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Blocks of Hypergraphs

Applied to Hypergraphs and Outerplanarity

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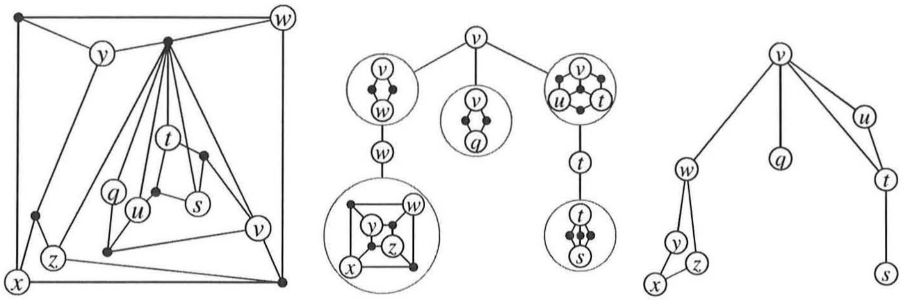
Abstract. A support of a hypergraph H is a graph with the same vertex set as H in which each hyperedge induces a connected subgraph. We show how to test in polynomial time whether a given hypergraph has a cactus support, i.e. a support that is a tree of edges and cycles. While it is $\mathcal{NP}$-complete to decide whether a hypergraph has a 2-outerplanar support, we show how to test in polynomial time whether a hypergraph that is closed under intersections and differences has an outerplanar or a planar support. In all cases our algorithms yield a construction of the required support if it exists. The algorithms are based on a new definition of biconnected components in hypergraphs.

1 Introduction

A *hypergraph* (see e.g. [2,28]) is a pair $H = (V, A)$ where V is a finite set and A is a (multi-)set of non-empty subsets of V. There are basically two different variants of drawing a hypergraph, the *edge-standard* (drawing each hyperedge $h \in A$ as a star or a tree whose leaves are the elements of h – see Fig. 1(a)) or the *subset standard* (drawing each hyperedge $h \in A$ as a simple closed region that contains exactly the vertices in h and no other vertices of V – see Fig. 2(b)). For drawings in the edge standard see, e.g., [7,11,18,20]. In this paper, we concentrate on the second variant which is also called the *Euler diagram* of the set of hyperedges. Simultaneous drawings of a graph and a hypergraph in the subset standard are called clustered graphs. Drawing graphs with overlapping clusters is discussed in [9,19]. There are different variants on when a hypergraph admits a nice drawing in the subset standard. Several of them are based on some graphs associated with the hypergraph.

A hypergraph $H = (V, E)$ is *Zykov-planar* [28] if and only if there is a plane multi-graph M with vertex set V such that each hyperedge equals the set of vertices of some face of M. The hypergraph H can be represented as a *bipartite graph* B_H with vertex set $V \cup A$ and an edge between a vertex $v \in V$ and $h \in A$ if and only if $v \in h$ (see Fig. 1(a)). A hypergraph is Zykov-planar if and only if its bipartite graph is planar [27]. Thus, Zykov-planarity can be tested in linear time [13].



(a) bipartite graph (b) block decomp. (c) cactus support

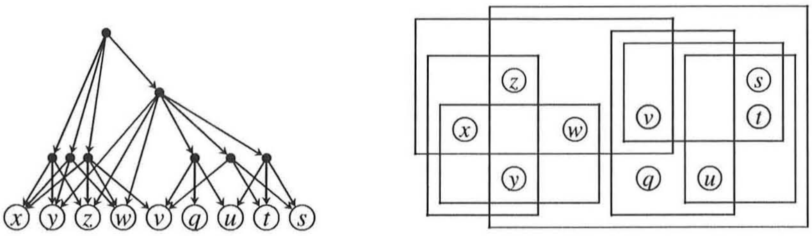
Fig. 1. Three representations of the hypergraph with hyperedges $\{s,t,v\}$, $\{s,t,u\}$, $\{q,u,v\}$, $\{w,x,z,v\}$, $\{x,y,z\}$, $\{w,x,y\}$, $\{q,s,t,u,v,w,z,y\}$

