# Folk Theorems on the Correspondence between State-Based and Event-Based Systems

Michel A. Reniers<sup>1</sup> and Tim A.C. Willemse<sup>2</sup>

<sup>1</sup> Department of Mechanical Engineering, Eindhoven University of Technology, P.O. Box 513, NL-5600 MB Eindhoven, The Netherlands

<sup>2</sup> Department of Computer Science, Eindhoven University of Technology, P.O. Box 513, NL-5600 MB Eindhoven, The Netherlands

Abstract. Kripke Structures and Labelled Transition Systems are the two most prominent semantic models used in concurrency theory. Both models are commonly believed to be equi-expressive. One can find many ad-hoc embeddings of one of these models into the other. We build upon the seminal work of De Nicola and Vaandrager that firmly established the correspondence between stuttering equivalence in Kripke Structures and divergence-sensitive branching bisimulation in Labelled Transition Systems. We show that their embeddings can also be used for a range of other equivalences of interest, such as strong bisimilarity, simulation equivalence, and trace equivalence. Furthermore, we extend the results by De Nicola and Vaandrager by showing that there are additional translations that allow one to use minimisation techniques in one semantic domain to obtain minimal representatives in the other semantic domain for these equivalences.

## 1 Introduction

Concurrency theory, and process theory in general, deal with the analysis and specification of behaviours of reactive systems, *i.e.*, systems that continuously interact with their environment. Over the course of the past decades, a rich variety of formal languages have been proposed for modelling such systems effectively. At the level of the semantics, however, consensus seems to have been reached over the models used to represent these behaviours. Two of the most pervasive models are the state-based model generally referred to as *Kripke Structures* and the event-based model known as *Labelled Transition Systems*, henceforth referred to as KS and LTS.

The common consensus is that both the KS and LTS models are on equal footing. This is supported by several embeddings of one model into the other that have been studied in the past, see below for a brief overview of the relevant literature. As far as we have been able to trace, in all cases embeddings of both semantic models were considered modulo a single behavioural equivalence. For instance, in their seminal work [8], De Nicola and Vaandrager showed that there are embeddings in both directions showing that stuttering equivalence [1] in KS coincides with divergence-sensitive branching bisimulation [4] in LTS. The

embeddings, however, look a bit awkward from the viewpoint of concrete equivalence relations.

On the basis of these results, one cannot arrive at the conclusion that the embeddings also work for a larger set of equivalences. For instance, it is very easy to come up with a mapping that reflects and preserves branching-time equivalences while breaking linear-time equivalences, by exposing observations of branching through the encodings. Note that it is equally easy to construct encodings that break branching-time equivalences while reflecting and preserving some linear-time equivalences, *e.g.*, by including some form of determinisation in the embeddings.

Our contributions are as follows. Using the KS-LTS embeddings lts and ks of De Nicola and Vaandrager in [7], in Section 3 we formally establish the following relations under these embeddings:

- 1. bisimilarity in KS reflects and preserves bisimilarity in LTS;
- 2. similarity in KS reflects and preserves similarity in LTS;
- 3. trace equivalence in KS reflects and preserves completed trace equivalence in LTS.

These results add to the credibility that indeed both worlds are on equal footing, and it may well be that the embeddings ks and lts are in fact canonical.

As already noted in [7], there is no immediate correspondence between the embeddings lts and ks. For instance, one cannot move between KS and LTS and back again by composing lts and ks. We mend this situation by introducing two additional translations, *viz.*,  $lts^{-1}$  and  $ks^{-1}$ , that can be used to this end. Moreover, we show that combining these with the original embeddings enables one to minimise with respect to an equivalence in KS by minimising the embedded artefact in LTS (and *vice versa*).

From a practical point of view, our contributions allow one to smoothly move between both semantic models using a single set of translations. This reduces the need for implementing dedicated software in one setting when one can take advantage of state-of-the-art machinery available in the other setting.

Related Work In their seminal paper (see [8]) on logics for branching bisimilarity, De Nicola and Vaandrager established, among others, a firm correspondence between the divergence-sensitive branching bisimilarity of [4], and stuttering equivalence [1]. Their results spawned an interest in the relation between temporal logics in the LTS and the KS setting, see *e.g.* [6, 7]. The latter both contain the embeddings that we use in this paper, differing slightly from the ones proposed in [8], which in turn were in part inspired by the (unpublished) embedding by Emerson and Lei [2]. The tight correspondence between stuttering equivalence and branching bisimilarity that was exposed, led Groote and Vaandrager to define algorithms for deciding said equivalences in [5]. Their algorithms (and their correctness proofs), however, are stated directly in terms of the appropriate setting, and do not appear to use the embeddings lts and ks (but they might have acted as a source of inspiration).

Apart from the few documented cases listed above, many ad-hoc embeddings are known to work for equivalences that are not sensitive to abstraction. For instance, one can model the state labelling in a Kripke Structure by means of labelled self-loops, or directly on the edges to the next states, thereby exposing the same information. Such embeddings, however, fail for equivalences that are sensitive to abstraction, such as stuttering equivalence, which basically compresses sequences of states labelled with the same state information.

Outline In Section 2, we formally introduce the computational models KS and LTS, along with the embeddings ks and lts. The latter are proved to preserve and reflect the additional three pairs of equivalences relations stated above. In Section 4, we introduce the inverses  $ks^{-1}$  and  $lts^{-1}$ , and we show that these can be combined with ks and lts, respectively, to obtain our minimisation results. We finish with a brief summary of our contributions and an outlook to some interesting open issues.

#### 2 Preliminaries

Central in both models of computation that we consider, *i.e.*, KS and LTS, are the notions of *states* and *transitions*. While the KS model emphasises the information contained in such states, the LTS model emphasises the state changes through some action modelling a real-life event. Let us first recall both models of computation.

**Definition 1.** A Kripke Structure is a structure  $\langle S, AP, \rightarrow, L \rangle$ , where

- -S is a set of states;
- AP is a set of atomic propositions;
- $\rightarrow \subseteq S \times S$  is a total transition relation, i.e., for all  $s \in S$ , there exists  $t \in S$ , such that  $(s,t) \in \rightarrow$ ; -  $L: S \to 2^{AP}$  is a state labelling.

By convention, we write  $s \to t$  whenever  $(s, t) \in \to$ .

*Remark 1.* The transition relation in the KS model is traditionally required to be total. Our results do not depend on the requirement of totality, but we choose to enforce totality in favour of a smoother presentation and more concise definitions. Without totality, slightly more complicated treatments of the notions of paths and traces (see also Section 3.4) are needed.

With the above restriction in mind, we define the LTS model with a similar restriction imposed on it.

**Definition 2** (Labelled Transition System). A structure  $\langle S, Act, \rightarrow \rangle$  is an LTS, where:

<sup>-</sup>S is a set of states;

- Act is a set of actions;

 $- \rightarrow \subseteq S \times (Act \cup \{\tau\}) \times S$  is a total transition relation, i.e., for all  $s \in S$ , there are  $a \in Act$ ,  $t \in S$ , such that  $(s, a, t) \in \rightarrow$ .

In lieu of the convention for KS, we write  $s \xrightarrow{a} t$  whenever  $(s, a, t) \in \rightarrow$ .

Note that in the setting of the LTS model, a special constant  $\tau$  is assumed outside the alphabet of the set of actions Act of any concrete transition system. This constant is used to represent so-called silent steps in the transition system, modelling events that are unobservable to any witness of the system.

In [7], De Nicola and Vaandrager considered embeddings called lts and ks, which allowed one to move from KS models to LTS models, and, *vice versa*, from LTS models to KS models. We repeat these embeddings below, starting with the embedding from KS into LTS.

