alpha = 0.01$ , then  $\Pr[\hat{x}_{j,S} = 1] \ge 0.00425 x_{j,S}^*$ , so  $\mathbb{E}\left[\sum_{j,S \in \mathcal{P}} v_j(S) \hat{x}_{j,S}\right] \ge 0.00425 \cdot \sum_{j,S \in \mathcal{P}} v_j(S) x_{j,S}^*$ . We thus obtain the following theorem as a corollary of Theorem 3.6.

**Theorem 3.8.** There is an O(1)-integrality-gap-verifying algorithm (for the new LP mentioned above). This yields an  $O(\log c_{\max})$ -approximation algorithm for the non-single-minded tollbooth problem on trees.

We remark that since the above algorithm satisfies the rounding property in Lemma 2.4, we can directly use it to round  $x^{(u)}$  (more efficiently) to a feasible allocation in step 3 of Algorithm 1, instead of using the Carr-Vempala decomposition procedure (which relies on the ellipsoid method).

# 4 Refinement for the non-single-minded highway problem

In this section, we describe a different approach that does not use  $OPT_{SWM}$  as an upper bound on the optimum profit. Instead our approach is based on using an exponential-size *configuration LP* to decompose the original instance into various smaller (and easier) instances. We use this to obtain an  $O(\log m)$ -approximation for the non-single-minded (non-SM) highway problem (recall that this is the tollbooth problem on a path, so customers desire intervals) with subadditive valuations, and arbitrary valuations but unlimited supply (Theorem 1.2). Note that this is *incomparable* to the  $O(\log n)$ -approximation obtained earlier for the tollbooth problem on trees (as  $c_{\max} \leq n$ ); the number of distinct sets is  $O(m^2)$  but the number of customers can be much larger (or smaller). Also, an  $O(\log m)$ -approximation is impossible to obtain using the approach in Section 3, and in general any approach that uses the optimum of the (integer or fractional) SWM problem as an upper bound, because, as mentioned earlier, there is a simple example with just one

item and  $c_{\max} = n$ , where the SWM-optimum is an  $H_{c_{\max}}$ -factor away from the optimum profit. Let  $\mathcal{P}$  be the set of all intervals on the line (with m edges). As with the non-SM tollbooth problem on trees (in Section 3.1), each customer j has a value for each subpath (which is now an interval). So we view  $v_j$  as a function  $v_j : \mathcal{P} \mapsto \mathbb{R}_+$ , and *subadditivity* means that for any two intervals A, B, where  $A \cup B$  is also an interval, we have  $v_j(A \cup B) \leq v_j(A) + v_j(B)$ .

We outline the proof of Theorem 1.2. First, we use a simple procedure (Proposition 4.1) to partition the intervals into  $O(\log m)$  disjoint sets, where each set is a union of item-disjoint "cliques". Here, a clique is a set of paths that share a common edge; two cliques  $\mathcal{P}_1$  and  $\mathcal{P}_2$  are item-disjoint, if  $A \cap B = \emptyset$  for all  $A \in \mathcal{P}_1$ ,  $B \in \mathcal{P}_2$ .

**Proposition 4.1** (see [7]). A set of k intervals on the line can be partitioned into at most  $\lfloor \log(k+1) \rfloor$  sets, each of which is a union of item-disjoint cliques.

Thus, we can decompose  $\mathcal{P}$  into  $O(\log m)$  sets; to get an  $O(\log m)$ -approximation algorithm, it suffices to give an O(1)-approximation algorithm when the intervals form a union of item-disjoint cliques. It is unclear how to achieve a near-optimal solution even in this structured setting, as there are various *dependencies* between the cliques in a set: a customer can only be assigned an interval in one of the cliques. We solve this "union-of-cliques" pricing problem as follows. We first trim each clique  $\mathcal{P}_i$  in our set randomly to a one-sided half-clique by (essentially) ignoring the items to the left or right of the common edge of  $\mathcal{P}_i$ . The details of this truncation are slightly different depending on whether we have subadditive or arbitrary valuations (see the proof of Lemma 4.2), but a key observation is that, in expectation, we only lose a factor of 2 by this truncation. We formulate an LP-relaxation for the pricing problem involving these half-cliques. Solving this LP requires the ellipsoid method, where the separation oracle is provided by the solution to another (easier) pricing problem, where the (half) cliques are now decoupled. We devise an algorithm based on dynamic programming (DP) to compute a near-optimal solution to this pricing problem, which then yields a near-optimal solution to the LP (Lemma 4.3). Finally, we argue that this near-optimal fractional solution can be rounded to an integer solution losing only an O(1)-factor (Lemma 4.4). Combining the various ingredients, we obtain the desired O(1)-approximation for the "union-of-cliques" pricing problem, which in turn yields an  $O(\log m)$ -approximation for our original non-single-minded highway problem.

