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OWA-based fuzzy m-ary adjacency relations in Social Network Analysis

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Abstract

In this paper we propose an approach to Social Network Analysis (SNA) based on fuzzy m-ary adjacency relations. In particular, we show that the dimension of the analysis can naturally be increased and interesting results can be derived. Therefore, fuzzy m-ary adjacency relations can be computed starting from fuzzy binary relations and introducing OWA-based aggregations. The behavioral assumptions derived from the measure and the exam of individual propensity to connect with other suggest that OWA operators can be considered particularly suitable in characterizing such relationships.

Keywords: reciprocal relation; fuzzy preference relation; priority vector; normalization.

1 Introduction

Social Network Analysis (SNA) is a relatively new and still developing subject that focuses on the study of social relations [30, 35] as a branch of the broader discipline named network analysis [26] whose main object is studying the relationships between objects belonging to one or more universal sets. SNA focuses its attention on social objects and has principally concerned with the structure and effects of relations between people, groups or organizations, rather than on individual psychological attributes. Nevertheless, as pointed out in [31], psychological attributes and behavioral issues are likely to influence the dynamics of networks of individuals. For instance, the role of individual differences in shaping organizational networks has been examined from several points of view [3], compelling as well the study of how similarity in personal relationships and social context affect each other [23]. For better understanding of roles played by actors in social networks the so called centrality indices have been introduced, accordingly a member is viewed as central whenever she or he has a high number of connections with a high number of different co-members [2, 4, 7, 15]. Since paths play a central role in the functioning of most of the networks, it is not surprising that a relevant number of centrality measures quantify importance with respect to the sharing of paths in the network. Betweenness centrality, as a measure of how many geodesic paths cross a given vertex, is one of the most popular and was introduced [14] to quantify the control of a given actor over the flow of information in the network. Therefore, this measure can be used to provide an ordering of the vertices in terms of their individual importance, but it does not provide any description of the way in which subsets of vertices influence the network as a whole. As pointed out in [21], vertex betweenness centrality can be naturally extended to sets of vertices either defining the betweenness of a set in terms of geodesic paths that pass through at least one of the vertices in the set, or in terms of geodesic paths that pass through all vertices in the set. Everett and Borgatti [12] introduced the first type of extension and called it group betweenness centrality, the second type was introduced in [21] showing that the two notions are intimately related. The relationship between the two approaches has been mathematically characterized showing how the betweenness of a group of an arbitrary number of vertices can be bounded above and below by quantities involving only the betweenness of the individual vertices and the co-betweenness of pairs of these vertices. In this way a direct insight into the composition of subgroups of vertices is provided and it can be used in evaluating the robustness of potential coalitions and the deploying of consensual dynamics. One of the most commonly used tool for representing social relationship among a set of actors in a network is the adjacency matrix representing a binary relation. The first limitation of binary relations is that they can be used only for representing pairwise adjacency, the second one is that the dichotomy is not suitable for shaping the strength of the adjacency relationship involving several social and individual attributes. One way to overcome the first limitation was introduced in [6] and it was based on the concept of multirelational systems [28]. The generalization of the definition of relation through the introduction of fuzziness opens the way to the extension of Social Network Analysis to contexts in which the network could be represented using fuzzy graphs [24], taking care of the vagueness influencing the relationships among the actors involved in the social dynamics and of the qualitative nature of the actors' attributes as well.

Fuzzy approaches to SNA provided so far are actually very few. In [25] a technique to model multi-modal social networks as fuzzy social networks is proposed. The technique is based on k-modal fuzzy graphs determined using the union operation on fuzzy graphs and a new operator called consolidation operator. The notion of regular equivalence [5] was generalized in [13] introducing the notion of regular similarity, represented by a fuzzy binary relation that describes the degree of similarity between actors in the social network. In [10] the problem of partitioning the nodes of a social network in overlapping groups allowing for multiple memberships and varied levels of membership was solved introducing the so called fuzzy groups. In [34], starting from the introduction of the natural connection between graph theory and granular computing, human-focussed concepts associated with social networks are formalized using set-based relational network theory and fuzzy sets. A softening of the concept of node importance (centrality of a node) is provided, considering the number of close connections. Kokabu et al. [20] proposed a model for evaluating reciprocity of networks represented by means of fuzzy binary relations. In literature, it was also proposed [8] to use fuzzy relations for defining a characterization for fuzzy m-ary relations and therefore expand the dimension of the analysis for $m > 2$.