**Definition 3.** The embedding lts :  $KS \rightarrow LTS$  is defined as  $lts(K) = \langle S', Act, \rightarrow \rangle$  for arbitrary Kripke Structures  $K = \langle S, AP, \rightarrow, L \rangle$ , where:

- $-S' = S \cup \{\bar{s} \mid s \in S\}, \text{ where it is assumed that } \bar{s} \notin S \text{ for all } s \in S; \\ -Act = 2^{AP} \cup \{\bot\};$
- $\rightarrow$  is the smallest relation satisfying:

$$\frac{s \to t \qquad L(s) = L(t)}{s \xrightarrow{\tau} t}$$

$$\frac{s \to t \qquad L(s) \neq L(t)}{s \xrightarrow{\tau} t}$$

$$\frac{s \to t \qquad L(s) \neq L(t)}{s \xrightarrow{L(t)} t}$$

The fresh symbol  $\perp$  is used to signal a forthcoming encoding of the state information of the Kripke Structure. Encoding the state information by means of a self-loop  $s \xrightarrow{L(s)} s$  introduces problems in preserving and reflecting equivalences that are sensitive to abstraction.

**Definition 4.** The embedding ks :  $LTS \rightarrow KS$  is formally defined as ks $(T) = \langle S', AP, \rightarrow, L \rangle$  for Labelled Transition System  $T = \langle S, Act, \rightarrow \rangle$ , where:

 $\begin{array}{l} - \ S' = S \cup \{(s, a, t) \in \rightarrow \ \mid a \neq \tau\}; \\ - \ AP = Act \cup \{\bot\}, \ where \ \bot \notin Act; \end{array}$ 

 $- \rightarrow$  is the least relation satisfying:

$$\frac{s \xrightarrow{\tau} t}{s \to (s, a, t)} \qquad \frac{s \xrightarrow{\tau} t}{(s, a, t) \to t} \qquad \frac{s \xrightarrow{\tau} t}{s \to t}$$
$$- L(s) = \{\bot\} \text{ for } s \in S, \text{ and } L((s, a, t)) = \{a\}.$$

In this embedding the fresh symbol  $\perp$  is used to label the states from the Labelled Transition System. The reason to treat  $\tau$ -transitions different from ordinary actions is that otherwise equivalences that abstract from sequences of  $\tau$ -transitions are not reflected well.

Observe that, as already stated in [7], due to the artefacts introduced by the embeddings, moving from LTS to KS and back again yields transition systems incomparable to the original ones. Consequently, in LTS, one cannot take advantage of tools for minimising in the setting of KS, and *vice versa*. We defer further discussions on this matter to Section 4.

## 3 Preservations and Reflections of Equivalences Under lts and ks

The embeddings lts and ks have already been shown to preserve and reflect stuttering equivalence [1] and divergence-sensitive branching bisimulation [4] by De Nicola and Vaandrager. In this section, we introduce three additional pairs of equivalences and show that these are also preserved by the embeddings lts and ks. Our choice for these four equivalences is motivated largely by the limited set of equivalence's available in the KS model (contrary to the LTS model, which offers a very fine-grained lattice of equivalence relations).

Remark 2. For reasons of brevity, throughout this paper we define equivalence relations on states within a single LTS (resp. KS) rather than equivalence relations between different models in LTS (resp. KS). Note that this does not incur a loss in generality, as it is easy to define the latter in terms of the former.

## 3.1 Similarity

Both KS and LTS have well-developed theories revolving around similarity. We first formally define both notions.

**Definition 5.** Let  $K = \langle S, AP, \rightarrow, L \rangle$  be a Kripke Structure. A relation  $B \subseteq S \times S$  is a simulation relation iff for every  $s, s' \in S$  satisfying  $(s, s') \in B$ :

$$- L(s) = L(s');$$

- for all  $t \in S$ , if  $s \to t$ , then  $s' \to t'$  for some  $t' \in S$  such that  $(t, t') \in B$ .

For states  $s, s' \in S$ , we say s is simulated by s' if there is a simulation relation B, such that  $(s, s') \in B$ . States  $s, s' \in S$  are said to be similar, denoted  $K \models s \simeq s'$  iff there are simulation relations B and B', such that  $(s, s') \in B$  and  $(s', s) \in B'$ .

Remark 3. It should be noted that when lifting our notion of similarity to an equivalence relation between different models in KS, the first requirement is sometimes stated as  $L(s) = L'(s') \cap AP$ , where L' is the state labelling of the second KS model, and AP is the set of atomic propositions of the first KS model. In this case, some form of abstraction is included already, and care should be taken to deal with such abstractions properly when lifting all our results to such a setting.

**Definition 6.** Let  $T = \langle S, Act, \rightarrow \rangle$  be a Labelled Transition System. A relation  $B \subseteq S \times S$  is a simulation relation iff for every  $s, s' \in S$  satisfying  $(s, s') \in B$ :

- for all  $t \in S$  and  $a \in Act \cup \{\tau\}$ , if  $s \xrightarrow{a} t$ , then  $s' \xrightarrow{a} t'$  for some  $t' \in S'$  such that  $(t, t') \in B$ .

State  $s \in S$  is said to be simulated by state  $s' \in S$  if there is a simulation relation B, such that  $(s,s') \in B$ . States  $s,s' \in S$  are similar, denoted  $T \models s \simeq s'$  iff there are simulation relations B and B', such that  $(s, s') \in B$  and  $(s', s) \in B'$ .

The theorems below state that indeed, embedding lts preserves and reflects KS-similarity through LTS-similarity (see Theorem 1), and vice versa, embedding ks preserves and reflects LTS-similarity through KS-similarity (Theorem 2).

**Theorem 1.** Let  $K = \langle S, AP, \rightarrow, L \rangle$  be an arbitrary Kripke Structure. Then, for all  $s, s' \in S$ , we have  $K \models s \simeq s'$  iff  $\mathsf{lts}(K) \models s \simeq s'$ .

Proof. See Appendix A.1.

**Theorem 2.** Let  $T = \langle S, Act, \rightarrow \rangle$  be a Labelled Transition System. Then for all  $s, s' \in S$ , we have  $T \models s \simeq s'$  iff  $ks(T) \models s \simeq s'$ .

Proof. See Appendix A.2.

#### 3.2Bisimilarity

A slightly stronger notion of equivalence that is rooted in the same concepts as similarity, is *bisimilarity*. Again, bisimilarity has been defined in both KS and LTS, and we here show that both definitions agree through the embeddings lts and ks.

**Definition 7.** Let  $K = \langle S, AP, \rightarrow, L \rangle$  be a Kripke Structure. States  $s, s' \in S$ are said to be bisimilar, denoted  $K \models s \leftrightarrow s'$  iff there is a symmetric simulation relation B, such that  $(s, s') \in B$ .

Similarly, we define bisimilarity in the setting of LTS as follows:

**Definition 8.** Let  $T = \langle S, Act, \rightarrow \rangle$  be a Labelled Transition System. States  $s, s' \in S$  are bisimilar, written  $T \models s \leftrightarrow s'$  iff there is a symmetric simulation relation B, such that  $(s, s') \in B$ .

**Theorem 3.** Let  $K = \langle S, AP, \rightarrow, L \rangle$  be a Kripke Structure. Then for all  $s, s' \in$ S, we have  $K \models s \leftrightarrow s'$  iff  $\mathsf{lts}(K) \models s \leftrightarrow s'$ .

*Proof.* The proof is along the lines of the proof for similarity. For details, see Appendix A.3. 

**Theorem 4.** Let  $T = \langle S, Act, \rightarrow \rangle$  be a Labelled Transition System. For all  $s, s' \in S$ , we have  $T \models s \leftrightarrow s'$  iff  $\mathsf{ks}(T) \models s \leftrightarrow s'$ .

*Proof.* Again, the proof is along the lines of the proof for similarity.

#### Stuttering Equivalence – Divergence-Sensitive Branching 3.3Bisimilarity

In this section, we merely repeat the definitions for stuttering equivalence and divergence-sensitive branching bisimilarity. In Section 4, we come back to these equivalence relations and state several new results for these.

