Specializations and Generalizations of the Stackelberg Minimum Spanning Tree Game*

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Abstract. Let be given a graph G = (V, E) whose edge set is partitioned into a set R of red edges and a set B of blue edges, and assume that red edges are weighted and form a spanning tree of G. Then, the Stackelberg Minimum Spanning Tree (StackMST) problem is that of pricing (i.e., weighting) the blue edges in such a way that the total weight of the blue edges selected in a minimum spanning tree of the resulting graph is maximized. StackMST is known to be APX-hard already when the number of distinct red weights is 2. In this paper we analyze some meaningful specializations and generalizations of StackMST, which shed some more light on the computational complexity of the problem. More precisely, we first show that if G is restricted to be complete, then the following holds: (i) if there are only 2 distinct red weights, then the problem can be solved optimally (this contrasts with the corresponding APX-hardness of the general problem); (ii) otherwise, the problem can be approximated within $7/4 + \epsilon$, for any $\epsilon > 0$. Afterwards, we define a natural extension of StackMST, namely that in which blue edges have a non-negative activation cost associated, and it is given a global activation budget that must not be exceeded when pricing blue edges. Here, after showing that the very same approximation ratio as that of the original problem can be achieved, we prove that if the spanning tree of red edges can be rooted so as that any root-leaf path contains at most h edges, then the problem admits a $(2h + \epsilon)$ -approximation algorithm, for any $\epsilon > 0$.

Keywords: Communication Networks, Minimum Spanning Tree, Stackelberg Games, Network Pricing Games.

1 Introduction

Leader-follower games, which were introduced by von Stackelberg in the far 1934 [13], have recently received a considerable attention from the computer science community.

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This is mainly due to the fact that the Internet is a vast electronic market composed of millions of independent end-users (i.e., the followers), whose actions are by the way influenced by a limited number of owners of physical/logical portions of the network (i.e., the leaders), that can set the price for using their own network links. In particular, in a scenario in which the leaders know in advance that the followers will allocate a communication subnetwork enjoying some criteria, a natural arising problem is that of analyzing how the leaders can optimize their pricing strategy. Games of this latter type are widely known as *Stackelberg Network Pricing Games* (SNPGs).

When only 2 players (i.e., a leader and a follower) are involved, a SNPG can be formalized as follows: We are given a graph G = (V, E), whose edge set is partitioned into a set R of red edges and a set B of blue edges, and an edge cost function $c:R\to\mathbb{R}^+$ for red edges only, while blue edges need instead to be priced by the leader. In the following, we assume that n = |V| and m = |R| + |B|. Then, the leader moves first and chooses a pricing function $p:B\to\mathbb{R}^+$ for her¹ edges, in an attempt to maximize her objective function $f_1(p, H(p))$, where H(p) denotes the decision which will be taken by the follower, consisting in the choice of a subgraph of G. This notation stresses the fact that the leader's problem is implicit in the follower's decision. Once observed the leader's choice, the follower reacts by selecting a subgraph H(p) = (V', E') of G which minimizes his objective function $f_2(p, H)$, parameterized in p. Note that the leader's strategy affects both the follower's objective function and the set of feasible decisions, while the follower's choice only affects the leader's objective function. Quite naturally, we assume that f_1 is price-additive, i.e., $f_1(p, H(p)) = \sum_{e \in B \cap E'} p(e)$. This means, the leader decides edge prices having in mind that her revenue equals the overall price of her selected edges. Therefore, the 2-player game can be equivalently thought (as we will do in the rest of the paper) as a bilevel optimization problem in which an optimal value of f_1 has to be computed.

Previous work. The most immediate SNPG is that in which we are given two specified nodes in G, say s, t, and the follower wants to travel along a shortest path in G between s and t (see [12] for a survey). This problem has been shown to be APX-hard [9], as well as not approximable within a factor of 2 - o(1) unless P = NP [5], while an $O(\log |B|)$ -approximation algorithm is provided in [11]. For the case of multiple followers (each with a specific source-destination pair), Labbé $et\ al.$ [10] derived a bilevel LP formulation of the problem (and proved NP-hardness), while Grigoriev $et\ al.$ [8] presented algorithms for a restricted shortest path problem on parallel edges. Furthermore, when all the followers share the same source node, and each node in G is a destination of a single follower, then the problem is known as the $Stackelberg\ single$ -

¹ Throughout the paper, we adopt the convention of referring to the leader and to the follower with female and male pronouns, respectively.

source shortest paths tree game. In this game, the leader's revenue for each selected edge is given by its price multiplied by the number of paths – emanating from the source – it belongs to, and in [1] it was proved that finding an optimal pricing for the leader's edges is NP-hard, as soon as $|B| = \Theta(n)$.

Another basic SNPG, which is of interest for this paper, is that in which the follower wants to use a minimum spanning tree (MST) of G. For this game, known as Stackelberg MST (StackMST) game, in [6] the authors proved the APX-hardness already when the number of red edge costs is 2, and gave a $\min\{k, 1 + \ln \beta, 1 + \ln \rho\}$ -approximation algorithm, where k is the number of distinct red costs, β is the number of blue edges selected by the follower in an optimal pricing, and ρ is the maximum ratio between red costs. In a further paper [7], the authors proved that the problem remains NP-hard even if G is planar, while it can be solved in polynomial time once that G has bounded treewidth. We point out that a structural property about StackMST, which will also hold for our generalized version we are going to present, is that the hardness in finding an optimal solution lies in the selection of the optimal set of blue edges that will be purchased by the follower, since once that a set of blue edges is part of the final MST, then their best possible pricing can be computed in polynomial time, as shown in [6].

Notice that all the above examples fall within the class of SNPGs handled by the general model proposed in [3], encompassing all the cases where each follower aims at optimizing a polynomial-time network optimization problem in which the cost of the network is given by the sum of prices and costs of contained edges. Nevertheless, SNPGs for models other than this one have been studied in [2, 4].

Our results. In this paper we analyze some meaningful specializations and generalizations of StackMST, which shed some more light on the computational complexity of the game. For the sake of presenting our results in a unifying framework, we start by defining the aforementioned generalized version of StackMST. First of all, notice that given any instance of StackMST, this can be simplified into an equivalent instance in which we compute a red MST of G, and then we discard all the red edges not belonging to it (see also [6]). Then, the budgeted StackMST game is a 2-player game defined as follows. We are given a tree T = (V, E(T)) of n nodes where each (red) edge $e \in E(T)$ has a fixed non-negative cost c(e). Moreover, we are given a non-negative activation cost $\gamma(e)$ for each (blue) edge $e = (u, v) \notin E(T)$, and a budget Δ . The game, denoted by StackMST(γ, Δ), consists of two phases. In the first phase the leader selects a set F of edges to add to T such that the budget is not exceeded, i.e., $\sum_{e \in F} \gamma(e) \leq \Delta$, and then prices them with a price function $p: F \to \mathbb{R}^+$ having in ming that, in the second phase, the follower will take the weighted graph $G = (V, E(T) \cup F)$ resulting from the first phase, and will compute a MST M(F, p)of G. Then, the leader will collect a revenue of $r(M(F,p)) = \sum_{e \in F \cap M(F,p)} p(e)$. Our

goal is to find a strategy for the leader which maximizes her revenue.² Notice that using this more general definition, the original StackMST game can be rephrased as a StackMST(γ , Δ) game in which T is any red MST of G, Δ is equal to 0, and the activation cost for an edge not in E(T) is equal to 0 if it belongs to B, otherwise it is equal to any positive value.