The approach proposed in this paper takes advantage of the ability of fuzzy relations [19, 36, 37] to model uncertainty permeating the relationships between the actors in the network, and of the OWA operators [32, 33] to move continuously from non-compensatory to full-compensatory situation and characterizing therefore the attitude of the actors to connect each other. The paper is outlined as follows. In section 2 we offer a presentation of SNA and adjacency matrix, which is the main tool to perform the analysis. In the same section we show that adjacency relations can be valued (cardinal) relations and that fuzzy adjacency relations are simply a special case of valued relations. Having presented that, in section 3 fuzzy m-ary adjacency relations are defined and a method based on aggregating functions for estimating them is presented. We claim, in section 4, that OWA operators satisfy some reasonable properties and that they can be employed as suitable aggregating functions to increase the dimension of the analysis. In section 5 we discuss an example and, finally, in section 6, we present our conclusions.

2 Crisp, valued and fuzzy adjacency relations in SNA

As already mentioned, SNA is the branch of network analysis devoted to studying and representing relationships between 'social' objects. To formalize, SNA mainly explores relationships between objects belonging to an universal set $X = \{x_1, \ldots, x_n\}$ and in order to achieve its aim, some mathematical properties of relations are utilized. More specifically, a binary relation on a single set, which is the most popular kind of relation used in the SNA, is a relation $A \subseteq X \times X$, whose characteristic function $\mu_A : X \times X \to \{0,1\}$ is defined as

$$\mu_A(x_i, x_j) = \begin{cases} 1, & \text{if } x_i \text{ is related to } x_j \\ 0, & \text{if } x_i \text{ is not related to } x_j \end{cases}$$

By definition [19], adjacency relations are reflexive, $\mu_A(x_i, x_i) = 1$, and symmetric, $\mu_A(x_i, x_j) = \mu_A(x_j, x_i)$. Note that no transitivity condition is required to hold. Moreover, if $a_{ij} := \mu_A(x_i, x_j)$ and X is reasonably not too large, then an adjacency matrix $\mathbf{A} = (a_{ij})_{n \times n}$ is a convenient way of representing a relation.

Some scholars in the field claim that $\mathbf{A}$ has its strong point in being a good synthesis of all the pairwise relations between elements of X. In contrast, according to some others, $\mathbf{A}$ is too poor of information, i.e. it does not contain information about the degree to which the relations between two elements hold. Therefore it may happen that it treats in the same way very different cases, without discriminating among situations where intensities of relationship may be very different. Indeed, many examples may be brought in order to support the latter point of view.

Some methods have already been proposed in order to overcome the problem related with the lack of information about the intensity of relationship between elements of a pair. For instance, a discrete scale can be adopted and a value be assigned to each entry a_{ij} to denote the intensity of relation between x_i and x_j. This approach, based on *valued adjacency relations*, is the most widely used in order to overcome the problem of unvalued relations.

Here, we want to propose an alternate approach based on *fuzzy sets* theory [36] in order to obtain a *fuzzy* adjacency relation. A binary fuzzy relation on a single set, $R_2 \subseteq X \times X$, is defined through the following membership function

$$\mu_{R2} : X \times X \to [0,1] \tag{1}$$

and also in this case, putting $r_{ij} := \mu_{R2}(x_i, x_j)$, a fuzzy relation can be conveniently represented by a matrix $\mathbf{R} = (r_{ij})_{n \times n}$ where the value of each entry is the degree to which the relation between x_i and x_j holds. In other words, the value of $\mu_{R2}(x_i, x_j)$ is the answer to the question: 'how strong is the relationship between x_i and x_j?'. Therefore, in the context of SNA

$$\mu_{R2}(x_i, x_j) = \begin{cases} 1, & \text{if } x_i \text{ has the strongest possible degree} \\ & \text{of relationship with } x_j \\ \gamma \in]0,1[& \text{if } x_i \text{ is, to some extent, related to } x_j \\ 0, & \text{if } x_i \text{ is not related with } x_j \end{cases}$$

