Unique perfect phylogeny is NP-hard

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Abstract. We answer, in the affirmitive, the following question proposed by Mike Steel as a $100 challenge: *"Is the following problem NP-hard? Given a ternary[†] phylogenetic X-tree $\mathcal{T}$ and a collection $\mathcal{Q}$ of quartet subtrees on X, is $\mathcal{T}$ the only tree that displays $\mathcal{Q}$?"* [25,27]

1 Introduction

One of the major efforts in molecular biology has been the computation of phylogenetic trees, or *phylogenies*, which describe the evolution of a set of species from a common ancestor. A phylogenetic tree for a set of species is a tree in which the leaves represent the species from the set and the internal nodes represent the (hypothetical) ancestral species. One standard model for describing the species is in terms of *characters*, where a character is an equivalence relation on the species set, partitioning it into different *character states*. In this model, we also assign character states to the (hypothetical) ancestral species. The desired property is that for each state of each character, the set of nodes in the tree having that character state forms a connected subgraph. When a phylogeny has this property, we say it is *perfect*. The Perfect Phylogeny problem [15] then asks *for a given set of characters defining a species set, does there exist a perfect phylogeny?* Note that we allow that states of some characters are unknown for some species; we call such characters *partial*, otherwise we speak of *full* characters. This approach to constructing phylogenies has been studied since the 1960s [4,19,20,21,30] and was given a precise mathematical formulation in the 1970s [9,10,11,12]. In particular, Buneman [3] showed that the Perfect Phylogeny problem reduces to a specific graph-theoretic problem, the problem of finding a chordal completion of a graph that respects a prescribed colouring. In fact, the two problems are polynomially equivalent [17]. Thus, using this formulation, it has been proved that the Perfect Phylogeny problem is NP-hard in [2] and independently in [28]. These two results rely on the fact that the input may contain partial characters. In fact, the characters in these constructions only have two states. If we insist on full characters, the situation is different as for any fixed number r of character states, the problem can be solved in time polynomial [1] in the size of the input

[†] The original formulation uses the term "binary", in the sense of "rooted binary tree", but in this contex the two are equivalent.

(and exponential in r). In fact, for $r = 2$ (or $r = 3$), the solution exists if and only if it exists of every pair (or triple) of characters [12,18]. Also, when the number of characters is k (even if there are partial characters), the complexity [22] is polynomial in the number of species (and exponential in k).

Another common formulation of this problem is the problem of a *consensus tree* [7,14,28], where a collection of subtrees with labeled leaves is given (for instance, the leaves correspond to species of a partial character). Here, we ask for a (phylogenetic) tree such that each of the input subtrees can be obtained by contracting edges from the tree (we say that the tree *displays* the subtree). It turns out that the problem is equivalent [25] even if we only allow particular input subtrees, the so-called *quartet trees* which have exactly six vertices and four leaves. In fact, any ternary phylogenetic tree can be uniquely described by a collection of quartet trees [25]. However, a collection of quartet trees does not necessarily uniquely describe a ternary phylogenetic tree.

This leads to a natural question: *what is the complexity of deciding whether or not a collection of quartet trees uniquely describes a (ternary) phylogenetic tree?* This question was posed in [25], later conjectured to be NP-hard and listed on M. Steel's personal webpage [27] where he offers \$100 for the first proof of NP-hardness. In this paper, we answer this question by showing that the problem is indeed NP-hard. In particular, we prove the following theorem.

Theorem 1. *It is NP-hard to determine, given a ternary phylogenetic X-tree $\mathcal{T}$ and a collection $\mathcal{Q}$ of quartet subtrees on X, whether or not $\mathcal{T}$ is the only phylogenetic tree that displays $\mathcal{Q}$.*

We prove the theorem by describing a polynomial-time reduction from the uniqueness problem for ONE-IN-THREE-3SAT, which is NP-hard by the following result of [16]. (Note that [16] gives a complete complexity characterization of uniqueness for boolean satisfaction problems similar to that of Shaefer [26].)

Theorem 2. [16] *It is NP-hard to decide, given an instance I to* ONE-IN-THREE-3SAT*, and a truth assignment σ that satisfies I, whether or not σ is the unique satisfying truth assignment for I.*

Our construction in the reduction is essentially a modification of the construction of [2] which proves NP-hardness of the Perfect Phylogeny problem. Recall that the construction of [2] produces instances $\mathcal{Q}$ that have a perfect phylogeny if and only if a particular boolean formula φ is satisfiable. We immediately observed that these instances $\mathcal{Q}$ have, in addition, the property that φ has a unique satisfying assignment if and only if there is a unique minimal restricted chordal completion of the partial partition intersection graph of $\mathcal{Q}$ (for definitions see Section 2). This is precisely one of the two necessary conditions for uniqueness of perfect phylogeny as proved by Semple and Steel in [24] (see Theorem 4). Thus by modifying the construction of [2] to also satisfy the other condition of uniqueness of [24], we obtained the construction that we present in this paper. Note that, however, unlike [2] which uses 3SAT, we had to use a different NP-hard problem in order for the construction to work correctly. Also, to prove that the construction is correct, we employ a variant of the characterization of [24] that uses the more general chordal sandwich problem [13] instead of the restricted chordal completion problem (see Theorem 7). In fact, by way of Theorems 5 and 6, we establish a direct connection between the problem of perfect phylogeny and the chordal sandwich problem, which apparently has not been yet observed. (Note that the connection to the (restricted) chordal completion problem of coloured graphs as mentioned above [3,17] is a special case of this.) Using this result, we are able to present a much simplified proof of Theorem 1.

Finally, as a corollary, we obtain the following result.

Corollary 1 (Chordal sandwich). *The Unique chordal sandwich problem is NP-hard. Counting the number of minimal chordal sandwiches is #P-complete.*

The first part follows directly from Theorems 2 and 8, while the second part follows from Theorem 8 and [5]. (Note that [5] gives a complete complexity characterization for the problem of counting satisfying assignments for boolean satisfaction problems, just like [16] gives for uniqueness as mentioned above).

The paper is structured as follows. First, in Section 2, we describe some preliminary definitions and results needed for our construction of the reduction. In particular, we describe, based on [24], necessary and sufficient conditions for the existence of a unique perfect phylogeny in terms of the minimal chordal sandwich problem (cf. [6,13]). The proof of this characterization is postponed until Section 5. In Section 3, we describe the actual construction and state one of the two uniqueness conditions (Theorem 8) relating minimal chordal sandwiches to satisfying assignments of an instance I of ONE-IN-THREE-3SAT. The proof is presented later in Section 6. In Section 4, we describe and prove the other uniqueness condition (Theorem 9) relating satisfying assignments of I to phylogenetic trees. In Section 7, we put these results together to prove Theorem 1.

2 Preliminaries

We mostly follow the terminology of [24,25] and graph-theoretical notions of [29].

Let X be a non-empty set. An *X-tree* is a pair (T, ϕ) where T is tree and $\phi : X \to V(T)$ is a mapping such that $\phi^{-1}(v) \neq \emptyset$ for all vertices $v \in V(T)$ of degree at most two. An X-tree (T, ϕ) is *ternary* if all internal vertices of T have degree three. Two X-trees (T_1, ϕ_1), (T_2, ϕ_2) are *isomorphic* if there exists an isomorphism $\psi : V(T_1) \to V(T_2)$ between T_1 and T_2 that satisfies $\phi_2 = \psi \circ \phi_1$.

An X-tree (T, ϕ) is a *phylogenetic X-tree* (or a *free X-free* in [24]) if ϕ is bijection between X and the set of leaves of T.

A *partial partition* of X is a partition of a non-empty subset of X into at least two sets. If A_1, A_2, …, A_t are these sets, we call them *cells* of this partition, and denote the partition $A_1|A_2|\dots|A_t$. If $t = 2$, we call the partition a *partial split*. A partial split $A_1|A_2$ is trivial if $|A_1| = 1$ or $|A_2| = 1$.

A *quartet tree* is a ternary phylogenetic tree with a label set of size four, that is, a ternary tree $\mathcal{T}$ with 6 vertices, 4 leaves labeled a, b, c, d, and with only one non-trivial partial split $\{a, b\}|\{c, d\}$ that it displays. Note that such a tree is unambiguously defined by this partial split. Thus, in the subseqent text, we identify the quartet tree $\mathcal{T}$ with the partial split $\{a,b\}|\{c,d\}$, that is, we say that $\{a,b\}|\{c,d\}$ is both a quartet tree and a partial split.

Let $\mathcal{T} = (T, \phi)$ be an X-tree, and let $\pi = A_1|A_2|\ldots|A_t$ be a partial partition of X. We say that $\mathcal{T}$ *displays* π if there is a set of edges F of T such that, for all distinct $i, j \in \{1 \ldots t\}$, the sets $\phi(A_i)$ and $\phi(A_j)$ are subsets of the vertex sets of different connected components of $T - F$. We say that an edge e of T is *distinguished* by π if every set of edges that displays π in $\mathcal{T}$ contains e.

Let $\mathcal{Q}$ be a collection of partial partitions of X. An X-tree $\mathcal{T}$ *displays* $\mathcal{Q}$ if it displays every partial partition in $\mathcal{Q}$. An X-tree $\mathcal{T} = (T, \phi)$ is *distinguished* by $\mathcal{Q}$ if every internal edge of T is distinguished by some partial partition in $\mathcal{Q}$; we also say that $\mathcal{Q}$ *distinguishes* $\mathcal{T}$. The set $\mathcal{Q}$ *defines* $\mathcal{T}$ if $\mathcal{T}$ displays $\mathcal{Q}$, and all other X-trees that display $\mathcal{Q}$ are isomorphic to $\mathcal{T}$. Note that if $\mathcal{Q}$ defines $\mathcal{T}$, then $\mathcal{T}$ is necessarily a ternary phylogenetic X-tree, since otherwise "resolving" any vertex either of degree four or more, or with multiple labels results in a non-isomorphic X-tree that also displays $\mathcal{Q}$ (also, see Proposition 2.6 in [24]).

The *partial partition intersection graph* of $\mathcal{Q}$, denoted by $\text{int}(\mathcal{Q})$, is a graph whose vertex set is $\{(A, \pi) \mid \text{where A is a cell of } \pi \in \mathcal{Q}\}$ and two vertices (A, π), (A', π') are adjacent just if the intersection of A and A' is non-empty.

A graph is *chordal* if it contains no induced cycle of length four or more. A *chordal completion* of a graph $G = (V, E)$ is a chordal graph $G' = (V, E')$ with $E \subseteq E'$. A *restricted chordal completion* of $\text{int}(\mathcal{Q})$ is a chordal completion G' of $\text{int}(\mathcal{Q})$ with the property that if A_1,A_2 are cells of $\pi \in \mathcal{Q}$, then (A_1, π) is not adjacent to (A_2, π) in G'. A restricted chordal completion G' of $\text{int}(\mathcal{Q})$ is *minimal* if no proper subgraph of G' is a restricted chordal completion of $\text{int}(\mathcal{Q})$.

The problem of perfect phylogeny is equivalent to the problem of determining the existence of an X-tree that display the given collection $\mathcal{Q}$ of partial partitions. In [3], it was given the following graph-theoretical characterization.

Theorem 3. [3,25,28] *Let $\mathcal{Q}$ be a set of partial partitions of a set X. Then there exists an X-tree that displays $\mathcal{Q}$ if and only if there exists a restricted chordal completion of* $\text{int}(\mathcal{Q})$.

Of course, the X-tree in the above theorem might not be unique. For the problem of uniqueness, Semple and Steel [24,25] describe necessary and sufficient conditions for when a collection of partial partitions defines an X-tree.

Theorem 4. [24] *Let $\mathcal{Q}$ be a collection of partial partitions of a set X. Let $\mathcal{T}$ be a ternary phylogenetic X-tree. Then $\mathcal{Q}$ defines $\mathcal{T}$ if and only if:*
(i) $\mathcal{T}$ displays $\mathcal{Q}$ and is distinguished by $\mathcal{Q}$, and
(ii) there is a unique minimal restricted chordal completion of $\text{int}(\mathcal{Q})$.

In order to simplify our construction, we now describe a variant of the above theorem that, instead, deals with the notion of chordal sandwich.

Let $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ be two graphs on the same set of vertices with $E_1 \cap E_2 = \emptyset$. A *chordal sandwich*† of (G_1,G_2) is a chordal graph $G' = (V, E')$

† In this formulation, E_1 are the *forced* edges and E_2 are the *forbidden* edges. See [13] for further details on different ways of specifying the input to this problem.

with $E_1 \subseteq E'$ and $E' \cap E_2 = \emptyset$. A chordal sandwich G' of (G_1,G_2) is *minimal* if no proper subgraph of G' is a chordal sandwich of (G_1,G_2).

The *cell intersection graph* of $\mathcal{Q}$, denoted by $\mathrm{int}^*(\mathcal{Q})$, is the graph whose vertex set is $\{A \mid \text{where A is a cell of } \pi \in \mathcal{Q}\}$ and two vertices A, A' are adjacent just if the intersection of A and A' is non-empty. Let $\mathrm{forb}(\mathcal{Q})$ denote the graph whose vertex set is that of $\mathrm{int}^*(\mathcal{Q})$ in which there is an edge between A and A' just if A,A' are cells of some $\pi \in \mathcal{Q}$.

The correspondence between the partial partition intersection graph and the cell intersection graph is captured by the following theorem.

Theorem 5. *Let $\mathcal{Q}$ be a collection of partial partitions of a set X. Then there is a one-to-one correspondence between the minimal restricted chordal completions of* $\mathrm{int}(\mathcal{Q})$ *and the minimal chordal sandwiches of* $(\mathrm{int}^*(\mathcal{Q}), \mathrm{forb}(\mathcal{Q}))$.

(The proof of this theorem is presented as Section 5.)

This combined with Theorem 3 yields that there exists a phylogenetic X-tree that displays $\mathcal{Q}$ if and only if there exists a chordal sandwich of $(\mathrm{int}^*(\mathcal{Q}), \mathrm{forb}(\mathcal{Q}))$. Conversely, we can express every instance to the chordal sandwich problem as a corresponding instance to the problem of perfect phylogeny as follows.

Theorem 6. *Let (G_1, G_2) be an instance to the chordal sandwich problem. Then there is a collection $\mathcal{Q}$ of partial splits such that there is a one-to-one correspondence between the minimal chordal sandwiches of (G_1, G_2) and the minimal restricted chordal completions of* $\mathrm{int}(\mathcal{Q})$. *In particular, there exists a chordal sandwich for (G_1, G_2) if and only if there exists a phylogenetic tree that displays $\mathcal{Q}$.*

Proof. Without loss of generality, we may assume that each connected component of G_1 has at least three vertices. (We can safely remove any component with two or less vertices without changing the number of minimal chordal completions, since every such component is already chordal.)

As usual, $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ where $E_1 \cap E_2 = \emptyset$. We define the collection $\mathcal{Q}$ of partial splits (of the set E_1) as follows: for every edge $xy \in E_2$, we construct the partial split $F_x|F_y$, where F_x are the edges of E_1 incident to x, and F_y are the edges of E_1 incident to y. By definition, the vertex set of the graph $\mathrm{int}^*(\mathcal{Q})$ is precisly $\{F_v \mid v \in V\}$. Further, it can be easily seen that the mapping ψ that, for each $v \in V$, maps v to F_v is an isomorphism between G_1 and $\mathrm{int}^*(\mathcal{Q})$. (Here, one only needs to verify that $F_u = F_v$ implies $u = v$; for this we use that each component of G_1 has at least three vertices.) Moreover, $\mathrm{forb}(\mathcal{Q})$ is precisely $\{\psi(x)\psi(y) \mid xy \in E_2\}$ by definition. Therefore, by Theorem 5, there is a one-to-one correspondence between the minimal chordal sandwiches of (G_1, G_2) are the minimal restricted chordal completions of $\mathrm{int}(\mathcal{Q})$. This proves the first part of the claim; the second part follows directly from Theorem 3. $\square$

As an immediate corollary, we obtain the following desired characterization.