We assume in the following that the edges of the line are numbered  $1, 2, \ldots, m$ , from left to right.

**Lemma 4.2.** There is a  $16(1+\frac{1}{m})$ -approx. algorithm for the non-SM highway problem when intervals form a union of item-disjoint cliques for (i) subadditive valuations with limited supply; (ii) arbitrary valuations with unlimited supply.

*Proof.* Let  $\mathcal{A} = \bigcup_i \mathcal{P}_i$  be a set of intervals where the  $\mathcal{P}_i$ s are item-disjoint cliques. Let  $e_i$  denote the common edge of  $\mathcal{P}_i$ , and  $\ell_i$  and  $r_i$  be the leftmost and rightmost edge used by some interval of  $\mathcal{P}_i$ . We first trim the cliques to one-sided half-cliques. For every clique  $\mathcal{P}_i$  independently, we discard one of the "halves" of  $\mathcal{P}_i$  with probability 1/2. More precisely, for subadditive valuations, discarding the right half means that we truncate each interval  $S \in \mathcal{P}_i$  to  $S \cap [\ell_i, e_i]$  to obtain the half-clique  $\mathcal{H}_i$  of truncated intervals; when discarding the left half we set  $\mathcal{H}_i = \{S \cap [e_i + 1, r_i] : S \in \mathcal{P}_i\}$ . For arbitrary valuations with unlimited supply, discarding the right half is defined to simulate the effect of pricing all edges in  $[e_i + 1, r_i]$  at 0 (discarding the left half is symmetric). So in this case, we define the half-clique  $\mathcal{H}_i$  to be  $\{S \cup [e_i + 1, r_i] : S \in \mathcal{P}_i\}$  (note that there are no capacity constraints).

A key observation is that for both subadditive and arbitrary valuations,  $E[opt(\mathcal{H}_i)] \ge opt(\mathcal{P}_i)/2$  for every *i*, where opt(S) denotes the optimum profit when players may only be assigned intervals from S.

We now consider the problem of setting *interval prices* for the intervals in  $\bigcup_i \mathcal{H}_i$  that of course obey the constraint that  $p(S) \leq p(T)$  if  $S \subseteq T$ . First, we discretize the space of interval prices. Let  $\mathcal{B}$  be the maximum price any player may pay in a feasible solution. We consider only positive prices of the form  $d_q = \mathcal{B}/2^q$ ,  $q \in \mathbb{Z}_{\geq 0}$  for  $d_q \geq \mathcal{B}/mn$ . We lose at most a factor of  $2(1 + \frac{1}{m})$  this way (since we have itemdisjoint half-cliques). Now we have  $O(\log n + \log m)$  different prices. Let  $\mathcal{R}_i$  denote the set of all possible solutions for  $\mathcal{H}_i$ , where a solution specifies a pricing of the intervals in  $\mathcal{H}_i$  (choosing non-zero prices from  $\{d_q\}$  or 0) and an allocation of intervals to customers satisfying the budget and capacity constraints. We introduce a variable  $y_{jp} \geq 0$  for each customer j and price p denoting if customer j buys a path at price p, and a variable  $x_{i,R}$  for each  $R \in \mathcal{R}_i$  denoting whether solution R has been chosen for  $\mathcal{H}_i$ . Let  $p_j(R)$  be the price that j pays under the solution R, and  $\mathcal{R}_{i,j,p} = \{R \in \mathcal{H}_i : p_j(R) = p\}$  be the set of solutions for  $\mathcal{H}_i$ where j pays price p. We consider the following LP. Here p indexes all the possible interval-prices.