Fuzzy adjacency relations, as well as crisp adjacency relations, are here assumed to be reflexive and symmetric. It is useful to spend some words about symmetry. A fuzzy binary relation is symmetric if and only if

$$\mu_{R2}(x_i, x_j) = \mu_{R2}(x_j, x_i) \quad i, j = 1, \ldots, n. \tag{2}$$

Although the assumption of symmetry is a simplification, it is of great help for the model because, due to it, such relations can be represented by means of undirected graphs and problems related with the so-called combinatorial explosion are partially avoided. Furthermore, in many real-world cases, symmetry is spontaneously satisfied by the nature of the relationship.

At this point we remind that:

- A fuzzy relation contains more information than a crisp one and the former can overcome some drawbacks of the latter. See for example [19], where fuzzy adjacency relations are called fuzzy compatibility or proximity relations.

- We can shift from the fuzzy approach to the crisp one thanks to the α-cuts. An α-cut is a crisp relation defined by

$$\mu_A(x_i, x_j) = \begin{cases} 1, & \text{if } \mu_{R2}(x_i, x_j) \geq \alpha \\ 0, & \text{if } \mu_{R2}(x_i, x_j) < \alpha\,. \end{cases}$$

  For instance, given

$$\mathbf{R} = \begin{pmatrix} 1 & 0.7 & 0.3 & 0.7 \\ 0.7 & 1 & 0.1 & 0.8 \\ 0.3 & 0.1 & 1 & 0.2 \\ 0.7 & 0.8 & 0.2 & 1 \end{pmatrix}, \tag{3}$$

  its α-cut with $\alpha = 0.5$ is

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} \tag{4}$$

- Applying fuzzy relations to SNA, we can extend most of the techniques employed for analyzing crisp adjacency matrices. A significant example, which will be used later on in this discussion, is the normalized index of local centrality, that is

$$C(x_i) = \frac{1}{n-1} \sum_{\substack{j=1 \\ j \neq i}}^{n} r_{ij}\,. \tag{5}$$

  If $c_i := C(x_i)$ and $\mathbf{c} = (c_1, \ldots, c_n)$, then we can refer to $\mathbf{R}$ in (3) and find that $\mathbf{c} = (\frac{17}{30}, \frac{3}{5}, \frac{7}{20}, \frac{185}{300})$. This result is more informative than the same index computed on $\mathbf{A}$ in (4), i.e. $\mathbf{c} = (\frac{2}{3}, \frac{2}{3}, 0, \frac{2}{3})$

- It is possible to exploit already known indices for fuzzy sets as, for instance, a measure of fuzziness [11] which would estimate how much information we would have lost if we had used crisp relations instead of fuzzy ones.

- Exploiting a fuzzy adjacency relation solves all the borderline cases, i.e. all the cases where it is difficult to establish whether x_i and x_j are related or not. To tell the truth, this property is, to some extent, shared with valued adjacency relations. However, a fuzzy relation is much easier to be interpreted from a logical point of view.

Let's note that the whole issue can be addressed thanks to graph theory. In this case there are n nodes $x_1, x_2, \ldots, x_n$ and $\frac{n(n-1)}{2}$ edges connecting them. Hence, nodes are nothing else but elements of the universe set X, weights of edges are $\mu_{R2}(x_i, x_j)$ and the graph is $G = \langle X, R \rangle$. Therefore, the problem can also be addressed in a graphical way with $\mu_{R2}(x_i, x_j)$ representing the "thickness" of the edge between x_i and x_j.

One might wonder how it would be possible to define a fuzzy adjacency matrix starting from real-world information. In all those cases where it is difficult to define it directly we can derive it from valued adjacency matrices, e.g. some evidence under the form of numerical data about the relationships is available. Let us assume that a valued adjacency matrix, $\mathbf{V} = (v_{ij})_{n \times n}$, exists with $v_{ij} \in \mathbb{R}_{\geq}$. If it is possible to define a maximal level for the valued graph, say v^*, such that it represents the maximum possible value of relationship, then, with v^* playing the role of the upper bound for entries v_{ij}, we can rescale each v_{ij} into a r_{ij} thanks to a suitable mapping $r_{ij} = h(v_{ij})$, $h : [0, v^*] \rightarrow [0, 1]$.

