# Emptiness and Universality Problems in Timed Automata with Positive Frequency*

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**Abstract.** The languages of infinite timed words accepted by timed automata are traditionally defined using Büchi-like conditions. These acceptance conditions focus on the set of locations visited infinitely often along a run, but completely ignore quantitative timing aspects. In this paper we propose a natural quantitative semantics for timed automata based on the so-called frequency, which measures the proportion of time spent in the accepting locations. We study various properties of timed languages accepted with positive frequency, and in particular the emptiness and universality problems.

## 1 Introduction

The model of timed automata, introduced by Alur and Dill in the 90's [2] is commonly used to represent real-time systems. Timed automata consist of an extension of finite automata with continuous variables, called clocks, that evolve synchronously with time, and can be tested and reset along an execution. Despite their uncountable state space, checking reachability, and more generally $\omega$-regular properties, is decidable *via* the construction of a finite abstraction, the so-called region automaton. This fundamental result made timed automata very popular in the formal methods community, and lots of work has been done towards their verification, including the development of dedicated tools like Kronos or Uppaal.

More recently a huge effort has been made for modelling quantitative aspects encompassing timing constraints, such as costs [3, 6] or probabilities [11, 5]. It is now possible to express and check properties such as: "the minimal cost to reach a given state is smaller than 3", or "the probability to visit infinitely often a given location is greater than 1/2". As a consequence, from qualitative verification, the emphasis is now put on quantitative verification of timed automata.

In this paper we propose a quantitative semantics for timed automata based on the proportion of time spent in critical states (called the *frequency*). Contrary

* This work has been partly supported by the ESF project GASICS, the ANR project ANR-2010-BLAN-0317, the Tournesol Hubert Curien partnership STP, the ARC project AUWB-2010–10/15-UMONS-3, the FRFC project 2.4515.11 and a grant from the National Bank of Belgium.

to probabilities or volume [4] that give a value to sets of behaviours of a timed automaton (or a subset thereof), the frequency assigns a real value (in $[0, 1]$) to each execution of the system. It can thus be used in a language-theoretic approach to define quantitative languages associated with a timed automaton, or boolean languages based on quantitative criteria *e.g.*, one can consider the set of timed words for which there is an execution of frequency greater than a threshold $\lambda$.

Similar notions were studied in the context of untimed systems. For finite automata, mean-payoff conditions have been investigated [10, 1, 9]: with each run is associated the limit average of weights encountered along the execution. Our notion of frequency extends mean-payoff conditions to timed systems by assigning to an execution the limit average of time spent in some distinguished locations. It can also be seen as a timed version of the asymptotic frequency considered in quantitative fairness games [7]. Concerning probabilistic models, a similar notion was introduced in constrained probabilistic Büchi automata yielding the decidability of the emptiness problem under the probable semantics [14]. Last, the work closest to ours deals with double-priced timed automata [8], where the aim is to synthesize schedulers which optimize on-the-long-term the reward of a system.

Adding other quantitative aspects to timed automata comes often with a cost (in terms of decidability and complexity), and it is often required to restrict the timing behaviours of the system to get some computability results, see for instance [13]. The tradeoff is then to restrict to single-clock timed automata. Beyond introducing the concept of frequency, which we believe very natural, the main contributions of this paper are the following. First of all, using a refinement of the region graph, we show how to compute the infimum and supremum values of frequencies in a given single-clock timed automaton, as well as a way to decide whether these bounds are realizable (*i.e.*, whether they are minimum and maximum respectively). The computation of these bounds together with their realizability can be used to decide the emptiness problem for languages defined by a threshold on the frequency. Moreover, in the restricted case of deterministic timed automata, it allows to decide the universality problem for these languages. Last but not least we discuss the universality problem for frequency-languages. Even under our restriction to one-clock timed automata, this problem is non-primitive recursive, and we provide a decision algorithm in the case of Zeno words when the threshold is 0. Our restriction to single-clock timed automata is crucial since at several points the techniques employed do not extend to two clocks or more. In particular, the universality problem becomes undecidable for timed automata with several clocks.

## 2 Definitions and preliminaries

In this section, we recall the model of timed automata, introduce the concept of frequency, and show how those can be used to define timed languages. We then compare our semantics to the standard semantics based on Büchi acceptance.

