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Adaptive Multiscale Schemes for Conservation Laws

With 58 Figures

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To my parents

Preface

During the last decade enormous progress has been achieved in the field of computational fluid dynamics. This became possible by the development of robust and high–order accurate numerical algorithms as well as the construction of enhanced computer hardware, e.g., parallel and vector architectures, workstation clusters. All these improvements allow the numerical simulation of *real world* problems arising for instance in automotive and aviation industry. Nowadays numerical simulations may be considered as an indispensable tool in the design of engineering devices complementing or avoiding expensive experiments. In order to obtain qualitatively as well as quantitatively *reliable* results the complexity of the applications continuously increases due to the demand of resolving more details of the real world configuration as well as taking better physical models into account, e.g., turbulence, real gas or aeroelasticity. Although the speed and memory of computer hardware are currently doubled approximately every 18 months according to Moore's law, this will not be sufficient to cope with the increasing complexity required by *uniform* discretizations.

The future task will be to *optimize* the utilization of the available resources. Therefore new numerical algorithms have to be developed with a computational complexity that can be termed nearly optimal in the sense that storage and computational expense remain proportional to the "inherent complexity" (a term that will be made clearer later) problem. This leads to adaptive concepts which correspond in a natural way to *unstructured* grids. The conclusion is justified by results of approximation theory which clearly indicate that nonlinear approximations, e.g., the positions of the discretization points are not a priori fixed, are more efficient than linear approximations, e.g., uniform discretizations. For details on nonlinear approximation theory see [DeV98]. Currently, numerous efforts of this type are made in different research fields such as image processing, data compression, partial differential equations. In this monograph, the adaptation concepts for partial differential equations are of special interest which shall be briefly reviewed. A naive technique is the *remeshing* of the grid where a fixed number of mesh points is relocated. Obviously this concept is aiming at balancing the error with a fixed number of points rather than reducing the error to a given tolerance. In order to meet a fixed error tolerance the grid adaptation has to allow for *mesh enrichment*, i.e., locally refining and coarsening the mesh. This may result in an unstructured grid with locally hanging nodes. Instead of refining the grid it is also possible to increase locally the approximation order p or apply a different discretization operator for a fixed grid. This leads to a *hybrid discretization*. Of course both strategies can be combined. More details on this subject can be found for instance in [Sch98]. For time–dependent problems one might also apply *local time steps*. In this case, the constraint for the time discretization due to a CFL number is locally weakened without causing instabilities. Hence the solution may evolve faster in time for coarse cells than for fine cells. Of course, the solution has to be synchronized in case of instationary problems but not necessarily for steady state problems. For details see e.g. [BO84]. Instead of adapting the discretization one might also locally change the underlying *model*, e.g. linearize the model or neglect higher order derivatives if the corresponding physical effects are small.

Although the above techniques differ in the adaptation strategy they have one problem in common, namely, the control of the adaptation. Two strategies that are applied in the context of grid refinement shall be briefly summarized. Here we distinguish between concepts based on *error indicators* and *error estimators*, respectively. In case of error indicators, the grid is remeshed, e.g., according to steep (discretely approximated) gradients of a physically relevant quantity or other indicators. However, this strategy provides only control on the grid refining and coarsening but no information about the error of the approximation. A reliable concept is the *error–balancing strategy*. The goal is to equilibrate the error. To this end, a tolerance *tol* and a maximal number of discretization points N_{max} are fixed. By means of residual–based a posteriori estimates the grid is locally refined until a local error estimator is proportional to the ratio tol/N_{max}. This leads to an optimal mesh size distribution. In practice, it cannot be realized. Therefore one is aiming at an almost quasi equidistribution of the error tolerances. Numerous results on a posteriori error estimates have been reported in the literature for elliptic problems, see [Ver95, EEHJ95, BR96, HR02], parabolic problems [EJ91, EJ95] and hyperbolic problems see [Tad91, CCL94, Vil94, JS95, CG96, Noe96, SH97, KO99]. During the last decade new strategies have been developed based on *multiscale techniques*. Here wavelet techniques have become very popular. The basic idea is to decompose the trial space into a coarser approximation space and a complement space spanned by so–called wavelet functions. This decomposition is recursively applied to the coarse approximation space. Finally, we obtain a decomposition of the trial space into the coarsest approximation space and a sequence of complement spaces representing the difference between the approximation spaces. Performing a change of basis the solution can now be equivalently represented in terms of the single–scale basis corresponding to the trial space of the finest approximation space and the multiscale or wavelet basis, respectively. Since the coefficients of the wavelet expansion, so–called wavelet coefficients or details, may become small whenever the solution is locally smooth, data compression can be performed applying threshold techniques. For instance, one only keeps the N largest coefficients. Here the objective is to minimize the error by N coefficients (see e.g. [CDD01]). This corresponds to the idea of *best N-term approximation*. Alternatively, a tolerance ε can be fixed and all details smaller than this threshold value are discarded. Here the idea is to reduce the total number of coefficients to a small number of significant coefficients where the error to the approximate solution of the underlying approximation space is proportional to ε (see e.g. [GM99a, CKMP01]). In order to control the threshold error we need to relate coefficient norms to function norms.