3 Fuzzy m-ary adjacency relations and the degree of social relationship

In this section we propose an extension of the analysis involving m-dimensional relations with $2 \leq m \leq n$. If we do so, then each element of the m-ary relation is the degree of social relationship among the m elements contained in the m-tuple which is taken into account. Analogously to the binary case, it is straightforward to define a fuzzy m-ary relation.

Definition 1. *A fuzzy m-ary relation R_m on a single set X is a fuzzy subset of X^m defined by means of the membership function*

$$\mu_{R_m} : X^m \rightarrow [0, 1] \tag{6}$$

Then, for $p_1, \ldots, p_m \in \{1, \ldots, n\}$, the membership function characterizing fuzzy m-ary relations is the following

$$\mu_{Rm}(x_{p_1},\ldots,x_{p_m}) = \begin{cases} 1, & \text{if } x_{p_1},\ldots,x_{p_m} \text{ are definitely related} \\ \gamma \in]0,1[& \text{if } x_{p_1},\ldots,x_{p_m} \text{ are, to some extent, related} \\ 0, & \text{if } x_{p_1},\ldots,x_{p_m} \text{ are definitely not related} \end{cases}$$

The logic underlying the membership function remains substantially unchanged and therefore properties of reflexivity and symmetry are extended to the m-dimensional case in the following way. A fuzzy m-ary relation is reflexive if and only if

$$\mu_{Rm}(x_i, x_i, \ldots, x_i) = 1, \quad i = 1, \ldots, n\,.$$

A fuzzy m-ary relation is symmetric if and only if for any $p_1, p_2, \ldots, p_m \in \{1, \ldots, n\}$ it is

$$\mu_{Rm}(x_{p_1},\ldots,x_{p_m}) = \mu_{Rm}(x_{q_1},\ldots,x_{q_m})$$

where $(x_{q_1}, x_{q_2}, \ldots, x_{q_m})$ is any permutation of $(x_{p_1}, x_{p_2}, \ldots, x_{p_m})$.
A fuzzy m-ary relation satisfying the reflexivity and symmetry properties is called a fuzzy m-ary adjacency relation.
At this point, having defined fuzzy m-ary adjacency relations, it is the case to highlight the difference between an element of a fuzzy m-ary adjacency relation and a clique [22, 29, 35]. Namely, a clique of a graph is a maximum complete subgraph whereas, if we deal with m-ary relations and the contrary is not made explicit, the value $\mu_{Rm}(x_{p_1},\ldots,x_{p_m})$ simply states, by means of the bounded unipolar scale $[0,1]$, the degree to which the relation holds, without taking into account any maximality condition.

However, problems arise when we try to elicit R_m in a direct way, as it is certainly not a trivial operation, especially when m is large enough. As we have seen in the previous section, in social network analysis adjacency relations in the form $R_2 \subseteq X \times X$ are often used and therefore degrees of relationship over pairs are known. That is why we propose an effective way to elicit R_m using the information embedded in the fuzzy binary adjacency relation on the same universal set X. We propose to recursively calculate the degree of relationship over m–tuples by means of the degree of relationship over pairs using aggregation functions $\rho_3, \ldots, \rho_m$ satisfying a fixed set of assumptions, as described in the next section. More precisely, from a given fuzzy binary adjacency relation R_2 we calculate the corresponding fuzzy ternary adjacency relation R_3, then from R_3 we calculate R_4, and so on. In general, to estimate $\mu_{R_m}(x_{p_1},\ldots,x_{p_m})$ we construct it recursively in the following way

$$\mu_{R_k}(x_{p_1},\ldots,x_{p_k}) = \rho_k(\mu_{R_{k-1}}(x_{p_1},\ldots,x_{p_{k-1}}),\ldots,\mu_{R_{k-1}}(x_{p_2},\ldots,x_{p_k})), \tag{7}$$