### 2.1 Timed automata and frequencies

We start with notations and useful definitions concerning timed automata [2].

Given $X$ a finite set of clocks, a *(clock) valuation* is a mapping $v : X \to \mathbb{R}_+$. We write $\mathbb{R}_+^X$ for the set of valuations. We note $\overline{0}$ the valuation that assigns 0 to all clocks. If $v$ is a valuation over $X$ and $t \in \mathbb{R}_+$, then $v+t$ denotes the valuation which assigns to every clock $x \in X$ the value $v(x)+t$. For $X' \subseteq X$ we write $v_{[X' \leftarrow 0]}$ for the valuation equal to $v$ on $X \setminus X'$ and to $\overline{0}$ on $X'$.

A *guard* over $X$ is a finite conjunction of constraints of the form $x \sim c$ where $x \in X$, $c \in \mathbb{N}$ and $\sim \in \{<, \leq, =, \geq, >\}$. We denote by $G(X)$ the set of guards over $X$. Given $g$ a guard and $v$ a valuation, we write $v \models g$ if $v$ satisfies $g$ (defined in a natural way).

**Definition 1.** *A* timed automaton *is a tuple $\mathcal{A} = (L, L_0, F, \Sigma, X, E)$ such that: $L$ is a finite set of locations, $L_0 \subseteq L$ is the set of initial locations, $F \subseteq L$ is the set of accepting locations, $\Sigma$ is a finite alphabet, $X$ is a finite set of clocks and $E \subseteq L \times G(X) \times \Sigma \times 2^X \times L$ is a finite set of edges.*

The semantics of a timed automaton $\mathcal{A}$ is given as a timed transition system $\mathcal{T}_\mathcal{A} = (S, S_0, S_F, (\mathbb{R}_+ \times \Sigma), \to)$ with set of states $S = L \times \mathbb{R}_+^X$, initial states $S_0 = \{(\ell_0, \overline{0}) \mid \ell_0 \in L_0\}$, final states $S_F = F \times \mathbb{R}_+^X$ and transition relation $\to \subseteq S \times (\mathbb{R}_+ \times \Sigma) \times S$, composed of moves of the form $(\ell, v) \xrightarrow{\tau, a} (\ell', v')$ with $\tau > 0$ whenever there exists an edge $(\ell, g, a, X', \ell') \in E$ such that $v + \tau \models g$ and $v' = (v+\tau)_{[X' \leftarrow 0]}$.

A *run* $\varrho$ of $\mathcal{A}$ is an infinite sequence of moves starting in some $s_0 \in S_0$, *i.e.*, $\varrho = s_0 \xrightarrow{\tau_0, a_0} s_1 \cdots \xrightarrow{\tau_k, a_k} s_{k+1} \cdots$. A *timed word* over $\Sigma$ is an element $(t_i, a_i)_{i \in \mathbb{N}}$ of $(\mathbb{R}_+ \times \Sigma)^\omega$ such that $(t_i)_{i \in \mathbb{N}}$ is increasing. The timed word is said to be *Zeno* if the sequence $(t_i)_{i \in \mathbb{N}}$ is bounded from above. The timed word associated with $\varrho$ is $w = (t_0, a_0) \dots (t_k, a_k) \dots$ where $t_i = \sum_{j=0}^{i} \tau_j$ for every $i$. A timed automaton $\mathcal{A}$ is *deterministic* whenever, given two edges $(\ell, g_1, a, X_1', \ell')$ and $(\ell, g_2, a, X_2', \ell')$ in $E$, $g_1 \wedge g_2$ cannot be satisfied. In this case, for every timed word $w$, there is at most one run reading $w$. An example of a (deterministic) timed automaton is given in Fig. 1. As a convention locations in $F$ will be depicted in black.