In this paper, we prove the following results:

- 1. $\mathsf{StackMST}(0,0)$ with only 2 distinct red costs can be solved optimally, where the first 0 in the argument is used to denote the fact that γ is identically equal to 0; in other words, this is a special case of $\mathsf{StackMST}$ with only two red edge costs in which the input graph is complete;
- 2. StackMST(0,0) can be approximated within $3/2 + \epsilon$, for any $\epsilon > 0$ when the red edges form a path;
- 3. StackMST(0,0) can be approximated within $7/4 + \epsilon$, for any $\epsilon > 0$, in general;
- 4. StackMST(γ , Δ) admits a min{k, 1+ln β , 1+ln ρ , 2h+ ϵ }-approximation algorithm, for any $\epsilon > 0$, where k, β and ρ are as previously defined for StackMST, and h denotes the radius of T w.r.t. the number of edges, once T is rooted at its center.

We point out that all the above problems have an application counterpart, since the $\mathsf{StackMST}(0,0)$ class of problems models the case in which the leader retains the potentiality to activate (at no cost) any missing connection in the network, while clearly result (4) complements the approximation ratio given in [6] whenever the radius of the red tree is bounded, which might well happen in practice. Finally, notice also that $\mathsf{StackMST}(0,0)$ is a specialization of the general $\mathsf{StackMST}$, for which however we were not able to prove whether the problem is in P or not. Therefore, this remains a challenging open problem.

The rest of the paper is organized by providing each of the above results in a corresponding section, followed by a concluding section listing some interesting problems left open.

2 Exact algorithm for $\mathsf{StackMST}(0,0)$ with costs in $\{a,b\}$

In this section we present an exact polynomial-time algorithm for $\mathsf{StackMST}(0,0)$ when the cost of any red edge belongs to the set $\{a,b\}$, with $0 \le a < b$. Notice that this case is already APX-hard for $\mathsf{StackMST}[6]$. For the sake of clarity, we will first present the algorithm and the analysis when the red tree is actually a path. The extension to the general case will be derived in the subsequent subsection.

² Throughout the paper, as usual we assume that when multiple optimal solutions are available for the follower, then he selects an optimal solution maximizing the leader's revenue.

2.1 Solving StackMST(0,0) with two red costs when T is a path

Now, we present an exact algorithm for $\mathsf{StackMST}(0,0)$ on a red path P with costs in $\{a,b\}$, with $0 \le a < b$. We call a subpath P' of P an a-block if P' has all edges of cost a, and P' is maximal (w.r.t. inclusion). We say that an a-block is good if its length is greater than or equal to 3, bad otherwise. Let σ be the number of bad blocks of P.

We first present an algorithm achieving a revenue of $c(P) - \min \left\{ \sigma a, \left\lfloor \frac{\sigma}{2} \right\rfloor (b-a) + 2 \left\lfloor \frac{\sigma}{2} \right\rfloor \right\}$ where c(P) denotes the sum of costs of edges of P, and

 $\left(\sigma - 2\left\lfloor \frac{\sigma}{2}\right\rfloor\right) \min\{a, b - a\}$, where c(P) denotes the sum of costs of edges of P, and then we show that such a revenue is actually an upper bound to the optimal revenue.

For technical convenience, we only consider instances where P has at least 5 edges. Clearly, the solutions for the remaining instances can be easily computed. The algorithm uses the following four rules. Each rule considers a subpath of P and specifies a feasible solution for the subpath, i.e., a set of blue edges incident to the vertices of the subpath with a corresponding pricing. The solutions corresponding to the rules are shown in Figure 1.

Rule 1: Let P' be a subpath of P containing only one a-block, and this a-block is good. We can obtain revenue c(P') from P' by adding blue edges only within P'.

Rule 2: Let P' be a subpath of P containing only one a-block and this a-block is bad. We can compute a solution with revenue c(P') - a from P'.

Rule 3: Let P' be a subpath of P containing one a-block, and this a-block is the last bad block of P. Moreover P' has at least one more edge of cost b that either precedes or follows the a-block. We can obtain a revenue of c(P') - (b-a) from P' by using a star of blue edges centered at the left or right endvertex of P, depending on the position of the edge of cost b. Notice that the endvertices of P' might be followed by other good blocks.

Rule 4: Let P_1, P_2 be two edge-disjoint subpaths of P each containing only one a-block. Assume that both a-blocks are bad and P_1 contains an edge of cost b whose removal separates the two a-blocks. We can obtain a revenue of $c(P_1) + c(P_2) - (b-a)$ from P_1 and P_2 . Notice that P_1 and P_2 do not need to be adjacent.

Our algorithm is as follows. If $b \ge 3a$ then we split P into subpaths each of them containing exactly one a-block. Then we apply Rule 1 or Rule 2 to each subpath, depending on whether the a-block in the subpath is good or bad. Hence, this solution yields a revenue of $c(P) - \sigma a$.

Now, consider the case b < 3a. Let B_1, \ldots, B_{σ} be the bad a-blocks contained in P from left to right w.r.t one of the endvertices. We first consider the case where $\sigma \geq 2$, i.e., there are at least two bad blocks. The algorithm splits P into subpaths such that (i) each subpath contains exactly one a-block, (ii) for every $i = 0, \ldots, \lfloor \sigma/2 \rfloor - 1$, subpath containing B_{2i+1} has an edge of cost b incident to its right endvertex, and

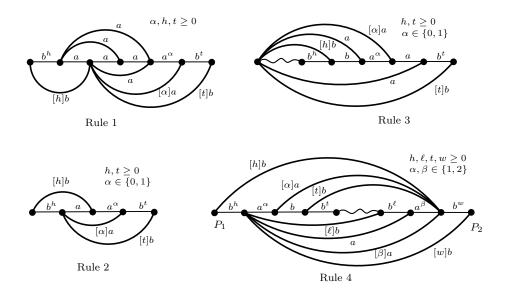


Fig. 1. Rules used by the algorithm to solve subpaths. We denote by η^{δ} a path of δ edges each having a cost of η . An edge with label $[i]\eta$ represents i blue edges each having a price of η . Observe that, except for Rule 2, all red (path) edges with cost a will be discarded by the follower. Concerning Rule 2, the follower will select only a single red edge of cost a (of the shown subpath). The left endvertex of the path of Rule 3 corresponds to one endvertex of P.