Theorem 7. *Let $\mathcal{Q}$ be a collection of partial partitions of a set X. Let $\mathcal{T}$ be a ternary phylogenetic X-tree. Then $\mathcal{Q}$ defines $\mathcal{T}$ if and only if:*

(i) $\mathcal{T}$ displays $\mathcal{Q}$ and is distinguished by $\mathcal{Q}$, and

(ii) there is a unique minimal chordal sandwich of $\Big(\mathrm{int}^*(\mathcal{Q}),\ \mathrm{forb}(\mathcal{Q})\Big)$.

3 Construction

Consider an instance I to ONE-IN-THREE-3SAT. That is, I consists of n variables $v_1, \ldots, v_n$ and m clauses $\mathcal{C}_1, \ldots, \mathcal{C}_m$ each of which is a disjunction of exactly three *literals* (i.e., variables v_i or their negations $\overline{v_i}$).

By standard arguments, we may assume that no variable appears twice in the same clause, since otherwise we can replace the instance I with an equivalent instance with this property. In particular, we can replace each clause of the form $v_i \vee \overline{v_i} \vee v_j$ by clauses $v_i \vee x \vee v_j$ and $\overline{v_i} \vee \overline{x} \vee v_j$ where x is a new variable, and replace each clause of the form $v_i \vee v_i \vee v_j$ by clauses $v_i \vee v_j \vee x$, $v_i \vee \overline{v_j} \vee \overline{x}$, and $\overline{v_i} \vee \overline{v_j} \vee x$ where x is again a new variable. Note that these two transformation preserve the number of satisfying assignments, since in the former the new variable x has always the truth value of $\overline{v_i}$ while in the latter x is always false in any satisfying assignment of this modified instance.

In what follows, we describe a collection $\mathcal{Q}_I$ of quartet trees arising from the instance I, and prove the following theorem. (We present the proof as Section 6.)

Theorem 8. *There is a one-to-one correspondence between satisfying assignments of the instance I and minimal chordal sandwiches of* $(\mathrm{int}^*(\mathcal{Q}_I), \mathrm{forb}(\mathcal{Q}_I))$.

To simplify the presentation, we shall denote literals by capital letters X, Y, etc., and indicate their negations by $\overline{X}$, $\overline{Y}$, etc. (For instance, if $X = v_i$ then $\overline{X} = \overline{v_i}$, and if $X = \overline{v_i}$ then $\overline{X} = v_i$.)

A *truth assignment* for the instance I is a mapping $\sigma : \{v_1, \ldots, v_n\} \to \{0, 1\}$ where 0 and 1 represent *false* and *true*, respectively. To simplify the notation, we write $v_i = 0$ and $v_i = 1$ in place of $\sigma(v_i) = 0$ and $\sigma(v_i) = 1$, respectively, and extend this notation to literals X,Y, etc., i.e., write $X = 0$ and $X = 1$ in place of $\sigma(X) = 0$ and $\sigma(X) = 1$, respectively. A truth assignment σ is a *satisfying assignment for I* if in each clause $\mathcal{C}_j$ exactly one the three literals evalues to true. That is, for each clause $\mathcal{C}_j = X \vee Y \vee Z$, either $X = 1$, $Y = 0$, $Z = 0$, or $X = 0$, $Y = 1$, $Z = 0$, or $X = 0$, $Y = 0$, $Z = 1$.

For each $i \in \{1 \ldots n\}$, we let Δ_i denote all indices j such that v_i or $\overline{v_i}$ appears in the clause $\mathcal{C}_j$. Let $\mathcal{X}_I$ be the set consisting of the following elements:

a) α_{v_i}, $\alpha_{\overline{v_i}}$ for each $i \in \{1 \ldots n\}$,
b) $\beta^j_{v_i}$, $\beta^j_{\overline{v_i}}$ for each $i \in \{1 \ldots n\}$ and each $j \in \Delta_i$,
c) γ^j_1, γ^j_2, γ^j_3, λ^j for each $j \in \{1 \ldots m\}$,
d) δ and μ.

Consider the following collection of 2-element subsets of $\mathcal{X}_I$:

a) $B = \Big\{\mu, \delta\Big\}$,
b) for each $i \in \{1, \ldots, n\}$:
$H_{v_i} = \{\alpha_{v_i}, \delta\Big\}$, $H_{\overline{v_i}} = \{\alpha_{\overline{v_i}}, \delta\Big\}$, $A_i = \Big\{\alpha_{v_i}, \alpha_{\overline{v_i}}\Big\}$,
$S^j_{v_i} = \Big\{\alpha_{v_i}, \beta^j_{v_i}\Big\}$, $S^j_{\overline{v_i}} = \Big\{\alpha_{\overline{v_i}}, \beta^j_{\overline{v_i}}\Big\}$ for all $j \in \Delta_i$

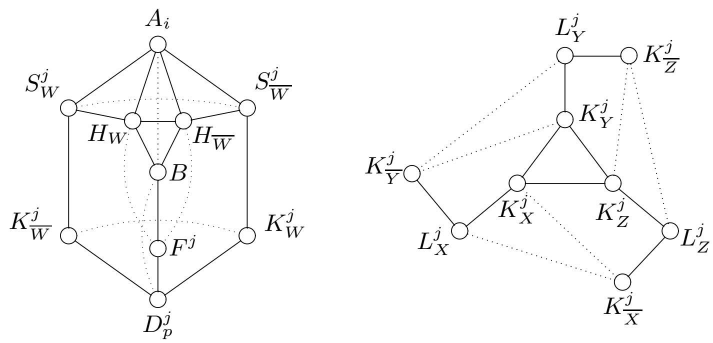

Fig. 1. Two configurations from of the graph $\mathrm{int}^*(\mathcal{Q}_I)$.

c) for each $j \in \{1 \ldots m\}$ where $C_j = X \vee Y \vee Z$:

$$K^j_{\overline{X}} = \left\{\beta^j_X, \gamma^j_1\right\},\ K^j_{\overline{Y}} = \left\{\beta^j_Y, \gamma^j_2\right\},\ K^j_{\overline{Z}} = \left\{\beta^j_Z, \gamma^j_3\right\},$$
$$K^j_X = \left\{\beta^j_{\overline{X}}, \lambda^j\right\},\ K^j_Y = \left\{\beta^j_{\overline{Y}}, \lambda^j\right\},\ K^j_Z = \left\{\beta^j_{\overline{Z}}, \lambda^j\right\},$$
$$L^j_X = \left\{\beta^j_{\overline{X}}, \gamma^j_2\right\},\ L^j_Y = \left\{\beta^j_{\overline{Y}}, \gamma^j_3\right\},\ L^j_Z = \left\{\beta^j_{\overline{Z}}, \gamma^j_1\right\},$$
$$D^j_1 = \left\{\gamma^j_1, \lambda^j\right\},\quad D^j_2 = \left\{\gamma^j_2, \lambda^j\right\},\quad D^j_3 = \left\{\gamma^j_3, \lambda^j\right\},\quad F^j = \left\{\lambda^j, \mu\right\}$$

The collection $\mathcal{Q}_I$ of quartet trees is defined as follows:

$$\begin{aligned}
\mathcal{Q}_I = & \bigcup_{i\in\{1\ldots n\}} \left\{A_i|B\right\} \cup \bigcup_{j\in\{1\ldots m\}} \left\{D^j_1|B, D^j_2|B, D^j_3|B\right\} \\
\cup & \bigcup_{\substack{i\in\{1\ldots n\}\\ j,j'\in\Delta_i}} \left\{S^j_{v_i}|S^{j'}_{\overline{v_i}}\right\} \cup \bigcup_{\substack{i\in\{1\ldots n\}\\ j,j'\in\Delta_i \text{ and } j<j'}} \left\{S^j_{v_i}|K^{j'}_{\overline{v_i}}, S^j_{\overline{v_i}}|K^{j'}_{v_i}\right\} \cup \bigcup_{\substack{i\in\{1\ldots n\}\\ j\in\Delta_i \text{ and } j<j'\le m}} \left\{K^j_{\overline{v_i}}|F^{j'}, K^j_{v_i}|F^{j'}\right\} \\
\cup & \bigcup_{\substack{1\le i'<i\le n\\ j\in\Delta_i}} \left\{H_{v_{i'}}|S^j_{v_i}, H_{\overline{v_{i'}}}|S^j_{v_i}, H_{v_{i'}}|S^j_{\overline{v_i}}, H_{\overline{v_{i'}}}|S^j_{\overline{v_i}}\right\} \cup \bigcup_{\substack{i\in\{1\ldots n\}\\ j\in\{1\ldots m\}}} \left\{H_{\overline{v_i}}|F^j, H_{v_i}|F^j\right\} \\
\cup & \bigcup_{\substack{j\in\{1\ldots m\}\\ \text{where } \mathcal{C}_j=X\vee Y\vee Z}} \left\{\begin{matrix} K^j_{\overline{X}}|K^j_X,\ K^j_{\overline{Y}}|K^j_Y,\ K^j_{\overline{Z}}|K^j_Z,\ K^j_{\overline{X}}|L^j_X,\ K^j_{\overline{Y}}|L^j_Y, K^j_{\overline{Z}}|L^j_Z \\ S^j_Y|K^j_X,\ \ S^j_Z|K^j_Y,\ \ S^j_X|K^j_Z,\ \ S^j_Z|L^j_X,\ \ S^j_X|L^j_Y, S^j_Y|L^j_Z \end{matrix}\right\}
\end{aligned}$$

Note that in each clause $\mathcal{C}_j = X \vee Y \vee Z$ there is a particular type of symmetry between the literals X, Y, and Z. In particular, if we replace, in the above, the incices X, Y, Z and 1, 2, 3 as follows: $X \to Y \to Z \to X$ and $1 \to 2 \to 3 \to 1$, we obtain precisely the same definition of $\mathcal{Q}_I$ as the above. We shall refer to this as the *rotational symmetry* between X, Y, Z.

4 Unique trees

Let T_I be the tree defined as follows: (for illustration, see Figures 2 and 3)

$$
\begin{aligned}
V(T_I) = &\Big\{y_0, y_1, y'_1, \ldots, y_n, y'_n\Big\} \cup \Big\{a_1, a'_1, \ldots, a_n, a'_n\Big\} \cup \Big\{u_0, u_1, \ldots, u_m\Big\} \\
&\cup \Big\{x^j_1, x^j_2, x^j_3, x^j_4, x^j_5, x^j_6, b^j_1, b^j_2, b^j_3, g^j_1, g^j_2, g^j_3, \ell^j\Big\}_{j=1}^{m} \cup \Big\{c^j_i, z^j_i \mid j \in \Delta_i\Big\}_{i=1}^{n} \\
E(T_I) = &\Big\{y_1y'_1, y_2y'_2, \ldots, y_ny'_n\Big\} \cup \Big\{a_1y'_1, a_2y'_2, \ldots a_ny'_n\Big\} \cup \Big\{c^j_i z^j_i \mid j \in \Delta_i\Big\}_{i=1}^{n} \\
\cup &\Big\{y_0y_1, y_1y_2, y_2y_3, \ldots, y_{n-1}y_n\Big\} \cup \Big\{y_nu_1, u_1u_2, u_2u_3, \ldots, u_{m-1}u_m, u_mu_0\Big\} \\
\cup &\Big\{u_jx^j_1, x^j_1x^j_2, x^j_2x^j_3, x^j_2x^j_4, x^j_4x^j_5, x^j_4x^j_6, b^j_1x^j_6, b^j_2x^j_3, b^j_3x^j_5, g^j_1x^j_6, g^j_2x^j_1, g^j_3x^j_3, \ell^j x^j_5\Big\}_{j=1}^{m} \\
\cup &\Big\{a'_i z^{j_1}_i, z^{j_1}_i z^{j_2}_i, \ldots, z^{j_{t-1}}_i z^{j_t}_i, z^{j_t}_i y'_i \mid \text{where } j_1 < j_2 < \ldots < j_t \text{ are elements of } \Delta_i\Big\}_{i=1}^{n}
\end{aligned}
$$

Let σ be a satisfying assignment for the instance I, and let ϕ_σ be the mapping of $\mathcal{X}_I$ to $V(T_I)$ defined as follows:

a) for each $i \in \{1 \ldots n\}$:
   if $v_i = 1$, then $\phi_\sigma(\alpha_{v_i}) = a_i$, $\phi_\sigma(\alpha_{\overline{v_i}}) = a'_i$, and $\phi_\sigma(\beta^j_{\overline{v_i}}) = c^j_i$ for all $j \in \Delta_i$,
   if $v_i = 0$, then $\phi_\sigma(\alpha_{\overline{v_i}}) = a_i$, $\phi_\sigma(\alpha_{v_i}) = a'_i$, and $\phi_\sigma(\beta^j_{v_i}) = c^j_i$ for all $j \in \Delta_i$,
b) for each $j \in \{1 \ldots m\}$ where $\mathcal{C}_j = X \vee Y \vee Z$:
   if $X = 1$, then $\phi_\sigma(\beta^j_X) = b^j_1$, $\phi_\sigma(\beta^j_{\overline{Y}}) = b^j_2$, $\phi_\sigma(\beta^j_{\overline{Z}}) = b^j_3$,
   $\phi_\sigma(\gamma^j_1) = g^j_1$, $\phi_\sigma(\gamma^j_2) = g^j_2$, $\phi_\sigma(\gamma^j_3) = g^j_3$, $\phi_\sigma(\lambda^j) = \ell_j$,
   if $Y = 1$, then $\phi_\sigma(\beta^j_Y) = b^j_1$, $\phi_\sigma(\beta^j_{\overline{Z}}) = b^j_2$, $\phi_\sigma(\beta^j_{\overline{X}}) = b^j_3$,
   $\phi_\sigma(\gamma^j_2) = g^j_1$, $\phi_\sigma(\gamma^j_3) = g^j_2$, $\phi_\sigma(\gamma^j_1) = g^j_3$, $\phi_\sigma(\lambda^j) = \ell_j$,
   if $Z = 1$, then $\phi_\sigma(\beta^j_Z) = b^j_1$, $\phi_\sigma(\beta^j_{\overline{X}}) = b^j_2$, $\phi_\sigma(\beta^j_{\overline{Y}}) = b^j_3$,
   $\phi_\sigma(\gamma^j_3) = g^j_1$, $\phi_\sigma(\gamma^j_1) = g^j_2$, $\phi_\sigma(\gamma^j_2) = g^j_3$, $\phi_\sigma(\lambda^j) = \ell_j$,
c) $\phi_\sigma(\delta) = y_0$ and $\phi_\sigma(\mu) = u_0$.

Theorem 9. *If σ is a satisfying assignment for I, then $\mathcal{T}_\sigma = (T_I, \phi_\sigma)$ is a ternary phylogenetic $\mathcal{X}_I$-tree that displays $\mathcal{Q}_I$ and is distinguished by $\mathcal{Q}_I$.*

Proof. Let σ be a satisfying assignment for I, i.e., for each clause $\mathcal{C}_j = X \vee Y \vee Z$, either $X = 1$, $Y = Z = 0$, or $Y = 1$, $X = Z = 0$, or $Z = 1$, $X = Y = 0$. For each $i \in \{1 \ldots n\}$, let $\mathcal{A}_i = \{a_i, a'_i, y'_i, z^{j_1}_i, \ldots, z^{j_t}_i, c^{j_1}_i, \ldots, c^{j_t}_i\}$ where $\Delta_i = \{j_1, \ldots, j_t\}$, and for each $j \in \{1 \ldots m\}$, let $\mathcal{B}_j = \{x^j_1, x^j_2, x^j_3, x^j_4, x^j_5, x^j_6, g^j_1, g^j_2, g^j_3, b^j_1, b^j_2, b^j_3, \ell^j\}$.

