# Relating $\mathcal{L}$ -Resilience and Wait-Freedom via Hitting Sets

Eli Gafni

Petr Kuznetsov\*

Computer Science Department, UCLA

Deutsche Telekom Laboratories/TU Berlin

#### Abstract

The condition of t-resilience stipulates that an n-process program is only obliged to make progress when at least n - t processes are correct. Put another way, the *live sets*, the collection of process sets such that progress is required if all the processes in one of these sets are correct, are all sets with at least n - t processes.

We show that the ability of arbitrary collection of live sets  $\mathcal{L}$  to solve distributed tasks is tightly related to the *minimum hitting set* of  $\mathcal{L}$ , a minimum cardinality subset of processes that has a non-empty intersection with every live set. Thus, finding the computing power of  $\mathcal{L}$  is NP-complete.

For the special case of *colorless* tasks that allow participating processes to adopt input or output values of each other, we use a simple simulation to show that a task can be solved  $\mathcal{L}$ -resiliently if and only if it can be solved (h-1)-resiliently, where h is the size of the minimum hitting set of  $\mathcal{L}$ .

For general tasks, we characterize  $\mathcal{L}$ -resilient solvability of tasks with respect to a limited notion of *weak* solvability: in every execution where all processes in some set in  $\mathcal{L}$  are correct, outputs must be produced for every process in some (possibly different) participating set in  $\mathcal{L}$ . Given a task T, we construct another task  $T_{\mathcal{L}}$  such that T is solvable weakly  $\mathcal{L}$ -resiliently if and only if  $T_{\mathcal{L}}$  is solvable weakly wait-free.

## 1 Introduction

One of the most intriguing questions in distributed computing is how to distinguish solvable from the unsolvable. Consider, for instance, the question of *wait-free* solvability of distributed tasks. Wait-freedom does not impose any restrictions on the scope of considered executions, i.e., a waitfree solution to a task requires every correct processes to output in every execution. However, most interesting distributed tasks cannot be solved in a wait-free manner [6, 19]. Therefore, much research is devoted to understanding how the power of solving a task increases as the scope of considered executions decreases. For example, *t-resilience* considers only executions where at least n-t processes are correct (take infinitely many steps), where n is the number of processes in the system. This provides for solving a larger set of tasks than wait-freedom, since in executions in which less than n-t processes are correct, no correct process is required to output.

<sup>\*</sup>Contact author: pkuznets@acm.org, TU Berlin, Sekr. TEL 16, Ernst-Reuter-Platz 7, 10587 Berlin, Fax: +49 391 534 783 47

What tasks are solvable t-resiliently? It is known that this question is undecidable even with respect to wait-free solvability, let alone t-resilient [9, 14]. But is the question about t-resilient solvability in any sense different than the question about wait-free solvability? If we agree that we "understand" wait-freedom [16], do we understand t-resilience to a lesser degree? The answer should be a resounding no if, in the sense of solving tasks, the models can be reduced to each other. That is, if for every task T we can find a task  $T_t$  which is solvable wait-free if and only if T is solvable t-resiliently. Indeed, [2, 4, 8] established that t-resilience can be reduced to wait-freedom. Consequently, the two models are unified with respect to task solvability.

In this paper, we consider a generalization of t-resilience, called  $\mathcal{L}$ -resilience. Here  $\mathcal{L}$  stands for a collection of subsets of processes. A set in  $\mathcal{L}$  is referred to as a *live set*. In the model of  $\mathcal{L}$ -resilience, a correct process is only obliged to produce outputs if all the processes in some live set are correct. Therefore, the notion of  $\mathcal{L}$ -resilience represents a restricted class of *adversaries* introduced by Delporte et al. [5], described as collections of exact correct sets.  $\mathcal{L}$ -resilience describes adversaries that are closed under the superset operation: if a correct set is in an adversary, then every superset of it is also in the adversary.

We show that the key to understanding  $\mathcal{L}$ -resilience is the notion of a minimum hitting set of  $\mathcal{L}$  (called simply hitting set in the rest of the paper). Given a set system  $(\Pi, \mathcal{L})$  where  $\Pi$  is a set of processes and  $\mathcal{L}$  is a set of subsets of  $\Pi$ , H is a hitting set of  $(\Pi, \mathcal{L})$  if it is a minimum cardinality subset of  $\Pi$  that meets every set in  $\mathcal{L}$ . Intuitively, in every  $\mathcal{L}$ -resilient execution, i.e., in every execution in which at least one set in  $\mathcal{L}$  is correct, not all processes in a hitting set of  $\mathcal{L}$  can fail. Thus, under  $\mathcal{L}$ -resilience, we can solve the k-set agreement task among the processes in  $\Pi$  where k is the hitting set size of  $(\Pi, \mathcal{L})$ . In k-set agreement, the processes start with private inputs and the set of outputs is a subset of inputs of size at most k. Indeed, fix a hitting set H of  $(\Pi, \mathcal{L})$  of size k. Every process in H simply posts its input value in the shared memory, and every other process returns the first value it witnesses to be posted by a process in H. Moreover, using a simple simulation based on [2, 4], we derive that  $\mathcal{L}$  does not allow solving (k-1)-set agreement or any other colorless task that cannot be solved (k-1)-resiliently. Thus, we can decompose superset-closed adversaries into equivalence classes, one for each hitting set size, where each class agrees on the set of colorless tasks it allows for solving.