(iii) if σ is odd, the subpath containing B_{σ} has an edge of cost b incident to its left endvertex.³ Let P_i be the subpath containing B_i . The algorithm uses Rule 4 for every pairs of subpaths $P_{2i+1}, P_{2i+2}, i = 0, \ldots, \lfloor \sigma/2 \rfloor - 1$, and Rule 1 for every subpath containing a good a-block. Finally, if σ is odd, we apply Rule 3 for P_{σ} when $b \leq 2a$ (we can apply Rule 3 since property (iii) above holds), while we use Rule 2 when b > 2a. It is easy to see that the revenue of this solution coincides with $c(P) - \min \left\{ \sigma a, \left\lfloor \frac{\sigma}{2} \right\rfloor (b-a) + \left(\sigma - 2 \left\lfloor \frac{\sigma}{2} \right\rfloor \right) \min \{a, b-a\} \right\}$.

Concerning the case $\sigma \leq 1$, then either P has no bad blocks, and then a revenue of c(P) can be obtained, or there exists only one bad block B_1 . In this latter case:

- if $b \ge 2a$, let P' be any subpath containing B_1 ; then, solve P' using Rule 2.
- Otherwise, if b < 2a, let P' be a subpath containing B_1 and a suitable additional edge of cost b; then, use Rule 3 on P'.

Finally, split $P \setminus P'$ into subpaths, each containing one good a-block, and solve them using Rule 1. By doing so we obtain a revenue of $c(P) - \min\{a, b - a\}$.

Now, we show that the revenue computed by the above algorithm is the optimal revenue r^* :

³ Property (iii) can always be guaranteed since $\sigma \geq 2$.

Lemma 1.
$$r^* \le c(P) - \min\left\{\sigma a, \left|\frac{\sigma}{2}\right| (b-a) + \left(\sigma - 2\left|\frac{\sigma}{2}\right|\right) \min\{a, b-a\}\right\}.$$

Proof. Let n_a be the number of red edges of cost a. Let T^* be the tree computed by the follower w.r.t. an optimal solution. Moreover, let B_1, \ldots, B_{σ} and $\hat{B}_1, \ldots, \hat{B}_{\sigma'}$ be the bad and the good blocks of P, respectively. We denote by m_i and \hat{m}_j the number of edges of B_i and \hat{B}_j , respectively. Moreover, for an edge e = (x, y), $T^*(e)$ will denote the unique path in T^* between x and y (observe that $T^*(e)$ may be the path containing only edge e). For each $i = 1, \ldots, \sigma$ and $j = 1, \ldots, \sigma'$, consider the bad tree $T_i = \bigcup_{e \in E(B_i)} T^*(e)$, and the good tree $\hat{T}_j = \bigcup_{e \in E(\hat{B}_j)} T^*(e)$. Let $\mathcal{T} = \{T_1, \ldots, T_{\sigma}\} \cup \{\hat{T}_1, \ldots, \hat{T}_{\sigma'}\}$. Observe that for each i, j, we have: (i) T_i and \hat{T}_j are trees and every edge has cost a, (ii) $V(B_i) \subseteq V(T_i)$ and $V(\hat{B}_j) \subseteq V(\hat{T}_j)$, and (iii) $E(T^*) \cap E(B_i) \neq \emptyset$, or T_i contains at least $m_i + 1$ edges.

Let us consider the following graph $H = (\bigcup_i V(T_i) \cup \bigcup_j V(\hat{T}_j), \bigcup_i E(T_i) \cup \bigcup_j E(\hat{T}_j))$, and let N be the number of nodes of H. Clearly, H is a forest. Moreover, each connected component of H is either a single tree of \mathcal{T} or it consists of the union of at least two trees in \mathcal{T} . Let us consider the set X of "unmerged" bad trees, i.e., $X = \{T_i \mid i = 1, \ldots, \sigma, V(T_i) \cap V(T) = \emptyset, \ \forall \ T \in \mathcal{T} \setminus \{T_i\}\}$. We define $\ell = |X|$. Observe that each tree in X is in the set \mathcal{C} of connected components of H. Let t be the number of the remaining connected components of H, i.e., $|\mathcal{C}| = t + \ell$. As each bad tree not in X has been merged with some other tree, we have $t \leq \sigma' + \left\lfloor \frac{\sigma - \ell}{2} \right\rfloor$.

In order to relate t to the number N of nodes of H, we define $\ell_1 = |\{\tilde{T}_i \mid \tilde{T}_i \in X, E(T^*) \cap E(B_i) \neq \emptyset\}|$. Notice that ℓ_1 is a lower bound to the number of red edges in H. We now give a lower bound to N. Since H spans all a-blocks (which are pairwise vertex disjoint), and since property (iii) holds, we have that $N \geq n_a + \sigma + \sigma' + \ell - \ell_1$. Therefore, since H has $N - \ell - t$ edges of cost a, and using $c(P) = n_a(a - b) + (n - 1)b$, we have:

$$r^* \leq \left(N - \ell - t\right)a - \ell_1 a + \left(n - 1 - \left(N - \ell - t\right)\right)b$$

$$= (N - \ell - t)(a - b) - \ell_1 a + (n - 1)b$$

$$\leq (n_a + \sigma + \sigma' - \ell_1 - t)(a - b) + (n - 1)b - \ell_1 a$$

$$= c(P) - \left(\sigma + \sigma' - \ell_1 - t\right)(b - a) - \ell_1 a$$

$$\leq c(P) - \left(\left(\sigma - \ell_1\right) - \left\lfloor\frac{\sigma - \ell}{2}\right\rfloor\right)(b - a) - \ell_1 a$$

$$\leq c(P) - \min\left\{\sigma a, \left\lfloor\frac{\sigma}{2}\right\rfloor(b - a) + \left(\sigma - 2\left\lfloor\frac{\sigma}{2}\right\rfloor\right)\min\left\{a, b - a\right\}\right\}.$$

To see why the latter inequality holds, one can consider the different parity of σ and ℓ for each of the following three cases: $b \geq 3a$, $2a \leq b < 3a$, and b < 2a.

⁴ Here the union symbol denotes the union of graphs.

Hence, from the above lemma, we have:

Theorem 1. StackMST(0,0) can be solved in polynomial time when the red edges form a path and their costs are in $\{a,b\}$.

2.2 Solving StackMST(0,0) with two red costs: the general case

We now extend the previous result by providing an optimal polynomial-time algorithm for $\mathsf{StackMST}(0,0)$ when the red costs belong to the set $\{a,b\}$, and T is a tree.

Let $0 \le a < b$, and let T be a red tree with costs in $\{a, b\}$. In a similar fashion as before, we call a subtree T' of T an a-block if T' has all edges of cost a, and T' is maximal (w.r.t. inclusion). We say that an a-block is bad if it is a star, good otherwise. Let σ be the number of bad blocks of T. As the upper bound to the maximum revenue r^* shown in Lemma 1 still holds, 5 we now present a general algorithm achieving a revenue equal to the given upper bound.

The four rules used by the algorithm are similar to the ones used in the algorithm for the path and they are shown in Figure 2 and 3, along with the corresponding revenues.