$$\max \quad \sum_{j,p} p \cdot y_{jp} \tag{P2}$$

$$\sum_{R \in \mathcal{R}_{i}} x_{i,R} = 1 \qquad \text{for all } i$$

$$\sum y_{jp} \le 1 \qquad \text{for all } j \qquad (4)$$

$$y_{jp} \le \sum_{i,R:R \in \mathcal{R}_{i,j,p}} x_{i,R} \quad \text{for all } j,p \tag{5}$$

$$x_{i,R}, y_{jp} \ge 0$$
 for all  $i, R, j, p$ 

Constraint (4) ensures that a customer only buys at at most one price, and constraint (5) ensures that j can only buy at price p if a solution  $R \in \bigcup_i \mathcal{R}_{i,j,p}$  has been selected. The arguments above establish that  $OPT_{(P2)}$  is at least  $1/4(1+\frac{1}{m})$ -fraction of the optimum for the instance  $\mathcal{A}$  (for both subadditive and arbitrary valuations). We show that one can obtain an integer solution to (P2) of objective value at least  $OPT_{(P2)}/4$ ; this will complete the proof.

(P2) has an exponential number of variables, so to solve it we consider the dual problem. The separation oracle for the dual amounts to solving a related pricing problem where the half-cliques are now decoupled. We give a 2-approximation algorithm for this problem, which then yields a 2-approximate dual solution, and hence, a 2-approximate solution to (P2) (Lemma 4.3). Lemma 4.4 states that this fractional solution can then be rounded to an integer solution losing at most another factor of 2. This completes the proof.

#### **Lemma 4.3.** One can compute a 2-approximate solution to (P2) in polynomial time.

s.t.

p

*Proof.* We first show how to get a 2-approximate solution for the dual problem. This will also yield a method to get a 2-approximate primal solution. Consider the dual program:

min 
$$\sum_{i} \alpha_i + \sum_{j} \beta_j$$
 (D2)

s.t.  $\beta_j + \gamma_{jp} \ge p$  for all j, price p (6)

$$\sum_{j,p:R\in\mathcal{R}_{i,j,p}}\gamma_{jp} \le \alpha_i \qquad \text{for all } i, R\in\mathcal{R}_i \tag{7}$$

$$\beta_j, \gamma_{jp} \ge 0$$
 for all  $j$ , price  $p$ . (8)

Note that (D2) has an exponential number of constraints. In order to solve (D2) efficiently, we use the ellipsoid method, which reduces the problem of solving the LP to finding a separation oracle that, given a candidate solution vector  $v = (\alpha, \beta, \gamma)$ , either produces a feasible solution (usually the input v) or returns a constraint violated by v. Constraints (6) and (8) can easily be checked in polynomial time. In the following, we will therefore assume that v satisfies these constraints. On the other hand, there are an exponential number of constraints (7). It turns out that checking whether one of these constraints is violated amounts

to solving a generalized non-single-minded pricing instance: we seek to find a solution  $R \in \mathcal{R}_i$ , such that  $\sum_{j,p:R \in \mathcal{R}_{i,j,p}} \gamma_{jp}$  is maximized, i.e., a customer j who is allotted an interval priced at p contributes  $\gamma_{jp}$  to the objective value. If the maximum achievable profit on some half-clique  $\mathcal{H}_i$  is larger than  $\alpha_i$ , then the corresponding solution yields a violated constraint. Otherwise all constraints are satisfied. Although it seems that we have not gained much by reducing the original non-single-minded pricing problem to another even more generalized pricing problem, the crucial point here is that we have *removed the dependencies* between different (half) cliques. Instead of solving this problem with the given  $\gamma$ , we move to a slightly more structured  $\tilde{v} = (\alpha, \beta, \tilde{\gamma})$  to be specified shortly, such that if  $\tilde{v}$  violates a constraint, then the same constraint is also violated by v. Note also that  $\tilde{v}$  and v have the same objective function value since they share the same  $\alpha$  and  $\beta$  values. We define  $\tilde{\gamma}_{jp} := \max\{0, p - \beta_j\}$ . Note that  $\gamma \geq \tilde{\gamma}$  since we have assumed that v satisfies the constraints (6) and (8). Hence, if  $\tilde{v}$  violates one of the constraints (7), v also violates it.