Informally, colorless tasks allow a process to adopt an input or output value of any other participating process. This restriction gives rise to simulation techniques in which dedicated simulators independently "install" inputs for other, possibly non-participating processes, and then take steps on their behalf so that the resulting outputs are still correct and can be adopted by any participant [2, 4]. The ability to do this is a strong simplifying assumption when solvability is analyzed.

For the case of general tasks, where inputs cannot be installed independently, the situation is less trivial. We address general tasks by considering a restricted notion of *weak solvability*, that requires every execution where all the processes in some set in  $\mathcal{L}$  are correct to produce outputs for every process in some (possibly different) participating set in  $\mathcal{L}$ . Note that for colorless tasks, weak solvability is equivalent to regular solvability that requires every correct process to output.

We relate between wait-free solvability and  $\mathcal{L}$ -resilient solvability. Given a task T and a collection of live sets  $\mathcal{L}$ , we define a task  $T_{\mathcal{L}}$  such that T is weakly solvable  $\mathcal{L}$ -resiliently if and only if  $T_{\mathcal{L}}$  is weakly solvable wait-free. Therefore, we *characterize*  $\mathcal{L}$ -resilient weak solvability, as wait-free solvability has already been characterized in [16]. Not surprisingly, the notion of a hitting set is crucial in determining  $T_{\mathcal{L}}$ .

The simulations that relate T and  $T_{\mathcal{L}}$  are interesting in their own right. We describe an agree-

ment protocol, called *Resolver Agreement Protocol* (or RAP), by which an agreement is immediately achieved if all processes propose the same value, and otherwise it is achieved if eventually a single correct process considers itself a dedicated *resolver*. This agreement protocol allows for a novel execution model of wait-free read-write protocols. The model guarantees that an arbitrary number of simulators starting with j distinct initial views should *appear* as j independent simulators and thus a (j - 1)-resilient execution can be simulated.

The rest of the paper is organized as follows. Section 2 briefly describes our system model. Section 3 presents a simple categorization of colorless tasks. Section 4 formally defines the wait-free counterpart  $T_{\mathcal{L}}$  to every task T. Section 5 describes RAP, the technical core of our main result. Sections 6 and 7 present two directions of our equivalence result: from wait-freedom to  $\mathcal{L}$ -resilience and back. Section 8 overviews the related work, and Section 9 concludes the paper by discussing implications of our results and open questions. Most proofs are delegated to the technical report [10].

## 2 Model

We adopt the conventional shared memory model [12], and only describe necessary details.

**Processes and objects.** We consider a distributed system composed of a set  $\Pi$  of n processes  $\{p_1, \ldots, p_n\}$   $(n \ge 2)$ . Processes communicate by applying atomic operations on a collection of *shared* objects. In this paper, we assume that the shared objects are registers that export only atomic readwrite operations. The shared memory can be accessed using atomic snapshot operations [1]. An execution is a pair  $(I, \sigma)$  where I is an initial state and  $\sigma$  is a sequence of process ids. A process that takes at least one step in an execution is called participating. A process that takes infinitely many steps in an execution is said to be *correct*, otherwise, the process is *faulty*.

**Distributed tasks.** A *task* is defined through a set  $\mathcal{I}$  of input *n*-vectors (one input value for each process, where the value is  $\perp$  for a non-participating process), a set  $\mathcal{O}$  of output *n*-vectors (one output value for each process,  $\perp$  for non-terminated processes) and a total relation  $\Delta$  that associates each input vector with a set of possible output vectors. A protocol *wait-free* solves a task T if in every execution, every correct process eventually outputs, and all outputs respect the specification of T.

**Live sets.** The correct set of an execution e, denoted correct(e) is the set of processes that appear infinitely often in e. For a given collection of live sets  $\mathcal{L}$ , we say that an execution e is  $\mathcal{L}$ -resilient if for some  $L \in \mathcal{L}$ ,  $L \subseteq correct(e)$ . We consider protocols which allow each process to produce output values for every other participating process in the system by posting the values in the shared memory. We say that a process terminates when its output value is posted (possibly by a different process).

**Hitting sets.** Given a set system  $(\Pi, \mathcal{L})$  where  $\mathcal{L}$  is a set of subsets of  $\Pi$ , a set  $H \subseteq \Pi$  is a hitting set of  $(\Pi, \mathcal{L})$  if it is a minimum cardinality subset of  $\Pi$  that meets every set in  $\mathcal{L}$ . We denote the set of hitting sets of  $(\Pi, \mathcal{L})$  by  $HS(\Pi, \mathcal{L})$ , and the size of a hitting set of  $(\Pi, \mathcal{L})$  by  $h(\Pi, \mathcal{L})$ . By  $(\Pi', \mathcal{L})$ ,  $\Pi' \subseteq \Pi$  we denote the set system that consists of the elements  $S \in \mathcal{L}$ , such that  $S \subseteq \Pi'$ .

The BG-simulation technique. In a *colorless* task (also called *convergence* tasks [4]) processes are free to use each others' input and output values, so the task can be defined in terms of input and output *sets* instead of vectors.

BG-simulation is a technique by which k + 1 processes  $q_1, \ldots, q_{k+1}$ , called *simulators*, can waitfree simulate a k-resilient execution of any asynchronous n-process protocol [2, 4] solving a colorless task. The simulation guarantees that each simulated step of every process  $p_j$  is either eventually agreed on by all simulators, or the step is *blocked* forever and one less simulator participates further in the simulation. Thus, as long there is a live simulator, at least n - k simulated processes accept infinitely many simulated steps. The technique has been later extended to tasks beyond colorless [8]. Weak  $\mathcal{L}$ -resilience. An execution is  $\mathcal{L}$ -resilient if some set in  $\mathcal{L}$  contains only correct processes. We say that a protocol solves a task T weakly  $\mathcal{L}$ -resiliently if in every  $\mathcal{L}$ -resilient execution, every process in some participating set  $L \in \mathcal{L}$  eventually terminates, and all posted outputs respect the specification of T. In the wait-free case, when  $\mathcal{L}$  consists of all n singletons, weak  $\mathcal{L}$ -resilient solvability stipulates that at least one participating process must be given an output value in every execution.

