Upward Point-Set Embeddability

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Abstract. We study the problem of Upward Point-Set Embeddability, that is the problem of deciding whether a given upward planar digraph D has an upward planar embedding into a point set S. We show that any switch tree admits an upward planar straight-line embedding into any convex point set. For the class of k-switch trees, that is a generalization of switch trees (according to this definition a switch tree is a 1-switch tree), we show that not every k-switch tree admits an upward planar straight-line embedding into any convex point set, for any $k \geq 2$. Finally we show that the problem of Upward Point-Set Embeddability is NP-complete.

1 Introduction

A *planar straight-line embedding* of a graph G into a point set S is a mapping of each vertex of G to a distinct point of S and of each edge of G to the straight-line segment between the corresponding end-points so that no two edges cross each other. Gritzmann *et al.* [8] proved that outerplanar graphs is the class of graphs that admit a planar straight-line embedding into every point set in general position or in convex position. Efficient algorithms are known to embed outerplanar graphs [3] and trees [4] into any point set in general or in convex position. From the negative point of view, Cabello [5] proved that the problem of deciding whether there exists a planar straight-line embedding of a given graph G into a point set P is NP-hard even when G is 2-connected and 2-outerplanar. For upward planar digraphs, the problem of constructing upward planar straight-line embeddings into point sets was studied by Giordano et al. [7], later on by Binucci et al. [2] and recently by Angelini et al. [1]. While some positive and negative results are known for the case of upward planar digraphs, the complexity of testing upward planar straight-line embeddability into point sets has not been known.

In this paper we continue the study of the problem of upward planar straight-line embedding of directed graphs into a given point set. Our results include:

- We extend the positive results given in [1,2] by showing that any directed switch tree admits an upward planar straight-line embedding into every point set in convex position.
- We study directed k-switch trees, a generalization of switch trees (a 1-switch tree is exactly a switch tree). From the construction given in [2] (Theorem 5), we know that for $k \geq 4$ not every k-switch tree admits an upward planar straight-line embedding into any convex point set. Then we fill the gap for 2 and 3-switch trees, by showing that, for any $k \geq 2$ there is a class of k-switch trees $\mathcal{T}_n^k$, and a point set S in convex position, such that any $T \in \mathcal{T}_n^k$ does not admit an upward planar straight-line embedding into S.
- We study the computational complexity of the upward embeddability problem. More specifically, given a n vertex upward planar digraph G and a set of n points on the plane S, we show that deciding whether there exists an upward planar straight-line embedding of G so that its vertices are mapped to the points of S is NP-Complete. The decision problem remains NP-Complete even when G has a single source and the longest simple cycle of G has length four and, moreover, S is an m-convex point set, for some integer $m > 0$.

2 Preliminaries

We mostly follow the terminology of [2]. Next, we give some definitions that are used throughout this paper.

Let l be a line on the plane, which is not parallel to the x-axis. We say that point p *lies to the right of* l (resp., *to the left of* l) if p lies on a semi-line that originates on l, is parallel with the x-axis and is directed towards $+\infty$ (resp., $-\infty$). Similarly, if l is a line on the plane, which is not parallel to the y-axis, we say that point p *lies above* l (resp., *below* l) if p lies on a semi-line that originates on l, is parallel with the y-axis and is directed towards $+\infty$ (resp., $-\infty$).

A *point set in general position*, or *general point set*, is a point set such that no three points lie on the same line and no two points have the same y-coordinate. The *convex hull* $H(S)$ of a point set S is the point set that can be obtained as a convex combination of the points of S. A *point set in convex position*, or *convex point set*, is a point set such that no point is in the convex hull of the others. Given a point set S, we denote by $b(S)$ and by $t(S)$ the lowest and the highest point of S, respectively. A *one-sided convex point set* S is a convex point set in which $b(S)$ and $t(S)$ are adjacent in the border of $H(S)$. A convex point set which is not one-sided, is called a *two-sided convex point set.* In a convex point set S, the subset of points that lie to the left (resp. right) of the line through $b(S)$ and $t(S)$ is called the *left (resp. right) part of* S. A one-sided convex point set S is called *left-heavy* (resp., *right-heavy*)convex point set if all the points of S lie to the left (resp., to the right) of the line through $b(S)$

and $t(S)$. Note that, a one-sided convex point set is either a left-heavy or a right-heavy convex point set.

Consider a point set S and its convex hull $H(S)$. Let $S_1 = S \setminus H(S)$, $S_2 = S_1 \setminus H(S_1)$, ..., $S_m = S_{m-1} \setminus H(S_{m-1})$. If m is the smallest integer such that $S_m = \emptyset$, we say that S is an *m-convex point set*. A subset of points of a convex point set S is called *consecutive* if its points appear consecutive as we traverse the convex hull of S in the clockwise or counterclockwise direction.

The graphs we study in this paper are directed. By (u, v) we denote an arc directed from u to v. A *switch-tree* is a directed tree T, such that, each vertex of T is either a source of a sink. Note that the longest directed path of a switch-tree has length one[3]. Based on the length of the longest path, the class of switch trees can be generalized to that of k-switch trees. A *k-switch tree* is a directed tree, such that its longest directed path has length k. According to this definition a switch tree is a 1-switch tree. A digraph D is called *path-DAG*, if its underlying graph is a simple path. A *monotone path* $(v_1, v_2, \ldots, v_k)$ is a path-DAG containing arcs (v_i, v_{i+1}), $1 \leq i \leq k-1$.

An *upward planar directed graph* is a digraph that admits a planar drawing where each edge is represented by a curve monotonically increasing in the y-direction. An *upward straight-line embedding* (*UPSE* for short) of a graph into a point set is a mapping of each vertex to a distinct point and of each arc to a straight-line segment between its end-points such that no two arcs cross and each arc (u, v) has $y(u) < y(v)$. The following results were presented in [2] and are used in this paper.