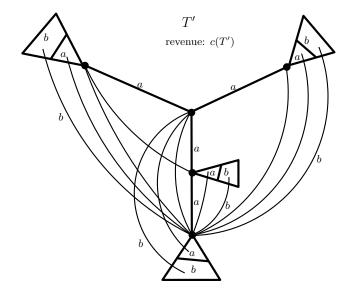
Our algorithm is as follows. If $b \geq 3a$, then we split T into subtrees, each of them containing exactly one a-block. Then we apply Rule 1 or Rule 2 to each subtree, depending on whether the a-block in the subtree is good or bad. Clearly, this solution yields a revenue of $c(T) - \sigma a$

Now, consider the case b < 3a. Let B_1, \ldots, B_{σ} be the bad a-blocks contained in T. The algorithm splits T into subtrees such that (i) each subtree contains exactly one a-block, (ii) there exists a permutation $B'_1, \ldots, B'_{\sigma}$ of the bad a-blocks such that for every $i = 0, \ldots, \lfloor \sigma/2 \rfloor - 1$, subtree containing B'_{2i+1} has an edge of cost b along the (unique) path joining B'_{2i+1} with B'_{2i+2} , and (iii) if σ is odd, the subtree containing B'_{σ} has an edge of cost b.

Let T_i be the subtree containing B'_i . The algorithm uses Rule 4 for every pair of subtrees $T_{2i+1}, T_{2i+2}, i = 0, \ldots, \lfloor \sigma/2 \rfloor$, Rule 1 for every subtree containing a good a-block. Finally, if σ is odd, we apply Rule 3 for T_{σ} when $b \leq 2a$, while we use Rule 2 when b > 2a. From this, we have:

Theorem 2. StackMST(0,0) can be solved in polynomial time when red edge costs are in $\{a,b\}$.

⁵ The proof is identical to the one previously shown.



Rule 1

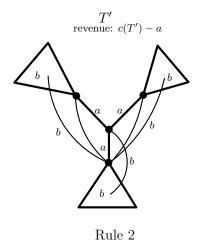
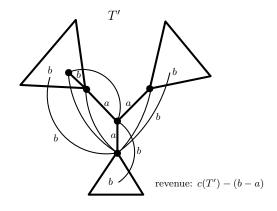


Fig. 2. Rules 1 and 2 used by the algorithm to solve subtrees. Edges without label are priced to b.



Rule 3

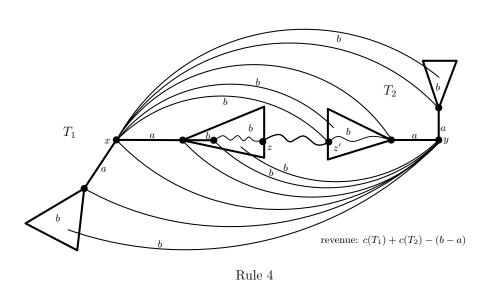


Fig. 3. Rules 3 and 4 used by the algorithm to solve subtrees. Edges without label are priced to b. Notice that in Rule 4, while there is a blue edge between x and z', there is no edge between y and z.

3 StackMST(0,0) can be approximated within $3/2 + \epsilon$ when the red edges form a path

Here we design a $(\frac{3}{2} + \epsilon)$ -approximation algorithm for $\mathsf{StackMST}(0,0)$ when the tree T is actually a path, say P.

Let then P be the path of red edges. The idea of the algorithm is to consider three possible solutions and pick the best one. We will argue that the revenue of such a solution is at least a fraction $\frac{2}{3}$ of the cost of almost the entire path. More precisely, we select a cheap subpath \bar{P} of P of length 2 or 3, and we then compute a solution achieving a revenue of at least $\frac{2}{3}(c(P) - c(\bar{P}))$.

Let m=n-1 be the length of P, and let e_1, \ldots, e_m be the red edges of P in the order of traversing P from an endpoint to the other one. Moreover, let us set ℓ to 2, if m is even, and to 3 otherwise. Let P_i be the subpath of P of length ℓ starting from e_i , i.e., P_i consists of the edges $e_i, \ldots, e_{i+\ell-1}$. Let \bar{P} be the subpath with minimum cost among P_{2j-1} , $j=1,\ldots,\lfloor m/2\rfloor$. If we remove \bar{P} from P, we obtain two paths of even length, say Q_1 and Q_2 . At most one of Q_1 and Q_2 may be empty. Let us assume for the ease of presentation that both paths are non-empty (similar arguments hold when this does not happen), and let 2h and 2k be the length of Q_1 and Q_2 , respectively. Moreover, let u_0, u_1, \ldots, u_{2h} and v_0, v_1, \ldots, v_{2k} be the nodes of Q_1 and Q_2 , respectively. The two end-nodes of \bar{P} are u_{2h} and v_0 . Let $x_i = c(u_{i-1}, u_i)$ and $y_i = c(v_{i-1}, v_i)$. Finally, let z be an internal node of \bar{P} (see Figure 4).

and $y_j = c(v_{j-1}, v_j)$. Finally, let z be an internal node of \bar{P} (see Figure 4). Let $A = \sum_{i=1}^h \max\{x_{2i-1}, x_{2i}\} + \sum_{i=1}^k \max\{y_{2i-1}, y_{2i}\}$, and let $B = \sum_{i=1}^h \min\{x_{2i-1}, x_{2i}\} + \sum_{i=1}^k \min\{y_{2i-1}, y_{2i}\}$. Notice that $c(Q_1) + c(Q_2) = A + B$. The first solution we consider is

$$F_1 = \{(u_{2i-2}, u_{2i}) \mid i = 1, \dots, h\} \cup \{(v_{2i-2}, v_{2i}) \mid i = 1, \dots, k\},\$$

and the price function is defined as $p(u_{2i-2}, u_{2i}) = \max\{x_{2i-1}, x_{2i}\}$, and $p(v_{2i-2}, v_{2i}) = \max\{y_{2i-1}, y_{2i}\}$. Notice that this solution obtains a revenue $r_1 = A$.