Weak solvability is sufficient to (strongly) solve every colorless task. For general tasks, however, weak solvability does not automatically implies strong solvability, since it only allows processes to adopt the output value of any terminated process, and does not impose any conditions on the inputs.

## 3 Colorless tasks

First recall the formal definition of a colorless task. Let val(U) denote the set of non- $\perp$  values in a vector U. In a colorless task, for all input vectors I and I' and all output vectors O and O', such that  $(I, O) \in \Delta$ ,  $val(I') \subseteq val(I)$ ,  $val(O') \subseteq val(O)$ , we have  $(I', O) \in \Delta$  and  $(I, O') \in \Delta$ .

**Theorem 1** A colorless task T is weakly  $\mathcal{L}$ -resiliently solvable if and only if T is  $(h(\Pi, \mathcal{L}) - 1)$ -resiliently solvable.

**Proof.** Let a colorless task T be (h-1)-resiliently solvable, where  $h = h(\Pi, \mathcal{L})$ , and let A be the corresponding algorithm. Let  $H = q_1, \ldots, q_h$  be a hitting set of  $(\Pi, \mathcal{L})$ . Since H is a hitting set of  $\mathcal{L}$ , in every  $\mathcal{L}$ -resilient execution, at least one simulator must be correct. Running BG-simulation [2, 4] of A on these h simulators, where each simulator tries to use its input value of T as an input value of every simulated process, results in an h-resilient simulated execution of A. By our assumption, every correct process must decide in this execution.

For the other direction, suppose, by contradiction that  $\mathcal{L}$  solves a task T that is not possible to solve (h-1)-resiliently. Let  $A_{\mathcal{L}}$  be the corresponding protocol.

Consider any (h-1)-resilient execution e of  $A_{\mathcal{L}}$ , and observe that e involves infinitely many steps of a set in  $\mathcal{L}$ . Indeed, otherwise, there is a hitting set that does not contain at least n-h+1processes (namely, the processes that appear infinitely often in e), and thus the hitting set size of  $\mathcal{L}$  is at most h-1.

Thus, every (h-1)-resilient execution is also  $\mathcal{L}$ -resilient, which implies an (h-1)-resilient solution to T — a contradiction.

Theorem 1 implies that  $\mathcal{L}$ -resilient adversaries can be categorized into n equivalence classes, class h corresponding to hitting sets of size h. Note that two adversaries that belong to the same class h agree on the set of colorless tasks they are able to solve, and the set includes h-set agreement.

## 4 Relating *L*-resilience and wait-freedom: definitions

Consider a set system  $(\Pi, \mathcal{L})$  and a task  $T = (\mathcal{I}, \mathcal{O}, \Delta)$ , where  $\mathcal{I}$  is a set of input vectors,  $\mathcal{O}$  is a set of output vectors, and  $\Delta$  is a total binary relation between them. In this section, we define the "wait-free" task  $T_{\mathcal{L}} = (\mathcal{I}', \mathcal{O}', \Delta')$  that characterizes  $\mathcal{L}$ -resilient solvability of T. The task  $T_{\mathcal{L}}$  is also defined for n processes. We call the processes solving  $T_{\mathcal{L}}$  simulators and denote them by  $s_1, \ldots, s_n$ .

Let X and X' be two *n*-vectors, and  $Z_1, \ldots, Z_n$  be subsets of  $\Pi$ . We say that X' is an *image of* X with respect to  $Z_1, \ldots, Z_n$  if  $\forall i$ , such that  $X'[i] \neq \bot$ , we have  $X'[i] = \{(j, X[j])\}_{j \in Z_i}$ .

Now  $T_{\mathcal{L}} = (\mathcal{I}', \mathcal{O}', \Delta')$  guarantees that for all  $(I', O') \in \Delta'$ , there exist  $(I, O) \in \Delta$  such that:

- (1)  $\exists S_1, \ldots, S_n \subseteq \Pi$ , each containing a set in  $\mathcal{L}$ :
  - (1a) I' is an image of I with respect to  $S_1, \ldots, S_n$ .
  - (1b)  $|\{I'[i]\}_i \{\bot\}| = m \Rightarrow h(\bigcup_{i,I'[i]\neq \bot} S_i, \mathcal{L}) \ge m.$

In other words, every process participating in  $T_{\mathcal{L}}$  obtains, as an input, a set of inputs of T for some live set, and all these inputs are consistent with some input vector I of T.

Also, if the number of distinct non- $\perp$  inputs to  $T_{\mathcal{L}}$  is m, then the hitting set size of the set of processes that are given inputs of T is at least m.

(2)  $\exists U_1, \ldots, U_n$ , each containing a set in  $\mathcal{L}$ : O' is an image of O with respect to  $U_1, \ldots, U_n$ .

In other words, the outputs of  $T_{\mathcal{L}}$  produced for input vector I should be consistent with  $O \in \mathcal{O}$  such that  $(I, O) \in \Delta$ .