Lemma 1 (Binucci at al. [2]). *Let T be an n-vertex tree-DAG and let S be any convex point set of size n. Let u be any vertex of T and let $T_1, T_2, \ldots, T_k$ be the subtrees of T obtained by removing u and its incident edges from T. In any UPSE of T into S, the vertices of T_i are mapped into a set of consecutive points of S, for each $i = 1, 2, \ldots, k$.*

Theorem 1 (Binucci at al. [2]). *For every odd integer $n \geq 5$, there exists a $(3n+1)$-vertex directed tree T and a convex point set S of size $3n+1$ such that T does not admit an UPSE into S.*

3 Embedding a switch-tree into a point set in convex position

In this section we enrich the positive results presented in [1,2] by proving that, any switch-tree has an UPSE into any point set in convex position. During the execution of the algorithms, presented in the following lemmata, which embed a tree T into a point set S, a *free point* is a point of

[3] The *length* of a directed path is the number of arcs in the path.

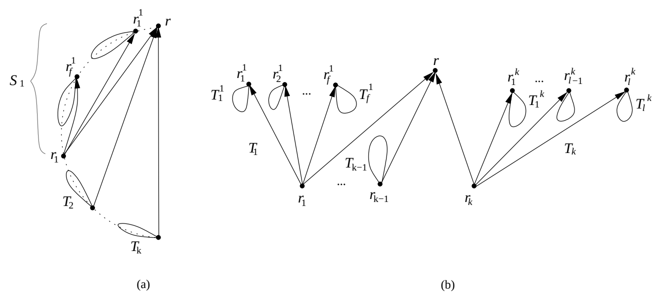


Fig. 1. The construction of Lemma 2 and Lemma 4

S to which no vertex of T has been mapped yet. The following lemma treats the simple case of a one-sided convex point set and is an immediate consequence of a result by Heath et al. [9].

Lemma 2. *Let T be a switch-tree, r be a sink of T, S be a one-sided convex point set so that $|S| = |T|$, and p be S's highest point. Then, T admits an UPSE into S so that vertex r is mapped to point p.*

Proof. Let $T_1, \ldots, T_k$ be the sub-trees of T that are connected to r by an arc and let $r_1, \ldots, r_k$ be the vertices of $T_1, \ldots, T_k$, respectively, that are connected to r (see Figure 1.b). Observe that, since T is a switch tree and r is a sink, vertices $r_1, \ldots, r_k$ are sources. We draw T as follows: We map r to p, then we map T_1 to the $|T_1|$ highest points of S, so that r_1 is mapped to the lowest of them. This can be trivially done if T_1 consists of a single vertex, i.e. of r_1. Assume now that T_1 contains more than one vertex. Denote by S_1 the $|T_1|$ highest free points of S. Let, $T_1^1, \ldots, T_f^1$ be the sub-trees of T_1, connected to r_1 by an arc, and let $r_1^1, \ldots, r_f^1$ be the vertices of $T_1^1, \ldots, T_f^1$, respectively, to which r_1 is connected. Since r_1 is a source, $r_1^1, \ldots, r_f^1$ are all sinks. Using the lemma recursively we draw T_1^1 on the $|T_1^1|$ consecutive highest points of S_1 so that r_1^1 is mapped to the highest point (Figure 1.a). Similarly we draw trees $T_2^1, \ldots, T_f^1$ on the remaining free points of S_1. Finally we map r_1 to the last free point of S_1, i.e. to its lowest point. Since all of $r_1^1, \ldots, r_f^1$ are drawn higher than r_1, the arcs $(r_1, r_1^1), \ldots, (r_1, r_f^1)$ are drawn in upward fashion. Since each of $T_1^1, \ldots, T_f^1$ is drawn on the consecutive points of S in an upward planar fashion we infer that the drawing of T_1 is upward planar and is placed on the consecutive points of S.

In a similar way, we map T_2 to the $|T_2|$ highest consecutive free points of S so that r_2 is mapped to the lowest of them. We continue mapping the rest of the trees in the same way on the remaining free points. Note that for any $i = 1, \ldots, k$, arc (r_i, r) does not intersect any of $H(P_j)$, where P_j is a point set where the vertices of the subtree T_j are mapped. Hence for any

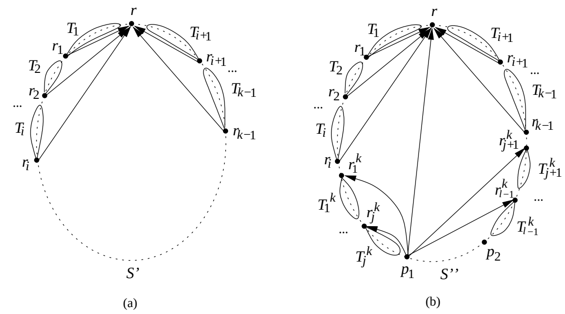


Fig. 2. The construction of Lemma 4

$i = 1, \dots, k$, arc (r_i, r) does not cross any other arc of the drawing. Since p is the highest point of S and r is mapped to p, we infer that arcs (r_i, r), $i = 1, \dots, k$ are drawn in upward fashion. Since, by construction, the drawings of $T_1, \dots, T_k$ are upward and planar, we infer that the resulting drawing of T is upward and planar. □

The following lemma is symmetrical to Lemma 2 and can be proved by a symmetric construction.

Lemma 3. *Let T be a switch-tree, r be a source of T, S be a one-sided convex point set so that $|S| = |T|$, and p be S's lowest point. Then, T admits an UPSE into S so that vertex r is mapped to point p.* □

Now we are ready to proceed to the main result of the section.