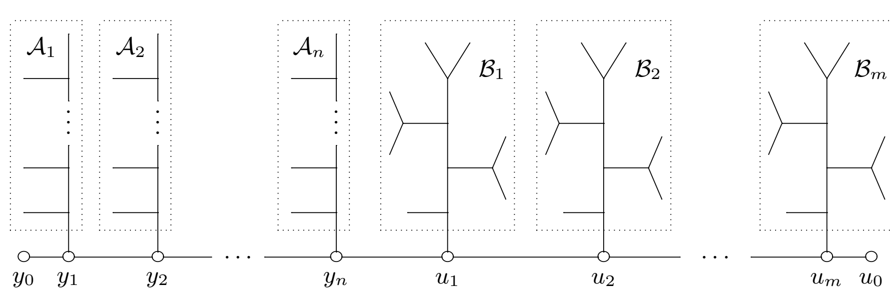

Fig. 2. The tree T_I.

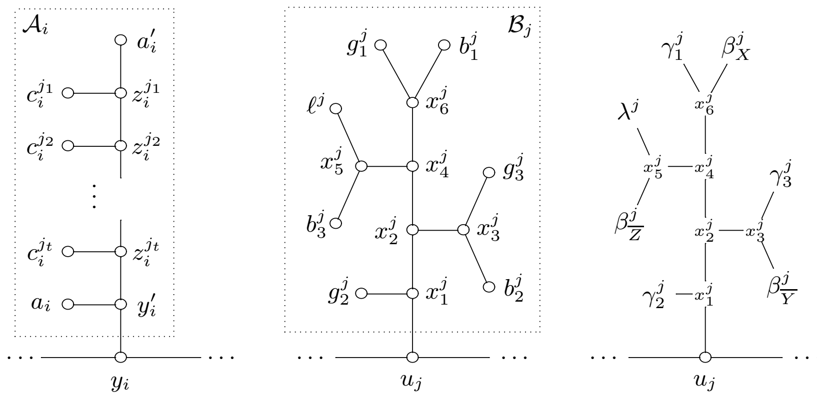

Fig. 3. *a)* the subtree $\mathcal{A}_i$ for the variable v_i, *b)* the subtree $\mathcal{B}_j$ for the clause $\mathcal{C}_j$, *c)* the subtree for $\mathcal{C}_j = X \vee Y \vee Z$ and assignment $\sigma(X) = 1$, $\sigma(Y) = \sigma(Z) = 0$

It is not difficult to see that ϕ_σ defines a bijection between the elements of $\mathcal{X}_I$ and the leaves of T_I. For instance, for each $i \in \{1 \ldots n\}$, we note that $\{\phi(\alpha_{v_i}),\ \phi(\alpha_{\overline{v_i}})\} = \{a_i,\ a_i'\}$, and for each $j \in \Delta_i$, either $\phi_\sigma(\beta^j_{v_i}) = c_i^j$ and $\phi_\sigma(\beta^j_{\overline{v_i}}) \in \{b_1^j, b_2^j, b_3^j\}$, or $\phi_\sigma(\beta^j_{\overline{v_i}}) = c_i^j$ and $\phi_\sigma(\beta^j_{v_i}) \in \{b_1^j, b_2^j, b_3^j\}$. Also, for each $j \in \{1 \ldots m\}$, we have $\phi_\sigma(\lambda^j) = \ell^j$, and $\{\phi_\sigma(\gamma_1^j), \phi_\sigma(\gamma_2^j), \phi_\sigma(\gamma_3^j)\} = \{g_1^j, g_2^j, g_3^j\}$. Further, it can be readily verified that T_I is a ternary tree. Thus, $\mathcal{T}_\sigma = (T_I, \phi_\sigma)$ is indeed a ternary phylogenetic $\mathcal{X}_I$-tree. First, we show that it displays $\mathcal{Q}_I$.

Consider $A_i|B$ for $i \in \{1 \ldots n\}$. Recall that $A_i = \{\alpha_{v_i}, \alpha_{\overline{v_i}}\}$, $B = \{\delta, \mu\}$, and that $\{\phi_\sigma(\alpha_{v_i}), \phi_\sigma(\alpha_{\overline{v_i}})\} = \{a_i, a_i'\}$. Also, $\phi_\sigma(\delta) = y_0$ and $\phi_\sigma(\mu) = u_0$. Observe that $a_i, a_i' \in \mathcal{A}_i$. Hence, both a_i, a_i' are in one connected component of $T_I - y_i y_i'$ whereas y_0, u_0 are in another component. Thus, $\mathcal{T}_\sigma$ indeed displays $A_i|B$.

Next, consider $D_p^j|B$ for $j \in \{1 \ldots m\}$ and $p \in \{1 \ldots 3\}$. Recall that $D_p^j = \{\gamma_p^j, \lambda^j\}$, and $\phi_\sigma(\gamma_p^j) \in \mathcal{B}_j$, $\phi_\sigma(\lambda^j) \in \mathcal{B}_j$. Also, $B = \{\delta, \mu\}$ and $\phi_\sigma(\delta) = y_0$, $\phi_\sigma(\mu) = u_0$. Thus both $\phi_\sigma(\gamma_p^j)$, $\phi_\sigma(\lambda^j)$ are in one component of $T_I - u_j x_1^j$ whereas y_0, u_0 are in another component. This shows that $\mathcal{T}_\sigma$ displays $D_p^j|B$.

Now, we look at $S^j_{v_i}|S^{j'}_{\overline{v_i}}$ where $i \in \{1 \ldots n\}$ and $j, j' \in \Delta_i$. Recall that $S^j_{v_i} = \{\alpha_{v_i}, \beta^j_{v_i}\}$ and $S^{j'}_{\overline{v_i}} = \{\alpha_{\overline{v_i}}, \beta^{j'}_{\overline{v_i}}\}$. By symmetry, we may assume that $v_i = 1$. Then $\phi_\sigma(\alpha_{v_i}) = a_i$, $\phi_\sigma(\alpha_{\overline{v_i}}) = a_i'$, $\phi_\sigma(\beta^j_{v_i}) \in \mathcal{B}_j$, and $\phi_\sigma(\beta^{j'}_{\overline{v_i}}) = c_i^{j'}$. Let j_t denote the largest element in Δ_i. Then, both $a_i', c_i^{j'}$ are in one component of $T_I - y_i' z_i^{j_t}$ whereas a_i and $\phi_\sigma(\beta^j_{v_i})$ are in a different component. Thus, $\mathcal{T}_\sigma$ displays $S^j_{v_i}|S^{j'}_{\overline{v_i}}$.

Next, consider $S^j_{v_i}|K^{j'}_{\overline{v_i}}$ and $S^j_{\overline{v_i}}|K^{j'}_{v_i}$ for $i \in \{1 \ldots n\}$ and $j, j' \in \Delta_i$ where $j < j'$. Recall that $K^{j'}_{\overline{v_i}} \subseteq \{\beta^{j'}_{v_i}, \gamma_1^{j'}, \gamma_2^{j'}, \gamma_3^{j'}, \lambda^{j'}\}$, $K^{j'}_{v_i} \subseteq \{\beta^{j'}_{\overline{v_i}}, \gamma_1^{j'}, \gamma_2^{j'}, \gamma_3^{j'}, \lambda^{j'}\}$, $S^j_{v_i} = \{\alpha_{v_i}, \beta^j_{v_i}\}$ and $S^j_{\overline{v_i}} = \{\alpha_{\overline{v_i}}, \beta^j_{v_i}\}$. Again, by symmetry, we assume $v_i = 1$. So, $\phi_\sigma(\alpha_{v_i}) = a_i$, $\phi_\sigma(\alpha_{\overline{v_i}}) = a_i'$, $\phi_\sigma(\beta^j_{\overline{v_i}}) = c_i^j$, $\phi_\sigma(\beta^{j'}_{\overline{v_i}}) = c_i^{j'}$, $\phi_\sigma(\beta^j_{v_i}) \in \mathcal{B}_j$, and $\{\phi_\sigma(\beta^{j'}_{v_i}), \phi_\sigma(\gamma_1^{j'}), \phi_\sigma(\gamma_2^{j'}), \phi_\sigma(\gamma_3^{j'}), \phi_\sigma(\lambda^{j'})\} \subseteq \mathcal{B}_{j'}$. Let $j_1 < j_2 < \ldots < j_t$ be the elements of Δ_i. Since $j \in \Delta_i$, let k be such that $j = j_k$. We conclude $k < t$, since

$j < j'$ and $j' \in \Delta_i$. Thus, the elements of $\phi_\sigma(S^j_{\overline{v_i}})$ and $\phi_\sigma(K^{j'}_{v_i})$, respectively are in different components of $T_I - z_i^{j_k} z_i^{j_{k+1}}$. Further, observe that $\phi_\sigma(K^{j'}_{\overline{v_i}}) \subseteq \mathcal{B}_{j'}$, and since $j \neq j'$, the elements of $\phi_\sigma(S^j_{v_i})$ and $\phi_\sigma(K^{j'}_{\overline{v_i}})$ are in different components of $T_I - u_{j'} x_1^{j'}$. This proves that $\mathcal{T}_\sigma$ displays both $S^j_{v_i}|K^{j'}_{\overline{v_i}}$ and $S^j_{\overline{v_i}}|K^{j'}_{v_i}$.

Now, consider $K^j_{\overline{v_i}}|F^{j'}$ and $K^j_{v_i}|F^{j'}$ for $i \in \{1 \dots n\}$ and $j < j'$ where $j \in \Delta_i$. Again, recall that $K^j_{\overline{v_i}} \subseteq \{\beta^j_{v_i}, \gamma^j_1, \gamma^j_2, \gamma^j_3, \lambda^j\}$, $K^j_{v_i} \subseteq \{\beta^j_{\overline{v_i}}, \gamma^j_1, \gamma^j_2, \gamma^j_3, \lambda^j\}$, and that $F^{j'} = \{\lambda^{j'}, \mu\}$. So, $\phi_\sigma(K^j_{\overline{v_i}}) \cup \phi_\sigma(K^j_{v_i}) \subseteq \mathcal{A}_i \cup \mathcal{B}_j$ whereas $\phi_\sigma(F^{j'}) \subseteq \mathcal{B}_{j'} \cup \{u_0\}$. Since $j < j' \leq m$, we conclude that $\phi_\sigma(K^j_{\overline{v_i}}) \cup \phi_\sigma(K^j_{v_i})$ and $\phi_\sigma(F^{j'})$ are in different components of $T_I - u_j u_{j+1}$. Thus $\mathcal{T}_\sigma$ displays both $K^j_{\overline{v_i}}|F^{j'}$ and $K^j_{v_i}|F^{j'}$.

Next, we consider $H_{v_{i'}}|S^j_{v_i}$, $H_{\overline{v_{i'}}}|S^j_{v_i}$, $H_{v_{i'}}|S^j_{\overline{v_i}}$, and $H_{\overline{v_{i'}}}|S^j_{\overline{v_i}}$ for $1 \leq i' < i \leq n$ and $j \in \Delta_i$. Recall that $H_{v_{i'}} = \{\alpha_{v_{i'}}, \delta\}$, $H_{\overline{v_{i'}}} = \{\alpha_{\overline{v_{i'}}}, \delta\}$, $S^j_{v_i} = \{\alpha_{v_i}, \beta^j_{v_i}\}$, and $S^j_{\overline{v_i}} = \{\alpha_{\overline{v_i}}, \beta^j_{\overline{v_i}}\}$. So, $\phi_\sigma(S^j_{v_i}) \cup \phi_\sigma(S^j_{\overline{v_i}}) \subseteq \mathcal{A}_i \cup \mathcal{B}_j$ whereas $\phi_\sigma(H_{v_{i'}}) \cup \phi_\sigma(H_{\overline{v_{i'}}}) \subseteq \mathcal{A}_{i'} \cup \{\delta\}$. Thus, since $i' < i \leq n$, we conclude that $\phi_\sigma(S^j_{v_i}) \cup \phi_\sigma(S^j_{\overline{v_i}})$ and $\phi_\sigma(H_{v_{i'}}) \cup \phi_\sigma(H_{\overline{v_{i'}}})$ are in different components of $T_I - y_{i'} y_{i'+1}$. This proves that $\mathcal{T}_\sigma$ displays all the four quartet trees $H_{v_{i'}}|S^j_{v_i}$, $H_{\overline{v_{i'}}}|S^j_{v_i}$, $H_{v_{i'}}|S^j_{\overline{v_i}}$ and $H_{\overline{v_{i'}}}|S^j_{\overline{v_i}}$.

Similarly, we consider $H_{\overline{v_i}}|F^j$ and $H_{v_i}|F^j$ for $i \in \{1 \dots n\}$ and $j \in \{1 \dots m\}$. Recall that $H_{v_i} = \{\alpha_{v_i}, \delta\}$, $H_{\overline{v_i}} = \{\alpha_{\overline{v_i}}, \delta\}$, and $F^j = \{\lambda^j, \mu\}$. Hence, it follows that $\{\phi_\sigma(H_{\overline{v_i}}) \cup \phi_\sigma(H_{v_i})\} \subseteq \mathcal{A}_i \cup \{\delta\}$ and $\phi_\sigma(F^j) \subseteq \mathcal{B}_j \cup \{\mu\}$. Thus, we conclude that $\phi_\sigma(H_{\overline{v_i}}) \cup \phi_\sigma(H_{v_i})$ and $\phi_\sigma(F^j)$ are in different components of $T_I - y_n u_1$. This proves that $\mathcal{T}_\sigma$ displays both $H_{\overline{v_i}}|F^j$ and $H_{v_i}|F^j$.