Intuitively, every group of simulators that share the same input value will act as a single process. According to the assumptions on the inputs to  $T_{\mathcal{L}}$ , the existence of m distinct inputs implies a hitting set of size at least m. The asynchrony among the m groups will be manifested as at most m-1failures. The failures of at most m-1 processes cannot prevent *all* live sets from terminating, as otherwise the hitting set in (1b) is of size at most m-1.

## 5 Resolver Agreement Protocol

We describe the principal building block of our constructions: the resolver agreement protocol (RAP). RAP is similar to consensus, though it is neither always safe nor always live. To improve liveness, some process may at some point become a *resolver*, i.e., take the responsibility of making sure that every correct process outputs. Moreover, if there is at most one resolver, then all outputs are the same.

Formally, the protocol accepts values in some set V as inputs and exports operations propose(v),  $v \in V$ , and resolve() that, once called by a process, indicates that the process becomes a resolver for RAP. The propose operation returns some value in V, and the following guarantees are provided: (i) Every returned value is a proposed value; (ii) If all processes start with the same input value or some process returns, then every correct process returns; (iii) If a correct process becomes a resolver, then every correct process returns; (iv) If at most one process becomes a resolver, then at most one value is returned.

A protocol that solves RAP is presented in Figure 1. The protocol uses the *commit-adopt* abstraction (CA) [7] exporting one operation propose(v) that returns (commit, v') or (adopt, v'),

```
Shared variables:
    D, initially \perp
Local variables:
    resolver, initially false
propose(v)
    (flag, est) := CA.propose(v)
1
    if flag = commit then
2
      D := est; return(est)
3
    repeat
      if resolver then D := est
5
6
    until D \neq \bot
    return(D)
7
resolve()
    resolver := true
```

Figure 1: Resolver agreement protocol: code for each process

for  $v, v' \in V$ , and guarantees that (a) every returned value is a proposed value, (b) if only one value is proposed then this value must be committed, (c) if a process commits a value v, then every process that returns adopts v or commits v, and (d) every correct process returns. The commit-adopt abstraction can be implemented wait-free [7].

In the protocol, a process that is not a resolver takes a finite number of steps and then either returns with a value, or waits on one posted in register D by another process or by a resolver. A process that waits for an output (lines 4-6) considers the agreement protocol *stuck*. An agreement protocol for which a value was posted in D is called *resolved*.

**Lemma 2** The algorithm in Figure 1 implements RAP.

**Proof.** Properties (i) and (ii) follow from the properties of CA and the algorithm: every returned value is a proposed value and if all inputs are v or some process returns (after writing a non- $\perp$  value v in D), then every process commits on v and returns v in line 3.

If there is a correct resolver, it eventually writes some value in D (line 5), and eventually every other process returns some value, and thus property (iii) holds.

Moreover, a returned value either was committed in an instance of CA or was written to D by a resolver. Even if some process returned a value v committed in CA, then by the properties of CA, the only value that a resolver can write in D is v. Thus, if there is at most one resolver, the protocol can return at most one value, and property (iv) holds.

## 6 From wait-freedom to $\mathcal{L}$ -resilience

Suppose that  $T_{\mathcal{L}}$  is weakly wait-free solvable and let  $A_{\mathcal{L}}$  be the corresponding wait-free protocol. We show that weak wait-free solvability of  $T_{\mathcal{L}}$  implies weak  $\mathcal{L}$ -resilient solvability of T by presenting an algorithm A that uses  $A_{\mathcal{L}}$  to solve T in every  $\mathcal{L}$ -resilient execution.

First we describe the *doorway* protocol (DW), the only  $\mathcal{L}$ -dependent part of our transformation. The responsibility of DW is to collect at each process a subset of the inputs of T so that all the collected subsets constitute a legitimate input vector for task  $T_{\mathcal{L}}$  (property (1) in Section 4). The doorway protocol does not require the knowledge of T or  $T_{\mathcal{L}}$  and depends only on  $\mathcal{L}$ . Shared variables:  $R_j, j = 1, \ldots, n$ , initially  $\perp$ Local variables:  $S_j, j = 1, \ldots, h(\Pi, \mathcal{L}),$  initially  $\emptyset$  $\ell_i, \ i = 1, \dots, h(\Pi, \mathcal{L}), \text{ initially } 0$  $R_i :=$ input value of T9 wait until  $snapshot(R_1, \ldots, R_n)$  contains inputs for some set in  $\mathcal{L}$ 10 while *true* do 11 $S := \{ p_i \in P, R_i \neq \bot \}$ {the current participating set} 12 $\{H^S \text{ is deterministically chosen in } HS(S, \mathcal{L})\}$ if  $p_i \in H^S$  then 13 $m := \text{the index of } p_i \text{ in } H^S$ 14  $RAP_m^{\ell_m}.resolve()$ 15 for each  $j = 1, \ldots, |H^S|$  do 16 17 if  $\ell_j = 0$  then  $S_i := S$ 18 take one more step of  $RAP_{i}^{\ell_{j}}.propose(S_{j})$ 19 if  $RAP_{i}^{\ell_{j}}$ .propose $(S_{j})$  returns v then 20 $(flag, S_j) := CA_j^{\ell_j}.propose(v)$ 21if (flag = commit) then 22 $return(\{(s, R_s)\}_{p_s \in S_i})$ {return the set of inputs of processes in  $S_i$ } 23 $\ell_i := \ell_i + 1$ 24



In contrast, the second part of the transformation described in Section 6.2 does not depend on  $\mathcal{L}$  and is implemented by simply invoking the wait-free task  $T_{\mathcal{L}}$  with the inputs provided by DW.