Some work on *Euler diagrams* and a definition on their well-formedness is summarized in [12]. The definition is associated with the *superdual* (or *combinatorial dual*) of H. Assuming that no two vertices of H are contained in the same set of hyperedges, the superdual is a graph on the vertex set V plus an artificial vertex that is not contained in any hyperedge. There is an edge between two vertices v and w if and only if the symmetric difference of the set of hyperedges containing v and the set of hyperedges containing w contains exactly one set h. Edge $\{v,w\}$ is then labeled h. Flower et al. [12] show that a hypergraph has a well-formed Euler diagram if and only if there is a plane subgraph of the super dual in which each hyperedge and its complement induces a connected subgraph and in which the labels around each face fulfill some condition. The superdual of the hypergraph H in Fig. 1 is highly non-connected and, hence, H has no well-formed Euler diagram. Verroust and Viaud [26] considered Euler diagrams for hypergraphs with at most 8 hyperedges. The complexity of Euler diagrams is discussed by Schaefer and Štefankovič [21]. Drawings of arbitray hypergraphs in an extended subset standart where the regions representing the hyperedges do not have to be connected are discussed by Simonetto and Auber [22,23].

A *support* [25,15] (or *host graph* [17]) of a hypergraph $H = (V,E)$ is a graph $G = (V,E)$ with the property that the subgraph of G induced by any hyperedge is connected. A hypergraph is *(vertex-)planar* [14] if it has a planar support. (The partial connectivity graphs of Chow [8] are planar supports of a dualized version of a hypergraph.) Planar hypergraphs are a generalization of both, Zykov-planar hypergraphs [25] and hypergraphs having a well-formed Euler-diagram [12]. It is $\mathcal{NP}$-complete to decide whether a hypergraph has a planar support [14] even if the set of hyperedges is closed under intersections and each hyperedge induces a path in the support. However, it can be decided in linear time whether a hypergraph has a support that is a tree [24], a path, or a cycle [6]. Tree supports with bounded degrees [6] and minimum weighted tree supports [16] can be constructed in polynomial time. Equivalent formulations for hypergraphs having a tree support can be found in [1].

To guarantee that each hyperedge can be drawn by a simple closed region, Kaufmann et al. [15] required *compact supports*. A support $G = (V, E)$ of a hypergraph is compact if G is planar, triangulated and no inner face of the subgraph of G induced by a hyperedge h contains a vertex not in h. It can be concluded from [14] that it is $\mathcal{NP}$-complete to decide whether a hypergraph has a compact support even if it is closed under intersections. However, a hypergraph has a compact support if it has an outerplanar support. So it would be interesting to know whether a hypergraph has an outerplanar support. So far the complexity of outerplanar supports is open. It is $\mathcal{NP}$-complete to decide whether a hypergraph has a 3-outerplanar support [6] or a 2-outerplanar support [5].

The *Hasse diagram* of a hypergraph $H = (V, A)$ is the directed acyclic graph with vertex set $A \cup V$ and there is an edge (h_1, h_2) (or (h_1, v) and $h_2 = \{v\}$) if and only if $h_2 \subsetneq h_1$ and there is no set $h \in A$ with $h_2 \subsetneq h \subsetneq h_1$. A hypergraph $H = (V, A)$ has an outerplanar support if its *based Hasse diagram*, i.e. the Hasse diagram of $A \cup \{V\}$ is planar [15].



(a) based Hasse diagram

(b) Euler diagram

Fig. 2. Two more representations of the hypergraph with hyperedges $\{s, t, v\}$, $\{s, t, u\}$, $\{q, u, v\}$, $\{w, x, z, v\}$, $\{x, y, z\}$, $\{w, x, y\}$, $\{q, s, t, u, v, w, z, y\}$

In this paper, we consider special cases of outerplanar supports. A graph is a *cactus* if it is connected and each edge is contained in at most one cycle. A cactus can be used to represent the set of all minimum cuts of a graph [10]. Cactus supports also have applications in hypergraph coloring [17]. In Sect. 3, we show that a hypergraph has a cactus support if its based Hasse diagram is planar but the converse is not true. Further, we show how to decide in polynomial time whether a hypergraph has a cactus support. The construction is based on a new definition of biconnected components of a hypergraph introduced in Sect. 2 (see Fig. 1(b) for an illustration).