Finally, we consider the clause $\mathcal{C}_j = X \vee Y \vee Z$ for $j \in \{1 \dots m\}$. Since σ is a satisfying assignment, and by the rotational symmetry between X, Y, and Z, we may assume that $X = 1$, $Y = 0$, and $Z = 0$. Let i_X be the index such that $X = v_{i_X}$ or $X = \overline{v_{i_X}}$, let i_Y be such that $Y = v_{i_Y}$ or $Y = \overline{v_{i_Y}}$, and let i_Z be such that $Z = v_{i_Z}$ or $Z = \overline{v_{i_Z}}$. Note that i_X, i_Y, i_Z are all distinct, since we assume that no variable appears more than once in each clause. Thus we have that $\phi_\sigma(\beta^j_X) = b^j_1$, $\phi_\sigma(\beta^j_{\overline{Y}}) = b^j_2$, $\phi_\sigma(\beta^j_{\overline{Z}}) = b^j_3$, $\phi_\sigma(\gamma^j_1) = g^j_1$, $\phi_\sigma(\gamma^j_2) = g^j_2$, $\phi_\sigma(\gamma^j_3) = g^j_3$, and $\phi_\sigma(\lambda^j) = \ell_j$. (See Figure 3c.) Also, $\{\phi_\sigma(\alpha_X), \phi_\sigma(\alpha_{\overline{X}}), \phi_\sigma(\beta^j_{\overline{X}})\} \subseteq \mathcal{A}_{i_X}$, $\{\phi_\sigma(\alpha_Y), \phi_\sigma(\alpha_{\overline{Y}}), \phi_\sigma(\beta^j_Y)\} \subseteq \mathcal{A}_{i_Y}$, and $\{\phi_\sigma(\alpha_Z), \phi_\sigma(\alpha_{\overline{Z}}), \phi_\sigma(\beta^j_Z)\} \subseteq \mathcal{A}_{i_Z}$. First, consider $K^j_{\overline{X}}|K^j_X$ and $K^j_{\overline{X}}|L^j_X$. Recall that $K^j_{\overline{X}} = \{\beta^j_X, \gamma^j_1\}$, $K^j_X = \{\beta^j_{\overline{X}}, \lambda^j\}$, and $L^j_X = \{\beta^j_{\overline{X}}, \gamma^j_2\}$. Also, recall that $\phi_\sigma(\beta^j_{\overline{X}}) \in \mathcal{A}_{i_X}$. Thus it follows that $\phi_\sigma(K^j_X) \cup \phi_\sigma(L^j_X)$ and $\phi_\sigma(K^j_{\overline{X}})$ are in different components of $T_I - x^j_4 x^j_6$. Now, consider $K^j_{\overline{Y}}|K^j_Y$ and $K^j_{\overline{Y}}|L^j_Y$. Recall that $K^j_{\overline{Y}} = \{\beta^j_Y, \gamma^j_2\}$, $K^j_Y = \{\beta^j_{\overline{Y}}, \lambda^j\}$, and $L^j_Y = \{\beta^j_{\overline{Y}}, \gamma^j_3\}$ where $\phi_\sigma(\beta^j_Y) \in \mathcal{A}_{i_Y}$. Thus, $\phi_\sigma(K^j_Y) \cup \phi_\sigma(L^j_Y)$ and $\phi_\sigma(K^j_{\overline{Y}})$ are in different components of $T_I - x^j_1 x^j_2$. Similarly, consider $K^j_{\overline{Z}}|K^j_Z$ and $K^j_{\overline{Z}}|L^j_Z$. Recall that $K^j_{\overline{Z}} = \{\beta^j_Z, \gamma^j_3\}$, $K^j_Z = \{\beta^j_{\overline{Z}}, \lambda^j\}$, and $L^j_Z = \{\beta^j_{\overline{Z}}, \gamma^j_1\}$ where $\phi_\sigma(\beta^j_Z) \in \mathcal{A}_{i_Z}$. Thus, $\phi_\sigma(K^j_Z) \cup \phi_\sigma(L^j_Z)$ and $\phi_\sigma(K^j_{\overline{Z}})$ are in different components of $T_I - x^j_2 x^j_4$. Now, consider $S^j_Y|K^j_X$ and $S^j_Y|L^j_Z$. Recall that $S^j_Y = \{\alpha_Y, \beta^j_Y\}$, $K^j_X = \{\beta^j_{\overline{X}}, \lambda^j\}$ and $L^j_Z = \{\beta^j_{\overline{Z}}, \gamma^j_1\}$. Also, $\{\phi_\sigma(\alpha_Y), \phi_\sigma(\beta^j_Y)\} \subseteq \mathcal{A}_{i_Y}$ whereas $\phi_\sigma(\beta^j_X) \in \mathcal{A}_{i_X}$. Thus, since $i_X \neq i_Y$, we conclude that $\phi_\sigma(S^j_Y)$

and $\phi_\sigma(K_X^j) \cup \phi_\sigma(L_Z^j)$ are in different components of $T_I - y_{i_Y} y'_{i_Y}$. Similarly, we consider $S_Z^j | K_Y^j$ and $S_Z^j | L_X^j$. Recall that $S_Z^j = \{\alpha_Z, \beta_Z^j\}$, $K_Y^j = \{\beta_{\overline{Y}}^j, \lambda^j\}$, and $L_X^j = \{\beta_{\overline{X}}^j, \gamma_2^j\}$. Also, $\{\phi_\sigma(\alpha_Z), \phi_\sigma(\beta_Z^j)\} \subseteq \mathcal{A}_{i_Z}$, and $\phi_\sigma(\beta_{\overline{X}}^j) \in \mathcal{A}_{i_X}$. Thus, since $i_X \neq i_Z$, we conclude that $\phi_\sigma(S_Z^j)$ and $\phi_\sigma(K_Y^j) \cup \phi_\sigma(L_X^j)$ are in different components of $T_I - y_{i_Z} y'_{i_Z}$. Finally, consider $S_X^j | K_Z^j$ and $S_X^j | L_Y^j$. Recall that $S_X^j = \{\alpha_X, \beta_X^j\}$, $K_Z^j = \{\beta_{\overline{Z}}^j, \lambda^j\}$ and $L_Y^j = \{\beta_{\overline{Y}}^j, \gamma_3^j\}$ where $\phi_\sigma(\alpha_X) \in \mathcal{A}_{i_X}$. Thus, $\phi_\sigma(S_X^j)$ and $\phi_\sigma(K_Z^j)$ are in different components of $T_I - x_4^j x_5^j$, whereas $\phi_\sigma(S_X^j)$ and $\phi_\sigma(L_Y^j)$ are in different components of $T_I - x_2^j x_3^j$.

This proves that $\mathcal{T}_\sigma$ displays $\mathcal{Q}_I$. It remains to prove that $\mathcal{T}_\sigma$ is distinguished by $\mathcal{Q}_I$. First, consider the edge $y_i y'_i$ for $i \in \{1 \dots n\}$. Recall that $A_i = \{\alpha_{v_i}, \alpha_{\overline{v_i}}\}$ and $B = \{\delta, \mu\}$. By definition, we have $\phi_\sigma(A_i) = \{a_i, a'_i\}$ and $\phi_\sigma(B) = \{y_0, u_0\}$. Note that every connected subgraph of T_I that contains both y_0 and u_0 must also contain y_i, since it lies on the path between u_0 and y_0 in T_I. Likewise, every connected subgraph of T_I that contains a_i, a'_i also contains y'_i. Thus, this shows that the edge $y_i y'_i$ is distinguished by $A_i | B$ which is in $\mathcal{Q}_I$. We similarly consider the edge $u_j x_1^j$ for $j \in \{1 \dots m\}$. By the definition of ϕ_σ, we observe that there exists $p \in \{1, 2, 3\}$ such that $\phi_\sigma(\gamma_p^j) = g_2^j$. We recall that $B = \{\delta, \mu\}$ and $D_p^j = \{\gamma_p^j, \lambda^j\}$. Thus, $\phi_\sigma(B) = \{y_0, u_0\}$ and $\phi_\sigma(D_p^j) = \{g_2^j, \ell^j\}$. Since g_j^2 is adjacent to x_1^j, and u_j lies on the path between y_0 and u_0, it follows that the edge $u_j x_1^j$ is distinguished by $D_p^j | B$ which is in $\mathcal{Q}_I$.

Now, consider $i \in \{1 \dots n\}$, and let $j_1 < j_2 < \dots < j_t$ be the elements of Δ_i. Let $W \in \{v_i, \overline{v_i}\}$ be such that $W = 1$. Then we have $\phi_\sigma(\alpha_W) = a_i$, $\phi_\sigma(\alpha_{\overline{W}}) = a'_i$, and $\phi_\sigma(\beta_{\overline{W}}^j) = c_i^j$ for all $j \in \Delta_i$. Recall that $S_{\overline{W}}^j = \{\alpha_{\overline{W}}, \beta_{\overline{W}}^j\}$ and $K_W^j \subseteq \{\beta_{\overline{W}}^j, \gamma_1^j, \gamma_2^j, \gamma_3^j, \lambda^j\}$ where $\{\phi_\sigma(\gamma_1^j), \phi_\sigma(\gamma_2^j), \phi_\sigma(\gamma_3^j), \phi_\sigma(\lambda^j)\} \subseteq \mathcal{B}_j$ for all $j \in \Delta_i$. Thus, for each $k \in \{1 \dots t-1\}$, it follows that $\phi_\sigma(\beta_{\overline{W}}^{j_k})$ is adjacent to $z_i^{j_k}$ whereas $\phi_\sigma(\beta_{\overline{W}}^{j_{k+1}})$ is adjacent to $z_i^{j_{k+1}}$. This proves that the edge $z_i^{j_k} z_i^{j_{k+1}}$ is distinguished by $S_{\overline{W}}^{j_k} | K_W^{j_{k+1}}$. Similarly, recall that $S_W^j = \{\alpha_W, \beta_W^j\}$ where $\phi_\sigma(\beta_W^j) \in \mathcal{B}_j$ and $\phi_\sigma(\alpha_W)$ is adjacent to y'_i. Thus, the edge $z_i^{j_t} y'_i$ is distinguished by $S_W^{j_t} | S_{\overline{W}}^{j_t}$. Further, if $i \geq 2$, then we recall that $H_{v_{i-1}} = \{\alpha_{v_{i-1}}, \delta\}$ where $\phi_\sigma(\alpha_{v_{i-1}}) \in \mathcal{A}_{i-1}$ and $\phi_\sigma(\delta) = y_0$. Thus $y_{i-1} y_i$ is distinguished by $H_{v_{i-1}} | S_W^{j_t}$.

Now, consider $j \in \{1, \dots m\}$ where $\mathcal{C}_j = X \vee Y \vee Z$. By the rotational symmetry, we may assume that $X = 1$ and $Y = Z = 0$. Thus $\phi_\sigma(\beta_X^j) = b_1^j$, $\phi_\sigma(\beta_{\overline{Y}}^j) = b_2^j$, $\phi_\sigma(\beta_{\overline{Z}}^j) = b_3^j$, $\phi_\sigma(\gamma_1^j) = g_1^j$, $\phi_\sigma(\gamma_2^j) = g_2^j$, $\phi_\sigma(\gamma_3^j) = g_3^j$, and $\phi_\sigma(\lambda^j) = \ell_j$. (Again see Figure 3c.) Recall that $K_Y^j = \{\beta_{\overline{Y}}^j, \lambda^j\}$ and $K_{\overline{Y}}^j = \{\beta_Y^j, \gamma_2^j\}$ where $\phi_\sigma(\beta_Y^j) \notin \mathcal{B}_j$. This shows that the edge $x_1^j x_2^j$ is distinguished by $K_{\overline{Y}}^j | K_Y^j$. Recall that $S_X^j = \{\alpha_X, \beta_X^j\}$, $L_Y^j = \{\beta_{\overline{Y}}^j, \gamma_3^j\}$, and $K_Z^j = \{\beta_{\overline{Z}}^j, \lambda^j\}$ where $\phi_\sigma(\alpha_X) \notin \mathcal{B}_j$. Thus, the edge $x_2^j x_3^j$ is distiguished by $S_X^j | L_Y^j$ whereas the edge $x_4^j x_5^j$ is distinguished by $S_X^j | K_Z^j$. Recall that $K_{\overline{Z}}^j = \{\beta_Z^j, \gamma_3^j\}$ and $L_Z^j = \{\beta_{\overline{Z}}^j, \gamma_1^j\}$ where $\phi_\sigma(\beta_Z^j) \notin \mathcal{B}_j$. Thus, the edge $x_2^j x_4^j$ is distinguished by $K_{\overline{Z}}^j | L_Z^j$. Recall that

$K^j_X = \{\beta^j_{\overline{X}}, \lambda^j\}$ and $K^j_{\overline{X}} = \{\beta^j_X, \gamma^j_1\}$ where $\phi_\sigma(\beta^j_X) \notin \mathcal{B}_j$. Thus, the edge $x^j_4 x^j_6$ is distinguished by $K^j_{\overline{X}}|K^j_X$. Further, if $j < m$, recall that $F^{j+1} = \{\lambda^{j+1}, \mu\}$ where $\phi_\sigma(\lambda^{j+1}) \in \mathcal{B}_{j+1}$ and $\phi_\sigma(\mu) = u_0$. Thus $u_j u_{j+1}$ is distinguished by $K^j_X|F^{j+1}$.

Finally, recall that $H_{v_n} = \{\alpha_{v_n}, \delta\}$ and $F^1 = \{\lambda^1, \mu\}$. So, $\phi_\sigma(H_{v_n}) \subseteq \mathcal{A}_n \cup \{y_0\}$ and $\phi_\sigma(F^1) \subseteq \mathcal{B}_j \cup \{u_0\}$. Thus, the edge $y_n u_1$ is distinguished by $H_{v_n}|F^1$.

This concludes the proof. □

5 Proof of Theorem 5

To prove Theorem 5, we need to introduce some additional tools. The following is a standard property of minimal chordal completions.

Lemma 1. *Let G' be a chordal completion of G. Then G' is a minimal chordal completion of G if and only if for all $uv \in E(G') \setminus E(G)$, the vertices u, v have at least two non-adjacent common neighbours in G'.*

Proof. Suppose that G' is a minimal chordal completion. Let $uv \in E(G') \setminus E(G)$, and let $G'' = G' - uv$. Since G' is a minimal chordal completion and $uv \notin E(G)$, we conclude that G'' is not chordal. Thus, there exists a set $C \subseteq V(G')$ that induces a cycle in G''. Since G' is chordal, C does not induce a cycle in G'. This implies $u, v \in C$, and hence, uv is the unique chord of $G'[C]$. So, we conclude $|C| = 4$, because otherwise $G'[C]$ contains an induced cycle. Let x, y be the two vertices of $C \setminus \{u, v\}$. Clearly, $xy \notin E(G')$ and both x and y are common neighbours of u, v as required.

Conversely, suppose that G' is not a minimal chordal completion. Then by [23], there exists an edge $uv \in E(G') \setminus E(G)$ such that $G' - uv$ is a chordal graph. Therefore, u, v do not have non-adjacent common neighbours x, y in G', since otherwise $\{u, x, v, y\}$ induces a 4-cycle in $G' - uv$, which is impossible since we assume that $G' - uv$ is chordal. That concludes the proof. □

Using this tool, we prove the following two important lemmas.

Lemma 2. *Let G be a graph and G' be a minimal chordal completion of G. If G contains vertices u, v with $N_G(u) \subseteq N_G(v)$, then also $N_{G'}(u) \subseteq N_{G'}(v)$.*

Proof. Let u, v be vertices of G with $N_G(u) \subseteq N(G_v)$. Let $B = N_{G'}(u) \setminus N_{G'}(v)$ and $A = N_{G'}(u) \cap N_{G'}(v)$. Assume that $B \neq \emptyset$, and let A_1 denote the vertices of A with at least one neighbour in B. Look at the graph $G_1 = G'[A_1 \cup B \cup \{v\}]$.

By the definition of A_1 and B, the vertex v is adjacent to each vertex of A_1 and non-adjacent to each vertex of B. Hence, no vertex of A_1 is simplicial in G_1, since it is adjacent to v and at least one vertex in B.

Now, consider $w \in B$. By the definition of B, we have that w is adjacent in G' to u but not v. Thus, uw is not an edge of G, since $N_G(u) \subseteq N_G(v)$ and $N_G(v) \subseteq N_{G'}(v)$. So, by Lemma 1, the vertices u, w have non-adjacent common neighbours x, y in G'. Since x, y are adjacent to u, we have $x, y \in A \cup B$. In fact, since w has no neighbours in $A \setminus A_1$, we conclude $x, y \in A_1 \cup B$. Thus, w is not a simplicial vertex in G_1, and hence, no vertex of B is simplicial in G_1.

This proves that no vertex of G_1, except possibly for v, is simplicial in G_1. Also, G_1 is not a complete graph, since $B \neq \emptyset$, and v has no neighbour in B. Recall that G_1 is chordal because G' is. Thus, by the result of Dirac [8], G_1 must contain at least two non-adjacent simplicial vertices, but that is impossible. Hence, we must conclude $B = \emptyset$. In other words, $N_{G'}(u) \subseteq N_{G'}(v)$. $\square$

Lemma 3. *Let G be a graph, and let H be a graph obtained from G by substituting complete graphs for the vertices of G. Then there is a one-to-one correspondence between minimal chordal completions of G and H.*

Proof. Let v_1,v_2,...,v_n be the vertices of G. Since H is obtained from G by substituting complete graphs, there is a partition $C_1 \cup \ldots \cup C_n$ of $V(H)$ where each C_i induces a complete graph in H, and for every distinct $i, j \in \{1 \ldots n\}$:

($\star$) each $x \in C_i$, $y \in C_j$ satisfy $v_i v_j \in E(G)$ if and only if $xy \in E(H)$.