We give a 2-approximation algorithm for this new pricing problem on a half-clique, which we call the *voucher-pricing problem*. (The rationale is that  $\beta_j$  can be viewed as a voucher that customer j can redeem and thereby decrease her price). We describe the algorithm shortly, but first we show that this implies the lemma. Using this approximation algorithm for the separation oracle yields a dual solution  $(\alpha, \beta, \gamma)$  that is potentially not feasible. In particular, it could violate the constraints (7), however only by a factor of at most 2. By scaling  $\alpha$  accordingly, we get a feasible solution  $(2\alpha, \beta, \gamma)$  whose objective function value is at most  $2 \cdot \text{OPT}_{(D2)}$ .

Now applying an argument similar to the one used by Jain et al. [22] shows that one can also compute a 2-approximate primal solution.

A 2-approximation algorithm for the voucher-pricing problem on a half-clique  $\mathcal{H}_i$ . The algorithm follows the dynamic-programming approach by Aggarwal et al. [1]. The main observation is that if we relax the constraint that a customer buys (i.e., is assigned) at most one interval and only prevent her from buying two intervals at the same price then we lose at most a factor of 2. To see this, note that since we have discretized our search space of prices, if p is the maximum price paid by a customer j in a solution to the relaxed problem (where she can buy multiple intervals), then j's contribution to the profit is at most

$$\sum_{q\geq 0} \max\{p \cdot 2^{-q} - \beta_j, 0\} \le \sum_{q\geq 0} 2^{-q} \cdot \max\{p - \beta_j, 0\} \qquad \text{since } \beta_j \ge 0$$
$$\le 2 \cdot \max\{p - \beta_j, 0\}$$

and if we assign j only the single interval at price p (note that this still satisfies the capacity constraints), we get profit  $\max\{p - \beta_j, 0\}$ .

We solve this relaxed problem using dynamic programming. To keep notation simple, let  $T_1 \supseteq T_2 \supseteq \ldots \supseteq T_\ell$  denote the intervals in  $\mathcal{H}_i$ . So we require that  $p(T_1) \ge \ldots \ge p(T_\ell)$ . Let  $d_Q$  be the lowest non-zero price in our discrete price-space, and let  $d_{Q+1} := 0$ .

Let F(q, i, U) denote the value of an optimal solution when customers are only assigned intervals from from  $\{T_i, \ldots, T_\ell\}$ , the prices of these intervals lie in  $\{d_q, \ldots, d_Q, d_{Q+1}\}$ , and U customers have been assigned intervals from  $\{T_1, \ldots, T_{i-1}\}$  (recall that a customer may be assigned multiple intervals if they are priced differently). Clearly F(0, 1, 0) is the optimal value we are looking for. The base cases are easy: we set F(Q + 1, i, U) = 0 for all i, U. For  $k \ge i$ , let C(i, k, q) denote the set of customers who can afford to buy an interval in  $\{T_i, \ldots, T_k\}$  priced at  $d_q$ . Set  $C(i, k, q) = \emptyset$  for k < i.

Suppose we decide to set the price of intervals  $T_i, \ldots, T_k$  to  $p = d_q$  and assign t customers to these intervals. Then, the best value that one can earn from intervals  $\{T_{k+1}, \ldots, T_\ell\}$  is F(q+1, k+1, U + t). Notice that this *does not* depend on which t customers are assigned intervals in  $T_i, \ldots, T_k$  or how these customers are allotted these intervals. Thus, we can compute the optimum assignment of intervals in  $T_i, \ldots, T_k$  to t customers separately. This is an interval packing problem which one can solve efficiently. For  $j \in C(i, k, q)$ , let  $i_j$  be the largest index  $i' \in \{i, \ldots, k\}$  such that j can afford to buy  $T_{i'}$  at price  $p = d_q$ . Notice that we may assume that in an optimal solution to this interval packing solution, if j is assigned an interval, it is assigned  $T_{i_j}$ . Now we can formulate the following integer program for solving this interval packing problem. For each  $j \in C(i, k, q)$ , let  $Z_j$  be an indicator variable that denotes if j is assigned interval  $T_{i_j}$ . Then, we want to solve the following integer program.

$$\max \sum_{j \in C(i,k,q)} Z_j \max\{d_q - \beta_j, 0\} \quad \text{s.t.} \quad \sum_j Z_j = t, \quad \sum_{j:e \in T_{i_j}} Z_j \le c_e - U \ \forall e, \quad Z_j \in \{0,1\} \ \forall j.$$