When drawing Euler diagrams it is desirable to visualize not only the hyperedges itself but also the intersection and the differences of two hyperedges. Motivated by this fact, we consider hypergraphs closed under intersections and differences (hcid) in Sect. 4. We show that it can be decided in polynomial time whether a hcid has an outerplanar or planar support.

In the remainder of the paper let $H = (V, A)$ be a hypergraph with $n = |V|$ vertices, $m = |A|$ hyperedges, and $N = \sum_{h \in A} |h|$ equals the sum of the sizes of all hyperedges. The *size of the hypergraph* is then $N + n + m$.

2 Biconnected Components

In this section, we show how to decompose a hypergraph into biconnected components that we will call blocks. This decomposition will be constructed in such a way that there is a support with the property that the blocks of the hypergraph correspond to the biconnected components of the support.

For a hypergraph $H = (V, A)$ and a subset $V' \subset V$ the hypergraph *induced* by V' is $H[V'] = (V', A[V'])$ with $A[V'] = \{h \cap V'; h \in A\} \setminus \{\emptyset, \{v\}; v \in V\}$. I.e., $A[V']$ contains from each hyperedge the part that is in V' omitting the empty set and the sets of size one to be consistent with the definition for ordinary graphs. Let $H|V' = (V', A|V')$ with $A|V' = \{h \in A;\ h \subseteq V'\}$. Note that $H[V']$ does not have to be planar if H is planar. However, $H|V'$ is planar if H is.

The sequence $p : v_0, h_1, v_1, \ldots, h_k, v_k$ is a *v_0v_k-path* in H if $h_1, \ldots, h_k \in A$, $v_0 \in h_1, v_k \in h_k$, and $v_i \in h_i \cap h_{i+1}, i = 1, \ldots k-1$. Vertices v_0 and v_k are the end vertices of p. Two vertices v, w of a hypergraph $H = (V, A)$ are *connected* if there is a vw-path in H. Connectivity is an equivalence relation on the set of vertices of a hypergraph and the hypergraphs induced by the equivalence classes are called *connected components* [28].

Let $v \in V$. The connected components of $H|(V \setminus \{v\})$ are the *parts* of v and v is an *articulation point* of H if v has more than one part. Note that v is an articulation point of H if and only if there is a support of H in which v is a cut vertex. E.g., vertex v is a cut vertex of the hypergraph in Fig. 1 and $\{w, x, y, z\}$, $\{q\}$, and $\{u, t, s\}$ are the parts of v.

A *decomposition into blocks* of a hypergraph $H = (V, A)$ is defined recursively. H is a block if and only if H is connected and does not contain an articulation point. If H is not connected then the blocks of H are the blocks of the connected components of H. If H is connected and contains an articulation point v, let $W_1, \ldots, W_k$ be the parts of v. Then the blocks of H are the blocks of $H[W_1 \cup \{v\}], \ldots, H[W_k \cup \{v\}]$.

Note that the blocks depend on the choices of the articulation points and are not uniquely defined. E.g., consider the hypergraph H in Fig. 1. Choosing the articulation points v, w, and t yields the subhypergraphs induced by the sets $\{v, w\}$, $\{w, x, y, z\}$, $\{v, q\}$, $\{t, u, v\}$, and $\{t, s\}$ as blocks. These are indicated within the circles of Fig. 1(b). Choosing s instead of t as an articulation point would yield the block $H[\{s, u, v\}]$ instead of $H[\{t, u, v\}]$.