Theorem 2. *Let T be a switch-tree and S be a convex point set such that $|S| = |T|$. Then, T admits an UPSE into S.*

The proof of the theorem is based on the following lemma, which extends Lemma 2 from one-sided convex point sets to convex point sets.

Lemma 4. *Let T be a switch-tree, r be a sink of T, S be a convex point set such that $|S| = |T|$. Then, T admits an UPSE into S so that vertex r is mapped to the highest point of S.*

Proof. Let $T_1, \dots, T_k$ be the sub-trees of T that are connected to r by an edge (Figure 1.b) and let $r_1, \dots, r_k$ be the vertices of $T_1, \dots, T_k$, respectively, that are connected to r. Observe that, since T is a switch tree and r is a sink, vertices $r_1, \dots, r_k$ are sources.
We draw T on S as follows. We start by placing the trees $T_1, T_2, \dots$ on the left side of the point set S as long as they fit, using the highest free points first. This can be done in an upward planar fashion by Lemma 3

(Figure 2.a). Assume that T_i is the last placed subtree. Then, we continue placing the trees $T_{i+1}, \dots, T_{k-1}$ on the right side of the point set S. This can be done due to Lemma 3. Note that the remaining free points are consecutive point of S, denote these points by S'. To complete the embedding we draw T_k on S'. Let $T_1^k, \dots, T_l^k$ the subtrees of T_k, that are connected to r_k by an arc. Let also $r_1^k, \dots, r_l^k$ be the vertices of $T_1^k, \dots, T_l^k$, respectively, that are connected to r_k (Figure 1.b). Note that $r_1^k, \dots, r_l^k$ are all sinks. We start by drawing $T_1^k, T_2^k, \dots$ as long as they fit on the left side of point set S', using the highest free points first. This can be done in an upward planar fashion by Lemma 2. Assume that T_j^k is the last placed subtree (Figure 2.b). Then, we continue on the right side of the point set S' with the trees $T_{j+1}^k, \dots, T_{l-1}^k$. This can be done again by Lemma 2. Note that there are exactly $|T_l^k|+1$ remaining free points since we have not yet drawn T_l^k and vertex r_k of T_k. Denote by S'' the remaining free points and note that S'' consists of consecutive points of S. If S'' is a one-sided point set then we can proceed by using the Lemma 2 again and the result follows trivially. Assume now that S'' is a two-sided convex point set and let p_1 and p_2 be the highest points of S'' on the left and on the right, respectively. W.l.o.g., let $y(p_1) < y(p_2)$. Then, we map r_k to p_1. By using the lemma recursively, we can draw T_l^k on $S'' \setminus \{p_1\}$ so that r_l^k is mapped to p_2. The proof is completed by observing that all edges connecting r_k to $r_1^k, \dots, r_l^k$ and $r_1, \dots, r_k$ to r are upward and do not cross each other. □

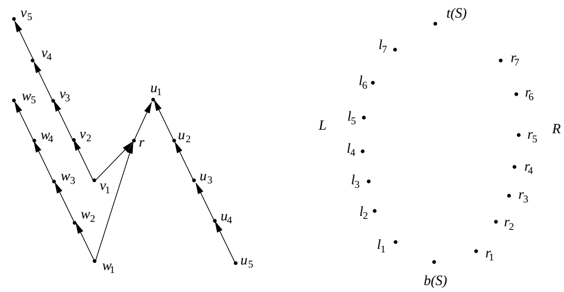
Theorem 2 follows immediately if we select any sink-vertex of T as r and apply Lemma 4.

4 *K*-switch trees

Binucci et al. [2] (see also Theorem 1) presented a class of trees and corresponding convex point sets, such that any tree of this class does not admit an UPSE into its corresponding point set.

The $(3n+1)$-size tree T constructed in the proof of Theorem 1[2] has the following structure (see Figure 3.a for the case $n=5$). It consists of: (i) one vertex r of degree three, (ii) three monotone paths of n vertices: $P_u = (u_n, u_{n-1}, \dots, u_1)$, $P_v = (v_1, v_2, \dots, v_n)$, $P_w = (w_1, w_2, \dots, w_n)$, (iii) arcs (r, u_1), (v_1, r) and (w_1, r).

The $(3n+1)$-convex point set S, used in the proof of Theorem 1[2], consists of two extremal points on the y-direction, $b(S)$ and $t(S)$, the set L of $(3n-1)/2$ points $l_1, l_2, \dots, l_{(3n-1)/2}$, comprising the left side of S and the set R of $(3n-1)/2$ points $r_1, r_2, \dots, r_{(3n-1)/2}$, comprising the right side of S. The points of L and R are located so that $y(b(S)) < y(r_1) < y(l_1) < y(r_2) < y(l_2) < \dots < y(r_{(3n-1)/2}) < y(l_{(3n-1)/2}) < y(t(S))$. See Figure 3.b for $n=5$.



(a) (b)

Fig. 3. (a-b) A 4-switch tree T and a point set S, such that T does not admit an UPSE into point set S.

Note that the $(3n+1)$-node tree T described above is a $(n-1)$-switch tree. Hence a straightforward corollary of Theorem 1[2] is the following statement.

Corollary 1 *For any $k \geq 4$, there exists a k-switch tree T and a convex point set S of the same size, such that T does not admit an UPSE into S.*

From Section 3, we know that any switch tree T, i.e. a 1-switch tree, admits an UPSE into any convex point set. The natural question raised by this result and Corollary 1 is whether an arbitrary 2-switch or 3-switch tree has an UPSE into any convex point set. This question is resolved by the following theorem.