The second solution is a star centered in the node z; more precisely:

$$F_2 = \{(z, u_i) \mid i = 0, \dots, 2h - 1\} \cup \{(z, v_i) \mid i = 1, \dots, 2k\},\$$

and the prices are defined as $p(z, u_0) = x_1, p(z, v_{2k}) = y_{2k}, p(z, u_i) = \min\{x_i, x_{i+1}\}$, and $p(z, v_i) = \min\{y_i, y_{i+1}\}$. Notice that this solution obtains a revenue of

$$r_2 = B + x_1 + y_{2k} + \sum_{i=0}^{h-2} \min\{x_{2i+2}, x_{2i+3}\} + \sum_{i=0}^{h-2} \min\{y_{2i+2}, y_{2i+3}\}.$$

Finally, the third solution is the following:

$$F_3 = \{(u_{2i+1}, u_{2i+3}) \mid i = 0, \dots, h-2\} \cup \{(v_{2i+1}, v_{2i+3}) \mid i = 0, \dots, k-2\} \cup \{(u_{2h-1}, z), (z, v_1)\},\$$



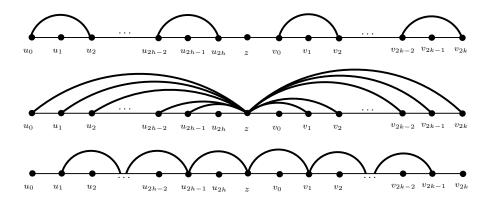


Fig. 4. The path and the three solutions considered by the algorithm. Blue edges are in bold. Here \bar{P} consists of 2 edges.

and the pricing is as follows: $p(u_{2h-1}, z) = x_{2h}, p(z, v_1) = y_1, p(u_{2i+1}, u_{2i+3}) = \max\{x_{2i+2}, x_{2i+3}\}$, and $p(v_{2i+1}, v_{2i+3}) = \max\{y_{2i+2}, y_{2i+3}\}$. Hence, the corresponding revenue is:

$$r_3 = x_{2h} + y_1 + \sum_{i=0}^{h-2} \max\{x_{2i+2}, x_{2i+3}\} + \sum_{i=0}^{h-2} \max\{y_{2i+2}, y_{2i+3}\}.$$

Hence, we have:

$$r_1 + r_2 + r_3 = A + B + x_1 + x_{2h} + y_1 + y_{2k} + \sum_{i=0}^{h-2} (\min\{x_{2i+2}, x_{2i+3}\} + \max\{x_{2i+2}, x_{2i+3}\}) + \sum_{i=0}^{k-2} (\min\{y_{2i+2}, y_{2i+3}\} + \max\{y_{2i+2}, y_{2i+3}\})$$

$$= 2(c(Q_1) + c(Q_2)),$$

from which it follows that the revenue $r = \max\{r_1, r_2, r_3\}$ is at least $\frac{2}{3}(c(Q_1) + c(Q_2))$. Now, observe that by construction we have

$$c(\bar{P}) \le \frac{c(P)}{\left|\frac{n}{\ell}\right|} \le \frac{3}{n-2} \left(c(\bar{P}) + c(Q_1) + c(Q_2)\right),$$

and hence $c(\bar{P}) \leq \frac{3}{n-5}(c(Q_1) + c(Q_2))$. Denoting by r^* the optimal revenue, and observing that the cost of the red tree is always an upper bound to r^* , we then have

$$\frac{r^*}{r} \le \frac{c(P)}{r} = \frac{c(Q_1) + c(Q_2)}{r} + \frac{c(\bar{P})}{r} \le \frac{3}{2} + \frac{\frac{3}{n-5}(c(Q_1) + c(Q_2))}{\frac{2}{3}(c(Q_1) + c(Q_2))} = \frac{3}{2} + \frac{9}{2n-10}.$$

We have proved the following:

Theorem 3. StackMST(0,0) can be approximated within a factor of $3/2 + \epsilon$, for any $\epsilon > 0$, when the red edges form a path.

We point out that our algorithm is asymptotically tight with respect to the adopted upper-bound scheme. An example is the path in which $c(e_1) = 1$, $c(e_2) = 2$, and $c(e_i) = 0$, for every i > 2. It is easy to see that for this path the revenue obtained by an optimal solution is 2, while the total cost of the path is 3.

4 StackMST(0,0) can be approximated within $7/4 + \epsilon$

In this section we design an algorithm that achieves an approximation ratio of $7/4 + \epsilon$ for the general StackMST(0,0) game.

The idea of the algorithm is to partition the red tree into suitable subtrees for which we can guarantee a revenue of at least 4/7 of the cost of each one of them. Let T = (V, E(T)) be the red tree. We say that $T_1 = (V_1, E_1), \ldots, T_\ell = (V_\ell, E_\ell)$ is a partition of T into ℓ subtrees if (i) each T_i is a subtree of T, (ii) $V = \bigcup_i V_i, E(T) = \bigcup_i E_i$, and (iii) for each $i, j, i \neq j, E_i \cap E_j = \emptyset$.

It is easy to see that once T is partitioned into subtrees as specified above, we can solve locally a $\mathsf{StackMST}(0,0)$ game for each red subtree of the partition, and then solve the original problem by joining together all the local solutions (by maintaining the corresponding pricing). Indeed, the union of all the trees associated with the local solutions is clearly a spanning tree of G. Hence, we can claim the following

Lemma 2. Let T_1, \ldots, T_ℓ be a partition of T into ℓ subtrees. For each i, let r_i be the revenue returned by a local solution of T_i . Then, the revenue which can be obtained for T is at least $\sum_{i=1}^{\ell} r_i$.

Moreover, we can prove the following:

Lemma 3. Let T be a tree rooted at a node s. There always exists a partition of T into ℓ subtrees T_1, \ldots, T_ℓ such that

- T_{ℓ} has at most 2 edges and at least one of them is incident to s;
- for every $1 \le j \le \ell 1$, T_j is either (i) a path of 3 or 4 edges, or (ii) a star with at least 3 edges.

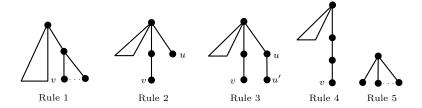


Fig. 5. The five rules of the decomposition algorithm.

Moreover, this partition can be found in polynomial time.

Proof. We provide a polynomial-time algorithm that finds the partition of the lemma. Let d(v) denote the depth of v in T, i.e., the number of edges of the path (in T) between s and v. We denote by S(v) the set of the children of v. Moreover, we use \bar{v} to denote the parent of v. We proceed in phases. In phase j, we find a subtree T_j by applying one of the rules below (we consider them in order), then we remove T_j from T and we move to the next phase. We stop when no rule can be applied. Let L be the set of leaves of T with depth equal to the current height of T. The rules are the following (see Figure 5):

Rule 1: if there exists a node $v \in L$ with $d(v) \geq 2$ and such that v has at least one sibling, then T_j is the star with edge set $\{(\bar{v}, \bar{v})\} \cup \{(\bar{v}, u) \mid u \in S(\bar{v})\};$

Rule 2 if there exists a node $v \in L$ with $d(v) \geq 2$ such that \bar{v} has a sibling u and u is a leaf, then T_j is the path with edge set $\{(v, \bar{v}), (\bar{v}, \bar{v}), (\bar{v}, u)\};$

Rule 3: if there exists a node $v \in L$ with $d(v) \ge 2$ such that \bar{v} has a sibling u and u is not a leaf, then let u' be the unique child of u (u' must be unique otherwise Rule 1 would apply). Then, T_j is the path with edge set $\{(v, \bar{v}), (\bar{v}, \bar{v}), (\bar{v}, u), (u, u')\}$;

Rule 4: if there exists a node $v \in L$ with $d(v) \geq 3$, then T_j is the path with edge set $\{(v, \bar{v}), (\bar{v}, \bar{\bar{v}}), (\bar{\bar{v}}, \bar{\bar{v}})\};$

Rule 5: if T is a star with at least 3 edges, then $T_j = T$.