#### 6.1 The doorway protocol

Formally, a DW protocol ensures that in every  $\mathcal{L}$ -resilient execution with an input vector  $I \in \mathcal{I}$ , every correct participant eventually obtains a *set* of inputs of T so that the resulting input vector  $I' \in \mathcal{T}_{\mathcal{L}}$  complies with property (1) in Section 4 with respect to I.

The algorithm implementing DW is presented in Figure 2. Initially, each process  $p_i$  waits until it collects inputs for a set of participating processes that includes at least one live set. Note that different processes may observe different participating sets. Every participating set S is associated with  $H^S \in HS(S, \mathcal{L})$ , some deterministically chosen hitting set of  $(S, \mathcal{L})$ . We say that  $H^S$  is a *resolver set for*  $\Pi$ : if S is the participating set, then we initiate  $|H^S|$  parallel sequences of agreement protocols with resolvers. Each sequence of agreement protocols can return at most one value and we guarantee that, eventually, every sequence is associated with a distinct resolver in  $H^S$ . In every such sequence j, each process  $p_i$  sequentially goes through an alternation of RAPs and CAs (see Section 5):  $RAP_j^1, CA_j^1, RAP_j^2, CA_j^2, \ldots$  The first RAP is invoked with the initially observed set of participants, and each next CA (resp., RAP) takes the output of the previous RAP (resp., CA) as an input. If some  $CA_i^\ell$  returns (*commit*, v), then  $p_i$  returns v as an output of the doorway protocol.

**Lemma 3** In every  $\mathcal{L}$ -resilient execution of the algorithm in Figure 2 starting with an input vector I, every correct process  $p_i$  terminates with an output value I'[i], and the resulting vector I' complies with property (1) in Section 4 with respect to I.

**Proof.** Consider any  $\mathcal{L}$ -resilient execution of the algorithm. We say that an agreement sequence j is *triggered* if some process  $p_i$  accessed  $RAP_i^1$ , the first RAP instance in the sequence, in line 19.

First, we observe that if a sequence j is triggered, then value  $S_j$  proposed to its first RAP instance by any process  $p_i$  (we simply say  $p_j$  proposes  $S_j$  to sequence j) is such that  $S_j$  is a set of participants containing a live set and  $h(S_j, \mathcal{L}) \geq j$  (lines 10 and 12–18). Recall that for all process subsets S and S' such that  $S \subseteq S'$ , we have  $h(S, \mathcal{L}) \leq h(S', \mathcal{L})|$ . By the properties of atomic snapshot (line 10), for every two sets  $S_j$  and  $S_\ell$  proposed to sequences j and  $\ell$  such that  $j < \ell$ , we have  $S_j \subset S_\ell$ .

Consider any  $\mathcal{L}$ -resilient execution of the algorithm. Let S be the set of participants in that execution. Every value returned by the protocol must be committed in some  $CA_j^{\ell_j}$ ,  $1 \leq j \leq |H^S|$ (line 21). By the properties of CA, every committed value is adopted by every process and then proposed to the next instance of RAP (line 19). By the properties of CA and RAP, every value returned by an instance of CA or RAP was previously proposed to the instance, and thus, no two different values can be returned in a given agreement sequence. Let m be the highest agreement sequence in which some value  $\bar{S}_m$  was returned. Thus, at most m distinct sets  $\bar{S}_1, \ldots, \bar{S}_m$  are returned in total and all of these sets are subsets of  $\bar{S}_m$ . Thus,  $\cup_{j,I'[j]\neq\perp}\bar{S}_j=\bar{S}_m$ . Recall that  $h(\cup_{j,I'[j]\neq\perp}\bar{S}_j,\mathcal{L})=h(\bar{S}_m,\mathcal{L})\geq m$  and property (1b) holds. Finally, resulting I' is an image of Iwith respect to some sequence  $S_1, \ldots, S_m$  where each  $S_j$  is a superset of a live set, and property (1a) also holds.

To show liveness, we first observe that in an  $\mathcal{L}$ -resilient execution, line 10 is non-blocking. Further, the body of the cycle in lines 12–24 contains no blocking statements. Thus, every correct process returns or goes through infinite number of cycles, trying to advance all triggered agreement sequences  $1, \ldots, |H^S|$ , where S is the participating set.

To prove that every correct process terminates, it is sufficient to show that at least one process returns. Indeed, suppose that a process  $p_i$  returns after having committed on a set  $S_j$  in some  $CA_j^{\ell_j}$ (line 21). If a process returns from an instance of RAP, then every correct process returns from the instance (property (ii) of RAP). Also, every correct process returns from each instance of CA. Thus, every correct process eventually reaches  $CA_j^{\ell_j}$ . By the properties of CA, every process that returns in  $CA_j^{\ell_j}$ , adopts or commits  $S_j$ . By properties (i) and (ii) of RAP, every correct process returns  $S_j$  in  $RAP_j^{\ell_j+1}$ . By the properties of CA, every correct process commits  $S_j$  in  $CA_j^{\ell_j+1}$ , and returns.