Note that this definition of articulation points and blocks is related to but different from the definition given in [1]. Further note that the sum of the sizes of all blocks is at most three times the size of the hypergraph itself.

We will use the terminology analogously for the bipartite graph B_H on the vertex set $V \cup A$ representing the hypergraph $H = (V, A)$. The connected components of H correspond to the connected components of B_H. Vertex v is an articulation point of B_H if $B[V \setminus \{v\} \cup A \setminus \{h \in A; v \in h\}]$ contains more than one connected component which will again be called the *parts* of v. The blocks of B_H are the bipartite graphs representing the blocks of H. Then the blocks of B_H and, hence, of H can be constructed by determining n times the connected components of a subgraph of B_H.

Lemma 1. *The blocks of the hypergraph H can be found in $\mathcal{O}(nN + n + m)$ time.*

Proof. Since the connected components of B_H can be computed in $\mathcal{O}(N+n+m)$ time, we may assume that H is connected. Let $v_1, \ldots, v_n$ be any ordering of the vertices of H. The algorithm BLOCKFINDER(B, k) takes as argument a subgraph B of B_H and a $k = 0, \ldots, n$ such that $v_1, \ldots, v_k$ are not articulation points of B. It outputs a link to the list of blocks of B.

BLOCKFINDER(B, k)

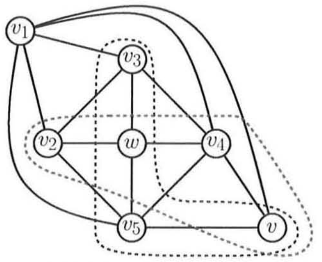
- If there is no $k' > k$ such that $v_{k'}$ is contained in B return B
- Let $k' > k$ be minimal such that $v_{k'}$ is contained in B
- Remove $v_{k'}$ and all its adjacent vertices $h_1, \ldots, h_j$ from B and compute the connected components $B_1, \ldots, B_\ell$ of this bipartite graph.
- For $i = 1, \ldots, \ell$, add $v_{k'}$ and those hyperedges among $h_1, \ldots, h_j$ that contain some vertices of B_i with the corresponding edges to B_i.
- Return BLOCKFINDER(B_1, k'), ..., BLOCKFINDER(B_ℓ, k').

Then BLOCKFINDER(B_H, 0) finds a partition of H into blocks represented as bipartite graphs: Assume that BLOCKFINDER returns a subgraph B_i of B_H that contains an articulation point $v_{k'}$. Let P_1 and P_2 be two parts of $v_{k'}$ in B_i. Consider the subgraph B of B_H such that k' was chosen while proceeding BLOCKFINDER(B, k). Since in the end P_1 and P_2 are both in B_i there is a path p in B connecting P_1 and P_2 that does not contain $v_{k'}$. Let p have minimum length among all such paths. Then p is a path in B_i: Otherwise let $p : w_0, h_1, \ldots, h_\ell, w_\ell$ and assume that w_j is the first vertex of p not in B_i. Let $j' > j$ be the smallest index such that $w_{j'}$ is in B_i. Then there is an articulation point $v_\ell, \ell > k'$ of B_i with $v_\ell \in h_j \cap h_{j'}$. Hence, $w_0, h_1, \ldots, w_{j-1}, h_j, v_\ell, h_{j'}, w_{j'}, \ldots, h_\ell, w_\ell$ is a shorter path than p connecting P_1 and P_2. □

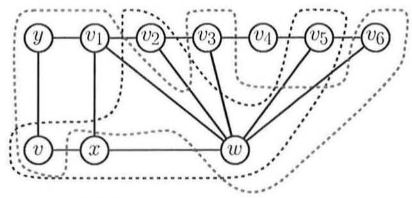
A decomposition of a hypergraph into blocks induces a "block-articulation-point tree" in the same way as block-cut-point trees for ordinary graphs: Let T be the bipartite graph that is constructed as follows. The vertices of T are the blocks of H and those vertices in V that are contained in more than one block. There is an edge between a vertex v and a block B if and only if v is contained in B. Then T is the *block-articulation-point tree* of the chosen decomposition of a hypergraph into blocks (see Fig. 1(b)).