Theorem 3. *For any $n \geq 5$ and for any $k \geq 2$, there exists a class $\mathcal{T}_n^k$ of $3n+1$-vertex k-switch trees and a convex point set S, consisting of $3n+1$ points, such that any $T \in \mathcal{T}_n^k$ does not admit an UPSE into S.*

Proof. For any $n \geq 5$ we construct the following class of trees (see Figure 4.a). Let P_u be an n-vertex path-DAG on the vertex set $\{u_1, u_2, \ldots, u_n\}$, enumerated in the order they are presented in the underlying undirected path of P_u, and such that arcs (u_3, u_2), (u_2, u_1) are present in P_u. Let also P_v and P_w be two n-vertex path-DAGs on the vertex sets $\{v_1, v_2, \ldots, v_n\}$ and $\{w_1, w_2, \ldots, w_n\}$ respectively, enumerated in the order they are presented in the underlying undirected path of P_v and P_w, and such that arcs $(v_1, v_2), (v_2, v_3)$ and $(w_1, w_2), (w_2, w_3)$ are present in P_v and P_w, respectively. Let $T(P_u, P_v, P_w)$ be a tree consisting of P_u, P_v, P_w, vertex r and arcs $(r, u_1), (v_1, r), (w_1, r)$.
Let $\mathcal{T}_n^k = \{T(P_u, P_v, P_w) \mid \text{the longest directed path in } P_u, P_v \text{ and } P_w \text{ has length } k\}$, $k \geq 2$. So, $\mathcal{T}_n^k$ is a class of $3n+1$-vertex k-switch trees. Let S be a convex point set as described in the beginning of the section. Next we show that any $T \in \mathcal{T}_n^k$ does not admit an UPSE into point set S.

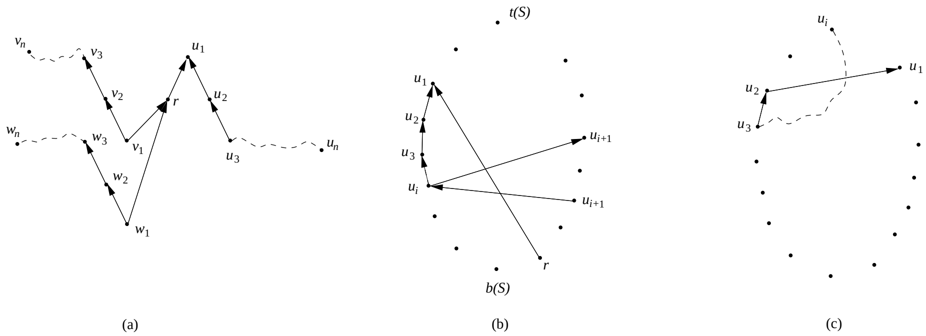


Fig. 4. (a) k-switch tree, $k \geq 2$. (b) The construction of the proof of Statement 1, Cases 1 to 2.

Let $T \in \mathcal{T}_n^k$. For the sake of contradiction, we assume that there exists an UPSE of T into S. By Lemma 1, each of the paths P_u, P_v and P_w of T is drawn on consecutive points of S. Denote by S_u, S_v and S_w the subsets of point set S, in which P_u, P_v and P_w are mapped to, respectively. Hence $|S_u| = |S_v| = |S_w| = n$. By construction of S, the largest subset of S which is a one-sided convex point set, contains two extremal points of S and has size $\lceil \frac{3n-1}{2} \rceil + 2 < 2n$, when $n \geq 5$. Thus, at least one of S_u, S_v and S_w is a two-sided convex point set. We denote by S_b and S_t any two-sided point sets, which consist of consecutive points of S, so that $|S_b| = |S_t| = n$, and $b(S) \in S_b$, $t(S) \in S_t$ respectively. Next, we show that in any UPSE of T on S, P_u can not be drawn on S_b, while P_v and P_w can not be drawn on S_t.

Statement 1 *For any upward drawing of P_u on S_t there is a crossing created by the arcs of T.*

Proof of Statement 1. Recall that $S_t \subset S$ is a two-sided convex point set, so that $t(S) \in S_t$. In any drawing of P_u on S_t, the vertices u_1, u_2, u_3 are mapped to some points of S_t. Next we consider four cases based on whether u_1, u_2, u_3 are drawn on the same side of S.

Case 1. Vertices u_1, u_2, u_3 are mapped to the same side of S, possibly including $t(S)$, say w.l.o.g. to the left side of S, see Figure 4.b. Let u_{i+1} be the first vertex of P_u that is mapped to the right side of S. Then, since r is mapped to a point of $S \setminus S_t$, arc (r, u_1) crosses arc (u_i, u_{i+1}) (or arc (u_{i+1}, u_i)).

Case 2. Vertices u_2, u_3 are mapped to the same side of S, possibly including $t(S)$, say w.l.o.g. to the left side of S, see Figure 4.c. Then, u_1 is mapped to the right side of S. Note that u_2 can not be mapped to $t(S)$, because then there is no point for u_1 to be mapped to, so that the drawing is upward. Hence, there is at least one point p higher than the

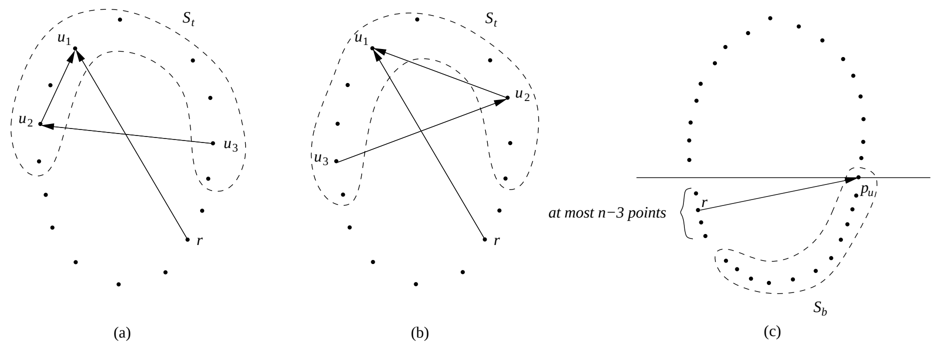


Fig. 5. (a-b) The construction of the proof of Statement 1, Cases 2 to 3. (c) The construction used in Statement 3.

end points of arc (u_2, u_1), that has to be visited by path P_u. Thus, path P_u crosses arc (u_2, u_1).