It is well known that an interval packing problem can be solved efficiently (e.g., by finding an optimal solution to its LP-relaxation). Let P(i, k, q, t) be the optimal value of the above program, and  $-\infty$  if the program is infeasible. Then we have the following recurrence.

$$F(q, i, U) = \max \Big\{ F(q+1, k+1, U+t) + P(i, k, q, t) : \quad i-1 \le k \le \ell, \ 0 \le t \le n \Big\}.$$

We need to compute  $O((\log m + \log n)\ell n)$  table entries for  $F(\cdot, \cdot, \cdot)$  to get the optimal value, and so this DP can be implemented in polynomial time. Note that we can easily record the corresponding solution along with the computation of each F(q, i, U).

# **Lemma 4.4.** One can round any solution (x, y) to (P2) to an integer solution of objective value at least $(1 - e^{-1}) \sum_{j} py_{jp}$ .

*Proof.* Let (x, y) denote a feasible solution to (P2). Recall that  $p_j(R)$  is the price j pays in the solution R. For each j and each solution  $R \in \bigcup_i \mathcal{R}_i$  we can define values  $y_{ij,R}$  such that  $y_{ij,R} \leq x_{i,R}$  for all  $i, j, R \in \mathcal{R}_i$ , and  $y_{jp} = \sum_{i,R \in \mathcal{R}_{i,j,p}} y_{ij,R}$  for all j, p. By making "clones" of a solution  $R \in \mathcal{R}_i$  if necessary, we can ensure that  $y_{ij,R}$  is either 0 or  $x_{i,R}$  for every  $R \in \mathcal{R}_i$ . The rounding is simple: independently, for each half-clique  $\mathcal{H}_i$ , we choose solution  $R \in \mathcal{R}_i$  with probability  $x_{i,R}$ . Let  $Q_i$  denote the solution selected for  $\mathcal{H}_i$ . Now we assign each customer j to the  $\mathcal{H}_i$  with maximum  $p_j(Q_i)$ . Notice that this yields a feasible solution for the instance composed of the union of the (half) cliques. The analysis is quite similar to the analysis in [14] and [1]; we reproduce it here for completeness.

Fix a customer j. Let  $\theta_i = \Pr[y_{ij,Q_i} > 0] = \sum_{R \in \mathcal{R}_i} y_{ij,R}$ . Let  $Z_{ij} = \left(\sum_{R \in \mathcal{R}_i} y_{ij,R} p_j(R)\right) / \left(\sum_{R \in \mathcal{R}_i} y_{ij,R}\right)$ . Note that  $\sum_i Z_{ij} \theta_i = \sum_i \sum_{R \in \mathcal{R}_i} y_{ij,R} p_j(R) = \sum_p py_{jp}$ . Consider the sub-optimal way of assigning j to a (random) R, where we assign j to the  $\mathcal{H}_i$  with maximum  $Z_{ij}$  for which  $y_{ij,Q_i} > 0$ ; if there is no such i then j is unassigned. Let k be the number of half-cliques, and let these be ordered so that  $Z_{1j} \ge Z_{2j} \ge \dots Z_{kj}$ . The expected price that j pays under this suboptimal assignment is

$$\theta_1 Z_{1j} + (1 - \theta_1) \theta_2 Z_{2j} + \ldots + (1 - \theta_1) \cdots (1 - \theta_{k-1}) \theta_k Z_{kj}$$

which following the analysis in [14] (for example) is at least  $(1 - \frac{1}{e}) \sum_{i} Z_{ij} \theta_i = (1 - \frac{1}{e}) \sum_{j,p} py_{jp}$ . Thus, the expected profit obtained is at least  $(1 - e^{-1}) \sum_{p} py_{jp}$ . The algorithm can be derandomized using a simple pipage rounding argument.

*Proof of Corollary 1.3.* As we mentioned in the introduction, the Max-Buy multi-product pricing problem can be viewed as a non-SM highway problem, where there are m disjoint edges and each bidder has values only on theses edges. As we did in the proof of Lemma 4.2, we write an LP of the form (P2), except that we do not have to round the prices (note that the number of relevant positive prices on each edge is at most the number of customer valuations on that edge). Then following the approach we used in the proof of Lemma 4.4, we can solve the dual problem (D2) in polynomial time since a separation oracle reduces to the voucher pricing problem on a *single* item which can be trivially solved in polynomial time.