for $k = 3, \ldots, m$. The k arguments of the function ρ_k themselves are functions of $k-1$ variables. For example, the first argument of the function ρ_k in (7) is

$$\begin{aligned} \mu_{R_{k-1}}(x_{p_1}, \ldots, x_{p_{k-1}}) =& \rho_{k-1}(\mu_{R_{k-2}}(x_{p_1}, \ldots, x_{p_{k-2}}), \ldots \\ & \ldots, \mu_{R_{k-2}}(x_{p_2}, \ldots, x_{p_{k-1}})) \,. \end{aligned} \tag{8}$$

Note that to calculate $\mu_{R_k}(x_{p_1}, \ldots, x_{p_k})$ in (7) we need to aggregate precisely k values of $\mu_{R_{k-1}}(\cdot)$. Since we assume that the symmetric property holds, the order of the arguments in $\mu_{R_m}(x_{p_1}, \ldots, x_{p_m})$ is not relevant and we can assume, without loss of generality, $x_{p_1} \leq \cdots \leq x_{p_m}$. Therefore, a fuzzy m–ary adjacency relation on a set X requires $\binom{n+m-1}{m}$ relationship values to be completely defined, i.e. the number of combinations with repetition of size m from a set of n elements.

The most important case for applications is that of $\mu_{R_m}(x_{p_1}, \ldots, x_{p_m})$ with all different arguments, i.e. allowing no repetition. This corresponds to take into account only groups of distinct social objects and in the following we will focus on this case. Under this assumption, a fuzzy m–ary adjacency relation on a set X requires only $\binom{n}{m}$ relationship values to be completely defined.

Let us introduce the notation which will be used hereafter.

Definition 2. *Given a finite non empty set $X = \{x_1, \ldots, x_n\}$, we denote by $\mathcal{F}_m(X)$ the family of subsets of X containing m elements,*

$$\mathcal{F}_m(X) = \{A \subseteq X; |A| = m\} \tag{9}$$

Example 1 Given $X = \{x_1, x_2, x_3, x_4\}$, it is

$$\begin{aligned} \mathcal{F}_1(X) &= \{\{x_1\}, \{x_2\}, \{x_3\}, \{x_4\}\} \\ \mathcal{F}_2(X) &= \{\{x_1, x_2\}, \{x_1, x_3\}, \{x_1, x_4\}, \{x_2, x_3\}, \{x_2, x_4\}, \{x_3, x_4\}\} \\ \mathcal{F}_3(X) &= \{\{x_1, x_2, x_3\}, \{x_1, x_2, x_4\}, \{x_1, x_3, x_4\}, \{x_2, x_3, x_4\}\} \\ \mathcal{F}_4(X) &= \{x_1, x_2, x_3, x_4\}. \end{aligned}$$

In general, with $|X| = n$ and $m \leq n$, the cardinality of $\mathcal{F}_m(X)$ is

$$|\mathcal{F}_m(X)| = \binom{n}{m}.$$

It turns out that in order to calculate the degree of relationship among the m distinct objects of a set $\{x_{p_1}, \ldots, x_{p_m}\} \subseteq X$, we need to calculate first the degrees of relationship over all its subsets with cardinality greater than one and less than m. The total number of these subsets is

$$\binom{m}{2} + \cdots + \binom{m}{m-1} = 2^m - m - 2 \; . \tag{10}$$

Let us now draw our attention again to the problem of aggregating relationship values in order to construct higher dimensional relations. The choice of aggregation functions $\rho_3, \rho_4, \ldots, \rho_m$ plays clearly a crucial role in determining the fuzzy m–ary adjacency relation μ_{R_m}. In the following section, therefore, we will focus on the suitable properties we require for these functions.