Suppose, by contradiction that no process ever returns. Eventually, all correct processes find the same set of participants S in line 12 and, thus, agree on the assigned hitting set  $H^S$  of  $(S, \mathcal{L})$ . In an  $\mathcal{L}$ -resilient execution, at most  $|H^S| - 1$  processes in  $H^S$  can fail. Otherwise,  $H^S$  is not a hitting set, since it does not meet every live set subset of S. In a given agreement sequence j, every  $RAP_j^{\ell_j}$  is eventually associated with a distinct resolver in  $H^S$ . Thus, by property (iii) of RAPs there exists an agreement sequence  $j \in \{1, \ldots, |H^S|\}$ , that is eventually associated with a distinct correct resolver  $p_r$  in  $H^S$ . Since, eventually,  $p_r$  is the only resolver of RAPs in sequence j and, by our assumption, agreement sequence j goes through an infinite number of RAP instances, there is an instance  $RAP_j^{\ell_j}$  in which  $p_r$  is the only resolver and, by property (iv) of RAPs, exactly one value  $S_j$  is returned to every correct process. Thus, every correct process commits on  $S_j$  in  $CA_j^{\ell_j}$ and returns — a contradiction.

### 6.2 Solving T through the doorway

Given the DW protocol described above, it is straightforward to solve T by simply invoking  $A_{\mathcal{L}}$  with the inputs provided by DW. Thus:

**Theorem 4** Task T is weakly  $\mathcal{L}$ -resiliently solvable if  $T_{\mathcal{L}}$  is weakly wait-free solvable.

**Proof.** By Lemma 3, every execution of DW starting with an input vector I makes sure that each process is assigned a set of inputs of T for some participating live set, and property (1) of  $T_{\mathcal{L}}$  is satisfied with respect to I and the resulting vector I'. Now we use  $A_{\mathcal{L}}$  with I', and, by the property (2) of  $T_{\mathcal{L}}$ , at least one participating set in  $\mathcal{L}$  obtains outputs.

## 7 From $\mathcal{L}$ -resilience to wait-freedom

Suppose T is weakly  $\mathcal{L}$ -resiliently solvable, and let A be the corresponding protocol. We describe a protocol  $A_{\mathcal{L}}$  that solves  $T_{\mathcal{L}}$  by wait-free simulating an  $\mathcal{L}$ -resilient execution of A.

For pedagogical reasons, we first present a simple *abstract simulation* (AS) technique. AS captures the intuition that a group of simulators sharing the initial view of the set of participating simulated codes should appear as a single simulator. Therefore, an arbitrary number of simulators starting with j distinct initial views should be able to simulate a (j - 1)-resilient execution.

Then we describe our specific simulation and show that it is an instance of AS, and thus it indeed generates a (j-1)-resilient execution of  $\mathcal{L}$ , where j is the number of distinct inputs of  $T_{\mathcal{L}}$ . By the properties of  $T_{\mathcal{L}}$ , we immediately obtain a desired  $\mathcal{L}$ -resilient execution of A.

#### 7.1 Abstract simulation

Suppose that we want to simulate a given *n*-process protocol, with the set of codes  $\{code_1, \ldots, code_n\}$ . Every instruction of the simulated codes (read or write) is associated with a unique *position* in  $\mathbb{N}$ . E.g., we can enumerate the instructions as follows: the first instructions of each simulated code, then the second instructions of each simulated code, etc.<sup>1</sup>

A state of the simulation is a map of the set of positions to colors  $\{U, IP, V\}$ , every position can have one of three colors: U (unvisited), IP (in progress), or V (visited). Initially, every position is unvisited. The simulators share a function next that maps every state to the next unvisited position to simulate. Accessing an unvisited position by a simulator results in changing its color to IP or V.

The state transitions of a position are summarized in Figure 3, and the rules the simulation follows are described below:

(AS1) Each process takes an atomic snapshot of the current state s and goes to position next(s) proposing state s. For each state s, the color of next(s) in state s is U.

- If an unvisited position is *concurrently* accessed by two processes proposing different states, then it is assigned color *IP*.

- If an unvisited position is accessed by every process proposing the same state, it may only change its color to V.

- If the accessed position is already V (a faster process accessed it before), then the process leaves the position unchanged, takes a new snapshot, and proceeds to the next position.

<sup>&</sup>lt;sup>1</sup>In fact, only read instructions of a read-write protocol need to be simulated since these are the only steps that may trigger more than one state transition of the invoking process [2, 4].



Figure 3: State transitions of a position in AS.

- (AS2) At any point in the simulation, the *adversary* may take an in-progress (IP) position and atomically turn it into V or take a *set* of unvisited (U) positions and atomically turn them into V.
- (AS3) Initially, every position is assigned color U. The simulation starts when the adversary changes colors of some positions to V.

We measure the progress of the simulation by the number of positions turning from U to V. Note that by changing U or IP positions to V, the adversary can potentially hamper the simulation, by causing some U positions to be accessed with different states and thus changing their colors to IP. However, the following invariant is preserved:

**Lemma 5** If the adversary is allowed at any state to change the colors of arbitrarily many IP positions to V, and throughout the simulation has j chances to atomically change any set of U positions to V, then at any time there are at most j - 1 IP positions.

**Proof.** Note that in the periods when the adversary does not move, every new accessed position may only become visited. Indeed, even though the processes run asynchronously, they march through the same sequence of snapshots. Every snapshot a process takes is either a fresh view that points to a currently unvisited position, or was previously observed by some process and it points to a visited position. In both cases, no new *IP* position can show up.

Now suppose that the adversary changed the color of a position from IP to V, thus decreasing the number of IP positions by one. This may result in one distinct inconsistent (not seen by any other simulator) state that points (through function *next*) to one distinct position. Thus, at most one position can be accessed with diverging states, resulting in at most one new IP position. Thus, in the worst case, the total number of IP positions remains the same.