Let G' be any graph with vertex set $V(G)$, and let $H' = \Psi(G')$ be the graph constructed from G' by, for each $i \in \{1 \ldots n\}$, substituting C_i for the vertex v_i, and making C_i into a complete graph. Thus, for every distinct $i, j \in \{1 \ldots n\}$

($\star\star$) each $x \in C_i$, $y \in C_j$ satisfy $v_i v_j \in E(G')$ if and only if $xy \in E(H')$.

We prove that Ψ is a bijection between the minimal chordal completions of G and H which will yield the claim of the lemma.

Let G' be a minimal chordal completion of G, and let $H' = \Psi(G')$. Clearly, H' is chordal, since G' is chordal, and chordal graphs are closed under the operation of substituting a complete graph for a vertex. Also, observe that $V(H) = V(H')$, and if $xy \in E(H)$, then either $x, y \in C_i$ for some $i \in \{1 \ldots n\}$, in which case $xy \in E(H')$, since C_i induces a complete graph in H', or we have $x \in C_i$, $y \in C_j$ for distinct $i, j \in \{1 \ldots n\}$ in which case $v_i v_j \in E(G)$ by ($\star$) implying $v_i v_j \in E(G')$, since $E(G) \subseteq E(G')$, and hence, $xy \in E(H')$ by ($\star\star$). This proves that $E(H) \subseteq E(H')$, and thus, H' is a chordal completion of H.

To prove that H' is a minimal chordal completion of H, it suffices, by Lemma 1, to show that for all $xy \in E(H') \setminus E(H)$, the vertices x, y have at least two non-adjacent common neighbours in H'. Consider $xy \in E(H') \setminus E(H)$, and let $i, j \in \{1 \ldots n\}$ be such that $x \in C_i$ and $y \in C_j$. Since $xy \notin E(H)$ and C_i induces a complete graph in H, we conclude $i \neq j$. Thus, by ($\star\star$), we have $v_i v_j \in E(G')$, and so, $v_i v_j \in E(G') \setminus E(G)$ by ($\star$). Now, recall that G' is a minimal chordal completion of G. Thus, by Lemma 1, the vertices v_i, v_j have non-adjacent common neighbours v_k, v_ℓ in G'. So, we let $w \in C_k$ and $z \in C_\ell$. By ($\star\star$), we conclude $wz \notin E(H')$, since $v_k v_\ell \notin E(G')$. Moreover, ($\star\star$) also implies that z, w are common neighbours of x, y, since v_k, v_ℓ are common neighbours of v_i, v_j. This proves that x, y have non-adjacent common neighbours, and thus shows that H' is a minimal chordal completion of H.

Conversely, let H' be a minimal chordal completion of H. Let G' be the graph with $V(G') = V(G)$ such that $v_i v_j \in E(G')$ if and only if there exists $x \in C_i$,

$y \in C_j$ with $xy \in E(H')$. Let $i \in \{1 \ldots n\}$ and consider vertices $x, y \in C_i$ in the graph H. Recall that C_i induces a complete graph in H. This implies that $xy \in E(H)$ and both x and y are adjacent in H to every $z \in C_i \setminus \{x, y\}$. Further, by $(\star)$, if $z \in C_j$ where $j \neq i$, then x, y are both adjacent to z if $v_i v_j \in E(G)$, and x, y are both non-adjacent to z if $v_i v_j \notin E(G)$. This shows that $N_H(x) = N_H(y)$, and hence, $N_{H'}(x) = N_{H'}(y)$ by Lemma 2 and the fact that H' is a minimal chordal completion of H. This proves that $H' = \Psi(G')$, and hence, G' is chordal. In fact, $E(G) \subseteq E(G')$ by $(\star)$ and $(\star\star)$. Thus G' is a chordal completion of G.

It remains to show that G' is a minimal chordal completion of G. Again, it suffices to show that for each $v_i v_j \in E(G') \setminus E(G)$, the vertices v_i, v_j have non-adjacent common neighbours in G'. Consider $v_i v_j \in E(G') \setminus E(G)$, and let $x \in C_i$ and $y \in C_j$. So, $i \neq j$ and $xy \in E(H')$ by $(\star\star)$. Further, $xy \in E(H') \setminus E(H)$ by $(\star)$ and the fact that $v_i v_j \notin E(G)$. So, the vertices x, y have non-adjacent common neighbours w, z in H' by Lemma 2 and the fact that H' is a minimal chordal completion of H. Let $k, \ell \in \{1 \ldots n\}$ be such that $w \in C_k$ and $z \in C_\ell$. Since $xz \in E(H')$ but $wx \notin E(H')$, we conclude by $(\star\star)$ that $i \neq k$. By symmetry, also $i \neq \ell$, $j \neq k$, and $j \neq \ell$. Further, $k \neq \ell$, since $wx \notin E(H')$ and C_k induces a complete graph in H'. Thus, $(\star\star)$ implies that v_k, v_ℓ are non-adjacent common neighbours of v_i, v_j, since w, z are non-adjacent common neighbours of x, y. This proves that G' is indeed a minimal chordal completion of G.

That concludes the proof. $\square$

Now, we are finally ready to prove Theorem 5.

Proof of Theorem 5. We observe that the graph $\mathrm{int}(\mathcal{Q})$ is obtained by substituting complete graphs for the vertices of $\mathrm{int}^*(\mathcal{Q})$. Thus, by Lemma 3, there is a bijection Ψ between the minimal chordal completions of $\mathrm{int}(\mathcal{Q})$ and $\mathrm{int}^*(\mathcal{Q})$.

By translating the condition $(\star\star)$ from the proof of Lemma 3, we obtain that if G' is a minimal chordal completion of $\mathrm{int}^*(\mathcal{Q})$, then $H' = \Psi(G')$ is the graph whose vertex set is that of $\mathrm{int}(\mathcal{Q})$ with the property that for all $A, A' \in V(G')$

$(\star\star)$ all meaningful $\pi, \pi' \in \mathcal{Q}$ satisfy $AA' \in V(G') \iff (A, \pi)(A', \pi') \in V(H')$.

We show that Ψ is a bijection between the minimal restricted chordal completions of $\mathrm{int}(\mathcal{Q})$ and the minimal chordal sandwiches of $(\mathrm{int}^*(\mathcal{Q}), \mathrm{forb}(\mathcal{Q}))$.

First, let H' be a minimal restricted chordal completion of $\mathrm{int}(\mathcal{Q})$. Then $G' = \Psi^{-1}(H')$ is a minimal chordal completion of $\mathrm{int}^*(\mathcal{Q})$. Consider two cells A_1, A_2 of $\pi \in \mathcal{Q}$. Since H' is a restricted chordal completion of $\mathrm{int}(\mathcal{Q})$, we have that (A_1, π) is not adjacent to (A_2, π) in H'. Thus, $A_1 A_2 \notin E(G')$ by $(\star\star)$. This shows that G' contains no edge of $\mathrm{forb}(\mathcal{Q})$. Thus G' is a minimal chordal sandwich of $(\mathrm{int}^*(\mathcal{Q}), \mathrm{forb}(\mathcal{Q}))$, since it is also a minimal chordal completion of $\mathrm{int}^*(\mathcal{Q})$.

Conversely, let G' be a minimal chordal sandwich of $(\mathrm{int}^*(\mathcal{Q}), \mathrm{forb}(\mathcal{Q}))$. Then $H' = \Psi(G')$ is a minimal chordal completion of $\mathrm{int}(\mathcal{Q})$. Consider two cells A_1, A_2 of $\pi \in \mathcal{Q}$. Since $A_1 A_2$ is an edge of $\mathrm{forb}(\mathcal{Q})$, and G' is a minimal chordal sandwich of $(\mathrm{int}^*(\mathcal{Q}),\mathrm{forb}(\mathcal{Q}))$, we have $A_1 A_2 \notin E(G')$. Thus, $(A_1, \pi)(A_2, \pi) \notin E(H')$ by $(\star\star)$. This shows that H' is a minimal restricted chordal completion of $\mathrm{int}(\mathcal{Q})$.

That concludes the proof. $\square$

6 Proof of Theorem 8

For the proof, we shall need the following simple properties of chordal graphs.

Lemma 4. *Let G be a chordal graph, and let a, b be non-adjacent vertices of G. Then every two common neighbours of a and b are adjacent.*

Lemma 5. *Let G be a chordal graph, and $C = \{a, b, c, d, e\}$ be a 5-cycle in G with edges ab, bc, cd, de, ae. Then*
(a) $bd, ce \notin E(G)$ implies $ac, ad \in E(G)$, and
(b) $bd, be \notin E(G)$ implies $ac \in E(G)$.

Lemma 6. *Let G be a chordal graph, and $C = \{a, b, c, d, e, f\}$ be a 6-cycle in G with edges ab, bc, cd, de, ef, af. Then*
(a) $bd, ce, df \notin E(G)$ implies $ac, ad, ae \in E(G)$,
(b) $bd, ce, cf \notin E(G)$ implies $ac, ad \in E(G)$, and
(c) $be, bf, ce, cf \notin E(G)$ implies $ad \in E(G)$.

To assist the reader in following the subsequent arguments, we list here the cliques of $\mathrm{int}^*(\mathcal{Q}_I)$ according to the elements from which they arise:

a) for each $i \in \{1 \dots n\}$ where $j_1, j_2, \dots, j_k$ are the elements of Δ_i:
α_{v_i}: H_{v_i}, A_i, $S^{j_1}_{v_i}$, $S^{j_2}_{v_i}$, $\dots$, $S^{j_t}_{v_i}$, $\alpha_{\overline{v_i}}$: $H_{\overline{v_i}}$, A_i, $S^{j_1}_{\overline{v_i}}$, $S^{j_2}_{\overline{v_i}}$, $\dots$, $S^{j_t}_{\overline{v_i}}$,

b) for each $j \in \{1 \dots m\}$ where $\mathcal{C}_j = X \vee Y \vee Z$:
λ^j: K^j_X, K^j_Y, K^j_Z, D^j_1, D^j_2, D^j_3, F^j
γ^j_1: $K^j_{\overline{X}}$, L^j_Z, D^j_1 $\quad$ γ^j_2: $K^j_{\overline{Y}}$, L^j_X, D^j_2 $\quad$ γ^j_3: $K^j_{\overline{Z}}$, L^j_Y, D^j_3
β^j_X: S^j_X, $K^j_{\overline{X}}$ $\quad$ β^j_Y: S^j_Y, $K^j_{\overline{Y}}$ $\quad$ β^j_Z: S^j_Z, $K^j_{\overline{Z}}$
$\beta^j_{\overline{X}}$: $S^j_{\overline{X}}$, K^j_X, L^j_X $\quad$ $\beta^j_{\overline{Y}}$: $S^j_{\overline{Y}}$, K^j_Y, L^j_Y $\quad$ $\beta^j_{\overline{Z}}$: $S^j_{\overline{Z}}$, K^j_Z, L^j_Z

c) δ: B, H_{v_1}, $\dots$, H_{v_n}, $H_{\overline{v_1}}$, $\dots$, $H_{\overline{v_n}}$
μ: B, F^1, $\dots$, F^m

We start with a useful lemma describing an important property of $\mathrm{int}^*(\mathcal{Q}_I)$.

Lemma 7. *Let G' be a chordal sandwich of $(\mathrm{int}^*(\mathcal{Q}_I), \mathrm{forb}(\mathcal{Q}_I))$, and $1 \le i \le n$.*
(a) there is $W \in \{v_i, \overline{v_i}\}$ such that for all $j \in \Delta_i$, K^j_W is adjacent to B.
(b) for each $j \in \Delta_i$, and each $W \in \{v_i, \overline{v_i}\}$, if K^j_W is adjacent to B, then the vertices S^j_W, K^j_W, L^j_W (if exists) are adjacent to B, A_i, H_W, $H_{\overline{W}}$, F^j.

Proof. Let $i \in \{1 \dots n\}$. First, we observe the following.

($\star$) *for each $j \in \Delta_i$, each $W \in \{v_i, \overline{v_i}\}$, at least one of $S^j_{\overline{W}}$, K^j_W is adjacent to B.*

We may assume that $S^j_{\overline{W}}$ is not adjacent to B, otherwise we are done. Observe that $S^j_{\overline{W}}$ is adjacent to K^j_W, since $\beta^j_{\overline{W}} \in K^j_W \cap S^j_{\overline{W}}$. Moreover, there exists $p \in \{1, 2, 3\}$ such that $K^j_W \cap D^j_p$ contains λ^j or γ^j_p, implying that K^j_W is adjacent to D^j_p. Also, F^j is adjacent to D^j_p and B, since $\lambda^j \in D^j_p \cap F^j$ and $\mu \in B \cap F^j$, respectively. Further, $H_{\overline{W}}$ is adjacent to $S^j_{\overline{W}}$ and B, since $\alpha_{\overline{W}} \in H_{\overline{W}} \cap S^j_{\overline{W}}$ and $\delta \in H_{\overline{W}} \cap B$. Finally, $H_{\overline{W}}$ is not adjacent to F^j, and B is not adjacent to D^j_p, since $H_{\overline{W}}|F^j$ and $D^j_p|B$ are in $\mathcal{Q}_I$. So, by Lemma 6 applied to the cycle $\{K^j_W$, $S^j_{\overline{W}}$, $H_{\overline{W}}$, B, F^j, $D^j_p\}$, we conclude that K^j_W is adjacent to B. This proves $(\star)$.

Now, to prove (a), suppose for contradiction that there are $j, j' \in \Delta_i$ such that both $K^j_{\overline{v_i}}$ and $K^{j'}_{v_i}$ are not adjacent to B. Then by $(\star)$, both $S^j_{v_i}$ and $S^{j'}_{\overline{v_i}}$ are adjacent to B. Note also that A_i is adjacent to both $S^j_{v_i}$, $S^{j'}_{\overline{v_i}}$, since $\alpha_{v_i} \in A_i \cap S^j_{v_i}$ and $\alpha_{\overline{v_i}} \in A_i \cap S^{j'}_{\overline{v_i}}$. Further, note that A_iB and $S^j_{v_i}S^{j'}_{\overline{v_i}}$ are not edges of G', since $A_i|B$ and $S^j_{v_i}|S^{j'}_{\overline{v_i}}$ are in $\mathcal{Q}_I$. But then G' contains an induced 4-cycle on $\{S^j_{v_i}$, A_i, $S^{j'}_{\overline{v_i}}$, $B\}$, which is impossible, since G' is chordal. This proves (a).

For (b), suppose that K^j_W is adjacent to B for $j \in \Delta_i$ and $W \in \{v_i, \overline{v_i}\}$. First observe that K^j_W is adjacent to $S^j_{\overline{W}}$, and the vertex $K^j_{\overline{W}}$ is adjacent to S^j_W, since $\beta^j_{\overline{W}} \in K^j_W \cap S^j_{\overline{W}}$ and $\beta^j_W \in K^j_{\overline{W}} \cap S^j_W$. Moreover, there exists $p \in \{1,2,3\}$ such that $K^j_W \cap D^j_p$ and $K^j_{\overline{W}} \cap D^j_p$ contain γ^j_p and λ^j, respectively, or λ^j and γ^j_p, respectively. This implies that K^j_W and $K^j_{\overline{W}}$ are adjacent to D^j_p. Also, A_i is adjacent to S^j_W and $S^j_{\overline{W}}$, since $\alpha_W \in A_i \cap S^j_W$ and $\alpha_{\overline{W}} \in A_i \cap S^j_{\overline{W}}$. Further, note that D^j_pB, A_iB, $K^j_WK^j_{\overline{W}}$, and $S^j_WS^j_{\overline{W}}$ are not edges of G', since $D^j_p|B$, $A_i|B$, $K^j_W|K^j_{\overline{W}}$, and $S^j_W|S^j_{\overline{W}}$ are in $\mathcal{Q}_I$. This implies that $K^j_{\overline{W}}$ is not adjacent to B, since otherwise G' contains an induced 4-cycle on $\{K^j_W$, B, $K^j_{\overline{W}}$, $D^j_p\}$. So, by $(\star)$, we have that $S^j_{\overline{W}}$ is adjacent to B. Thus, Lemma 5 applied to $\{K^j_W$, $S^j_{\overline{W}}$, A_i, S^j_W, $B\}$ yields that K^j_W is adjacent to A_i and S^j_W. So, by Lemma 4 applied to $\{S^j_W$, K^j_W, D^j_p, $K^j_{\overline{W}}\}$, we have that S^j_W is adjacent to D^j_p.