4 Some properties of the OWA-based aggregating function

This section is devoted to present and justify the assumptions that we make regarding $\rho_3, \ldots, \rho_m$. First of all, as we already said, we require $\rho_3, \ldots, \rho_m$ to be 'aggregation functions'. We recall the corresponding definition [1]

Definition 3 (Aggregation function)**.** *An aggregation function is a function of $m > 1$ arguments that maps the (m-dimensional) unit cube onto the unit interval, $f : [0,1]^m \to [0,1]$, with the properties*

- $f(0, \ldots, 0) = 0$ *and* $f(1, \ldots, 1) = 1$
- $\mathbf{a} \leq \mathbf{b}$ *implies* $f(\mathbf{a}) \leq f(\mathbf{b})$ *for all* $\mathbf{a}, \mathbf{b} \in [0,1]^m$ *(monotonicity)*

Moreover, we are going to propose some other properties which, in our opinion, should be satisfied by every ρ_m, $m = 3, \ldots, n$.

1. idempotency $\rho_m(a, \ldots, a) = a$. Therefore, if the objects of some set are pairwise related with degree a, then we assume that the intensity of relation computed on the tuple containing those objects has value a as well.
2. commutativity, $\rho_m(a_1, \ldots, a_m) = \rho_m(a_{q_1}, \ldots, a_{q_m})$ where $(q_1, \ldots, q_m)$ is any permutation of the indices. This property is required to hold because fuzzy adjacency relations are symmetrically defined for all $m = 2, \ldots, n$.
3. *strict* monotonicity: $\rho_m(a_1, \ldots, a_m) > \rho_m(b_1, \ldots, b_m)$ if $a_i \geq b_i \; \forall i$ and there exists at least one j such that $a_j > b_j$. Strict monotonicity is asked to hold in order to overcome some evaluation problems which would arise if we used non-strictly monotonically increasing functions

as, for instance, the geometric mean $g(\cdot)$. To give an example, substituting g to ρ_m we would have $g(1,\ldots,1,0) = g(0,\ldots,0)$, which is not a desirable result from the social analysis point of view

4. continuity. This is essentially a technical assumption.

These four assumptions lead us to choose within a restricted class of averaging operators. Namely, ρ_m should be an aggregating function respecting properties 1–4. It is possible to see that the geometric mean, as mentioned above, is excluded because it is not strictly monotone. The weighted arithmetic mean is also excluded because it is not commutative.

Conversely, provided that $0 < w_i < 1$, any OWA operator [32, 33] satisfies the listed properties [9]. Choosing between OWA operators would be anything but arbitrary as an index of orness is associated to each OWA and several approaches has been developed to find an OWA operator with a given level of orness and optimizing some other properties as, for instance, entropy and variance. Moreover, OWA operators cover a range of some well known aggregating functions, as they can be meant as trade offs between the min and the max operators.

Hence, in our case, as we use OWA operators, they should be defined such that $w_i \in]0,1[$ so that they are strictly monotonically increasing functions in all the terms. Let us therefore give the following modified definition of OWA.

Definition 4 (Strictly monotone OWA operator)**.** *A strictly monotone OWA operator of dimension m is a mapping $F : \mathbb{R}^m \to \mathbb{R}$, that has an associated weighting vector $\mathbf{w} = (w_1,\ldots,w_m)$ such that $0 < w_i < 1$ and*

$$\sum_{i=1}^{m} w_i = 1$$

Furthermore

$$F(a_1,\ldots,a_m) = w_1 b_1 + \cdots + w_m b_m = \sum_{j=1}^{m} w_j b_j$$

where b_j is the j-th largest element of the bag $A = \langle a_1,\ldots,a_m\rangle$.

5 Example

A number of examples explaining the utility of m-ary relations can be brought. Let us, for example, consider the following fuzzy binary adjacency relation

$$\begin{pmatrix} 1 & 0.8 & 0.8 & 0.6 & 0.2 & 0.4 & 0.3 \\ 0.8 & 1 & 0.9 & 0.2 & 0.1 & 0.3 & 0.3 \\ 0.8 & 0.9 & 1 & 0.3 & 0.2 & 0.3 & 0.3 \\ 0.6 & 0.2 & 0.3 & 1 & 0.7 & 0.7 & 0.4 \\ 0.2 & 0.1 & 0.2 & 0.7 & 1 & 0.9 & 0.7 \\ 0.4 & 0.3 & 0.3 & 0.7 & 0.9 & 1 & 0.5 \\ 0.3 & 0.3 & 0.3 & 0.4 & 0.7 & 0.5 & 1 \end{pmatrix} \tag{11}$$