The following definition for stuttering equivalence is taken from [8], where it is shown to coincide with the original definition by Brown, Clarke and Grumberg [1]. We prefer the former phrasing because of its coinductive nature.

**Definition 9.** Let  $K = \langle S, AP, \rightarrow, L \rangle$  be a Kripke Structure. A symmetric relation  $B \subseteq S \times S$  is a divergence-blind stuttering equivalence iff for all  $(s, s') \in B$ :

- -L(s) = L(s');
- for all  $t \in S$ , if  $s \to t$ , then there exist  $s'_0, \ldots, s'_n \in S$ , such that  $s' = s'_0$  and  $(t, s'_n) \in B$ , and for all  $i < n, s'_i \to s'_{i+1}$  and  $(s, s'_i) \in B$ .

**Definition 10.** Let  $K = \langle S, AP, \rightarrow, L \rangle$  be a Kripke Structure. Let the Kripke Structure  $K_d = \langle S_d, AP_d, \rightarrow_d, L_d \rangle$  be defined as follows:

- $S_d = S \cup \{s_d\}$  for some fresh state  $s_d \notin S$ ;
- $\begin{array}{l} AP_d = AP \cup \{d\} \text{ for some fresh proposition } d \notin AP; \\ \rightarrow_d = \rightarrow \cup \{(s, s_d) \mid s \text{ is on an infinite path of states labelled } L(s), \text{ or } s = s_d\}; \end{array}$ - for all  $s \in S$ ,  $L_d(s) = L(s)$ , and  $L_d(s_d) = \{d\}$ .

States  $s, s' \in S$  are said to be stuttering equivalent, notation:  $K \models s \approx_s s'$  iff there is a divergence-blind stuttering equivalence relation B on  $S_d$  of  $K_d$ , such that  $(s, s') \in B$ .

The origins of divergence-sensitive branching bisimilarity can be traced back to [4]. In [9], Van Glabbeek *et al* demonstrate that various incomparable phrasings of the divergence property all coincide with the original definition. For our purposes the following formulation is most suitable.

**Definition 11.** Let  $T = \langle S, Act, \rightarrow \rangle$  be a Labelled Transition System. A symmetric relation  $B \subseteq S \times S'$  is a divergence-sensitive branching simulation relation iff for all  $(s, s') \in B$ :

- if there is an infinite sequence of states  $s_0 \ s_1 \ s_2 \cdots$  such that  $s = s_0$  and  $s_i \xrightarrow{\tau} s_{i+1}$  for all *i*, then there exist a mapping  $\sigma : \mathbb{N} \to \mathbb{N}$ , and an infinite sequence of states  $s'_0 s'_1 s'_2 \cdots$  such that  $s' = s'_0, s'_k \xrightarrow{\tau} s'_{k+1}$  and  $(s_{\sigma(k)}, s'_k) \in$ B for all  $k \in \mathbb{N}$ ;
- for all  $t \in S$  and  $a \in Act \cup \{\tau\}$ , if  $s \xrightarrow{a} t$ , then  $a = \tau$  and  $(t, s') \in B$ , or  $s' \xrightarrow{\tau^*} s^* \xrightarrow{a} t'$  for some  $s^*, t' \in S$  such that  $(s, s^*) \in B$  and  $(t, t') \in B$ .

States  $s, s' \in S$  are divergence-sensitive branching bisimilar, notation  $s \leftrightarrow_{dsb} s'$  iff there is a symmetric divergence-sensitive branching simulation relation B, such that  $(s, s') \in B$ .

### 3.4 Trace Equivalence – Completed Trace Equivalence

Trace equivalence and completed trace equivalence are the only linear-time equivalence relations that we consider in this paper. In defining these equivalence relations, we require some auxiliary notions, basically defining what a *computation* is in our respective models of computation.

**Definition 12.** Let  $K = \langle S, AP, \rightarrow, L \rangle$  be a Kripke Structure. A path starting in state  $s \in S$  is an infinite sequence  $s_0 \ s_1 \ \ldots$ , such that  $s_i \rightarrow s_{i+1}$  for all i, and  $s = s_0$ . The set of all paths starting in s is denoted Paths(s).

Basically, a path formalises how a single computation evolves in time. Actually, it is the information contained in the states that are visited along such a computation that is often of interest, as it shows how the state information evolves in time. This is exactly captured by the notion of a *trace*.

**Definition 13.** Let  $K = \langle S, AP, \rightarrow, L \rangle$  be a Kripke Structure. Let  $\pi = s_0 s_1 \ldots$  be a path starting in  $s_0$ . The trace of  $\pi$ , denoted  $\mathsf{Trace}(\pi)$ , is the infinite sequence  $L(s_0) \ L(s_1) \ \ldots$  For a set of paths  $\Pi$ , we set

$$\mathsf{Traces}(\Pi) = \{\mathsf{Trace}(\pi) \mid \pi \in \Pi\}$$

States  $s, s' \in S$  are trace equivalent, denoted  $K \models s \simeq_{t} s'$ , if  $\mathsf{Traces}(\mathsf{Paths}(s)) = \mathsf{Traces}(\mathsf{Paths}(s'))$ .

*Remark 4.* In the presence of non-totality of the transition relation of a Kripke Structure, it no longer suffices to consider only the infinite paths as the basis for defining trace equivalence. Instead, *maximal* paths are considered, which in addition to the infinite paths, also contains paths made up of sequences of states that end in a sink-state, *i.e.*, a state without outgoing edges.

For models in LTS, we define similar-spirited concepts; for the origins of the definition, we refer to Van Glabbeek's lattice of equivalences [3].

**Definition 14.** Let  $T = \langle S, Act, \rightarrow \rangle$  be a Labelled Transition System. A run starting in a state  $s \in S$  is an infinite, alternating sequence of states and actions  $s_0 \ a_0 \ s_1 \ a_1 \ \ldots$  satisfying  $s_i \xrightarrow{a_i} s_{i+1}$  for all i, and  $s = s_0$ . The set of all runs starting in  $s_0$  is denoted Runs $(s_0)$ .

**Definition 15.** Let  $T = \langle S, Act, \rightarrow \rangle$  be a Labelled Transition System. The trace of a run  $\rho = s_0 \ a_0 \ s_1 \ a_1 \ \ldots$ , denoted  $\mathsf{Trace}(\rho)$ , is the infinite sequence  $a_0 \ a_1 \ \cdots$ . For a set of runs R, we define

$$\mathsf{Traces}(R) = \{\mathsf{Trace}(\rho) \mid \rho \in R\}$$

States  $s, s' \in S$  are completed trace equivalent, denoted by  $T \models s \simeq_{t} s'$  iff  $\operatorname{Traces}(\operatorname{Runs}(s)) = \operatorname{Traces}(\operatorname{Runs}(s')).$ 

**Theorem 5.** Let  $K = \langle S, AP, \rightarrow, L \rangle$  be a Kripke Structure. For all  $s, s' \in S$ , we have  $K \models s \simeq_t s'$  iff  $\mathsf{lts}(K) \models s \simeq_t s'$ .

*Proof.* See Appendix A.3 for details.

In a similar vein, we obtain that completed trace equivalence in LTS is preserved and reflected by trace equivalence in KS.

**Theorem 6.** Let  $T = \langle S, Act, \rightarrow \rangle$  be a Labelled Transition System. Let  $s, s' \in S$  be arbitrary states. We have  $T \models s \simeq_{t} s'$  iff  $ks(T) \models s \simeq_{t} s'$ .

*Proof.* Along the lines of the proof for Theorem 5.

## 4 Minimisations in LTS and KS

As we concluded in Section 2, the mappings lts and ks cannot be used to freely move to and fro the computational models. Instead, we introduce two additional mappings, viz,  $lts^{-1}$  and  $ks^{-1}$  that act as inverses to lts and ks, respectively, and we show that these can be used to come to our results for minimisation. Here, we focus on the computationally most attractive equivalences, viz, *bisimilarity* and *stuttering equivalence*.