Case 3. Vertices u_1, u_2 are mapped to the same side of S, possibly including $t(S)$, say w.l.o.g. to the left side of S. Then, u_3 is mapped to the right side of S (Figure 5.a) and, as a consequence, arcs (r, u_1) and (u_3, u_2) cross.

Case 4. Vertices u_1, u_3 are mapped to the same side of S, possibly including $t(S)$, say w.l.o.g. to the left side of S. Then, u_2 is mapped to the right side of S (Figure 5.b) and, as a consequence, arcs (r, u_1) and (u_3, u_2) cross. □

The proof of following statement is symmetrical to the proof of Statement 1.

Statement 2 *For any upward drawing of P_u or P_w on S_b there is a crossing created by the arcs of T.* □

So, we have proved that there is no upward planar mapping of T into S so that P_u is mapped to a set S_t, or such that P_v or P_w is mapped to a set S_b. Next, we prove that there is also no upward planar mapping of T on S so that P_u is mapped to S_b, and such that P_v or P_w is mapped to S_t.

Statement 3 *There is no upward drawing of T on point set S, such that P_u is mapped to the points of S_b.*

Proof of Statement 3. Denote by p_u the point of S_b with the largest y-coordinate, see Figure 5.c. By the construction of S and since S_b is a two-sided point-set which contains n points, we infer that $S \setminus S_b$ contains at most $n-3$ points lower than p_u. Moreover, all of these points are on the side opposite to p_u. We observe the following: (i) r has to be placed lower than p_u, and hence r is placed on the opposite side of that of p_u, (ii) v_1 has to be placed lower than r, and since P_v has to be mapped to consecutive points of S, the whole P_v is mapped to the points on the same side with r and lower than r. But, there are at most $n-4$ free points, a clear contradiction since $|P_v| = n$. □

The following statement is symmetrical to Statement 3.

Statement 4 *There is no upward drawing of T on point set S, such that P_v or P_w is mapped to the points of S_t.* □

As we observed in the beginning of the proof of the theorem, at least one of P_u, P_v, P_w is mapped to a two-sided point set containing either $b(S)$ or $t(S)$. But, as it is proved in Statements 1 to 4 this is impossible. So, the theorem follows. □

5 Upward planar straight-line point set embeddability is NP-complete

In this section we examine the complexity of testing whether a given n-vertex upward planar digraph G admits an UPSE into a point set S. We show that the problem is NP-complete even for a single source digraph G having longest simple cycle of length at most 4. This result is optimal for the class of cyclic graphs[4], since Angelini et al. [1] showed that every single-source upward planar directed graph with no cycle of length greater than three admits an UPSE into every point set in general position.

Theorem 4. *Given an n-vertex upward planar digraph G and a planar point set S of size n in general position, the decision problem of whether there exists an UPSE of G into S is NP-Complete. The decision problem remains NP-Complete even when G has a single source and the longest simple cycle of G has length at most 4 and, moreover, S is an m-convex point set for some $m > 0$.*

Proof. The problem is trivially in NP. In order to prove the NP-completeness, we construct a reduction from the 3-Partition problem.

> *Problem:* 3-Partition
> *Input:* A bound $B \in \mathbb{Z}^+$, and a set $A = \{a_1, \ldots, a_{3m}\}$ with $a_i \in \mathbb{Z}^+$, $\frac{B}{4} < a_i < \frac{B}{2}$.
> *Output:* m disjoint sets $A_1, \ldots, A_m \subset A$ with $|A_i| = 3$ and $\sum_{a \in A_i} a = B,\ 1 \le i \le m$.

We use the fact that 3-Partition is a strongly NP-hard problem, i.e. it is NP-hard even if B is bounded by a polynomial in m [6]. Let A and B

[4] A digraph is *cyclic* if its underling undirected graph contains at least one cycle.

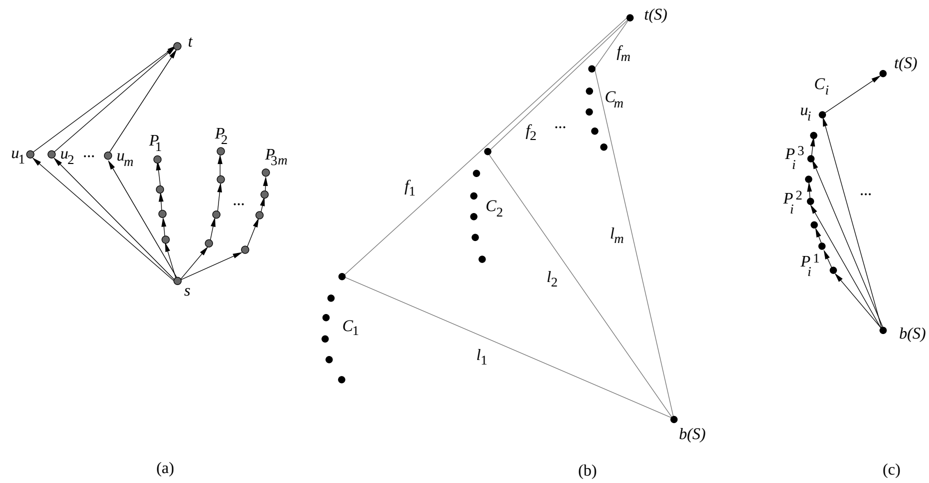


Fig. 6. (a) The graph G of the construction used in the proof of NP-completeness. (b) The point set S of the construction. (c) An UPSE of G on S. (d) The construction of Statement 5.

be the set of the $3m$ positive integers and the bound, respectively, that form the instance (A, B) of the 3-Partition problem. Based on A and B, we show how to construct an upward planar digraph G and a point set S such that G has an UPSE on point set S if and only if the instance (A, B) of the 3-partition problem has a solution.