Lemma 2. *A hypergraph has an (outer-)planar support if all its blocks have an (outer-)planar support.*

Proof. Let $B_1, \ldots, B_k$ be the blocks of a hypergraph $H = (V, A)$. Let $G_i = (V_i, E_i)$ be a support of B_i for $i = 1, \ldots, k$. Then $G = (V, E_1 \cup \ldots \cup E_k)$ is a support of H and $G_1, \ldots, G_k$ are the 2-connected components of G. Proceeding from the leaves of the block-articulation-point tree one can choose the embedding of the support of each block such that the articulation point with the parent block is on the outer face. Hence, if all G_i have an (outer-)planar support then so does G. □



(a) blocks not planar



(b) blocks not outerplanar

Fig. 3. Illustration of some examples. Solid edges indicate a support, dashed curves indicate hyperedges that contain more than two vertices.

The converse of Lemma 2 is not true. Let H be the hypergraph with hyperedges $\{v, v_1\}$, $\{v, v_4\}$, $\{v, v_5\}$, $\{v_2, v_4, v, w\}$, $\{v_3, v_5, v, w\}$, $\{v_1, v_2\}$, $\{v_1, v_3\}$, $\{v_1, v_4\}$, $\{v_1, v_5\}$, $\{v_2, v_3\}$, $\{v_3, v_4\}$, $\{v_4, v_5\}$, $\{v_2, v_5\}$. Then H is planar, v is an articulation point of H and $H[\{v_1, v_2, v_3, v_4, v_5, v\}]$ is a block of H that is not planar. See Fig. 3(a) for an illustration. In the outerplanar case consider the hyperedges $\{v_1, v_2\}$, $\{v_2, v_3\}$, $\{v_3, v_4\}$, $\{v_4, v_5\}$, $\{v_5, v_6\}$, $\{v, y\}$, $\{y, v_1\}$, $\{v, x\}$, $\{x, v_1\}$, $\{v, x, w, v_2, v_5\}$, and $\{v, y, v_1, w, v_3, v_6\}$ and the articulation point v. See Fig. 3(b) for an illustration. For hypergraphs closed under intersections, however, we have equivalence. A hypergraph $H = (V, A)$ is *closed under intersections* if $h_1 \cap h_2 \in A \cup \{\emptyset\} \cup \{\{v;\ v \in V\}\}$ for $h_1, h_2 \in A$.

Lemma 3. *A hypergraph that is closed under intersections has an (outer-) planar support if and only if each block has an (outer-) planar support.*

Proof. Let $H = (V, A)$ be a hypergraph that is closed under intersections and let $G = (V, E)$ be an (outer-)planar support of H. Let $v \in V$ and let W be a part of v. We show by induction on the number of vertices of $V \setminus W$ that $H[W \cup \{v\}]$ has an (outer-)planar support. There is nothing to show if $V = W \cup \{v\}$.

So let $w \in V \setminus (W \cup \{v\})$. We construct an (outer-)planar support G' of $H' = (V \setminus \{w\}, \{h' \in A; w \notin h'\} \cup \{h' \setminus \{w\}; v \in h' \in A\})$. If there is no hyperedge containing v and w let G' be the graph that results from G by deleting w and all its incident edges. Otherwise let h be the intersection of all hyperedges that contain v and w. Then there is a wv-path in $G[h]$. Let w' be the neighbor of w on this path. Then G' is constructed from G by merging w and w'. I.e., for each neighbor $u \neq w'$ of w add $\{u, w'\}$ to the edge set of G. Finally, remove w and all its incident edges from G.

If $V \setminus \{w\} = W \cup \{v\}$ then $H' = H[W \cup \{v\}]$. Otherwise v is an articulation point and W is a part of v in H'. Hence, by the inductive hypothesis $H'[W \cup \{v\}] = H[W \cup \{v\}]$ has an (outer-)planar support. □

3 Cactus Supports

A cactus is a connected graph that has an outerplanar embedding such that each edge is incident to the outer face. In this section, we relate cactus supports to planar based Hasse diagrams and we show how to utilize the decomposition into blocks to construct a cactus support if one exists.