Let  $\sim \in \{\underline{\leftrightarrow}, \approx_s\}$  and  $\leftrightarrow \in \{\underline{\leftrightarrow}, \underline{\leftrightarrow}_{dsb}\}$  be arbitrary equivalence relations on KS and LTS, respectively. For a given model K in KS, its *quotient* with respect to  $\sim$  is denoted  $K_{/\sim}$ . Similarly, for a given model T in LTS, its *quotient* with respect to  $\leftrightarrow$  is denoted  $T_{/\leftrightarrow}$ . We assume unique functions  $\sim \min_{KS}$  for KS, and  $\leftrightarrow \min_{LTS}$  for LTS that uniquely determine transition systems that are isomorphic to the quotient. If, from the equivalence relation  $\sim$ , the setting is clear, we drop the subscripts and write  $\sim$ -min instead.

### 4.1 Minimisation in KS via minimisation in LTS

We first characterise a subset of models of LTS for which we can define our inverse  $lts^{-1}$  of lts.

**Definition 16.** Let  $T = \langle S, Act, \rightarrow \rangle$  be a Labelled Transition System. Then T is reversible iff

- 1.  $Act = 2^{AP} \cup \{\bot\}$ , for some set AP;
- 2. for all  $s, s' \in S$  and  $a \in Act \cup \{\tau\}$ , if  $s \xrightarrow{a} s'$ , then  $s' \xrightarrow{\perp}$ ;
- 3. for all  $s, s', s'' \in S$  such that  $s \xrightarrow{\perp} s'$  and  $s \xrightarrow{\perp} s''$ , we require that  $s' \xrightarrow{a}$  and  $s'' \xrightarrow{a'}$  implies a = a' for all actions  $a, a' \in Act$ .

Note that any embedding  $\mathsf{lts}(K)$  of a Kripke Structure K is a reversible Labelled Transition System. Reversibility is preserved by the quotients for  $\underline{\leftrightarrow}$  and  $\underline{\leftrightarrow}_{dsb}$ , as stated by the following proposition.

**Proposition 1.** Let T be an arbitrary reversible Labelled Transition System. Then  $T_{/\leftrightarrow}$ , for  $\leftrightarrow \in \{ \underline{\leftrightarrow}, \underline{\leftrightarrow}_{dsb} \}$ , is reversible.

The embedding lts introduces a fresh, *a priori* known action label  $\perp$ . We treat this constant differently from all other actions in our reverse embedding.

**Definition 17.** Let  $T = \langle S, Act, \rightarrow \rangle$  be a reversible Labelled Transition System. We define the Kripke Structure  $|ts^{-1}(T)|$  as the structure  $\langle S', AP, \rightarrow, L \rangle$ , where:

 $\begin{aligned} -S' &= \{ s \in S \mid s \xrightarrow{\perp} \}; \\ -AP \text{ is such that } Act &= 2^{AP} \cup \{ \bot \}; \end{aligned}$ 

 $- \rightarrow$  is the least relation satisfying the single rule:

$$\frac{s \xrightarrow{a} s' \qquad a \neq \bot \qquad s \xrightarrow{\bot}}{s \rightarrow s'}$$

-L(s) = a for the unique a such that  $s \xrightarrow{\perp} s' \xrightarrow{a}$ .

The following proposition establishes that  $\mathsf{lts}^{-1}$  is the inverse of embedding  $\mathsf{lts}$ .

**Proposition 2.** We have 
$$\mathsf{lts}^{-1} \circ \mathsf{lts} = \mathsf{ld}$$
.

*Proof.* Establishing the isomorphism follows immediately from the definitions and the observation that lts(K) is reversible. See Appendix B.1. 

Note that reversibility of a Labelled Transition System T is too weak to obtain  $(\mathsf{lts} \circ \mathsf{lts}^{-1})(T) = T$ , as the following example illustrates:

Example 1. Consider the Labelled Transition System left below.

Clearly, the Labelled Transition System is reversible, so the mapping  $|ts^{-1}|$  is applicable. Its result is given by the Kripke Structure in the middle. Applying Its to the middle Kripke Structure yields the Labelled Transition System at the right. It is clear that the latter is not isomorphic to the original Labelled Transition System. 

**Lemma 1.** We have  $\leftrightarrow -min_{LTS} \circ \mathsf{lts} \circ \leftrightarrow -min_{KS} = \mathsf{lts} \circ \leftrightarrow -min_{KS}$ .

*Proof.* See Appendix B.2.

**Lemma 2.** We have  $\leftrightarrow_{dsb}$ -min<sub>LTS</sub>  $\circ$  Its  $\circ \approx_s$  -min<sub>KS</sub> = Its  $\circ \approx_s$  -min<sub>KS</sub>.

*Proof.* See Appendix B.3.

Before we present the main theorems concerning the minimisations in KS through minimisations in LTS, we first show that it suffices to prove such results for Kripke Structures that are already minimal; see the lemma below.

**Lemma 3.** Let  $\sim \in \{\underline{\leftrightarrow}, \approx_{\rm s}\}$  and  $\leftrightarrow \in \{\underline{\leftrightarrow}, \underline{\leftrightarrow}_{\rm dsb}\}$  such that  $\mathsf{lts}$  preserves and reflects ~ through  $\leftrightarrow$ . Then

$$\begin{array}{l} \sim -min = \mathsf{lts}^{-1} \circ \leftrightarrow -min \circ \mathsf{lts} \circ \sim -min \\ implies \\ \sim -min = \mathsf{lts}^{-1} \circ \leftrightarrow -min \circ \mathsf{lts} \end{array}$$

*Proof.* Assume that we have

$$\sim -\min = \mathsf{lts}^{-1} \circ \leftrightarrow -\min \circ \mathsf{lts} \circ \sim -\min \qquad (*)$$

By definition of  $\sim$ -min, we find  $\forall K : \sim$ -min $(K) \sim K$ . Since, by assumption, lts preserves and reflects  $\sim$  through  $\leftrightarrow$ , we derive  $\forall K : \mathsf{lts}(K) \leftrightarrow \mathsf{lts}(\sim-\min(K))$ . By definition of  $\leftrightarrow$ -min, this means that we have:

 $\leftrightarrow -\min \circ \mathsf{lts} \ = \ \leftrightarrow -\min \circ \mathsf{lts} \circ \sim -\min$ 

As  $\mathsf{lts}^{-1}$  is functional, and  $\leftrightarrow$ -min preserves reversibility, we immediately obtain:

$$\mathsf{lts}^{-1} \circ \leftrightarrow -\min \circ \mathsf{lts} = \mathsf{lts}^{-1} \circ \leftrightarrow -\min \circ \mathsf{lts} \circ \sim -\min \qquad (**)$$

The desired conclusion then follows by combining \* and \*\*.

We finally state the two main theorems in this section.

**Theorem 7.** We have  $\underline{\leftrightarrow}$ -min<sub>KS</sub> =  $\mathsf{lts}^{-1} \circ \underline{\leftrightarrow}$ -min<sub>LTS</sub>  $\circ \mathsf{lts}$ .

Proof. Lemma 1 guarantees

ŀ

 $\underline{\leftrightarrow}\text{-}\mathrm{min}_{\mathsf{KS}} \circ \mathsf{lts} \circ \underline{\leftrightarrow}\text{-}\mathrm{min}_{\mathsf{KS}} = \mathsf{lts} \circ \underline{\leftrightarrow}\text{-}\mathrm{min}_{\mathsf{KS}}$ 

Functionality of  $|ts^{-1}|$ , combined with Proposition 1, we find:

$$\mathsf{ts}^{-1} \circ \underline{\leftrightarrow} - \min_{\mathsf{LTS}} \circ \mathsf{lts} \circ \underline{\leftrightarrow} - \min_{\mathsf{KS}} = \mathsf{lts}^{-1} \circ \mathsf{lts} \circ \underline{\leftrightarrow} - \min_{\mathsf{KS}}$$

By Lemma 3, we then have our desired conclusion:

$$\underline{\leftrightarrow} \operatorname{-min}_{\mathsf{KS}} = \mathsf{lts}^{-1} \circ \underline{\leftrightarrow} \operatorname{-min}_{\mathsf{LTS}} \circ \mathsf{lts}$$

**Theorem 8.** We have  $\approx_{s} - min_{KS} = |ts^{-1} \circ \leftrightarrow_{dsb} - min_{LTS} \circ |ts.$ 

*Proof.* Similar to Theorem 7, using Lemma 2 instead of Lemma 1.