which we can suppose be representing of fuzzy adjacency relations between decision makers. Although we bring an example, it is easy to imagine several other possible applications indeed. Following our proposal, it is possible to estimate a ternary fuzzy adjacency relation simply by applying function ρ_3 according to (7). Let us further assume that, hereafter, function ρ_m is univocally determined as an OWA operator of dimension m with maximal entropy and $orness(w) = 0.4$, which, in the special case with $m = 3$, is $w \simeq (0.238371, 0.323257, 0.438371)$. Thus, the result, would be

$$\begin{aligned} \mu_{R3}(x_1, x_2, x_3) &= \rho_3(\mu_{R2}(x_1, x_2), \mu_{R2}(x_1, x_3), \mu_{R2}(x_2, x_3)) \simeq 0.823837 \\ \mu_{R3}(x_1, x_2, x_4) &= \rho_3(\mu_{R2}(x_1, x_2), \mu_{R2}(x_1, x_4), \mu_{R2}(x_2, x_4)) \simeq 0.472326 \\ \vdots &= \vdots \\ \mu_{R3}(x_4, x_6, x_7) &= \rho_3(\mu_{R2}(x_4, x_6), \mu_{R2}(x_4, x_7), \mu_{R2}(x_6, x_7)) \simeq 0.503837 \\ \mu_{R3}(x_5, x_6, x_7) &= \rho_3(\mu_{R2}(x_5, x_6), \mu_{R2}(x_5, x_7), \mu_{R2}(x_6, x_7)) \simeq 0.66 \end{aligned}$$

It could be particularly interesting to pick the element of $\mathcal{F}_m(X)$ such that the value of its membership function is maximal,

$$\max\{\mu_{R_m}(x_{p_1}, x_{p_2}, \ldots, x_{p_m})|\ p_1, \ldots, p_m \in \{1, \ldots, n\},\ p_1 < p_2 < \cdots < p_m\}\ , \tag{12}$$

In our case the maximum value of membership function is 0.823837 and it is achieved by the triplet (x_1, x_2, x_3).

A rather special case is that involving the $m^\star$-ary relation where $m^\star$ is defined as the integer part of $n/2 + 1$, more formally $m^\star = \lfloor n/2 + 1 \rfloor$, because the associated subset of X would be a minimum winning coalition. In our case $m^\star = 4$ and $w = (0.167087, 0.213266, 0.272208, 0.34744)$ with

$$\begin{aligned} \mu_{R4}(x_1, x_2, x_3, x_4) &= \rho_4(\mu_{R3}(x_1, x_2, x_3), \ldots, \mu_{R3}(x_2, x_3, x_4)) \simeq 0.514996 \\ \mu_{R4}(x_1, x_2, x_3, x_5) &= \rho_4(\mu_{R3}(x_1, x_2, x_4), \ldots, \mu_{R3}(x_2, x_3, x_5)) \simeq 0.402686 \\ \vdots &= \vdots \\ \mu_{R4}(x_3, x_5, x_6, x_7) &= \rho_4(\mu_{R3}(x_3, x_5, x_6), \ldots, \mu_{R3}(x_5, x_6, x_7)) \simeq 0.41189 \\ \mu_{R4}(x_4, x_5, x_6, x_7) &= \rho_4(\mu_{R3}(x_4, x_5, x_6), \ldots, \mu_{R3}(x_5, x_6, x_7)) \simeq 0.595482 \end{aligned}$$

with the maximum being 0.595482, achieved by (x_4, x_5, x_6, x_7).