Now, observe that H_W, $H_{\overline{W}}$ are adjacent to both A_i and B, since $\alpha_W \in H_W \cap A_i$, $\alpha_{\overline{W}} \in H_{\overline{W}} \cap A_i$, and $\delta \in B \cap H_W \cap H_{\overline{W}}$. Thus, by Lemma 4 applied to $\{u$, A_i, u', $B\}$ where $u \in \{S^j_W, K^j_W\}$ and $u' \in \{H_W, H_{\overline{W}}\}$, we conclude that S^j_W and K^j_W are adjacent to both H_W and $H_{\overline{W}}$. Similarly, we observe that F^j is adjacent to B and D^j_p, since $\mu \in F^j \cap B$ and $\lambda^j \in D^j_p \cap F^j$. Thus, Lemma 4 applied to $\{u$, B, F^j, $D^j_p\}$ yields that S^j_W and K^j_W are also adjacent to F^j.

Lastly, suppose that L^j_W exists. Then there exists $q \in \{1,2,3\}$ such that $\gamma^j_q \in D^j_q \cap L^j_W$ implying that L^j_W is adjacent to D^j_q. Moreover, F^j is adjacent to D^j_q and B, since $\lambda^j \in D^j_q \cap F^j$ and $\mu \in F^j \cap B$. Also, $H_{\overline{W}}$ is adjacent to B, $S^j_{\overline{W}}$, and the vertex $S^j_{\overline{W}}$ is adjacent to L^j_W, since $\delta \in B \cap H_{\overline{W}}$, $\alpha_{\overline{W}} \in H_{\overline{W}} \cap S^j_{\overline{W}}$, and $\beta^j_{\overline{W}} \in S^j_{\overline{W}} \cap L^j_W$. Further, $H_{\overline{W}}F^j$ and D^j_qB are not edges of G', since $H_{\overline{W}}|F^j$ and $D^j_q|B$ are in $\mathcal{Q}_I$. Also, $S^j_{\overline{W}}B$ is not an edge of G', since otherwise G' contains an induced 4-cycle on $\{S^j_W$, B, $S^j_{\overline{W}}$, $A_i\}$. Thus, by Lemma 5 applied to $\{L^j_W$, $S^j_{\overline{W}}$, $H_{\overline{W}}$, B, F^j, $D^j_q\}$, we conclude that L^j_W is adjacent to $H_{\overline{W}}$, B, and F^j. Moreover, by Lemma 5 applied to $\{L^j_W$, B, S^j_W, A_i, $S^j_{\overline{W}}\}$, we conclude that L^j_W

is adjacent to A_i. Finally, recall that H_W is adjacent to both A_i and B. Thus, Lemma 4 applied to $\{L^j_W,\ A_i,\ H_W,\ B\}$ yields that L^j_W is also adjacent to H_W.

That concludes the proof. $\square$

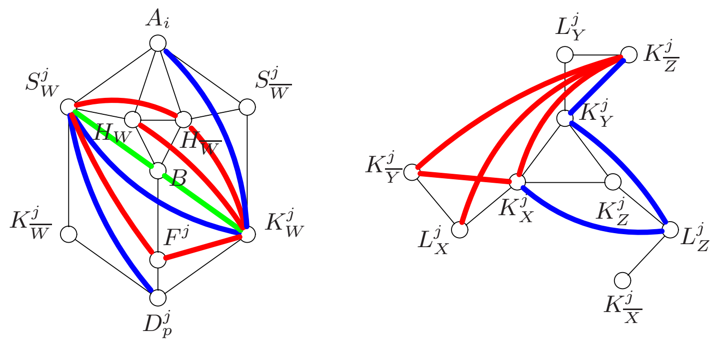

Fig. 4. The fill-in edges for *a)* $W = 1$, *b)* $X = 1$, $Y = 0$, $Z = 0$.

Let σ be a truth assignment for the instance I. Recall that, for simplicity, we write $X = 0$ and $X = 1$ in place of $\sigma(X) = 0$ and $\sigma(X) = 1$, respectively.

To facilitate the arguments in the proof, we introduce a naming convention for the vertices in $\mathrm{int}^*(\mathcal{Q}_I)$ similar to that of [2]. The vertices S^j_W for all meaningful choices of j and W are called *shoulders*. For a fixed j, we call them *shoulders of the clause* $\mathcal{C}_j$, and for a fixed W, we call them *shoulders of the literal* W. A shoulder is a *a true shoulder* if $W = 1$. Otherwise, it is a *false shoulder*. The vertices K^j_W, L^j_W for all meaningful choices of j and W are called *knees*. For a fixed j, we call them *knees of the clause* $\mathcal{C}_j$, and for a fixed W, we call them *knees of the literal* W. A knee is a *true knee* if $W = 1$. Otherwise, it is a *false knee*. The vertices A_i, D^j_p, H_W, F^j for all meaningful choices of indices are called *A-vertices*, *D-vertices*, *H-vertices*, and *F-vertices*, respectively.

Let G_σ be the graph constructed from $\mathrm{int}^*(\mathcal{Q}_I)$ by performing the following:

(i) make B adjacent to all true knees and true shoulders

Let G'_σ be the graph constructed from G_σ by performing the following steps:

(ii) make {true knees, true shoulders} into a complete graph
(iii) for all $i \in \{1 \dots n\}$, make A_i adjacent to all true knees of the literals v_i,$\overline{v_i}$,
(iv) for all $1 \le i' \le i \le n$, make H_{v_i}, $H_{\overline{v_i}}$ adjacent to all true knees and true shoulders of the literals $v_{i'}$, $\overline{v_{i'}}$
(v) for all $1 \le j \le j' \le m$, make F^j adjacent to all true knees and true shoulders of the clause $\mathcal{C}_{j'}$,
(vi) for all $1 \le i \le n$ and all $j, j' \in \Delta_i$ such that $j \le j'$:
  a) if $v_i = 1$, make $S^{j'}_{\overline{v_i}}$ adjacent to $K^j_{v_i}$, $L^j_{v_i}$ (if exists)
  b) if $v_i = 0$, make $S^{j'}_{v_i}$ adjacent to $K^j_{\overline{v_i}}$, $L^j_{\overline{v_i}}$ (if exists)

Finally, let G^*_σ be constructed from G'_σ by adding the following edges.

(vii) for all $j \in \{1 \dots m\}$ where $\mathcal{C}_j = X \vee Y \vee Z$:
  a) if $X = 1$, then add edges $F^j L^j_Z$, $K^j_X L^j_Z$, $K^j_Y K^j_{\overline{Z}}$, $D^j_2 K^j_{\overline{Z}}$, $D^j_2 S^j_{\overline{Y}}$, $D^j_3 S^j_{\overline{Y}}$ and make $\{D^j_1, D^j_2, D^j_3, S^j_X, S^j_{\overline{Z}}, L^j_Z, K^j_Y\}$ into a complete graph
  b) if $Y = 1$, then add edges $F^j L^j_X$, $K^j_Y L^j_X$, $K^j_Z K^j_{\overline{X}}$, $D^j_3 K^j_{\overline{X}}$, $D^j_3 S^j_{\overline{Z}}$, $D^j_1 S^j_{\overline{Z}}$ and make $\{D^j_1, D^j_2, D^j_3, S^j_Y, S^j_{\overline{X}}, L^j_X, K^j_Z\}$ into a complete graph
  c) if $Z = 1$, then add edges $F^j L^j_Y$, $K^j_Z L^j_Y$, $K^j_X K^j_{\overline{Y}}$, $D^j_1 K^j_{\overline{Y}}$, $D^j_1 S^j_{\overline{X}}$, $D^j_2 S^j_{\overline{X}}$ and make $\{D^j_1, D^j_2, D^j_3, S^j_Z, S^j_{\overline{Y}}, L^j_Y, K^j_X\}$ into a complete graph

Lemma 8. *G'_σ is a subgraph of every chordal sandwich of $(G_\sigma, \mathrm{forb}(\mathcal{Q}_I))$.*

Proof. Let G' be a chordal sandwich of $(G_\sigma, \mathrm{forb}(\mathcal{Q}_I))$. We prove the claim by showing that G' contains all edges defined in (ii)-(vi).

For (ii), let us consider true shoulders S^j_W, $S^{j'}_{W'}$ and true knees K^j_W, $K^{j'}_{W'}$ and L^j_W, $L^{j'}_{W'}$ (if they exist). We allow that W is possibly equal to W' and possibly $j = j'$. First, we observe that, by (i), the true knees K^j_W and $K^{j'}_{W'}$ are adjacent to B. Therefore, by Lemma 7, the vertices S^j_W, K^j_W, L^j_W are adjacent to H_W and F^j, whereas $S^{j'}_{W'}$, $K^{j'}_{W'}$, $L^{j'}_{W'}$ are adjacent to $H_{W'}$ and $F^{j'}$. Also, H_W is adjacent to $H_{W'}$ and F^j is adjacent to $F^{j'}$, since $\delta \in H_W \cap H_{W'}$ and $\mu \in F^j \cap F^{j'}$, respectively. Further, $H_W F^j$, $H_W F^{j'}$, $H_{W'} F^j$, $H_{W'} F^{j'}$ are not edges of G', since $H_W|F^j$, $H_W|F^{j'}$, $H_{W'}|F^j$, $H_{W'}|F^{j'}$ are in $\mathcal{Q}_I$. Thus, if $j = j'$ and W is equal to W', then, by Lemma 4 applied to cycles $\{u, H_W, u', F^j\}$ where $u, u' \in \{S^j_W, S^{j'}_{W'}, K^j_W, K^{j'}_{W'}, L^j_W, L^{j'}_{W'}\}$, we conclude that $\{S^j_W, S^{j'}_{W'}, K^j_W, K^{j'}_{W'}, L^j_W, L^{j'}_{W'}\}$ forms a complete graph in G'. If $j \neq j'$ and W is not equal to W', we reach the same conclusion by Lemma 6 applied to the cycles $\{u, H_W, H_{W'}, u', F^{j'}, F^j\}$. Otherwise, we obtain the conclusion by applying Lemma 5 either to cycles $\{u, H_W, u', F^{j'}, F^j\}$ or cycles $\{u, F^j, u', H_{W'}, H_W\}$. This proves (ii).

For (iii), consider the vertex A_i for $i \in \{1 \dots n\}$. Let $W \in \{v_i, \overline{v_i}\}$ be such that $W = 1$. Then, for each $j \in \varDelta_i$, the vertex K^j_W is adjacent to B by (i). Thus, by Lemma 7, both K^j_W and L^j_W (if exists) are adjacent to A_i. This proves (iii).

For (iv), we consider $1 \leq i' \leq i \leq n$. Let $W' \in \{v_{i'}, \overline{v_{i'}}\}$ be such that $W' = 1$. Then, for all $j \in \varDelta_{i'}$, the vertex $K^j_{W'}$ is adjacent to B by (i), and hence, the vertices $S^j_{W'}$, $K^j_{W'}$ and $L^j_{W'}$ (if exists) are adjacent by Lemma 7 to $H_{v_{i'}}$, $H_{\overline{v_{i'}}}$. In other words, the vertices $H_{v_{i'}}$, $H_{\overline{v_{i'}}}$ are adjacent to all true knees and true shoulders of the literals $v_{i'}$, $\overline{v_{i'}}$. Thus, we may assume that $i' < i$. Now, the vertex $H_{v_{i'}}$ is adjacent to H_{v_i},$H_{\overline{v_i}}$, since $\delta \in H_{v_i} \cap H_{\overline{v_i}} \cap H_{v_{i'}}$. Let $W \in \{v_i, \overline{v_i}\}$ be such that $W = 1$. Then K^j_W is adjacent to B by (i), and hence, S^j_W is adjacent to H_{v_i}, $H_{\overline{v_i}}$ by Lemma 7. Also, S^j_W is adjacent to all true knees and true shoulders of the literals $v_{i'}$, $\overline{v_{i'}}$, by (ii). Further, the vertex S^j_W is not adjacent to $H_{v_{i'}}$, since $H_{v_{i'}}|S^j_W$ is in $\mathcal{Q}_I$. Thus, by Lemma 4, both H_{v_i} and $H_{\overline{v_i}}$ are adjacent to all true knees and true shoulders of the literals $v_{i'}$, $\overline{v_{i'}}$. This proves (iv).

For (v), consider $1 \leq j \leq j' \leq m$. Again, we observe that if $K_W^{j'}$ is a true knee, then $K_W^{j'}$ is adjacent to B by (i), and hence, $S_W^{j'}$, $K_W^{j'}$, and $L_W^{j'}$ (if exists) are adjacent to $F^{j'}$ by Lemma 7. In other words, the vertex $F^{j'}$ is adjacent to all true knees and true shoulders of the clause $\mathcal{C}_{j'}$. So, we may assume that $j < j'$. Now, let K_W^j be any true knee of the clause $\mathcal{C}_j$. Then K_W^j is adjacent to B, and hence, to F^j by (i) and Lemma 7, respectively. Also, K_W^j is adjacent to all true shoulders and true knees of $\mathcal{C}_{j'}$ by (ii). Further, F^j is adjacent to $F^{j'}$, since $\mu \in F^j \cap F^{j'}$, and the vertex K_W^j is not adjacent to $F^{j'}$, since $K_W^j|F^{j'}$ is in $\mathcal{Q}_I$. Thus, by Lemma 4, the vertex F^j is adjacent to all true knees and true shoulders of the clause $\mathcal{C}_{j'}$. This proves (v).

For (vi), let $i \in \{1 \dots n\}$ and consider $j, j' \in \Delta_i$ with $j \leq j'$. Let $W \in \{v_i, \overline{v_i}\}$ be such that $W = 1$. Observe that K_W^j is adjacent to $S_{\overline{W}}^j$, since $\beta_{\overline{W}}^j \in S_{\overline{W}}^j \cap K_W^j$. If L_W^j exists, also L_W^j is adjacent to $S_{\overline{W}}^j$, since then $\beta_{\overline{W}}^j \in S_{\overline{W}}^j \cap L_W^j$. Thus, we may assume that $j < j'$. Now, $S_{\overline{W}}^{j'}$ is adjacent to $S_{\overline{W}}^j$ and $K_W^{j'}$, since $\alpha_{\overline{W}} \in S_{\overline{W}}^j \cap S_{\overline{W}}^{j'}$, and $\beta_{\overline{W}}^{j'} \in S_{\overline{W}}^{j'} \cap K_W^{j'}$. Also, K_W^j and L_W^j (if exists) are adjacent to $K_W^{j'}$ by (ii). Further, $S_{\overline{W}}^j K_W^{j'}$ is not an edge of G', since $S_{\overline{W}}^j | K_W^{j'}$ is in $\mathcal{Q}_I$. Thus, by Lemma 4, the vertices K_W^j, L_W^j (if exists) are adjacent to $S_{\overline{W}}^{j'}$. This proves (vi).