We first show how to construct G (see Figure 6.a for illustration). We start the construction of G by first adding two vertices s and t. Vertex s is the single source of the whole graph. We then add m disjoint paths from s to t, each of length two. The degree-2 vertices of these paths are denoted by $u_i,\ i = 1, \ldots, m$. For each $a \in A$, we construct a monotone directed path P_i of length a that has a new vertices and s at its source. Totally, we have $3m$ such paths $P_1, \ldots, P_{3m}$.

We proceed to the construction of point set S. Let $b(S)$ and $t(S)$ be the lowest and the highest points of S (see Figure 6.b). In addition to $b(S)$ and $t(S)$, S also contains m one-sided convex point sets $C_1, \ldots, C_m$, each of size $B + 1$, so that the points of S satisfy the following properties:

- $C_i \cup \{b(S), t(S)\}$ is a left-heavy convex point set, $i \in \{1, \ldots, m\}$.
- The points of C_{i+1} are higher than the points of C_i, $i \in \{1, \ldots, m-1\}$.
- Let l_i be the line through $b(S)$ and $t(C_i)$, $i \in \{1, \ldots, m\}$. $C_1, \ldots, C_i$ lie to the left of line l_i and $C_{i+1}, \ldots, C_m$ lie to the right of line l_i.
- Let f_i be the line through $t(S)$ and $t(C_i)$, $i \in \{1, \ldots, m\}$. C_j, $j \geq i$, lie to the right of line f_i.
- $\{t(C_i) : i = 1, \ldots, m\}$ is a left-heavy convex point set.

The next statement follows from the properties of point set S.

Statement 5 *Let C_i be one of the left-heavy convex point sets comprising S and let $x \in C_j$, $j > i$. Then, set $C_i \cup \{b(S), x\}$ is also a left-heavy convex point set, with $b(S)$ and x consecutive on its convex hull.* □

Statement 6 *We can construct a point set S that satisfies all the above requirements so that the area of S is polynomial on B and m.*

Proof of Statement 6: For each $i \in \{0, \ldots, m-1\}$ we let C_{m-i} to be the set of $B+1$ points

$$C_{m-i} = \left\{(-j - i(B+2), j^2 - (i(B+2))^2) \mid \; j = 1, 2, \ldots, B+1 \;\right\}$$

Then, we set the lowest point of the set S, called $b(S)$, to be point $(-(B+1)^2 + ((m-1)(B+2))^2, (B+1)^2 - (m(B+2))^2)$ and the highest point of S, called $t(S)$, to be point $(0, (m(B+2))^2)$.

It is easy to verify that all the above requirements hold and that the area of the rectangle bounding the constructed point set is polynomial on B and m. □

Statement 7 $|S| = |V(G)| = m(B+1) + 2.$ □

We now proceed to show how from a solution for the 3-Partition problem we can derive a solution for the upward point set embeddability problem. Assume that there exists a solution for the instance of the 3-Partition problem and let it be $A_i = \{a_i^1, a_i^2, a_i^3\}$, $i = 1 \ldots m$. Note that $\sum_{j=1}^{3} a_i^j = B$. We first map s and t to $b(S)$ and $t(S)$, respectively. Then, we map vertex u_i on $t(C_i)$, $i = 1 \ldots m$. Note that the path from s to t through u_i is upward and $C_1, \ldots, C_i$ lie entirely to the left of this path, while $C_{i+1}, \ldots, C_m$ lie to the right of this path. Now each C_i has B free points. We map the vertices of paths P_i^1, P_i^2 and P_i^3 corresponding to a_i^1, a_i^2, a_i^3 to the remaining points of C_i in an upward fashion (see Figure 6.c). It is easy to verify that the whole drawing is upward and planar.

Assume now that there is an UPSE of G into S. We prove that there is a solution for the corresponding 3-Partition problem. The proof is based on the following statements.

Statement 8 *In any UPSE of G into S, s is mapped to $b(S)$.* □

Statement 9 *In any UPSE of G into S, only one vertex from set $\{u_1, \ldots u_m\}$ is mapped to point set C_i, $i = 1 \ldots m$.*

Proof of Statement 9: For the sake of contradiction, assume that there are two distinct vertices u_j and u_k that are mapped to two points of the same point set C_i (see Figures 7). W.l.o.g. assume that u_k is mapped to

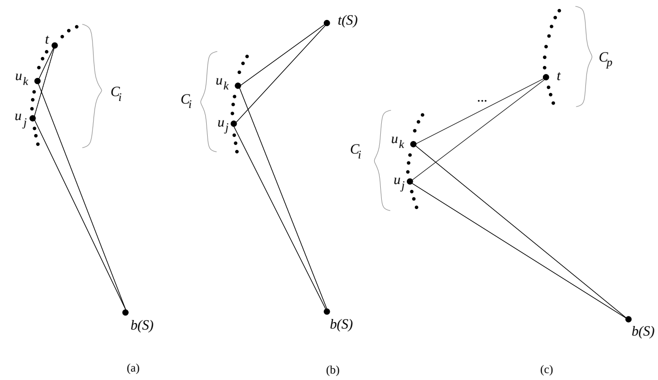


Fig. 7. Mappings used in the proof of Statement 9

a point higher than the point u_j is mapped to. We consider three cases based on the placement of the sink vertex t.

Case 1: t is mapped to a point of C_i (Figure 7.a). It is easy to see that arc (s, u_k) crosses arc (u_j, t), a clear contradiction to the planarity of the embedding.