## 4.2 Minimisation in LTS via minimisation in KS

In the previous section, we showed that one can minimise in KS with respect to bisimilarity or stuttering equivalence, using the embedding lts, a matching equivalence relation in LTS and converting to KS again. In a similar vein, we propose a reverse translation for ks, which allows one to return to LTS from KS. We first characterise a set of Kripke Structures that are amenable to translating to Labelled Transition Systems.

**Definition 18.** Let  $K = \langle S, AP, \rightarrow, L \rangle$  be a Kripke Structure. Then K is reversible iff

1.  $AP = Act \cup \{\bot\}$  for some set Act;

- 2. |L(s)| = 1 for all  $s \in S$ ;
- 3. for all s for which  $\perp \notin L(s)$ , we require that for all s', s",  $s \to s'$  and  $s \to s''$ implies both s' = s'' and  $L(s') = \{\bot\}$ .

**Proposition 3.** Let K be an arbitrary reversible Kripke Structure. Then  $K_{/\sim}$ , for  $\sim \in \{ \underline{\leftrightarrow}, \approx_{s} \}$ , is reversible.

**Definition 19.** Let  $K = \langle S, AP, \rightarrow, L \rangle$  be a reversible Kripke Structure. The Labelled Transition System  $ks^{-1}(K)$  is the structure  $\langle S', Act, \rightarrow \rangle$ , where:

- $S' = \{ s \in S \mid L(s) = \{ \bot \} \};$
- Act is such that  $Act = AP \setminus \{\bot\};$
- $\rightarrow$  is the least relation satisfying:

$$\frac{s \to s' \qquad L(s) = L(s')}{s \xrightarrow{\tau} s'} \qquad \qquad \frac{s \to s'' \qquad a \in L(s'') \setminus \{\bot\} \qquad s'' \to s'}{s \xrightarrow{a} s'}$$

**Proposition 4.** We have  $ks^{-1} \circ ks = Id$ .

*Proof.* Similar to the proof of Proposition 2.

Without further elaboration, we state the final results.

**Theorem 9.** We have  $\underline{\leftrightarrow} - min_{LTS} = ks^{-1} \circ \underline{\leftrightarrow} - min_{KS} \circ ks.$ 

**Theorem 10.** We have  $\underline{\leftrightarrow}_{dsb}$ -min<sub>LTS</sub> = ks<sup>-1</sup>  $\circ \approx_s$ -min<sub>KS</sub>  $\circ$  ks.

## 5 Conclusions

Our results in Section 3 naturally extend the fundamental results obtained by De Nicola and Vaandrager in [7, 8]. In a sense, we can now state that their embeddings ks and lts are canonical for four commonly used equivalence relations.

While the stated embeddings have traditionally been used to come to results about the correspondence between logics, the question whether they support minimisation modulo behavioural equivalences was never answered. Thus, in addition to the above stated results, we proved that indeed the embeddings ks and lts can serve as basic tools in the problem of minimising modulo a behavioural equivalence relation. To this end, we defined inverses of the embeddings to compensate for the fact that composing ks and lts does not lead to transition systems that are comparable (in whatever sense) to the one before applying the embeddings. The latter results are clearly interesting from a practical perspective, allowing one to take full advantage of state-of-the-art minimisation tools available for one computational model, when minimising in the other.

Our minimisation results are for two of the most commonly used equivalence relations that are, arguably, still efficiently computable. However, we do intend to extend our results also in the direction of (completed) trace equivalence and similarity. As a slightly more esoteric research topic, one could look for improving on the embedding lts, as, compared to the embedding ks, it introduces more "noise". For instance, it yields Labelled Transition Systems that have runs that cannot sensibly be related to paths in the original Kripke Structure.

## References

- 1. M.C. Browne, E.M. Clarke, and O. Grumberg. Characterizing finite Kripke structures in propositional temporal logic. *Theor. Comput. Sci.*, 59:115–131, 1988.
- E.A. Emerson and C.L. Lei. Model checking under generalized fairness constraints. Technical report, 1984.
- R.J. van Glabbeek. The linear time branching time spectrum I. In Jan A. Bergstra, Alban Ponse, and Scott A. Smolka, editors, *Handbook of Process Algebra, Chapter* 1, pages 3–100. Elsevier Science, Dordrecht, The Netherlands, 2001.
- R.J. van Glabbeek and W.P. Weijland. Branching time and abstraction in bisimulation semantics. Journal of the ACM (JACM), 43(3):555–600, 1996.
- J.F. Groote and F.W. Vaandrager. An efficient algorithm for branching bisimulation and stuttering equivalence. In Mike Paterson, editor, *ICALP*, volume 443 of *Lecture Notes in Computer Science*, pages 626–638. Springer, 1990.
- R. De Nicola, A. Fantechi, S. Gnesi, and G. Ristori. An action-based framework for verifying logical and behavioural properties of concurrent systems. *Computer Networks and ISDN Systems*, 25(7):761–778, 1993.
- R. De Nicola and F.W. Vaandrager. Action versus state based logics for transition systems. In Irène Guessarian, editor, Semantics of Systems of Concurrent Processes, volume 469 of Lecture Notes in Computer Science, pages 407–419. Springer, 1990.
- R. De Nicola and F.W. Vaandrager. Three logics for branching bisimulation. J. ACM, 42(2):458–487, 1995.
- R.J. van Glabbeek, B. Luttik, and N. Trcka. Branching bisimilarity with explicit divergence. Fundam. Inform., 93(4):371–392, 2009.

## A Proofs for Section 3

#### A.1 Proof of Theorem 1

Consider states s and s' in a Kripke structure  $\langle S, A, \rightarrow, L \rangle$ . Assume that  $K \models s \simeq s'$  and that this is witnessed by the simulation relations B and C with  $(s, s') \in B$  and  $(s', s) \in C$ . We show that, with respect to the Labelled Transition System associated with the Kripke structure, the relation  $B' = B \cup \{(\bar{s}, \bar{s}') \mid (s, s') \in B\}$  is a simulation relation with  $(s, s') \in B$ . In a similar way a simulation relation C' with  $(s', s) \in C'$  can be defined. This part is omitted.

First consider an arbitrary pair  $(\bar{s}, \bar{s}') \in B'$ . This is due to the fact that  $(s, s') \in B$ . By construction the only transitions for  $\bar{s}$  and  $\bar{s}'$  are  $\bar{s} \xrightarrow{L(s)} s$  and  $\bar{s}' \xrightarrow{L(s')} s'$ . From the fact that  $(s, s') \in B$  it follows that L(s) = L(s'). This suffices to satisfy all transfer conditions for the pair  $(\bar{s}, \bar{s}')$ .

Next, consider an arbitrary pair  $(s, s') \in B'$ . This is due to the fact that  $(s, s') \in B$ . Let us consider all transitions from s.