It was shown by Kaufmann et al. [15] that a hypergraph $H = (V, A)$ has an outerplanar support if its based Hasse diagram is planar. In fact, in that case H has even a cactus support. In the construction of Kaufmann et al. [15] some unnecessary edges on the outer face have to be omitted. We briefly sketch their construction and our modification.

Theorem 1. *A hypergraph has a cactus support if its based Hasse diagram is planar.*

Proof. Let $H = (V, A)$ be a hypergraph, let $V \in A$, and let its Hasse diagram D be planar. Assume that a planar embedding of D is given. Let T be the DFS tree resulting from a directed left-first DFS and replace each non-tree arc $e = (h_1, h_2)$ in D by an arc (h_1, v) for some $v \in h_2$. According to Kaufmann et al. [15], this can be done by "sliding down" the arcs and thus maintaining planarity. Let D' be the thus constructed Hasse diagram and let A' be the set of vertices of D' that are not sinks. Let $H' = (V, \{\{v \in V;$ there is a directed hv-path in $D'\};\ h \in A'\}$. Then T remains a left-first DFS-tree of D' and any support of H' is a support of H.

Consider a simple closed curve C that visits the sequence $v_1, \ldots, v_n$ of leaves of T from left to right. We may assume that the vertex V of D is in the exterior of C, that C intersects no tree edges and that it intersects non-tree edges at most once. The *support sequence* $\sigma : w_1, \ldots, w_\ell$ is the sequence of vertices or targets of intersecting edges as they occur on C. Note that σ contains only vertices of V and that a vertex of V may occur several times in σ. As mentioned by Kaufmann et al. [15], each set $h \in A'$ corresponds then to a subsequence of σ.

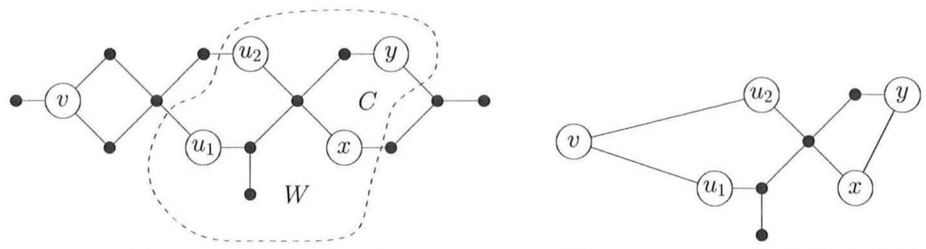
Let now $w_{\ell+1} = w_1$. Then $G = (V, \{\{w_i, w_{i+1}\};\ i = 1, \ldots, \ell\})$ is a cactus support of H' and, hence, of H. In fact, the edges can be routed along C and the pieces of the arcs between C and $v_1, \ldots, v_n$. Then G has a planar embedding in which each edge is on the outer face. Further, each subsequence of W corresponds to a walk in G. Hence, G is a cactus support for H'. □

However, not only hypergraphs with a planar Hasse diagram have a cactus support. E.g., $A = \{\{i, i+1\}, i = 1, \ldots, 6; \{1, \ldots, 5\}, \{2, \ldots, 6\}, \{3, \ldots, 7\}\}$. In the following, we will show how to test efficiently whether any hypergraph has a cactus support and if so how to construct it in the same asymptotic run time.

Lemma 4. *A hypergraph has a support that is a cactus if and only if each block has a support that is a cycle or an edge.*

Proof. The if-part is analogous to Lemma 2. For the only-if-part let $H = (V, A)$ be a hypergraph and let $G = (V, E)$ be a cactus support of H. Let v be an articulation point and W a part of v. We show that $H[W \cup \{v\}]$ has a support that is a cactus.