Case 2: t is mapped to $t(S)$ (Figure 7.b). Similar to the previous case since $C_i \cup \{b(S), t(S)\}$ is a one-sided convex point set.

Case 3: t is mapped to a point of $C_p,\ p > i$, denote it by p_t (Figure 7.c). By Statement 5 $C_i \cup \{b(S), p_t\}$ is a convex point set and points p_t, $b(S)$ are consecutive points of $C_i \cup \{b(S), p_t\}$. Hence, arc (s, u_k) crosses arc (u_j, t), a contradiction. □

By Statement 9, we have that each $C_i, i = 1 \ldots m$, contains exactly one vertex from set $\{u_1, \ldots u_m\}$. W.l.o.g., we assume that u_i is mapped to a point of C_i.

Statement 10 *In any UPSE of G into S, vertex t is mapped to either a point of C_m or to $t(S)$.*

Proof of Statement 10: t has to be mapped higher than any $u_i,\ i = 1 \ldots m$, and hence higher than u_m, which is mapped to a point of C_m. □

Statement 11 *In any UPSE of G into S, vertex u_i is mapped to $t(C_i)$, $1 \leq i \leq m-1$, moreover, there is no arc (v, w) so that v is mapped to a point of C_i and w is mapped to a point of C_j, $j > i$.*

Proof of Statement 11: We prove this statement by induction on i, $i = 1 \ldots m-1$. For the basis, assume that u_1 is mapped to a point p_1 different from $t(C_1)$ (see Figure 8.a). Let p_t be the point where vertex t is mapped. By Statement 10, p_t can be either $t(S)$ or a point of C_m. In both cases, point set $C_1 \cup \{b(S), p_t\}$ is a convex point set, due to the construction of

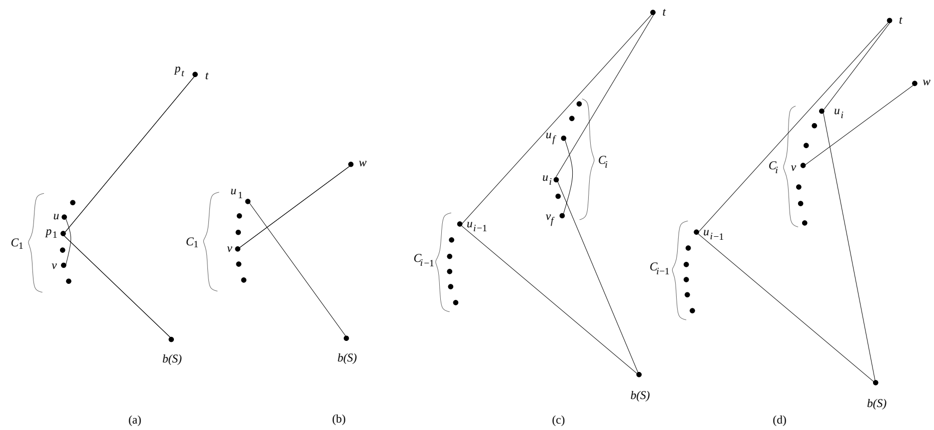


Fig. 8. Mappings used in the proof of Statement 11.

the point set S and the Statement 5. Moreover, the points $b(S)$ and p_t are consecutive on the convex hull of point set $C_1 \cup \{b(S), p_t\}$.

Denote by p the point of C_1 that is exactly above the point p_1. From Statement 9, we know that no u_j, $j \neq 1$ is mapped to the point p. Due to Statement 10, t cannot be mapped to p. Hence there is a path P_k, $1 \leq k \leq 3m$, so that one of its vertices is mapped to p. Call this vertex u. We now consider two cases based on whether u is the first vertex of P_k of not.

Case 1: Assume that there is a vertex v of P_k, such that there is an arc (v, u). Since the drawing of S is upward, v is mapped to a point lower than p and lower than p_1. Since $C_1 \cup \{b(S), p_t\}$ is a convex point set, arc (v, u) crosses arc (u_1, t). A clear contradiction.

Case 2: Let u be the first vertex of P_k. Then, arc (s, u) crosses the arc (u_1, t) since, again, $C_1 \cup \{b(S), p_t\}$ is a convex point set, a contradiction.

So, we have that u_1 is mapped to $t(C_1)$, see Figure 8.b. Observe now that any arc (v, w), such that v is mapped to a point of C_1 and w is mapped to a point $x \in C_2 \cup \ldots \cup C_m \cup \{t(S)\}$ crosses arc (s, u_1), since $C_1 \cup \{b(S), x\}$ is a convex point set. So, the statement is true for $i = 1$.

For the induction step, we assume that the statement is true for C_g and u_g, $g \leq i - 1$, i.e. vertex u_g is mapped to $t(C_g)$ and there is no arc connecting a point of C_g to a point of C_k, $k > g$ and this holds for any $g \leq i - 1$. We now show that it also holds for C_i and u_i. Again, for the sake of contradiction, assume that u_i is mapped to a point p_i different from $t(C_i)$ (see Figure 8.c).

Denote by q the point of C_1 that is exactly above point p_i. From Statement 9, we know that no u_l, $l \neq i$, is mapped to the point q. Due to Statement 10, t can not be mapped to q. Hence, there is a path P_f,

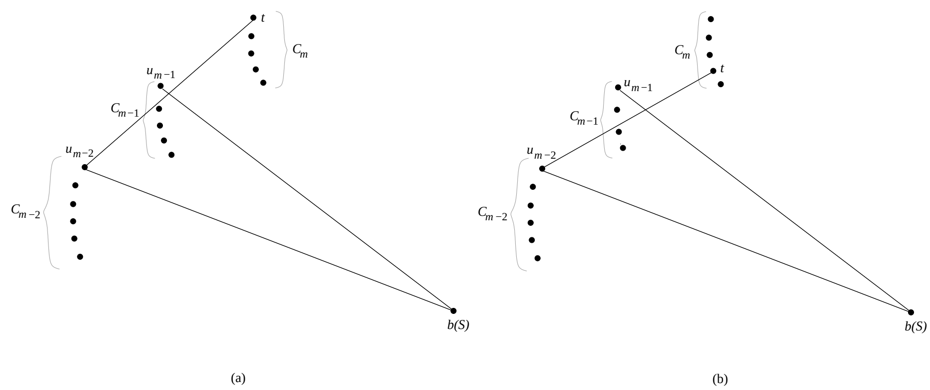


Fig. 9. (a-b) Mappings used in the proof of Statement 13.

so that one of its vertices is mapped to q. Call this vertex u_f. We now consider two cases based on whether u_f is the first vertex of P_f of not.