- $-s \xrightarrow{\perp} \bar{s}$ . Since  $s' \in S$  we have  $s' \xrightarrow{\perp} \bar{s}'$ . Since  $(s, s') \in B$  it also follows that  $(\bar{s}, \bar{s}') \in B'$ .
- $-s \xrightarrow{\tau} t$  for some  $t \in S$  such that  $s \to t$  and L(s) = L(t). Since  $(s, s') \in B$  and B is a simulation, it follows that L(s) = L(s') and  $s' \to t'$  for some  $t' \in S$  such that  $(t, t') \in B$ . Since  $(t, t') \in B$  we have L(t) = L(t'), and therefore L(s') = L(t') as well. Thus, by construction  $s' \xrightarrow{\tau} t'$ . From  $(t, t') \in B$  we obtain  $(t, t') \in B'$ .
- $\begin{array}{l} -s \xrightarrow{L(t)} t \text{ for some } t \in S \text{ such that } s \to t \text{ and } L(s) \neq L(t). \text{ Since } (s,s') \in B \\ \text{and } B \text{ is a simulation, it follows that } L(s) = L(s') \text{ and } s' \to t' \text{ for some } t' \in S \\ \text{ such that } (t,t') \in B. \text{ Since } (t,t') \in B \text{ we have } L(t) = L(t'), \text{ and therefore } \\ L(s') \neq L(t') \text{ as well. Thus, by construction } s' \xrightarrow{L(t')} t'. \text{ From } (t,t') \in B \text{ we obtain } (t,t') \in B'. \end{array}$

### A.2 Proof of Theorem 2

Consider states s and s' in a Labelled Transition System  $T = \langle S, A, \rightarrow \rangle$ . Assume that  $T \models s \simeq s'$  and that this is witnessed by the simulations B and C with  $(s, s') \in B$  and  $(s', s) \in C$ . We show that, with respect to the Kripke structure associated with the Labelled Transition System, the relation  $B' = B \cup \{((s, a, t), (s', a, t')) \mid (s, a, t), (s', a, t') \in S' \land (s, s'), (t, t') \in B\}$  is a simulation relation with  $(s, s') \in B$ . Here S' is the set of states of that Kripke structure as prescribed by Definition 4. Similarly, a simulation relation C' with  $(s', s) \in C$  can be defined.

First consider a pair (s, s') that is present in B' due to its presence in B. Since  $s, s' \in S$  we have by definition that  $L(s) = \{\bot\} = L(s')$ . We consider all possible transitions from s. By construction the only possible transitions for s are the following.

- $-s \to t$  for some  $t \in S$  such that  $s \xrightarrow{\tau} t$ . Since  $(s, s') \in B$  and B is a simulation relation, we have  $s' \xrightarrow{\tau} t'$  for some  $t' \in S$  such that  $(s', t') \in B$ . By construction then also  $s' \to t'$  and  $(s', t') \in B'$ .
- $-s \rightarrow (s, a, t)$  for some  $a \in A$  and  $t \in S$  such that  $a \neq \tau$  and  $s \xrightarrow{a} t$ . Since  $(s, s') \in B$  and B is a simulation relation, we have  $s' \xrightarrow{a} t'$  for some  $t' \in S$  such that  $(t, t') \in B$ . By construction then also  $s' \rightarrow (s', a, t')$ . Note that  $((s, a, t), (s', a, t')) \in B'$  since  $(s, s') \in B$  and  $(t, t') \in B$ .

Next, consider a pair ((s, a, t), (s', a, t')) that is present in B' due to presence of both (s, s') and (t, t') in B. By construction  $L((s, a, t)) = \{a\} = L((s', a, t'))$ . Let us consider all transitions from (s, a, t). The only possible transition is  $(s, a, t) \rightarrow t$ . Since  $(s, s') \in B$ ,  $s \xrightarrow{a} t$  and B is a simulation relation it follows that  $s' \xrightarrow{a} t'$  for some  $t' \in S$  such that  $(t, t') \in B$ . By construction then also  $(s', a, t') \rightarrow t'$  and  $(t, t') \in B'$ .

### A.3 Proof of Theorem 3

Consider states s and s' in a Kripke structure  $\langle S, A, \rightarrow, L \rangle$ . Assume that  $K \models s \leftrightarrow s'$  and that this is witnessed by the bisimulation relation B with  $(s, s') \in B$ . Thus B is a simulation relation with  $(s, s') \in B$  and with  $(s', s) \in B$ . We define the relation  $B' = B \cup \{(\bar{s}, \bar{s}') \mid (s, s') \in B\}$ . It follows from the proof of Theorem 1 that B' is a simulation relation for (s, s') and for (s', s).

#### A.4 Proof of Theorem 5

Before we prove Theorem 5 in this section, we establish an intermediate result concerning the relation between paths —and their prefixes— of a Kripke Structure and the subset of bare runs, defined below —and their prefixes— in the LTS-embedding of the same Kripke Structure.

**Definition 20 (Bare run).** A run  $\rho$  is said to be a bare run iff the labels occurring on the run differ from  $\bot$ . The set of bare runs starting in a state s, for  $s \xrightarrow{\perp}$  is given by the set  $\mathsf{Runs}_{\mathsf{b}}(s)$ .

Let  $T = \langle S, Act, \rightarrow \rangle$ . Let  $\operatorname{Runs}^p(s) \subseteq S(Act S)^*$  be the set of prefixes of runs starting in states  $s \in S$ ; likewise,  $\operatorname{Runs}^p_b(s) \subseteq S((Act \setminus \{\bot\}) S)^*$  is the set of prefixes of bare runs starting in  $s \in S$  satisfying  $s \xrightarrow{\perp}$ .

Given a (finite) trace  $\sigma \in \operatorname{Traces}(\operatorname{Runs}^p(s))$ , we write  $s \xrightarrow{\sigma} t$  if there is some  $\rho_p \in \operatorname{Runs}^p(s)$  ending in state t such that  $\sigma = \operatorname{Trace}(\rho_p)$ .

**Definition 21.** Let  $\sigma \in \text{Traces}(\text{Runs}^p(s))$ . Denote the sequence  $\beta(\sigma)$ , obtained from  $\sigma$  by deleting

- all subsequences of the form  $\perp l$ ;
- $-\perp$  in case  $\sigma$  ends as such.

It is not hard to see that  $\beta(\sigma) \in \operatorname{Traces}(\operatorname{Runs}^p(s))$  implies  $\beta(\sigma) \in \operatorname{Traces}(\operatorname{Runs}^p_b(s))$ , *i.e.*, any trace  $\beta(\sigma)$  is generated by a bare run.

**Lemma 4.** For all  $\sigma \in \text{Traces}(\text{Runs}^p(s))$ ,  $s \in S$  such that  $s \xrightarrow{\perp}$  and all  $t \in S \cup \overline{S}$ , we have  $s \xrightarrow{\sigma} t$  iff

1. 
$$s \xrightarrow{\beta(\sigma)} t \text{ and } \forall \sigma' : \sigma \neq \sigma' \bot$$
, or  
2.  $s \xrightarrow{\beta(\sigma) \bot} t \text{ and } \exists \sigma' : \sigma = \sigma' \bot$ .

*Proof.* By induction on the length of  $\sigma$ .

The above lemma firmly establishes a connection between a trace  $\sigma$  of a run starting in a state s and the trace  $\beta(\sigma)$ . Intuitively, as bare runs only pass through states that can perform a  $\perp$  transition, any trace generated by a bare run can be "pumped up" to generate an arbitrary trace that can lead to the same state as its corresponding bare run, simply by following the loop  $\perp l$ , for some action label l.

**Lemma 5.** Let  $K = \langle S, AP, \rightarrow, L \rangle$  be a Kripke Structure. Then  $s_0 \ s_1 \ \cdots \in$ Paths $(s_0)$  implies  $s_0 \ l_0 \ s_1 \ l_1 \ \cdots \in \text{Runs}_b(s_0)$  in Its(K) for precisely one infinite sequence  $l_0 \ l_1 \ \cdots$ . Vice versa, if for some infinite sequence  $l_0 \ l_1 \ \cdots$ , we have  $s_0 \ l_0 \ s_1 \ l_1 \ \cdots \in \text{Runs}_b(s_0)$  in Its(K), then  $s_0 \ s_1 \ \cdots \in \text{Paths}(s_0)$ .

*Proof.* Follows by definition of lts.

Informally, the above lemma states that for each path in a Kripke Structure, there is a unique matching bare run in its LTS embedding, and, *vice versa*, for every bare run in its LTS embedding, there is a unique path in the Kripke Structure.