We say that $u \in W$ is close to v if and only if there is a path in G from v to u not containing any edge of $G[W]$. Note that $G[W]$ is a connected subgraph of a cactus not containing v, hence there are at most two vertices in W that are



(a) cactus support of H (b) cactus support of $H[W \cup \{v\}]$

Fig. 4. Illustration of the proof of Lemma 4. Vertices inside the dashed curve are contained in a part W of v. Vertices u_1 and u_2 are close to v. Vertices x and y are end vertices of p_C.

close to v. A cactus support $G_W = (V_W, E_W)$ of $H[W \cup \{v\}]$ can be constructed as follows (see Fig. 4 for an illustration):

- Start with $G_W \leftarrow G[W \cup \{v\}]$
- For each $u \in W$ that is close to v, add $\{u, v\}$ to E_W
- For each cycle of G, let $C = \{e_1, e_2, \dots, e_k\}$ be its set of edges . If $E[W] \cap C \neq \emptyset$ and $C \not\subseteq E[W]$ then $G[W \cap C]$ is a path p_C. If the end vertices x and y of p_C are not both close to v, add $\{x, y\}$ to E_W. □

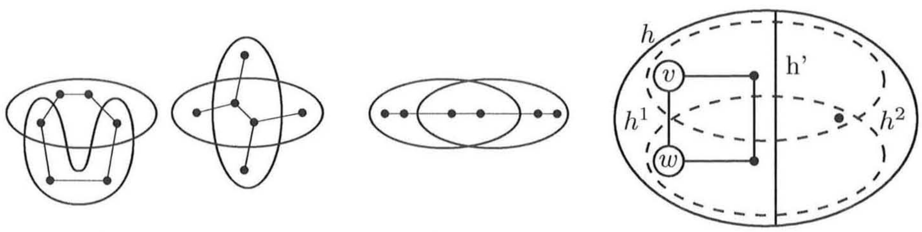
A hypergraph $H = (V, A)$ has a support that is a cycle if and only if it has the *circular consecutive ones property*, i.e. if and only if there is an ordering $v_1, \dots, v_n$ of the vertices such that for each hyperedge $h \in A$ there are $1 \leq j \leq k \leq n$ such that $h = \{v_j, \dots, v_k\}$ or $V \setminus h = \{v_j, \dots, v_k\}$. Summarizing, we have the following theorem.

Theorem 2. *It can be tested in $\mathcal{O}(nN + n + m)$ time whether a hypergraph has a support that is a cactus.*

Proof. Compute all blocks in $\mathcal{O}(nN + n + m)$ time. Test all blocks in linear time for the circular consecutive ones property [4]. □

4 Hypergraphs Closed under Intersections and Differences

Two hyperedges h_1, h_2 *overlap* if $h_1 \cap h_2 \neq \emptyset$, $h_1 \setminus h_2 \neq \emptyset$, and $h_2 \setminus h_1 \neq \emptyset$. An Euler diagram of two overlapping hyperedges is usually drawn such that the intersection of the two regions representing the two hyperedges is connected and such that the part of one of the regions that is not contained in the other is also connected. See Fig. 5 for an illustration. This motivates the following definition. A hypergraph $H = (V, A)$ is *closed under intersections and differences* if $h_1 \cap h_2 \in A \cup \{\{v\};\ v \in V\}$ and $h_1 \setminus h_2 \in A \cup \{\{v\};\ v \in V\}$ for two overlapping hyperedges $h_1, h_2 \in A$. In the remainder of this section we show that it is easy to decide whether a hypergraph closed under intersections and differences has a planar or an outerplanar support.



(a) undesired (b) desired (c) illustration of Lemma 5

Fig. 5. (a) Undesired and (b) desired drawings of two overlapping hyperedges and (c) an illustration of the proof of Lemma 5. In (a) the intersection or the difference of two hyperedges is not connected, while in (b) it is.

For a hypergraph $H = (V, A)$ let $H_2 = (V, \{h \in A;\ |h| = 2\})$ be the graph of all hyperedges of H that contain exactly two vertices. We will show that H_2 is a support of H if H is a block.