Now suppose that j sets of positions changed their colors from U to V, one set at a time. The change of colors of the very first group starts the simulation and thus does not introduce IP positions. Again, every subsequent group of changes can result in at most one inconsistent state, which may bring up to j - 1 new IP positions in total.

## 7.2 Solving $T_{\mathcal{L}}$ through AS

Now we show how to solve  $T_{\mathcal{L}}$  by simulating a protocol A that weakly  $\mathcal{L}$ -resiliently solves T. First, we describe our simulation and show that it instantiates AS, which allows us to apply Lemma 5.

Every simulator  $s_i \in \{s_1, \ldots, s_n\}$  posts its input in the shared memory and then continuously simulates participating codes in  $\{code_1, \ldots, code_n\}$  of algorithm A in the breadth-first manner: the first command of every participating code, the second command of every participating code, etc. (A code is considered participating if its input value has been posted by at least one simulator.) The procedure is similar to BG-simulation, except that the result of every *read* command in the code is agreed upon through a distinct RAP instance. Simulator  $s_i$  is statically assigned to be the only resolver of every *read* command in *code<sub>i</sub>*.

The simulated read commands (and associated RAPs) are treated as positions of AS. Initially, all positions are U (unvisited). The outcome of accessing a RAP instance of a position determines its color. If the RAP is resolved (a value was posted in D in line 3 or 5), then it is given color V (visited). If the RAP is found stuck (waiting for an output in lines 4-6) by some process, then it is given color IP (in progress). Note that no RAP accessed with identical proposals can get stuck (property (ii) in Section 5). After accessing a position, the simulator chooses the first not-yet executed command of the next participating code in the round-robin manner (function *next*). For the next simulated command, the simulator proposes its current view of the simulated state, i.e., the snapshot of the results of all commands simulated so far (AS1).

Further, if a RAP of  $code_i$  is observed stuck by a simulator (and thus is assigned color IP), but later gets resolved by  $s_i$ , we model it as the adversary spontaneously changing the position's color from IP to V. Finally, by the properties of RAP, a position can get color IP only if it is concurrently accessed with diverging states (AS2).

We also have n positions corresponding to the input values of the codes, initially unvisited. If an input for a simulated process  $p_i$  is posted by a simulator, the initial position of  $code_i$  turns into V. This is modeled as the intrusion of the adversary, and if simulators start with j distinct inputs, then the adversary is given j chances to atomically change sets of U positions to V. The simulation starts when the first set of simulators post their inputs concurrently take identical snapshots (AS3).

Therefore, our simulation is an instance of AS, and thus we can apply Lemma 5 to prove the following result:

**Lemma 6** If the number of distinct values in the input vector of  $T_{\mathcal{L}}$  is j, then the simulation above blocks at most j-1 simulated codes.

**Proof.** Every distinct value S in an input vector of  $T_{\mathcal{L}}$  posted by a participating simulator results in the adversary changing some set of initial positions from U to V. (Note that the set can be empty if the inputs for set S has been previously posted by another simulator.) By Lemma 5, at any time there are at most j - 1 IP positions, i.e., at most j - 1 RAPs for read steps that are stuck. Thus, in the worst case, at most j - 1 simulated codes can block forever.

The simulated execution terminates when some simulator observes outputs of T for at least one participating live set. Finally, using the properties of the inputs to task  $T_{\mathcal{L}}$  (Section 4), we derive that eventually, some participating live set of simulated processes obtain outputs. Thus, using Theorem 4, we obtain:

### **Theorem 7** T is weakly $\mathcal{L}$ -resiliently solvable if and only if $T_{\mathcal{L}}$ is weakly wait-free solvable.

**Proof.** Suppose that we are given an input vector I' of  $T_{\mathcal{L}}$  with j distinct values, each value consists of inputs of T for a set of processes containing a live set. By property (1a) of  $T_{\mathcal{L}}$  (Section 4), these input sets are consistent with some input vector I of T. We call the set of simulated processes that obtain inputs of T the *participating set* of T, and denote by  $\Pi'$ .

Since every simulated step goes through a RAP with a single resolver, by property (iv) of RAP (Section 5), simulators agree on the result of every simulated read command, and thus we simulate a correct execution of algorithm A (solving T).

By Lemma 6, at most j-1 processes can fail in the simulated execution of A. By property (1b) of  $T_{\mathcal{L}}$ , the size of the hitting set of the participating set  $H(\Pi', \mathcal{L})$  is at least j. Thus, there is at least one live set in  $\Pi'$  that contains no faulty simulated process. This live set accepts infinitely many steps in the simulated execution of A and, by weak  $\mathcal{L}$ -resilient solvability, must eventually output. This set of outputs constitutes the output of  $T_{\mathcal{L}}$ . Since the output comes from an execution of A starting with I, the output satisfies property (2) of  $T_{\mathcal{L}}$ .

Thus, the algorithm indeed solves  $T_{\mathcal{L}}$ .

## 8 Related work

The equivalence between t-resilient task solvability and wait-free task solvability has been initially established for colorless tasks in [2, 4], and then extended to all tasks in [8]. In this paper, we consider a wider class of assumptions than simply t-resilience, which can be seen as a strict generalization of [8].