The present work is concerned with developing and analyzing an adaptive finite volume scheme (FVS) for the approximation of multidimensional hyperbolic conservation laws. The concept is based on multiscale techniques which have already been mentioned above. First work on this subject has been reported by Harten [Har94, Har95]. Here the goal is the acceleration of a *given* FVS on a grid of *uniform* resolution by a *hybrid* flux computation. The core ingredient is the *multiscale decomposition* of a sequence of averages corresponding to a grid of finest resolution into a sequence of *details* and *coarse grid averages*. This decomposition is performed on a sequence of *nested* grids with decreasing resolution. It can be utilized in order to distinguish smooth regions of the flow field from regions with locally strong variations in the solution. In particular, the hybrid flux evaluation can be controlled by the decomposition, i.e., expensive upwind discretizations based on Riemann solvers are only applied near discontinuities of the solution. Elsewhere cheaper linear combinations of already computed numerical fluxes on coarser scales are used instead. These correspond to finite difference approximations. In the meantime this originally one–dimensional concept has been extended to multidimensional problems on Cartesian grids [BH97, CD01], curvilinear patches [DGM00] and triangulations [SSF00, Abg97, CDKP00].

The bottleneck of Harten's strategy is the fact that the *computational complexity*, i.e., the number of floating point operations as well as the memory requirements, corresponds to the globally finest grid. In view of multidimensional applications, this is a severe disadvantage. Recently, a real *adaptive* approach has been presented in [GM99a] and has been investigated in [CKMP01] where the computational complexity is proportional to the problem–inherent degrees of freedom. The basic idea of this concept is to determine an *adaptive grid* by means of a sequence of *truncated details*. The set of significant details can be interpreted as a *tree*. Then the adaptive grid is constructed by locally refining the grid according to the tree of *significant details*. This leads to an unstructured grid with hanging nodes. In order to restrict the computational complexity to the number of significant details the multiscale transformation is only performed on the set of significant details and the averages corresponding to the adaptive grid. It turned out that the *grading* of the tree simplifies the local transformation without increasing the complexity. In particular, the leaves of the graded tree directly correspond to the adaptive grid.

In order to preserve the accuracy of the reference FVS with respect to the finest grid the numerical fluxes on the adaptive grid have to be evaluated judiciously. No error at all is introduced when locally performing the flux evaluation by means of the averages on the *finest* scale. However, this requires a local reconstruction process by which the computational complexity is increased for multidimensional problems. Investigations for a one–dimensional scalar equation verify that for first order approximations the accuracy of the adaptive FVS is much less than that of the reference FVS (see [CKMP01]). However, parameter studies show that in case of higher order accurate FVS based on reconstruction techniques this constraint can be weakened. Here it is possible to utilize the given local averages directly instead of computing the averages on the finest scale. The target accuracy is still preserved by means of the solver–inherent reconstruction step.

A point of special interest is the reliability of the scheme, i.e., the perturbation error introduced by the truncation process can be controlled over all time levels. For this purpose analytically rigorous estimates have to be derived by which the details on the *new* time level can be estimated by those already computed in the *previous* time step. For the one–dimensional scalar case this prediction has been analytically investigated in [CKMP01]. The results derived there justify for the first time the heuristic approach suggested by Harten.

By now the new adaptive multiresolution concept has been applied by several groups with great success to different applications, e.g., 2D–steady state computations of compressible fluid flow around air wings modeled by the Euler and Navier–Stokes equations, respectively, on block–structured curvilinear grid patches [BGMH$^+$01], non–stationary shock–bubble interactions on 2D Cartesian grids for Euler equations [Mül02], backward–facing step on 2D triangulations [CKP02] and simulation of a flame ball modeled by reaction–diffusion equations on 3D Cartesian grids [RS02].

This book presents a self-contained account of the above adaptive concept for conservation laws. The main objectives are the construction and the analysis of the local multiscale transformation, the derivation of the adaptive FVS and a rigorous error analysis. New applications on Cartesian and curvilinear grids for the 2D Euler equations are presented which verify that the solver can be applied to real world problems. According to this the outline of the present work is as follows: In Chap. 1 the governing equations are presented and some of the characteristic properties are summarized. This is concluded by a brief introduction to Godunov–type schemes which form an important class of FVS frequently applied to approximate the solution of conservation laws. The multiscale setting is outlined in Chap. 2. It is based on a *hierarchy of nested grids*. As a simple but important example the Haar basis is presented to outline the basic principles and the goal of the multiscale setting. This motivates the general framework of *biorthogonal wavelets* and *stable completions.* Modifying the Haar basis appropriately leads to a new basis with "good" cancellation properties which is utilized in the adaptive scheme. In Chap. 3 the local multiscale analysis is introduced by means of the modified basis. In particular, the tree of significant details, the grading of the tree and the construction of the adaptive grid are investigated in some detail. The performance of the local multiscale transformation is analyzed in detail which results in sufficient conditions for the grading of the details. The construction of the adaptive FVS is presented in Chap. 4. In particular, several strategies for the evaluation of the numerical fluxes are discussed and the construction of the prediction set of significant details on the new time level is outlined. An error analysis is presented in Chap. 5. It is based on an ansatz originally considered by Harten [Har95] in the context of his hybrid scheme and the results derived in [CKMP01]. An efficient implementation of the adaptive scheme crucially depends on the data structures by which the algorithm is realized. This is no longer a trivial task as it is for schemes based on structured meshes. In order to realize optimal computational complexity the data structures have to be adapted judiciously to the underlying adaptive algorithm. Such appropriate data structures are discussed in Chap. 6. Finally, in Chap. 7, some relevant numerical examples illustrate the computational complexity and accuracy behavior of the scheme and problems arising in engineering applications are presented.

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