Now, assume that the last phase is phase $\ell-1$, then we set T_ℓ equal to the remaining tree T. If there is no edge left, we set T_ℓ equal to the empty subtree. It is easy to see that if T_ℓ is non-empty, it must have at most 2 edges, and one of them must be incident to s. Moreover, since each phase takes polynomial time and each T_j with $j < \ell$ contains at least one edge, the claim follows.

The following lemmas allow us to obtain a revenue of at least $\frac{4}{7}c(T_i)$ for each subtree $T_i, i = 1, \ldots, \ell - 1$, of the decomposition.

Lemma 4. Let S be a star with at least 3 edges, then we can obtain a revenue of at least $\frac{2}{3}c(S)$.

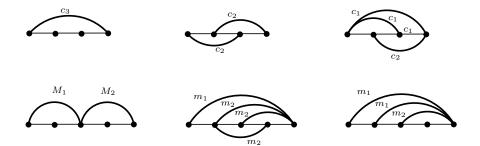


Fig. 6. Three possibile solutions for paths of 3 or 4 edges.

Proof. Let s be the center of the star, and let u_1, \ldots, u_t be the leaves ordered such that $c(s, u_1) \le c(s, u_2) \le \cdots \le c(s, u_t)$. The set of blue edges $F = \{(u_1, u_j) \mid j = 2, \ldots, t\}$ yields a revenue of $\sum_{j=2}^t c(s, u_j) \ge \frac{2}{3}c(S)$, since $t \ge 3$.

Lemma 5. Let P be a path of 3 or 4 edges, then we can obtain a revenue of at least $\frac{4}{7}c(P)$.

Proof. Let us consider the path of 3 edges first. Let $0 \le c_1 \le c_2 \le c_3$ be the edge costs. If the cost of the middle edge is c_1 , we can easily obtain a revenue of $c_2 + c_3 \ge \frac{2}{3}c(P)$. Assume that the cost of the middle edge is not c_1 . In Figure 6 three solutions are shown. The corresponding revenues are: $c_3, 2c_2, 2c_1 + c_2$. A trivial calculation shows that the maximum of the three revenues is at least $\frac{4}{7}c(P)$.

Now, we consider a path of 4 edges. Let x_1, \ldots, x_4 be the costs of the edges from left to right. We set $M_1 = \max\{x_1, x_2\}, m_1 = \min\{x_1, x_2\}, M_2 = \max\{x_3, x_4\}, m_2 = \min\{x_3, x_4\}$. Assume w.l.o.g. that $m_1 \geq m_2$. Three solutions are shown in Figure 6. The corresponding revenues are: $M_1 + M_2, m_1 + 3m_2, 2m_1 + m_2$, the maximum of which is easy to see to be at least $\frac{4}{7}c(P)$.

We are now ready to prove the following:

Theorem 4. StackMST(0,0) can be approximated within a factor of $7/4 + \epsilon$, for any constant $\epsilon > 0$.

Proof. W.l.o.g., we can restrict ourselves to the case $n \geq \frac{7}{2\epsilon} + 1$, since otherwise to find an optimal solution we can always use an exhaustive search algorithm that tries all the possible sets of blue edges and prices them at the optimum (remember this can be done in polynomial time [6]). For each v, let $\mu(v) = \max_{u|(u,v)\in E(T)} c(u,v)$. We root T at a node s minimizing μ . Then we decompose T using the algorithm given in Lemma 3, and we solve locally each T_j with $j \leq \ell$. Let r_j be the corresponding obtained revenue, and observe that $r_\ell \geq c(T_\ell) - \min_{e \in E(T_\ell)} c(e) \geq c(T_\ell) - \mu(s)$.

As Lemma 2 together with Lemmas 4 and 5 implies that the total revenue r is at least $\sum_{j=1}^{\ell} r_j \geq \frac{4}{7}(c(T) - \mu(s))$, and since $c(T) \geq \frac{1}{2} \sum_{v \in V} \mu(v) \geq \frac{n}{2} \mu(s)$, we obtain

$$\frac{r^*}{r} \le \frac{c(T)}{r} \le 7/4 + \frac{7}{2n-4} \le 7/4 + \epsilon.$$

5 StackMST (γ, Δ) on trees of bounded radius

In this section, we study the general $\mathsf{StackMST}(\gamma, \Delta)$. First, we will argue that for this generalized version, the very same approximation ratio as that of the original game can be achieved, since the single-price algorithm defined in [6] can be easily adapted to provide an approximation of $\min\{k, 1 + \ln \beta, 1 + \ln \rho\}$ for $\mathsf{StackMST}(\gamma, \Delta)$ as well, where k is the number of distinct red costs, β is the number of blue edges selected by the follower in an optimal solution, and ρ is the maximum ratio between red costs. Then we focus on the case in which T is a tree of radius h (measured w.r.t. the number of edges) once rooted at its center. For this case, we show that the problem remains APX -hard even for constant values of h, as well as approximable within a factor of $2h + \epsilon$.

Let k denote the number of distinct red costs, and let $c_1 < c_2 < \cdots < c_k$ denote these costs. To extend the single-price algorithm, we proceed as follows. We consider the complete graph consisting of the union of the red tree and all the potential blue edges. For each j between 1 and k, we set the price of every potential blue edge to c_j , and we compute a spanning tree by a slightly modification of Kruskal's algorithm as follows. In the phase in which the algorithm considers all the edges of cost c_j , we break tightness in favor of blue edges, and among the blue edges, we prefer those with smaller activation cost. As soon as we consider a blue edge exceeding the budget Δ , we delete that edge and all the remaining blue edges, and we go on with Kruskal's algorithm. The solution for a given j will be the set of all picked blue edges which will be priced to c_j . Then we pick j such that the corresponding revenue is maximum, and we return the corresponding solution. It turns out that the same analysis given in [6] can be applied here. Hence, we have:

Theorem 5. The above algorithm achieves an approximation ratio of $\min\{k, 1 + \ln \beta, 1 + \ln \rho\}$ for $\mathsf{StackMST}(\gamma, \Delta)$.

We now study $\mathsf{StackMST}(\gamma, \Delta)$ when T is a tree that once rooted at its center, say v_0 , has height/radius h. First, we observe that the reduction the authors in [6] used to prove that $\mathsf{StackMST}$ is APX -hard already when T is a path can be modified to show the following:

Theorem 6. StackMST is APX-hard even if T is a star.

Proof. We show an approximation-preserving reduction from $\mathsf{StackMST}$ for the case in which T is a path to $\mathsf{StackMST}$ for the case in which T is a star. Our reduction works only for the hard instances constructed in [6].