We next establish that the embedding lts is such that for the trace equivalence of two states in a Labelled Transition System resulting from the embedding lts, it suffices to prove that the traces of all bare runs coincide. Formally, we have:

**Lemma 6.** Let  $K = \langle S, AP, \rightarrow, L \rangle$  be a Kripke Structure. Let  $s, s' \in S$ , with L(s) = L(s') be such that  $\operatorname{Traces}(\operatorname{Runs}_{\mathsf{b}}(s)) = \operatorname{Traces}(\operatorname{Runs}_{\mathsf{b}}(s'))$  in  $\operatorname{Its}(K)$ . Then also  $\operatorname{Traces}(\operatorname{Runs}(s)) = \operatorname{Traces}(\operatorname{Runs}(s'))$  in  $\operatorname{Its}(K)$ .

*Proof.* By induction on the length of the traces.

Since all runs are infinite, in the limit, any trace  $\sigma \in \mathsf{Traces}(\mathsf{Runs}(s))$  is also in the set  $\mathsf{Traces}(\mathsf{Runs}(s'))$ .

Proof (Theorem 5). Let states  $s, s' \in S$  in a Kripke Structure be trace equivalent. Suppose  $\pi = s_0 \ s_1 \ s_2 \ \ldots \in \mathsf{Paths}(s)$  and  $\pi' = s'_0 \ s'_1 \ s'_2 \ \ldots \in \mathsf{Paths}(s')$  are such that  $\mathsf{Trace}(\pi) = \mathsf{Trace}(\pi')$ . Because of Lemma 5, we find that there must be unique  $\rho \in \mathsf{Runs}_{\mathsf{b}}(s)$  and  $\rho' \in \mathsf{Runs}_{\mathsf{b}}(s')$  passing through the exact same states as the paths  $\pi$  and  $\pi'$  respectively. That is:

$$\begin{cases} \rho = s_0 \ l_0 \ s_1 \ l_1 \ s_2 \ \dots \\ \rho' = s'_0 \ l'_0 \ s'_1 \ l'_1 \ s'_2 \ \dots \end{cases}$$

By construction of Its, we have  $l_i = \tau$  if  $L(s_i) = L(s_{i+1})$  and  $l_i = L(s_{i+1})$ otherwise (and similarly for  $l'_i$ ). But from the fact that  $\text{Trace}(\pi) = \text{Trace}(\pi')$ , we find that  $L(s_i) = L(s'_i)$  for all *i*. Hence, also  $l_i = l'_i$  for all *i*. But this means that  $\text{Trace}(\rho) = \text{Trace}(\rho')$ . Appealing to Lemma 5, *all* bare runs correspond to paths in the Kripke Structure. Hence, we find that  $\text{Traces}(\text{Runs}_b(s)) =$  $\text{Traces}(\text{Runs}_b(s'))$ . Since L(s) = L(s'), Lemma 6 yields the desired conclusion that Traces(Runs(s)) = Traces(Runs(s')).

For the other direction, we assume that states  $s, s' \in S$  are trace equivalent in  $\mathsf{lts}(K)$ . In short, this means that the set of bare runs starting in s and s'produce the same traces. Let  $\rho = s_0 \ l_0 \ s_1 \ l_1 \ \ldots$  be a bare run starting in s, and  $\rho' = s'_0 \ l'_0 \ s'_1 \ l'_1 \ \ldots$  be a bare run starting in s', such that  $l_i = l'_i$ . Using Lemma 5, we find that there are unique matching paths  $\pi = s_0 \ s_1 \ s_2 \ \ldots$ and  $\pi' = s'_0 \ s'_1 \ s'_2 \ \ldots$  By construction of lts, we find that this implies that  $L(s_i) = L(s'_i)$  for all i satisfying that  $i \ge j$  for the least j such that  $l_j \ne \tau$ . For all 0 < i < j, we observe that  $l_i = l_{i-1} = \tau$ , which can only be if  $L(s_i) = L(s_{i-1})$ for 0 < i < j. Likewise,  $L(s'_i) = L(s'_{i-1})$ . Since all traces starting in s and s'are the same in  $\mathsf{lts}(K)$ , also  $\mathsf{Trace}(s \perp L(s) \ \rho) = \mathsf{Trace}(s' \perp L(s') \ \rho')$ , which can only be the case when L(s) = L(s'). But then also  $L(s_i) = L(s'_i)$  for all  $i \ge 0$ . Hence,  $\mathsf{Trace}(\pi) = \mathsf{Trace}(\pi')$ . Since all paths starting in s and s' correspond to unique bare runs starting in s and s' in  $\mathsf{lts}(K)$ , this means we have considered all possible paths and therefore all possible traces.

## **B** Proofs for Section 4

#### **B.1** Proof of Proposition 2

This theorem follows directly from the definitions. Consider arbitrary Kripke structure  $K = \langle S, AP, \rightarrow, L \rangle$ . Let  $\mathsf{lts}(K) = T = \langle S', A', \rightarrow \rangle$  and  $\mathsf{lts}^{-1}(T) = K' = \langle S'', AP', \rightarrow', L' \rangle$ . We will show that S'' = S, AP' = AP,  $\rightarrow' = \rightarrow$  and L' = L, thus establishing the isomorphism of K and K'.

From the definition of lts (applied to K) it follows that

$$\begin{array}{l} - \ S' = S \cup \{ \bar{s} \mid s \in S \}; \\ - \ A' = 2^{AP} \cup \{ \bot \}; \\ - \ \stackrel{=}{\longrightarrow} \ \{ (s, \bot, \bar{s}), (\bar{s}, L(s), s) \mid s, s' \in S \} \cup \\ \quad \{ (s, \tau, s') \mid s, s' \in S \land L(s) = L(s') \land s \to s' \} \cup \\ \quad \{ (s, L(s'), s') \mid s, s' \in S \land L(s) \neq L(s') \land s \to s' \} \end{array}$$

and application of  $\mathsf{lts}^{-1}$  (applied to T) gives

 $- S'' = \{s' \mid s' \in S' \land s' \xrightarrow{\perp}\} = \{s' \mid s' \in S \cup \{\bar{s} \mid s \in S\} \land s' \xrightarrow{\perp}\}. \text{ Since } s' \xrightarrow{\perp} \text{ iff } s' \in S \text{ we obtain } S'' = \{s' \mid s' \in S\} = S.$ - AP' = AP.

$$\begin{array}{l} - \rightarrow' = \{(s',t') \mid (s',a,t') \in \rightarrow \land a \neq \bot \land s' \xrightarrow{\bot} \} \\ = \{(s',t') \mid (s',a,t') \in \rightarrow \land a \neq \bot \land s' \in S \} \\ = \{(s',t') \mid s',t' \in S \land L(s') = L(t') \land s' \rightarrow t' \} \\ \cup \{(s',t') \mid s',t' \in S \land L(s') \neq L(t') \land s' \rightarrow t' \} \\ = \{(s',t') \mid s',t' \in S \land s' \rightarrow t' \} \\ = \rightarrow \end{array}$$

-L'(s') = a where a is such that  $s' \xrightarrow{\perp} t' \xrightarrow{a}$  for some t'. Therefore  $s' \in S$  and t' = s'. From this it follows that a = L(s'). So L'(s') = L(s').

### B.2 Proof of Lemma 1

Consider a Kripke structure  $K = \langle S, AP, \rightarrow, L \rangle$  that is minimal w.r.t. strong bisimilarity (on KS). We have to show that  $\mathsf{lts}(K)$  is minimal w.r.t. strong bisimilarity (on LTS). We show that (1) the identity relation on the states of  $\mathsf{lts}(K)$ is a bisimulation relation, and (2) that this bisimulation relation is maximal.

We know, since K is minimal, that the identity relation on S is a maximal bisimulation relation. From this it follows that the identity relation on S' (the states of  $\mathsf{lts}(K)$ ) is a bisimulation relation as well.