This latter proposal can be refined if we assume that every element $x_i \in X$ has a specific weight ω_i denoting its relative importance. Let us consider the weight vector

$$\boldsymbol{\omega} = (\omega_1, \ldots, \omega_n) \quad \text{s.t.} \sum_{i=1}^{n} \omega_i = 1 \ , \omega_i \geq 0 \ \forall i. \tag{13}$$

Then, we can perform an analysis similar to that described above by assuming that parameter m is free, not necessarily equal to $m^\star$, and by requiring that the sum of the weights associated to the considered m elements is equal or greater than a given majority threshold $0 < t \leq 1$. In light of these observations, the optimization problem is

$$\begin{aligned} \max\{\mu_{R_m}(x_{p_1}, \ldots, x_{p_m}) \mid p_1, \ldots, p_m \in \{1, \ldots, n\}, p_1 < \cdots < p_m, \\ \sum_{i=1}^{m} \omega_{p_i} > t, \ m = 2, \ldots, n-1\} \ . \end{aligned} \tag{14}$$

Some comments on (14) could be useful to better understand the involved optimization. In (14) we are still interested in the strongest coalition, but the constraint of having a fixed number of elements is replaced by a constraint on a majority threshold t to be satisfied by the sum of the weights of the coalition's elements. That is, coalitions with different number m of elements are taken into account, provided that they fulfil threshold t. Note that large values of μ_{R_m} can be easily achieved if the number m of elements is small, while the constraint $\sum_{i=1}^{m} \omega_{p_i} > t$ is satisfied by the coalitions with a sufficiently large number of strong elements. Therefore, the optimal solution of (14) arises by taking into account the two conflicting criteria: power of the coalition and degree of relationship among the coalition's elements. We stress again that the number m of the coalition's elements is optimally determined only after having solved (14).

Although the example proposed here is not based on a real world case, the problem solved in (14) could be applied to economics and political sciences. In fact, it is possible to see vector $\boldsymbol{\omega}$ as a collection of weights for political parties. At this point, if we are able to establish some distance measures between any two parties (i.e. a relationship degree), then we can apply (14) and find the strongest winning coalition.

Note that vector $\boldsymbol{\omega}$ defining the relative importance of each $x_i \in X$ must not be confused with vector $\mathbf{w}$ of an OWA operator, which is used in this paper to assign weights to degrees of relationships among elements in X.

Another problem that can be addressed is that of maximizing the number m of elements in a subset satisfying a fixed majority threshold. Namely, let us fix a threshold $\delta \in [0, 1]$ such that $\mu_{R_m}(x_{p_1}, x_{p_2}, \ldots, x_{p_m}) > \delta$ and leave the dimension m of our analysis free. In this way, progressively increasing m and calculating $\mu_{Rm}(x_{p_1}, x_{p_2}, \ldots, x_{p_m})$ at every stage, we can detect the largest $B \subseteq X$ such that $\mu_{Rm}(B) > \delta$. Let $\hat{m}$ denote this maximal cardinality,

$$\hat{m} = \max\{m| \ \mu_{Rm}(B) > \delta, \ m = |B|, \ B \subseteq X\} \ . \tag{15}$$

It may occur that set B is not unique, since there exist ν different subsets $B_j, \ \ j = 1, \ldots, \nu$ satisfying inequality $\mu_{Rm}(B_j) > \delta$ with the same maximal cardinality $\hat{m}$. In this case, it is possible to define a winner as the subset B_i with the strongest degree of relationship, $\mu_{Rm}(B_i) \geq \mu_{Rm}(B_j), \ j = 1, \ldots, \nu$. If again the solution is not unique, the multiple solutions are considered equivalent for our analysis.

The very last observation concerns the dimension of the analysis. If $m = n$, then the degree to which this particular relation holds is a measure of how strong the relation among all the $x_i \in X$ is. It can be interpreted as the degree of social relationship computed on the entire network.

6 Conclusions

We provided a new approach to the analysis of social networks based on m–ary fuzzy adjacency relations and OWA operators. Our aim was to show that from the combined use of these two mathematical tools, the vagueness pervading the relationships between the actors involved in the social network and their attitude to connect each other can be represented more effectively. Through the introduction of the strictly monotone OWA operator, we provided a representation of fuzzy m-ary adjacency relations using the information embedded in the fuzzy binary relations defined on the same universal set. Hopefully starting from the results of this paper it will be possible to provide further representations of the interactions characterizing the dynamics of social networks involving linguistically based evaluations as well.

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