The hard instances given in [6] are constructed from instances of the Set Cover Problem in the following way. Let $U = \{u_1, \ldots, u_\ell\}$ be a set of objects and let $\{S_1, \ldots, S_t\}$ be a set of subsets of U such that $u_\ell \in S_i$, for every $i = 1, \ldots, t$. T is a path of $\ell + t$ vertices $\{u_1, \ldots, u_\ell\} \cup \{S_1, \ldots, S_t\}$ with edge set $E(T) = \{(u_i, u_{i+1}) \mid i = 1, \ldots, \ell - 1\} \cup \{(S_i, S_{i+1}) \mid i = 1, \ldots, t - 1\} \cup \{(u_\ell, S_1)\}$. The fixed cost c(e) of an edge $e \in E(T)$ is 2 if $e = (S_i, S_{i+1})$ or $e = (u_\ell, S_1)$, 1 otherwise. Let $B = \{(u_i, S_j) \mid u_i \in S_j, i = 1, \ldots, \ell, j = 1, \ldots, t\}$ be the set of blue edges.

Our reduction works as follows. We take the above hard instance for StackMST on red paths, we add a vertex v_0 and we replace the red tree T by a star of red edges centered at v_0 . Let T' denote the star of red edges. The fixed cost c'(e) of an edge e is 2 if $e = (v_0, S_i)$, 1 otherwise. First observe that for every $F \subseteq B$, (V(T), F) is acyclic iff (V(T'), F) is acyclic. Let $F \subseteq B$ be a set of blue edges such that (V(T), F) is acyclic. The revenue yielded by F in both instances of StackMST is the same, as the price of an edge $(S_i, u_j) \in F$ in both instances is 2 iff (S_i, u_j) is the only edge in F which is incident to S_i . The claim follows.

In the remaining of the section we will show the existence of a $(2h+\epsilon)$ -approximation algorithm. Before starting, recall that once that a set F of activated edges is part of the final MST, then the optimal pricing for each $e \in F$ can be computed in polynomial time, as observed in (see [6]). More precisely, this can be done by computing efficiently

$$p_F(e) := \min_{H \in \mathsf{cycle}(F,e)} \max_{e' \in E(H) \cap E(T)} c(e') \tag{1}$$

where $\operatorname{cycle}(F, e)$ is the set of (simple) cycles containing edge e in the graph $(V(T), E(T) \cup F)$. With a little abuse of notation, in the following we will denote by r(F) the revenue yielded by the above optimal pricing p_F .

The main idea of the algorithm is to reduce the problem instance to h instances in which the red trees are stars. With a little abuse of notation, in each of the h instances, the leader is sometimes allowed to activate edges which are parallel to red edges. We denote by $V_i = \{v_1, \ldots, v_{\ell_i}\}$ the set of vertices at level i in T, and by E_i the set of edges in T going from vertices in V_i to their parents. Let T_i be a red star obtained by identifying all red edges in T but those in E_i . With a little abuse of notation, when edge (u, v) is identified, and w.l.o.g. u is the parent of v, we assume that the corresponding vertex is labeled with u. Thus, according to this assumption, we have that T_i is a star centered at v_0 with v_1, \ldots, v_{ℓ_i} as leaves. The

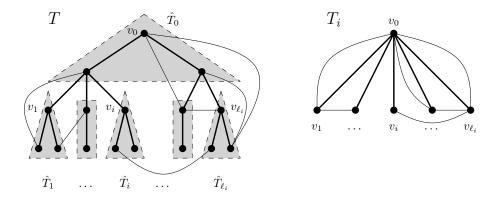


Fig. 7. Auxiliary instance for $\mathsf{StackMST}(\gamma, \Delta)$ corresponding to level i. Vertices in the tree \hat{T}_q are identified. When two or more edges between the same two trees are present, only the one with smallest activation cost is preserved.

cost of a red edge (v_0, v) in T_i is $c_i(v_0, v) = c(u, v)$, where u is the parent of v in T. Let $\hat{T}_0, \hat{T}_1, \ldots, \hat{T}_{\ell_i}$ be the connected components in $T - E_i$. W.l.o.g., assume $v_i \in V(\hat{T}_i)$. Let $e_{j,q}$ be a blue edge connecting \hat{T}_j and \hat{T}_q with cheapest activation cost. Let $\mathtt{blue}_i := \{e_{j,q} \mid j, q = 0, \ldots, \ell_i, j \neq q\}$. Notice that this set contains edges that the leader can activate in the original instance of the problem. We now map them to their counterpart in T_i , namely let $B_i := \{\bar{e}_{j,q} := (v_j, v_q) \mid e_{j,q} \in \mathtt{blue}_i\}$ be the set of blue edges the leader is allowed to activate in T_i . The activation cost of an edge $\bar{e}_{j,q} \in B_i$ is $\gamma_i(\bar{e}_{j,q}) := \gamma(e_{j,q})$. The auxiliary instance corresponding to level i is shown in Figure 7.

Let F^* be an optimal solution for the leader on input instance T and let $F_i^* := \{(v_j, v_q) \in B_i \mid (u, v) \in F^*, u \in V(\hat{T}_j), v \in V(\hat{T}_q), j \neq q\}$ be the corresponding edges in T_i . Let $G_i^* := (\{v_0, \ldots, v_{\ell_i}\}, F_i^*)$, and denote by $\mathsf{comp}(G_i^*)$ the set of connected components of G_i^* . We start by proving an upper bound on the revenue yielded by F^* .

Lemma 6.
$$r(F^*) \le c(T) - \sum_{i=1}^h \sum_{H \in \text{comp}(G_i^*)} \min_{v \in V(H)} c_i(v_0, v)$$
.

Proof. Observe that for every $H \in \text{comp}(G_i^*)$ not containing vertex v_0 , at least one red edge (v_0, v) , for some $v \in V(H)$, has to be contained in any MST of $(V(T_i), E(T_i) \cup F_i^*)$. Thus, for some $v \in V(H)$, at least one edge (u, v) where u is the parent of v in T has to be contained in any MST of $G = (V(T), E(T) \cup F^*)$. As $c_i(v_0, v) = c(u, v)$, the claim follows by summing over all components $H \in \text{comp}(G_i^*)$ for all i's.

⁶ With a slight abuse of notation, we assume $c_i(v_0, v_0) = 0$.

The key idea of our algorithm is to find a set F of blue edges whose overall activation cost does not exceed the budget, and such that (V, F) is a forest of stars. More precisely, for every i = 1, ..., h, the algorithm first finds a set $\hat{F}_i \subseteq B_i$ such that $\sum_{e \in \hat{F}_i} \gamma_i(e) \leq \Delta$ and $\hat{G}_i := (V(T_i), \hat{F}_i)$ is a forest of stars; then, it considers the set $F_i := \{e_{j,q} \mid \bar{e}_{j,q} \in \hat{F}_i\}$ of the corresponding blue edges for the original instance. Observe that (i) $G_i := (V(T), F_i)$ is still a forest of stars and (ii) the overall activation cost of the edges in F_i equals that of the edges in \hat{F}_i . Furthermore, using Equation (1), we can derive the following lemma, which claims that when we map \hat{F}_i back to F_i the obtained revenue cannot decrease:

Lemma 7.
$$r(F_i) \geq r(\hat{F}_i)$$
.