Now assume that the identity relation on S' is not the maximal bisimulation relation, i.e., there exists a bisimulation relation  $B \subseteq S' \times S'$  that relates at least one pair of different states. First, we show that it has to be the case that at least one pair of different states from S is related by B.

This can be seen as follows. Consider a pair of different states s and t related by B. Suppose that  $s \in S$  and  $t \notin S$ . In this case, by definition of  $\mathsf{lts}$ ,  $s \xrightarrow{\perp}$ , but  $t \xrightarrow{\perp}$ . Hence s and t cannot be related by a bisimulation relation. The case that  $s \notin S$  and  $t \in S$  is similar. In case both  $s \notin S$  and  $t \notin S$ , by definition  $s = \bar{s'}$ and  $t = \bar{t'}$  for some  $s', t' \in S$  with  $s' \neq t'$ . Then, by definition of  $\mathsf{lts}$ , the only transitions of s and t are  $s \xrightarrow{L(s')} s'$  and  $t \xrightarrow{L(t')} t'$ . In order for s and t to be related by B necessarily s' and t' need to be related by B. Thus we can safely conclude that B relates a pair of different states s and t, both from S.

Now we show that  $B \cap (S \times S)$  is a bisimulation relation on KS, thus contradicting the assumption that the identity relation on S is the maximal bisimulation relation.

Consider a pair of different states s and t, both from S, that are related by B. We show that L(s) = L(t). This follows from the following observations. Both s and t each have a single  $\perp$ -transition:  $s \xrightarrow{\perp} \overline{s}$  and  $t \xrightarrow{\perp} \overline{t}$ . Then, also  $\overline{s}$  and  $\overline{t}$  are related by B. These states each have only one transition:  $\overline{s} \xrightarrow{L(s)} s$  and  $\overline{t} \xrightarrow{L(t)} t$ . From this it follows that L(s) = L(t).

Assume that  $s \to s'$  for some  $s' \in S$ . We distinguish two cases:

-L(s) = L(s'). Then  $s \xrightarrow{\tau} s'$ . Then  $t \xrightarrow{\tau} t'$  for some t' such that  $(s', t') \in B$ . Since B cannot relate states from S (such as s') with states outside S, also  $t' \in S$ . Therefore, by definition of lts,  $t \to t'$ .  $-L(s) \neq L(s')$ . Then  $s \xrightarrow{L(s')} s'$ . Then  $t \xrightarrow{L(s')} t'$  for some t' such that  $(s', t') \in B$ . Since B cannot relate states from S (such as s') with states outside S, also  $t' \in S$ . Then, by definition of lts it has to be the case that L(s') = L(t') and  $t \to t'$ .

In each case it follows that  $t \to t'$  and s' and t' are related by  $B \cap (S \times S)$ , which was to be shown.

The case that  $t \to t'$  for some  $t' \in S$  needs to be mimicked is similar.

From the contradiction obtained it can be concluded that the identity relation on S' is the maximal bisimulation relation.

## B.3 Proof of Lemma 2

Consider a Kripke structure  $K = \langle S, AP, \rightarrow, L \rangle$  that is minimal w.r.t. stuttering equivalence (on KS). We have to show that  $\mathsf{lts}(K)$  is minimal w.r.t. divergencesensitive branching bisimilarity (on LTS). We show that (1) the identity relation on the states of  $\mathsf{lts}(K)$  is a divergence-sensitive branching bisimulation relation, and (2) that this bisimulation relation is maximal.

We know, since K is minimal, that  $K_d$  is minimal with respect to divergenceblind stuttering equivalence. Denote the states of  $K_d$  by  $S \cup \{s_d\}$ . Hence, the identity relation on  $S \cup \{s_d\}$  is a maximal divergence-blind stuttering bisimulation relation with respect to the Kripke structure  $K_d$ . From this it follows that the identity relation on S' (the states of  $\mathsf{lts}(K)$ ) is a divergence-sensitive branching bisimulation relation as well.

Now assume that the identity relation on S' is not the maximal bisimulation relation, i.e., there exists a divergence-sensitive branching bisimulation relation B' such that there are different states s and t from S' with  $(s,t) \in B'$ . We distinguish four cases:

- $-s \in S$  and  $t \notin S$ . In this case, by definition of  $\mathsf{lts}, s \xrightarrow{\perp}$ , but  $t \xrightarrow{\perp}$  and  $t \xrightarrow{\tau}$ . Therefore the transition from s cannot be mimicked from t. So this case cannot occur.
- $-s \notin S$  and  $t \in S$ . Similar to the previous case.
- $-s \in S$  and  $t \in S$ . We have to show that there exists a divergence-blind stuttering bisimulation relation B'' with  $(s,t) \in B''$ .

First we consider the case that  $s \to s'$  for some  $s' \in S \cup \{s_d\}$ . We can distinguish two cases

• Suppose that  $s' \in S$ . By definition  $s \xrightarrow{a} s'$ . Then, by definition of divergence-sensitive branching bisimulation, we have  $a = \tau$  and  $(s', t) \in B'$ , or the existence of  $t_i$  and t' such that

$$t \xrightarrow{\tau} \cdots \xrightarrow{\tau} t_i \xrightarrow{\tau} \cdots \xrightarrow{\tau} t_n \xrightarrow{a} t'$$

with  $(s, t_i) \in B'$  (using the Stuttering Lemma) and  $(s', t') \in B'$ . In the first case we have  $(s', t) \in B'$  and in the second case we have

$$t \to \cdots \to t_i \to \cdots \to t_n \to t'$$

with  $(s, t_i) \in B'$  and  $(s', t') \in B'$ .

• Suppose that  $s' = s_d$ . By definition there is an infinite sequence

$$s \to \dots \to s_i \to \dots$$

of states with the same label. Therefore, in  $\mathsf{lts}(K)$  there is an infinite sequence

$$s \xrightarrow{\tau} \cdots \xrightarrow{\tau} s_i \xrightarrow{\tau} \cdots$$

where all states have the same label  $L(s) = L(s_i)$ . Hence, there is an infinite sequence

$$t \xrightarrow{\tau} \cdots \xrightarrow{\tau} t_j \xrightarrow{\tau} \cdots$$

Therefore, in K, there is an infinite sequence

$$t \to \cdots \to t_i \to \cdots$$

where  $L(t) = L(t_j)$  for all j. Thus  $t \to s_d$  as required.

 $-s \notin S$  and  $t \notin S$ . By definition the only transition of s is of the form  $s \xrightarrow{L(s')} s'$  for some  $s' \in S$ . Since  $t \xrightarrow{\tau}$  obviously the only way to mimic the transition is by means of  $t \xrightarrow{L(s')} t'$  for some  $t' \in S$  with L(t') = L(s'). Necessarily  $(s', t') \in B$ . We have established in the previous item that such s' and t' cannot be related. Therefore, also s and t cannot be related.

Second, we show that L(s) = L(t). Since  $s \in S$  we have  $s \xrightarrow{\perp} \bar{s} \xrightarrow{L(s)} s$ . Then,  $t \xrightarrow{\tau^*} t^* \xrightarrow{\perp} t'$  and  $t' \xrightarrow{\tau^*} t^{**} \xrightarrow{L(s)} t''$  with  $(s,t^*) \in B'$ ,  $(\bar{s},t') \in B'$ ,  $(\bar{s},t^{**}) \in B'$ and  $(s,t'') \in B'$ . It follows that  $L(t) = L(t^*)$  and from the fact that  $t' = \bar{t}$  it follows that L(t') = L(t) as well. Similarly,  $L(t^{**}) = L(t')$ . Since  $t'' = t^{\bar{*}*}$  it also follows that  $L(s) = L(t'') = L(t^{**})$ . Thus we have obtained L(s) = L(t).

We have shown that  $K_d$  was not minimal. Therefore the assumption that  $\mathsf{lts}(K)$  is not minimal is flawed, which completes the proof.