The proof is now complete. □

Lemma 9. *If σ is a satisfying assignment for I, then G_σ^* is a subgraph of every chordal sandwich of $(G_\sigma, \mathrm{forb}(\mathcal{Q}_I))$.*

Proof. Let G' be a chordal sandwich of $(G_\sigma, \mathrm{forb}(\mathcal{Q}_I))$, and assume that σ is a satisfying assignment for I. That is, in each clause $\mathcal{C}_j = X \vee Y \vee Z$, either $X = 1$, $Y = Z = 0$, or $Y = 1$, $X = Z = 0$, or $Z = 1$, $X = Y = 0$.

By Lemma 8, the graph G' contain all edges defined in (ii)-(vi). Thus it remains to prove that it also contains the edges defined in (vii).

Consider $j \in \{1 \dots m\}$ where $\mathcal{C}_j = X \vee Y \vee Z$. By the rotational symmetry between X, Y, and Z, we may assume that $X = 1$, $Y = 0$, and $Z = 0$. Observe that K_Z^j is adjacent to K_X^j and L_Z^j, since $\lambda^j \in K_Z^j \cap K_X^j$ and $\beta_{\overline{Z}}^j \in K_Z^j \cap L_Z^j$. Further, $K_{\overline{X}}^j$ is adjacent to L_Z^j and S_X^j, since $\gamma_1^j \in L_Z^j \cap K_X^j$ and $\beta_X^j \in K_{\overline{X}}^j \cap S_X^j$. By (ii), also K_X^j is adjacent to S_X^j. Moreover, $S_X^j K_Z^j$ and $K_X^j K_{\overline{X}}^j$ are not edges of G', since $S_X^j | K_Z^j$, $K_X^j | K_{\overline{X}}^j$ are in $\mathcal{Q}_I$. Thus, by Lemma 5 applied to the cycle $\{L_Z^j$, K_Z^j, K_X^j, S_X^j, $K_{\overline{X}}^j\}$, we conclude that L_Z^j is adjacent to S_X^j and K_X^j. Now, observe that L_Y^j is adjacent to K_Y^j and $K_{\overline{Z}}^j$, since $\beta_{\overline{Y}}^j \in L_Y^j \cap K_Y^j$ and $\gamma_3^j \in L_Y^j \cap K_{\overline{Z}}^j$. Recall that K_Z^j is adjacent to L_Z^j and also to K_Y^j, since $\lambda^j \in K_Z^j \cap K_Y^j$. Moreover, S_X^j is adjacent to $K_{\overline{Z}}^j$ and L_Z^j by (ii) and the above. Further, $K_{\overline{Z}}^j L_Z^j$, $S_X^j L_Y^j$, $S_X^j K_Z^j$ are not edges of G', since $K_{\overline{Z}}^j | L_Z^j$, $S_X^j | L_Y^j$, $S_X^j | K_Z^j$ are in $\mathcal{Q}_I$. Thus, by Lemma 6 applied to the cycle $\{K_Y^j$, L_Y^j, $K_{\overline{Z}}^j$, S_X^j, L_Z^j, $K_Z^j\}$, we conclude that K_Y^j is adjacent to $K_{\overline{Z}}^j$, S_X^j, and L_Z^j. Next, observe that $S_{\overline{Z}}^j$ is adjacent to $K_{\overline{Z}}^j$ and

K^j_Z by (ii) and since $\beta^j_{\overline{Z}} \in S^j_{\overline{Z}} \cap K^j_Z$. Recall that K^j_Y is adjacent to $K^j_{\overline{Z}}$ and K^j_Z. Further, $K^j_Z K^j_{\overline{Z}}$ is not an edge of G', since $K^j_Z|K^j_{\overline{Z}}$ is in $\mathcal{Q}_I$. Thus, by Lemma 4, the vertex $S^j_{\overline{Z}}$ is adjacent to K^j_Y. Now, recall that L^j_Z is adjacent to S^j_X and K^j_Z, and $S^j_X K^j_Z$ is not an edge of G'. Also, F^j is adjacent to S^j_X and K^j_Z by (v) and since $\lambda^j \in F^j \cap K^j_Z$. Thus, by Lemma 4, the vertex L^j_Z is adjacent to F^j. Now, observe that D^j_1 is adjacent to K^j_X, $K^j_{\overline{X}}$, since $\lambda^j \in D^j_1 \cap K^j_X$ and $\gamma^j_1 \in D^j_1 \cap K^j_{\overline{X}}$. Recall that also S_X is adjacent to both K^j_X and $K^j_{\overline{X}}$, and that $K^j_X K^j_{\overline{X}}$ is not an edge of G'. Thus, by Lemma 4, we have that D^j_1 is adjacent to S^j_X. Next, observe that D^j_2 is adjacent to K^j_Y, $K^j_{\overline{Y}}$, since $\lambda^j \in D^j_2 \cap K^j_Y$ and $\gamma^j_2 \in D^j_2 \cap K^j_{\overline{Y}}$. Recall that K^j_Y is adjacent to $K^j_{\overline{Z}}$ and S^j_X. Also, $K^j_{\overline{Y}}$ is adjacent to S^j_X, $S^j_{\overline{Y}}$, $K^j_{\overline{Z}}$ by (ii), and K^j_Y is adjacent to $S^j_{\overline{Y}}$, since $\beta^j_{\overline{Y}} \in K^j_Y \cap S^j_{\overline{Y}}$. Further, $K^j_Y K^j_{\overline{Y}}$ is not an edge of G', since $K^j_Y|K^j_{\overline{Y}}$ is in $\mathcal{Q}_I$. Thus, by Lemma 4, the vertices S^j_X, $S^j_{\overline{Y}}$, $K^j_{\overline{Z}}$ are adjacent to D^j_2. Now, observe that D^j_1, D^j_2 are adjacent to K^j_Z, since $\lambda^j \in D^j_1 \cap D^j_2 \cap K^j_Z$. Also, recall that S^j_X is adjacent to D^j_1, D^j_2, L^j_Z, the vertex K^j_Z is adjacent to $S^j_{\overline{Z}}$, L^j_Z, and $S^j_X K^j_Z$ is not an edge of G'. Further, S^j_X is adjacent to $S^j_{\overline{Z}}$ by (ii). Thus, by Lemma 4, both D^j_1 and D^j_2 are adjacent to $S^j_{\overline{Z}}$ and L^j_Z. Next, observe that D^j_3 is adjacent to K^j_Z, $K^j_{\overline{Z}}$, since $\lambda^j \in D^j_3 \cap K^j_Z$ and $\gamma^j_3 \in D^j_3 \cap K^j_{\overline{Z}}$. Recall that also $S^j_{\overline{Z}}$ is adjacent to K^j_Z, $K^j_{\overline{Z}}$, and that $K^j_Z K^j_{\overline{Z}}$ is not an edge of G'. Thus, by Lemma 4, the vertex D^j_3 is adjacent to $S^j_{\overline{Z}}$. Further, recall that L^j_Z is adjacent to K^j_Z, S^j_X, the vertex $K^j_{\overline{Z}}$ is adjacent to S^j_X, and $S^j_X K^j_Z$ and $K^j_{\overline{Z}} L^j_Z$ are not edges of G'. Thus, Lemma 5 applied to $\{D^j_3, K^j_Z, L^j_Z, S^j_X, K^j_{\overline{Z}}\}$ yields that D^j_3 is adjacent to both L^j_Z and S^j_X. Moveover, $S^j_{\overline{Y}}$ is also adjacent to S^j_X by (ii), and L^j_Y is also adjacent to D^j_3, $S^j_{\overline{Y}}$, since $\gamma^j_3 \in D^j_3 \cap L^j_Y$ and $\beta^j_{\overline{Y}} \in S^j_{\overline{Y}} \cap L^j_Y$. Further, recall that $S^j_X L^j_Y$ is not an edge of G'. Thus, by Lemma 4 applied to $\{D^j_3, L^j_Y, S^j_{\overline{Y}}, S^j_X\}$, the vertex D^j_3 is adjacent to $S^j_{\overline{Y}}$.

To prove (vii), we observe that the above analysis yields that G' contains edges $F^j L^j_Z$, $K^j_X L^j_Z$, $K^j_Y K^j_{\overline{Z}}$, $D^j_2 K^j_{\overline{Z}}$, $D^j_2 S^j_{\overline{Y}}$, and $D^j_3 S^j_{\overline{Y}}$. It remains to show that $\{D^j_1, D^j_2, D^j_3, S^j_X, S^j_{\overline{Z}}, L^j_Z, K^j_Y\}$ forms a complete graph. By the above paragraph, we have that S^j_X, $S^j_{\overline{Z}}$, L^j_Z are adjacent to D^j_1, D^j_2, D^j_3. Also, D^j_1, D^j_2, D^j_3 and K^j_Y are pair-wise adjacent, since $\lambda^j \in D^j_1 \cap D^j_2 \cap D^j_3 \cap K^j_Y$. Further, L^j_Z is adjacent to S^j_X, and K^j_Y is adjacent to S^j_X, $S^j_{\overline{Z}}$, L^j_Z, by the above paragraph. Finally, $S^j_{\overline{Z}}$ is adjacent to S^j_X and L^j_Z by (ii) and since $\beta^j_{\overline{Z}} \in S^j_{\overline{Z}} \cap L^j_Z$. This proves (vii).

The proof is now complete. $\square$

Lemma 10. *If σ is a satisfying assignment for I, then G^*_σ is chordal.*

Proof. Again, assume that σ is a satisfying assignment for I. That is, for each clause $\mathcal{C}_j = X \vee Y \vee Z$, either $X = 1$, $Y = Z = 0$, or $Y = 1$, $X = Z = 0$, or $Z = 1$, $X = Y = 0$. Consider the following partition $V_1 \cup V_2 \cup V_3 \cup V_4 \cup V_5$ of $V(G^*_\sigma)$ where

$V_1 = \{$false knees, D-vertices$\}$, $V_2 = \{$false shoulders$\}$, $V_3 = \{A$-vertices$\}$, $V_4 = \{H$-vertices, F-vertices$\}$, and $V_5 = \{$true knees, true shoulders, the vertex $B\}$.

Let π be an enumeration of $V(G^*_\sigma)$ constructed by listing the elements of V_1, V_2, V_3, V_4, V_5 in that order such that:

- (•) the elements of V_1 are listed by considering each clause $\mathcal{C}_j = X \vee Y \vee Z$ and listing vertices (based on the truth assignment) as follows:
  - a) if $X = 1$, then list $K^j_{\overline{X}}$, K^j_Z, L^j_Y, L^j_Z, D^j_1, K^j_Y, D^j_3, D^j_2 in that order,
  - b) if $Y = 1$, then list $K^j_{\overline{Y}}$, K^j_X, L^j_Z, L^j_X, D^j_2, K^j_Z, D^j_1, D^j_3 in that order,
  - c) if $Z = 1$, then list $K^j_{\overline{Z}}$, K^j_Y, L^j_X, L^j_Y, D^j_3, K^j_X, D^j_2, D^j_1 in that order,
- (•) the elements of V_2 (the false shoulders) are listed by listing the false shoulders of the clauses $\mathcal{C}_1$, $\mathcal{C}_2$, $\ldots$, $\mathcal{C}_m$ in that order,
- (•) the elements of V_4 are listed as follows: first the vertices H_{v_1},$H_{\overline{v_1}}$, H_{v_2},$H_{\overline{v_2}}$, $\ldots H_{v_n}$,$H_{\overline{v_n}}$ in that order, then F^m, F^{m-1}, $\ldots$, F^1 in that order,
- (•) the elements of V_3 and V_5 are listed in any order.

We show that π is a perfect elimination ordering of G^*_σ which implies the claim.

First, consider V_1. Let $j \in \{1 \ldots m\}$ and let $\mathcal{C}_j = X \vee Y \vee Z$. By the rotational symmetry between X, Y, Z, assume that $X = 1$ and $Y = Z = 0$. So, π lists the false knees and D-vertices of $\mathcal{C}_j$ as $K^j_{\overline{X}}$, K^j_Z, L^j_Y, L^j_Z, D^j_1, K^j_Y, D^j_3, D^j_2.

First, consider the vertex $K^j_{\overline{X}}$. Recall that $K^j_{\overline{X}} = \{\beta^j_X, \gamma^j_1\}$. Observe that S^j_X is the only other vertex containing β^j_X, and L^j_Z, D^j_1 are the only other vertices containing γ^j_1. Moreover, none of the rules (i)-(vii) adds edges incident to $K^j_{\overline{X}}$. Thus, S^j_X, L^j_Z, D^j_1 are the only neighbours of $K^j_{\overline{X}}$, and they are pair-wise adjacent by (vii). This proves that $K^j_{\overline{X}}$ is indeed simplicial in G^*_σ.

Next, consider K^j_Z. Since $K^j_Z = \{\beta^j_Z, \lambda^j\}$, we conclude that K^j_Z is adjacent to $S^j_{\overline{Z}}$, L^j_Z, K^j_X, K^j_Y, D^1_j, D^2_j, D^3_j, and F^j. Moreover, K^j_Z has no other neighbours by observing the rules (i)-(vii). Now, by (vii), we conclude that $S^j_{\overline{Z}}$, L^j_Z, K^j_Y, D^j_1, D^j_2, D^j_3 are pair-wise adjacent. Also, the vertices F^j, K^j_X, K^j_Y, D^j_1, D^j_2, D^j_3 are pair-wise adjacent, since they all contain λ^j. Further, F^j is adjacent to $S^j_{\overline{Z}}$ and L^j_Z by (v) and (vii), respectively, and K^j_X is adjacent to $S^j_{\overline{Z}}$ and L^j_Z by (ii) and (vii), respectively. This proves that K^j_Z is simplicial in G^*_σ.

Now, consider L^j_Y. The neighbours of L^j_Y are $S^j_{\overline{Y}}$, K^j_Y, $K^j_{\overline{Z}}$, and D^j_3. So, $S^j_{\overline{Y}}$ is adjacent to $K^j_{\overline{Z}}$, D^j_3, and K^j_Y by (ii), (vii), and since $\beta^j_{\overline{Y}} \in S^j_{\overline{Y}} \cap K^j_Y$. Similarly, K^j_Y is adjacent to $K^j_{\overline{Z}}$ and D^j_3 by (vii) and since $\lambda^j \in K^j_Y \cap D^j_3$. Finally, $K^j_{\overline{Z}}$ is adjacent to D^j_3, since $\gamma^j_3 \in K^j_{\overline{Z}} \cap D^j_3$. Thus L^j_Y is simplicial in G^*_σ.

Next, consider L^j_Z. The neighbours of L^j_Z are F^j, K^j_X, K^j_Y, K^j_Z, D^j_1, D^j_2, D^j_3, S^j_X, $S^j_{\overline{Z}}$, and $K^j_{\overline{X}}$. By (vii), the vertices D^j_1, D^j_2, D^j_3, S^j_X, $S^j_{\overline{Z}}$, K^j_Y are pair-wise adjacent. Also, F^j, K^j_X, K^j_Y, D^j_1, D^j_2, D^j_3 are pair-wise adjacent, since they all contain λ_j. Further, K^j_X and F^j are adjacent to S^j_X, $S^j_{\overline{Z}}$ by (ii) and (v), respectively. This proves that L^j_Z is simplicial in $G^*_\sigma - \{K^j_{\overline{X}}, K^j_Z\}$.

Now, consider D_1^j. The neighbours of D_1^j are F^j, K_X^j, K_Y^j, K_Z^j, D_2^j, D_3^j, S_X^j, $S_{\overline{Z}}^j$, L_Z^j, and $K_{\overline{X}}^j$. By (vii), the vertices D_2^j, D_3^j, S_X^j, $S_{\overline{Z}}^j$, K_Y^j are pair-wise adjacent. Also, F^j, K_X^j, K_Y^j, D_2^j, D_3^j are pair-wise adjacent, since they all contain λ^j. Further, K_X^j and F^j are adjacent to S_X^j, $S_{\overline{Z}}^j$ by (ii) and (v), respectively. This proves that D_1^j is simplicial in $G_\sigma^* - \{K_{\overline{X}}^j, K_Z^j, L_Z^j\}$.