The claim then follows immediately by combining this with the rounding procedure of Lemma 4.4.

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# A Proof of Theorem 3.8

We prove that the rounding algorithm described in Section 3.1, "Non-single minded tollbooth problem on trees" is an O(1)-integrality-gap verifying algorithm for the new LP relaxation. Recall that this new LP-relaxation for the SWM problem is derived from (P) by retaining only variables  $x_{j,S}$  where S is a path of the tree. More specifically, we prove that with  $\alpha = 0.01$  (as defined in the algorithm), the algorithm returns a random integer solution  $\hat{x}$  such that  $\Pr[\hat{x}_{j,S} = 1] \ge 0.00425 x_{j,S}^*$ . This implies that  $\mathbb{E}\left[\sum_{j,S\in\mathcal{P}} v_j(S)\hat{x}_{j,S}\right] \ge 0.00425 \cdot \sum_{j,S\in\mathcal{P}} v_j(S)x_{j,S}^*$ . (Recall that  $\mathcal{P}$  is the collection of all paths of the tree.) Denote by " $\leq$ " the ordering defined in step 2 of the rounding procedure. This ordering has the following useful property.

**Fact A.1.** If two sets  $A, B \in \mathcal{P}$  with  $A \leq B$  share a common edge e, then they also share the path from e up to the highest edge in B, i.e., the edge of B that is closest to the root.

Let  $X_{j,S}$  and  $Y_{j,S}$  for  $j \in [n], S \in \mathcal{P}$  to be two  $\{0, 1\}$ -random variables defined as follows:  $X_{j,S} := 1$  if and only if  $S = S_j$  was assigned to customer j in step 1 of the procedure and and  $Y_{j,S} = 1$  if and only if S survives in Step 2, that is,  $S_j = S$  and  $j \in W$ . Note that, for  $S \neq \emptyset$ ,  $\Pr[X_{j,S} = 1] = \alpha x_{j,S}^*$ .

Now  $\hat{x}$  is defined by  $\hat{x}_{j,S_j} = 1$  and  $\hat{x}_{j,S} = 0$  for all  $S \neq S_j, j \in W$ . We have

$$\Pr[Y_{j,S} = 1] = \Pr[X_{j,S} = 1] \cdot \Pr[Y_{j,S} = 1 \mid X_{j,S} = 1]$$
  
=  $\alpha x_{j,S}^* \cdot \Pr[Y_{j,S} = 1 \mid X_{j,S} = 1]$   
=  $\alpha x_{j,S}^* \cdot (1 - \Pr[Y_{j,S} = 0 \mid X_{j,S} = 1]).$ 

We will show that  $\Pr[Y_{j,S} = 0 | X_{j,S} = 1]$ , the probability of rejecting *S* in step 2, is bounded from above by a constant.

Recall that path S is rejected in step 2 if its inclusion violates a capacity constraint at some of its edges. It is natural now to apply a simple union bound on these events. Unfortunately, this bound turns out to be too weak to prove a constant rejection probability. Instead, we only consider a small subset  $S' \subseteq S$  and show that the rejection probability is bounded from above by the probability that some capacity constraint on S' is violated by the sets chosen in step 1. Let v be the node in S closest to the root. We consider the two branches of S that are split by v, separately. Let  $\ell = (\ell_1, \ell_2, ...)$  denote one branch and  $r = (r_1, r_2, ...)$  the other one, where  $\ell_1$  and  $r_1$  denote the edges of S incident to v. Note that r or  $\ell$  could be empty if the path only consists of a single branch. The edges of S' along the first branch are now defined recursively:  $\ell'_1 := \ell_1$  and  $\ell'_i = \ell_j$  where  $j = \min\{k \mid c_{\ell_k} \leq c_{\ell'_{i-1}}/2\}$ , i.e.,  $\ell'_i$  is the first edge after  $\ell'_{i-1}$  along the branch with less than half the capacity of  $\ell'_{i-1}$ . Similarly, we define the edges along the second branch:  $r'_1 := r_1$  and  $r'_i = r_j$  such that  $j = \min\{k \mid c_{r_k} \leq c_{r'_{i-1}}/2\}$ . So  $S' = (\bigcup_i \ell'_i) \cup (\bigcup_i r'_i)$ .