Lemma 5. *If the hypergraph H is closed under intersections and differences and does not contain an articulation point then the hypergraph H_2 induced by all hyperedges of size two is a support of H.*

Proof. Let $H = (V, A)$ be a hypergraph that is closed under intersections and differences and assume that H does not contain an articulation point. Let h by a hyperedge of H. By induction on the size of h, we show that $H_2[h]$ is connected. There is nothing to show if $|h| \leq 2$. So assume that $|h| > 2$.

We first assume that $h \neq V$. Since H does not contain any articulation point there are at least two hyperedges h_1, h_2 with $h_1 \cap h \neq h_2 \cap h$ that overlap with h. We have $h \cap h_i, h \setminus h_i \in A \cup \{\{v\};\ v \in V\}, i = 1, 2$. By the inductive hypothesis, $H_2[h \cap h_i]$ and $H_2[h \setminus h_i], i = 1, 2$ are all four connected. If $h \cap h_1 \neq h \setminus h_2$ then it follows that $H_2[h]$ is connected.

So assume that for all pairs h_1, h_2 of hyperedges with $h \cap h_1 \neq h \cap h_2$ that overlap with h it holds that $h \cap h_1 = h \setminus h_2$. Hence there is a bisection h^1, h^2 of h such that for all hyperedges h_1 that overlap with h it holds that $h \cap h_1 = h^1$ or $h \cap h_1 = h^2$. See Fig. 5 for an illustration of this part of the proof. Note again that by the inductive hypothesis $H_2[h^i], i = 1, 2$ are both connected. Since h contains more than two vertices, we may assume without loss of generality that h^1 contains at least two vertices. If $|h^2| = 1$ there has to be a hyperedge $h' \subset h$ that overlaps h^1 and contains h^2. Otherwise every vertex in h^1 would be an articulation vertex. Similarly, if $|h^2| > 1$ there has to be a hyperedge h' that overlaps both, h^1 and h^2. Let h' be the smallest hyperedge with this property. Assume that $|h' \cap h^i| > 1$ for $i = 1$ or $i = 2$. Since $H_2[h^i]$ is connected there have to be vertices $v \in h^i \cap h', w \in h^i \setminus h'$ such that $\{v, w\}$ is a hyperedge. But then $h' \setminus \{v, w\} \in A$ is a smaller hyperedge than h' with the required property – a contradiction. It follows that $|h'| = 2$. Hence, $H_2[h]$ contains the connected subgraphs $H_2[h^i], i = 1, 2$ and the edge h' connecting them. Thus, $H_2[h]$ is connected.

Assume finally that $h = V$. If H contains more than two vertices then the hypergraph $(V, A\backslash\{V\})$ has to be connected. Otherwise all but at most one vertex of H would be articulation points. Since $H_2[h']$ is connected for all hyperedges $h' \neq V$ it thus follows that also $H_2[V]$ is connected. □

Note that the hyperedges of size two have to be contained in every support of a hypergraph. So we have the following corollary.

Corollary 1. *It can be decided in $\mathcal{O}(nN + n + m)$ time whether a hypergraph closed under intersections and differences has a planar or outerplanar support.*

Proof. First, decompose the hypergraph into blocks. Then test for each block whether the graph induced by the hyperedges of size two is planar or outerplanar, respectively (Lemma 3). □

5 Conclusions

In this paper, we newly defined a decomposition of a hypergraph into blocks. For any such decomposition there is a support with the property that the blocks of the hypergraph correspond to the biconnected components of the support. We then give two applications of the decomposition into blocks. A hypergraph has a cactus support if and only if each block has the cyclic consecutive one's property. A hypergraph that is closed under intersections and differences has an (outer-)planar support if and only if for each block the graph induced by the hyperedges of size two is (outer-)planar.

As a future work, we want to improve the run time of the decomposition into blocks and to solve the problem of testing whether an outerplanar support exists in more general cases.

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