Next, consider K_Y^j. The neighbours of K_Y^j are F^j, K_X^j, K_Z^j, D_1^j, D_2^j, D_3^j, S_X^j, $S_{\overline{Y}}^j$, $S_{\overline{Z}}^j$, $K_{\overline{Z}}^j$, L_Y^j, and L_Z^j. By (vii), the vertices D_2^j, D_3^j, S_X^j, $S_{\overline{Z}}^j$ are pair-wise adjacent. Also, F, K_X^j, D_2^j, D_3^j are pair-wise adjacent, since they all contain λ^j. Further, by (ii), the vertices S_X^j, $S_{\overline{Y}}^j$, $S_{\overline{Z}}^j$, K_X^j, and $K_{\overline{Z}}^j$ are pair-wise adjacent, and are adjacent to F^j by (v). Moreover, by (vii), both $S_{\overline{Y}}^j$ and $K_{\overline{Z}}^j$ are adjacent D_2^j, and are also adjacent to D_3^j by (vii) and since $\gamma_3^j \in K_{\overline{Z}}^j \cap D_3^j$, respectively. This proves that K_Y^j is simplicial in $G_\sigma^* - \{K_Z^j, L_Y^j, L_Z^j, D_1^j\}$.

Now, consider D_j^3. The neighbours of D_j^3 are F^j, K_X^j, K_Y^j, K_Z^j, D_1^j, D_2^j, S_X^j, $S_{\overline{Y}}^j$, $S_{\overline{Z}}^j$, $K_{\overline{Z}}^j$, L_Z^j, and L_Y^j. By (ii), the vertices S_X^j, $S_{\overline{Y}}^j$, $S_{\overline{Z}}^j$, K_X^j, $K_{\overline{Z}}^j$ are pair-wise adjacent. Also, F^j, K_X^j, D_2^j are pair-wise adjacent, since they all contain λ^j. Further, F^j and D_2^j are adjacent to S_X^j, $S_{\overline{Y}}^j$, $S_{\overline{Z}}^j$, $K_{\overline{Z}}^j$ by (v) and (vii), respectively. Thus D_j^3 is simplicial in $G_\sigma^* - \{K_Z^j, L_Y^j, L_Z^j, D_1^j, K_Y^j\}$.

Finally, consider D_j^2. The neighbours of D_j^2 are F^j, K_X^j, K_Y^j, K_Z^j, D_j^1, D_j^3, S_X^j, $S_{\overline{Y}}^j$, $S_{\overline{Z}}^j$, $K_{\overline{Z}}^j$, $K_{\overline{Y}}^j$, L_X^j and L_Z^j. By (ii), the vertices S_X^j, $S_{\overline{Y}}^j$, $S_{\overline{Z}}^j$, K_X^j, L_X^j, $K_{\overline{Y}}^j$, $K_{\overline{Z}}^j$ are pair-wise adjacent, and are adjacent to F by (v). Thus D_j^2 is simplicial in $G_\sigma^* - \{K_Z^j, L_Z^j, D_1^j, K_Y^j, D_3^j\}$. This concludes the vertices in V_1.

We now consider V_2. Let $j \in \{1 \ldots m\}$ and consider a false shoulder S_W^j for some $W = 0$. Let i be such that $W = v_i$ or $W = \overline{v_i}$. Then the neighbours of S_W^j are the vertices H_W, A_i, and the elements of the following sets:

$$\mathcal{S}^- = \{S_W^{j'} \mid j' \in \Delta_i \text{ and } j' < j\} \qquad \mathcal{S}^+ = \{S_W^{j'} \mid j' \in \Delta_i \text{ and } j < j'\}$$
$$\mathcal{K}^- = \{K_{\overline{W}}^{j'}, L_{\overline{W}}^{j'} \text{ (if exists)} \mid j' \in \Delta_i \text{ and } j' \leq j\}$$

By (ii), the elements of $\mathcal{K}^-$ are pair-wise adjacent. Similarly, the elements of $\{H_W, A_i\} \cup \mathcal{S}^+$ are pair-wise adjacent, since they all contain α_W. Further, each element of $\mathcal{S}^+$ is adjacent to every element of $\mathcal{K}^-$ by (vi), and each element of $\mathcal{K}^-$ is adjacent to A_i and H_W by (iii) and (iv), respectively. This proves that S_W^j is simplicial in $G_\sigma^* - \mathcal{S}^-$. Finally, note that the elements of $\mathcal{S}^-$ are false shoulders in clauses $\mathcal{C}_1, \ldots, \mathcal{C}_{j-1}$. This concludes the elements of V_2.

For V_3, let $i \in \{1 \ldots n\}$ and consider the vertex A_i. The neighbours of A_i are the vertices H_{v_i}, $H_{\overline{v_i}}$, all shoulders of the literals v_i, $\overline{v_i}$, and all true knees of $v_i, \overline{v_i}$. By (ii), the true knees and true shoulders of v_i, $\overline{v_i}$ are pair-wise adjacent, and are adjacent to both H_{v_i} and $H_{\overline{v_i}}$ by (iv). Also, H_{v_i} is adjacent to $H_{\overline{v_i}}$, since $\delta \in H_{v_i} \cap H_{\overline{v_i}}$. Thus A_i is simplicial in $G_\sigma^* - V_2$. This concludes V_3.

Now, we consider V_4. Let $i \in \{1 \ldots n\}$ and consider H_{v_i}, $H_{\overline{v_i}}$. The vertices H_{v_i}, $H_{\overline{v_i}}$ are adjacent to the vertices B, A_i, the elements of the following sets

$$\mathcal{H}^- = \{H_{v_{i'}}, H_{\overline{v_{i'}}} \mid i' < i\} \qquad \mathcal{H}^+ = \{H_{v_{i'}}, H_{\overline{v_{i'}}} \mid i < i'\}$$

and all true knees, true shoulders of $v_{i'}$, $\overline{v_{i'}}$ for all $i' \in \{1 \dots i\}$. Further, H_{v_i} is adjacent to $H_{\overline{v_i}}$, to all shoulders of v_i and to no other vertices, whereas $H_{\overline{v_i}}$ is adjacent H_{v_i}, to all shoulders of $\overline{v_i}$ and to no other vertices. Now, by (ii), the true knees and true shoulders of $v_{i'}$, $\overline{v_{i'}}$ for all $i' \in \{1 \dots i\}$, are pair-wise adjacent, and are adjacent to B and each element of $\mathcal{H}^+$ by (i) and (iv), respectively. Also, the elements of $\{B\} \cup \mathcal{H}^+$ are pair-wise adjacent, since they all contain δ. Finally, observe that the false shoulders of v_i, $\overline{v_i}$ belong to V_2. This proves that both H_{v_i} and $H_{\overline{v_i}}$ are simplicial in $G^*_\sigma - (V_2 \cup V_3 \cup \mathcal{H}^-)$ as required.

Next, let $j \in \{1 \dots m\}$ and consider F^j. Let $\mathcal{C}_j = X \vee Y \vee Z$, and by the rotational symmetry, assume that $X = 1$ and $Y = Z = 0$. Then the neighbours of F^j are B, K^j_Y, K^j_Z, D^j_1, D^j_2, D^j_3, L^j_Z, the elements of the following sets

$$\mathcal{F}^- = \{F^{j'} \mid j' < j\} \qquad \mathcal{F}^+ = \{F^{j'} \mid j < j'\}$$

and all true knees and true shoulders of the clause $\mathcal{C}_{j'}$ for all $j' \in \{j \dots m\}$. By (ii), the true knees and true shoulders of the clause $\mathcal{C}_{j'}$ for all $j' \in \{j \dots m\}$, are pair-wise adjacent, and are adjacent to B and each elements of $\mathcal{F}^-$ by (i) and (v), respectively. Also, the vertices of $\{B\} \cup \mathcal{F}^-$ are pair-wise adjacent, since they all contain μ. Thus F^j is simplicial in $G^*_\sigma - (V_1 \cup \mathcal{F}^+)$. This concludes V_4.

Finally, observe that all vertices of V_5 are pair-wise adjacent by (i) and (ii). That concludes the proof. $\square$

Lemma 11. *For every chordal sandwich G' of $(\mathrm{int}^*(\mathcal{Q}_I), \mathrm{forb}(\mathcal{Q}_I))$, there is σ such that G_σ is a subgraph of G', and such that σ is a satisfying assignment for I.*

Proof. By Lemma 7, for each $i \in \{1 \dots n\}$, there is $W \in \{v_i, \overline{v_i}\}$ such that for all $j \in \Delta_i$, the vertices S^j_W, K^j_W, and L^j_W(if exists) are adjacent to B. Set $\sigma(v_i) = 1$ if $W = v_i$, and otherwise set $\sigma(v_i) = 0$. For such a mapping σ, the graph G' clearly contains all edges of G_σ. Thus, by Lemma 9, the graph G'_σ is a subgraph of G', that is, G' contains the edges defined in (ii)-(vi).

It remains to prove that σ is a satisfying assignment for I. Let $j \in \{1 \dots m\}$ and consider the clause $\mathcal{C}_j = X \vee Y \vee Z$. If $X = Y = 1$, then the vertex S^j_Y is a true shoulder, and K^j_X is a true knee. Thus, by (ii), we conclude that S^j_Y is adjacent K^j_X. However, this is impossible, since $S^j_Y | K^j_X$ is in $\mathcal{Q}_Y$. Similarly, if $X = Z = 1$, we have that S^j_X is adjacent to K^j_Z by (ii) while $S^j_X | K^j_Z$ is in $\mathcal{Q}_I$, and if $Y = Z = 1$, then S^j_Z is adjacent to K^j_Y by (ii) while $S^j_Z | K^j_Y$ is in $\mathcal{Q}_I$.

Now, suppose that $X = Y = Z = 0$. First, observe that K^j_X is adjacent to L^j_X, K^j_Z, and the vertex L^j_Z is adjacent to K^j_Z, $K^j_{\overline{X}}$, since $\beta^j_{\overline{X}} \in K^j_X \cap L^j_X$, $\lambda^j \in K^j_X \cap K^j_Z$, $\beta^j_{\overline{Z}} \in L^j_Z \cap K^j_Z$, and $\gamma^j_1 \in L^j_Z \cap K^j_{\overline{X}}$. Also, $K^j_{\overline{X}}$ is adjacent to $K^j_{\overline{Z}}$ by (ii). Further, $K^j_{\overline{Z}} K^j_Z$, $K^j_{\overline{Z}} L^j_Z$ and $K^j_{\overline{X}} L^j_X$ are not edges of G', since $K^j_{\overline{Z}} | K^j_Z$, $K^j_{\overline{Z}} | L^j_Z$, and $K^j_{\overline{X}} | L^j_X$ and in $\mathcal{Q}_I$. Thus, if L^j_X is adjacent to $K^j_{\overline{Z}}$, then by Lemma 6 applied to $\{K^j_X,\ L^j_X,\ K^j_{\overline{Z}},\ K^j_{\overline{X}},\ L^j_Z,\ K^j_Z\}$, we conclude that K^j_X is adjacent to $K^j_{\overline{X}}$, which is impossible since $K^j_{\overline{X}} | K^j_X$ is in $\mathcal{Q}_I$. Similarly, if K^j_X is adjacent to $K^j_{\overline{Z}}$, then by Lemma 5 applied to $\{K^j_X,\ K^j_{\overline{Z}},\ K^j_{\overline{X}},\ L^j_Z,\ K^j_Z\}$, we again conclude that K^j_X is adjacent to $K^j_{\overline{X}}$, a contradiction. So, we may assume that both K^j_X

and L^j_X are not adjacent to $K^j_{\overline{Z}}$. Now, observe that L^j_Y is adjacent to $K^j_{\overline{Z}}$, K^j_Y, and the vertex K^j_X is adjacent to L^j_X, K^j_Y, since $\gamma^j_3 \in K^j_{\overline{Z}} \cap L^j_Y$, $\beta^j_{\overline{Y}} \in L^j_Y \cap K^j_Y$, $\beta^j_{\overline{X}} \in K^j_X \cap L^j_X$, and $\lambda^j \in K^j_Y \cap K^j_X$. Also, $K^j_{\overline{Y}}$ is adjacent to $K^j_{\overline{Z}}$ and L^j_X by (ii) and since $\gamma^j_2 \in K^j_{\overline{Y}} \cap L^j_X$. Further, $K^j_{\overline{Y}}K^j_Y$ and $K^j_{\overline{Y}}L^j_Y$ are not edges of G', since $K^j_{\overline{Y}}|K^j_Y$ and $K^j_{\overline{Y}}|L^j_Y$ are in $\mathcal{Q}_I$. Recall that K^j_X and L^j_X are not adjacent to $K^j_{\overline{Z}}$. Then this contradicts Lemma 6 when applied to $\{K^j_X, L^j_X, K^j_{\overline{Y}}, K^j_{\overline{Z}}, L^j_Y, K^j_Y\}$.

Thus, it is not the case that $X = Y = Z = 0$, and by the above also not $X = Y = 1$, nor $X = Z = 1$, nor $Y = Z = 1$. Therefore, either $X = 1$, $Y = Z = 0$, or $Y = 1$, $X = Z = 0$, or $Z = 1$, $X = Y = 0$. This proves that σ is indeed a satisfying assignment for I, which concludes the proof. $\square$

We are finally ready to prove Theorem 8.

Proof of Theorem 8. Let G' be a minimal chordal sandwich of $(\text{int}^*(\mathcal{Q}_I), \text{forb}(\mathcal{Q}_I))$. By Lemma 11, there exists σ, a satisfying assignment for I, such that G_σ is a subgraph fo G'. Thus, G' is also a chordal sandwich of $(G_\sigma, \text{forb}(\mathcal{Q}_I))$, and hence, G^*_σ is a subgraph of G' by Lemma 9. But by Lemma 10, G^*_σ is chordal, and so G' is isomorphic to G^*_σ by the minimality of G'.

Conversely, if σ is a satisfying assignment for I, then the graph G^*_σ is chordal by Lemma 10. Moreover, $\text{int}^*(\mathcal{Q}_I)$ is a subgraph of G^*_σ, by definition, and G^*_σ contains no edges of $\text{forb}(\mathcal{Q}_I)$, also by definition. Thus, G^*_σ is a chordal sandwich of $(\text{int}^*(\mathcal{Q}_I), \text{forb}(\mathcal{Q}_I))$, and it is minimal by Lemma 9.

This proves that by mapping each satisfying assigment σ to the graph G^*_σ, we obtain the required bijection. That concludes the proof. $\square$

Finally, we have all the pieces to prove Theorem 1.

7 Proof of Theorem 1

Consider an instance I to ONE-IN-THREE-3SAT and a satisfying assignment for I. We construct the collection $\mathcal{Q}_I$ of quartet trees, as well as the ternary phylogenetic tree $\mathcal{T}_\sigma$ as described in Sections 3 and 4, respectively. Clearly, constructing $\mathcal{Q}_I$ and $\mathcal{T}_\sigma$ takes polynomial time. By combining Theorem 7 with Theorems 8 and 9, we obtain that σ is the unique satisfying assignment of I if and only if $\mathcal{T}_\sigma$ is the only phylogenetic tree that displays $\mathcal{Q}_I$. Since, by Theorem 2, it is NP-hard to determine if an instance to ONE-IN-THREE-3SAT has a unique satisfying assignment, it is therefore NP-hard to decide, for a given phylogenetic tree $\mathcal{T}$ and a collection of quartet trees $\mathcal{Q}$, whether or not $\mathcal{Q}$ defines $\mathcal{T}$.

That concludes the proof.

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