Case 1: Assume that there is a vertex v_f of P_k such that there is an arc (v_f, u_f). By the induction hypothesis, we know that v_f is not mapped to any C_l, $l < i$. Then, since the drawing of S is upward, v_f is mapped to a point lower than q and lower than p_i. Since $C_i \cup \{b(S), p_t\}$ is a convex point set, arc (v_f, u_f) crosses arc (u_i, t). A clear contradiction.

Case 2: Let u_f be the first vertex of P_k. Then, arc (s, u_f) crosses the arc (u_i, t) since, again, $C_i \cup \{b(S), p_t\}$ is a convex point set, a contradiction.

So, we have shown that u_i is mapped to $t(C_i)$, see Figure 8.d. Observe now that, any arc (v, w), such that v is mapped to a point of C_i and w is mapped to a point $x \in C_{i+1} \cup \ldots \cup C_m \cup \{t(S)\}$ crosses arc (s, u_i), since $C_i \cup \{b(S), x\}$ is a convex point set. So, the statement holds for i. □

A trivial corollary of the previous statement is the following:

Statement 12 *In any UPSE of G into S, any directed path P_j of G originating at s, $j \in \{1, \ldots, 3m\}$, has to be drawn entirely in C_i, for $i \in \{1, \ldots, m\}$.* □

The following statement completes the proof of the theorem.

Statement 13 *In any UPSE of G into S, vertex t is mapped to point $t(S)$.*

Proof of Statement 13: For the sake of contradiction, assume that t is not mapped to $t(S)$. By Statement 10 we know that t has to be mapped to a point in C_m. Assume first that t is mapped to point $t(C_m)$ (see Figure 9.a). Recall that u_{m-2} and u_{m-1} are mapped to $t(C_{m-2})$ and $t(C_{m-1})$, respectively, and that $\{t(C_i) : i = 1 \ldots m\}$ is a left-heavy convex point set. Hence, points $\{t(C_{m-2}), t(C_{m-1}), t(C_m), b(S)\}$ form a convex point set.

It follows that segments $(t(C_{m-2}), t(C_m))$ and $(t(C_{m-1}), b(S))$ cross each other, i.e. edges (s, u_{m-1}) and (u_{m-2}, t) cross, contradicting the planarity of the drawing.

Consider now the case where t is mapped to a point of C_m, say p, different from $t(C_m)$ (see Figure 9.b). Since point p does not lie in triangle $t(C_{m-2}), t(C_{m-1}), b(S)$ and point $t(C_{m-1})$ does not lie in triangle $t(C_{m-2}), p, b(S)$, points $\{t(C_{m-2}), t(C_{m-1}), p, b(S)\}$ form a convex point set. Hence, segments $(t(C_{m-2}), p)$ and $(t(C_{m-1}), b(S))$ cross each other, i.e. edges (s, u_{m-1}) and (u_{m-2}, t) cross; a clear contradiction. □

Let us now combine the above statements in order to derive a solution for the 3-Partition problem when we are given an UPSE of G into S. By Statement 8 and Statement 13, vertices s and t are mapped to $b(S)$ and $t(S)$, respectively. By Statement 9, for each $i = 1 \dots m$, point set C_i contains exactly one vertex from $\{u_1, \dots, u_m\}$, say u_i and, hence, the remaining points of C_i are occupied by the vertices of some paths $P_i^1, P_i^2, \dots, P_i^c$. By Statement 12, $P_i^1, P_i^2, \dots, P_i^c$ are mapped entirely to the points of C_i. Since C_i has $B+1$ points, the highest of which is occupied by u_i, we infer that $P_i^1, P_i^2, \dots, P_i^c$ contain exactly B vertices. We set $A_i = \{a_i^1, a_i^2, \dots, a_i^c\}$, where a_i^j is the size of path P_i^j, $1 \leq j \leq c$. Since $\frac{B}{4} < a_i^j < \frac{B}{2}$ we infer that $c = 3$. The subsets A_i are disjoint and their union produces A.

Finally, we note that G has a single source s and the longest simple cycle of G has length 4, moreover the point set S is an m-convex point set for some $m > 1$. This completes the proof. □

6 Open Problems

In this paper, we continued the study of the upward point-set embeddability problem, initiated in [1,2,7]. We showed that the problem is NP-complete, even if some restrictions are posed on the digraph and the point set. We also extended the positive and the negative results presented in [1,2] by resolving the problem for the class of k-switch trees, $k \in \mathbb{N}$. The partial results on the directed trees presented in [1,2] and in the present work, may be extended in two ways: (i) by presenting the time complexity of the problem of testing whether a given directed tree admits an upward planar straight-line embedding (UPSE) to a given general/convex point set and (ii) by presenting another classes of trees, that admit/do not admit an UPSE to a given general/convex point set. It would be also interesting to know whether there exists a class of upward planar digraphs $\mathcal{D}$ for which the decision problem whether a digraph $D \in \mathcal{D}$ admits an UPSE into a given point set P remains NP-complete even for a convex point set P.

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