We now give a lower bound of the revenue that can be obtained from \hat{F}_i . The bound trivially follows from (1):

Lemma 8. Let $L_i := \{v \mid v \in V(T_i), v \text{ is a leaf of some star in } \hat{G}_i\}$. Then, $r(\hat{F}_i) \ge \sum_{v \in L_i} c_i(v_0, v)$.

Next lemma essentially shows that there exists a solution for T_i which is a forest of stars yielding a revenue of at lest a half of the optimal revenue for T_i .

Lemma 9. Let $B' \subseteq B_i$ and let $U = \{v \mid v \text{ is an endvertex of some edge in } B'\}$. There exists a polynomial time algorithm that finds two sets F^1 and F^2 such that (i) $F^1, F^2 \subseteq B'$, (ii) both $(V(T_i), F^1)$ and $(V(T_i), F^2)$ are forests of stars, and (iii) $r(F^1) + r(F^2) \ge \sum_{v \in U} c_i(v_0, v)$.

Proof. Let D be the graph induced by edge set B'. Let D^j be any of the t connected components in D, and let T^j be any spanning tree in D^j . As T^j is a bipartite graph, it is possible to partition the set of its vertices into two sets V_1^j and V_2^j in polynomial time. Moreover, by the connectivity of T^j , every vertex $v \in V_\ell^j$ ($\ell \in \{1, 2\}$) is adjacent to some vertex in $V_{3-\ell}^j$, and thus it is easy to find a set E_ℓ^j of edges in T^j such that $(V(D^j), E_\ell^j)$ is a forest of stars with centers in V_ℓ^j and leaves in $V_{3-\ell}^j$. Therefore, for $\ell = 1, 2, F^\ell = \bigcup_{j=1}^t E_\ell^j$ are two sets of edges satisfying (i) and (ii). Furthermore, $\bigcup_{j=1}^t (V_1^j \cup V_2^j) = \bigcup_{j=1}^t V(D^j) = V(D)$. As a consequence, from Lemma 8, (iii) is also satisfied.

To compute \hat{F}_i , the algorithm does the following. Our algorithm uses the well-known FPTAS for the $Knapsack\ Problem$ to compute a $(1 + \epsilon/(2h))$ -approximate solution S_i for the following instance of knapsack. For each v_j , consider the blue edge $e \in B_i$ incident to v_j with cheapest activation cost. We create an object o_j^i of profit

⁷ If a star contains only one edge, then let any of its vertices to be a leaf.

 $c_i(v_0, v_j)$ and volume $\gamma_i(e)$; we say that e is associated with the object o_j^i . Finally, the volume of the knapsack is Δ . Denote by K_i the input instance of knapsack. The solution S_i for K_i identifies a set of blue edges, namely $B' = \{e \in B_i \mid \exists o_j^i \in S_i \text{ s.t. } e \text{ is associated with } o_j^i\}$. The algorithm then uses the decomposition algorithm described in Lemma 9 to find two subsets of edges F^1 and F^2 , and then it sets \hat{F}_i to F^1 if $r(F^1) \geq r(F^2)$, and to F^2 otherwise. The pseudocode of the algorithm is given in Algorithm 1.

Algorithm 1

```
1: for i=1 to h do

2: compute a (1+\epsilon/(2h))-approximate solution S_i for the knapsack instance K_i

3: B'=\{e\in B_i\mid o_j^i\in S_i, e \text{ is associated with }o_j^i\}

4: compute F^1 and F^2 w.r.t. B' as explained in Lemma 9

5: if r(F^1)\geq r(F^2) then \hat{F}_i:=F^1 else \hat{F}_i:=F^2 end if

6: F_i:=\{e_{j,q}\mid \bar{e}_{j,q}\in \hat{F}_i\}

7: end for

8: return the best of the F_i's
```

Theorem 7. Algorithm 1 computes a $(2h + \epsilon)$ -approximate solution in polynomial time for StackMST (γ, Δ) , for any constant $\epsilon > 0$.

Proof. Remind that $G_i^* = (\{v_0, \dots, v_{\ell_i}\}, F_i^*)$, where F_i^* are obtained by mapping the edges of an optimal solution F^* to the blue edges of the auxiliary instance T_i . In order to show a lower bound for the profit of the optimal solution of K_i , we define a feasible solution S_i^* as follows: for each connected component H of G_i^* , let $v \in V(H)$ be the vertex that minimizes $c(v_0, v)$, and consider any spanning tree \mathcal{T}_H of H rooted at v. Notice that for each $v_j \in V(H) \setminus \{v\}$ we have an object o_j^i whose volume is at most $\gamma(\bar{v}_j, v_j)$, where \bar{v}_j denotes the parent of v_j in \mathcal{T}_H . Add such objects to the solution S_i^* . Once we have considered all the connected components of G_i^* , the solution S_i^* contains a set of objects of total volume at most Δ (since the solution F^* is feasible). Moreover, by construction, the profit of S_i^* must be at least

$$\mathtt{profit}(S_i^*) \ge c(T_i) - \sum_{H \in \mathtt{comp}(G_i^*)} \min_{v \in V(H)} c_i(v_0, v).$$

We now bound the revenue $r(\hat{F}_i)$. Since, from Lemma 7 and Lemma 9, $r(F_i) \ge r(\hat{F}_i) \ge \frac{1}{2} \operatorname{profit}(S_i)$, we have that

$$\begin{split} c(T_i) - \sum_{H \in \mathsf{comp}(G_i^*)} \min_{v \in V(H)} c_i(v_0, v) &\leq \mathsf{profit}(S_i^*) \\ &\leq \left(1 + \frac{\epsilon}{2h}\right) \mathsf{profit}(S_i) \leq \left(2 + \frac{\epsilon}{h}\right) r(\hat{F}_i) \leq \left(2 + \frac{\epsilon}{h}\right) r(F_i). \end{split}$$

By summing over all levels and using Lemma 6, we obtain

$$\begin{split} r(F^*) & \leq c(T) - \sum_{i=1}^h \sum_{H \in \mathsf{comp}(G_i^*)} \min_{v \in V(H)} c_i(v_0, v) \\ & \leq \left(2 + \frac{\epsilon}{h}\right) \sum_{i=1}^h r(F_i) \leq (2h + \epsilon) \max_{i=1,\dots,h} r(F_i). \end{split}$$

This completes the proof.

6 Open problems

In this paper we have presented a collection of results concerning some interesting variants of the classic StackMST game. Many intriguing problems are left open. Among the others, we list the following: (i) Is StackMST(0,0) NP-hard? (ii) Can we design a better approximation algorithm for StackMST(0,0)? (iii) Can we prove a stronger inapproximability result for StackMST(γ , Δ) than the one holding for StackMST? (iv) What can we say about StackMST(γ , Δ) for instances with uniform activation costs? Is the problem NP-hard? Can we design and can we extend our results for StackMST(0,0)? (v) Finally, and most importantly, does StackMST admit a constant factor approximation algorithm?

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