Generalizing t-resilience, Janqueira and Marzullo [18] considered the case of dependent failures and proposed describing the allowed executions through cores and survivor sets which roughly translate to our hitting sets and live sets. Note that the set of survivor sets (or, equivalently, cores) exhaustively describe only superset-closed adversaries. More general adversaries introduced by Delporte et al. [5] are defined as a set of exact correct sets. It is shown in [5] that the power of an adversary  $\mathcal{A}$  to solve colorless tasks is characterized by  $\mathcal{A}$ 's disagreement power, the highest k such that k-set agreement cannot be solved assuming  $\mathcal{A}$ : a colorless task T is solvable with adversary  $\mathcal{A}$ of disagreement power k if and only if it is solvable k-resiliently. Herlihy and Rajsbaum [15] (concurrently and independently of this paper) derived this result for a restricted set of superset-closed adversaries with a given core size using elements of modern combinatorial topology. Theorem 1 in this paper derives this result directly, using very simple algorithmic arguments.

Considering only colorless tasks is a strong restriction, since such tasks allow for definitions that only depend on *sets* of inputs and *sets* of outputs, regardless of which processes actually participate. (Recall that for colorless tasks, solvability and our weak solvability are equivalent.) The results of this paper hold for all tasks. On the other hand, as [15], we only consider the class of superset-closed adversaries. This filters out some popular liveness properties, such as *obstruction-freedom* [13]. Thus, our contributions complement but do not contain the results in [5]. A protocol similar to our RAP was earlier proposed in [17].

## 9 Side remarks and open questions

**Doorways and iterated phases.** Our characterization shows an interesting property of weak  $\mathcal{L}$ -resilient solvability: To solve a task T weakly  $\mathcal{L}$ -resiliently, we can proceed in two logically synchronous phases. In the first phase, processes wait to collect "enough" input values, as prescribed by  $\mathcal{L}$ , without knowing anything about T. Logically, they all finish the waiting phase simultaneously. In the second phase, they all proceed wait-free to produce a solution. As a result, no process is waiting on another process that already proceeded to the wait-free phase. Such phases are usually referred to as iterated phases [3]. In [8], some processes are waiting on others to produce an output and consequently the characterization in [8] does not have the iterated structure.

 $\mathcal{L}$ -resilience and general adversaries. The power of a general adversary of [5] is not exhaustively captured by its hitting set. In a companion paper [11], we propose a simple characterization of the set consensus power of a general adversary  $\mathcal{A}$  based on the hitting set sizes of its recursively proper subsets. Extending our equivalence result to general adversaries and getting rid of the weak solvability assumption are two challenging open questions.

## References

- Y. Afek, H. Attiya, D. Dolev, E. Gafni, M. Merritt, and N. Shavit. Atomic snapshots of shared memory. J. ACM, 40(4):873–890, 1993.
- [2] E. Borowsky and E. Gafni. Generalized FLP impossibility result for t-resilient asynchronous computations. In STOC, pages 91–100. ACM Press, May 1993.
- [3] E. Borowsky and E. Gafni. A simple algorithmically reasoned characterization of wait-free computation (extended abstract). In PODC '97: Proceedings of the sixteenth annual ACM symposium on Principles of distributed computing, pages 189–198, New York, NY, USA, 1997. ACM Press.
- [4] E. Borowsky, E. Gafni, N. A. Lynch, and S. Rajsbaum. The BG distributed simulation algorithm. Distributed Computing, 14(3):127–146, 2001.
- [5] C. Delporte-Gallet, H. Fauconnier, R. Guerraoui, and A. Tielmann. The disagreement power of an adversary. In *DISC*, pages 8–21, 2009.
- [6] M. J. Fischer, N. A. Lynch, and M. S. Paterson. Impossibility of distributed consensus with one faulty process. J. ACM, 32(2):374–382, Apr. 1985.
- [7] E. Gafni. Round-by-round fault detectors (extended abstract): Unifying synchrony and asynchrony. In Proceedings of the 17th Symposium on Principles of Distributed Computing, 1998.
- [8] E. Gafni. The extended BG-simulation and the characterization of t-resiliency. In STOC, pages 85–92, 2009.
- [9] E. Gafni and E. Koutsoupias. Three-processor tasks are undecidable. SIAM J. Comput., 28(3):970–983, 1999.
- [10] E. Gafni and P. Kuznetsov. L-resilient adversaries and hitting sets. CoRR, abs/1004.4701, 2010. http://arxiv.org/abs/1004.4701.
- [11] E. Gafni and P. Kuznetsov. Turning adversaries into friends: Simplified, made constructive, and extended. In OPODIS, 2011.
- [12] M. Herlihy. Wait-free synchronization. ACM Trans. Prog. Lang. Syst., 13(1):123-149, Jan. 1991.
- [13] M. Herlihy, V. Luchangco, and M. Moir. Obstruction-free synchronization: Double-ended queues as an example. In *ICDCS*, pages 522–529, 2003.
- [14] M. Herlihy and S. Rajsbaum. The decidability of distributed decision tasks (extended abstract). In STOC, pages 589–598, 1997.
- [15] M. Herlihy and S. Rajsbaum. The topology of shared-memory adversaries. In PODC, 2010.
- [16] M. Herlihy and N. Shavit. The topological structure of asynchronous computability. J. ACM, 46(2):858– 923, 1999.
- [17] D. Imbs and M. Raynal. Visiting gafni's reduction land: From the bg simulation to the extended bg simulation. In SSS, pages 369–383, 2009.
- [18] F. Junqueira and K. Marzullo. A framework for the design of dependent-failure algorithms. Concurrency and Computation: Practice and Experience, 19(17):2255–2269, 2007.

[19] M. Loui and H. Abu-Amara. Memory requirements for agreement among unreliable asynchronous processes. Advances in Computing Research, 4:163–183, 1987.