A bad event  $\mathcal{E}_e$  at edge e occurs when  $\sum_{j,A:e\in A} X_{j,A} \ge c_e/2$ . The next lemma shows that it is sufficient to only consider bad events at the previously selected edges in S'.

**Lemma A.2.** For every customer j, we have

$$Pr[Y_{j,S} = 0 \mid X_{j,S} = 1] \le \sum_{e \in S'} Pr[\mathcal{E}_e].$$

*Proof.* Assume that S was rejected in step 2. In that case, there has to be an edge  $e \in S$  such that  $\sum_{j,A:e\in A} Y_{j,A} = c_e$ , i.e., the number of paths picked prior to S that contain e equals  $c_e$  and therefore the inclusion of S would violate the capacity constraint on e. Let e' be the next ancestor of e that is in S' (an edge is also an ancestor of itself). Since the highest edge along each branch of S was included in S', such an edge e' has to exist. Moreover,  $c_e \ge c_{e'}/2$ : if  $e \in S'$ , then e = e'; otherwise by definition, we have  $c_e \ge c_{e'}/2$ . Now by Fact A.1, every set A that was considered in step 2 before S also contains e'. Hence, we have

$$\sum_{j,A:e'\in A} X_{j,A} \ge \sum_{j,A:e'\in A} Y_{j,A} \ge \sum_{j,A:e\in A} Y_{j,A} = c_e \ge c_{e'}/2.$$

Thus the bad event  $\mathcal{E}_{e'}$  occurs at edge e'. The result then follows from a simple union bound.

It remains to bound the probability of a bad event.

**Lemma A.3.** For  $\alpha = 0.01$ , we have  $\sum_{e \in S'} \Pr[\mathcal{E}_e] \leq 0.575$ .

*Proof.* Consider an edge *e*. Define  $Z_j = \sum_{A:e \in A} X_{j,A}$ . Note that the random variables  $Z_1, \ldots, Z_n$  are independent. Then we have  $\Pr[\mathcal{E}_e] = \Pr[\sum_j Z_j \ge c_e/2]$ . By standard Chernoff bounds (see, e.g., [25, Theorem 4.1]), we get (with  $\mu = \sum_j \mathbb{E}[Z_j] \le \sum_j \sum_{A:e \in A} \mathbb{E}[X_{j,A}] = \sum_j \sum_{a \ge j} \sum_{A:e \in A} \mathbb{E}[X_{j,A}]$ 

 $\alpha \sum_{j} \sum_{A:e \in A} x_{j,A}^* \le \alpha c_e)$ 

$$\Pr\left[\sum_{j} Z_{j} \ge c_{e}/2\right] = \Pr\left[\sum_{j} Z_{j} \ge \left(1 + \frac{c_{e} - 2\mu}{2\mu}\right)\mu\right]$$

$$< \frac{(2\mu)^{c_{e}/2}e^{c_{e}/2 - \mu}}{c_{e}^{c_{e}/2}}$$

$$< \left((2\alpha)^{1/(2\alpha)}e^{\frac{1-2\alpha}{2\alpha}}\right)^{\alpha c_{e}}$$

$$< 0.231^{c_{e}} \qquad (since \alpha = 0.01)$$

$$(9)$$

where inequality (9) is valid for  $\alpha < \frac{1}{2}$  since the left hand side is monotonically increasing in  $\mu$  for such small  $\mu$ . Finally, since the capacities of edges in S' decrease by a factor of 2 along each of the two branches, we get  $\sum_{e \in S'} \Pr[\mathcal{E}_e] \leq 2 \sum_{i \geq 0} 0.231^{2^i} \leq 0.575$ .

Combining Lemma A.2 and Lemma A.3, we get  $\Pr[Y_{j,S} = 1] \ge 0.00425 x_{j,S}^*$ . Hence, the expected weight of  $\hat{x}$  is at least 0.00425 times the weight of  $x^*$ .

Remark A.4. In the case of uniform capacities, Lemma A.2 simplifies to just two summands on the right hand side, that is, it is sufficient only to consider bad events on the first edge along each branch of S. Using this observation, the approximation factor can be improved to 0.3.