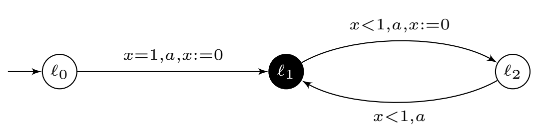



**Fig. 1.** Example of a timed automaton $\mathcal{A}$ with $L_0 = \{\ell_0\}$ and $F = \{\ell_1\}$.

**Definition 2.** *Given $\mathcal{A} = (L, L_0, F, \Sigma, X, E)$ a timed automaton and a run $\varrho = (\ell_0, v_0) \xrightarrow{\tau_0, a_0} (\ell_1, v_1) \xrightarrow{\tau_1, a_1} (\ell_2, v_2) \cdots$ of $\mathcal{A}$, the* frequency *of $F$ along $\varrho$, denoted $\mathsf{freq}_\mathcal{A}(\varrho)$, is defined as $\limsup_{n \to \infty} (\sum_{i \leq n \mid \ell_i \in F} \tau_i)/(\sum_{i \leq n} \tau_i)$.*

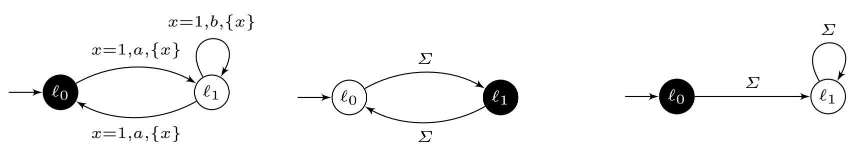
Note that the choice of lim sup is arbitrary, and the choice of lim inf would be as relevant. Furthermore notice that the limit may not exist in general.

A timed word $w$ is said *accepted with positive frequency* by $\mathcal{A}$ if there exists a run $\varrho$ which reads $w$ and such that $\mathsf{freq}_{\mathcal{A}}(\varrho)$ is positive. The *positive-frequency language* of $\mathcal{A}$ is the set of timed words that are accepted with positive frequency by $\mathcal{A}$. Note that we could define more generally languages where the frequency of each word should be larger than some threshold $\lambda$, but even though some of our results apply to this more general framework we prefer focusing on languages with positive frequency.

*Example 3.* We illustrate the notion of frequency on runs of the deterministic timed automaton $\mathcal{A}$ of Fig. 1. First, the only run in $\mathcal{A}$ 'reading' the word $(1,a).((\frac{1}{3},a).(\frac{1}{3},a))^*$ has frequency $\frac{1}{2}$ because the sequence $\frac{n/3}{1+(2n)/3}$ converges to $\frac{1}{2}$. Second, the Zeno run reading $(1,a).(((\frac{1}{2^k},a).(\frac{1}{2^k},a))^k)_{k\geq 1}$ in $\mathcal{A}$ has frequency $\frac{1}{3}$ since the sequence $\frac{\sum_{k\geq 1} 1/2^k}{1+\sum_{k\geq 1} 1/2^{k-1}}$ converges to $\frac{1}{3}$. Finally, the run in $\mathcal{A}$ reading the word $(1,a).(((\frac{1}{2},a).(\frac{1}{4},a))^{2^{2k}}.((\frac{1}{4},a).(\frac{1}{2},a))^{2^{2k+1}})_{k\geq 1}$ has frequency $\frac{4}{9}$. Note that the sequence under consideration does not converge, but its lim sup is $\frac{4}{9}$.

### 2.2 A brief comparison with usual semantics

The usual semantics for timed automata considers a Büchi acceptance condition. We naturally explore differences between this usual semantics, and the one we introduced based on positive frequency. The expressiveness of timed automata under those acceptance conditions is not comparable, as witnessed by the automaton represented in Fig. 2(a): on the one hand, its positive-frequency language is not timed-regular (*i.e.* accepted by a timed automaton with a standard Büchi acceptance condition), and on the other hand, its Büchi language cannot be recognized by a timed automaton with a positive-frequency acceptance condition.




(a) Expressiveness. (b) Universality (non-Zeno). (c) Universality (Zeno).

**Fig. 2.** Automata for the comparison with the usual semantics.

The contribution of this paper is to study properties of the positive-frequency languages. We will show that we can get very fine information on the set of frequencies of runs in *single-clock* timed automata, which implies the decidability of the emptiness problem for positive-frequency languages. We also show that our technics do not extend to multi-clock timed automata.

We will also consider the universality problem and variants thereof (restriction to Zeno or non-Zeno timed words). On the one hand, clearly enough, a (non-Zeno)-universal timed automaton with a positive-frequency acceptance condition is (non-Zeno)-universal for the classical Büchi-acceptance. The timed automaton of Fig. 2(b) is a counterexample to the converse. On the other hand, a Zeno-universal timed automaton under the classical semantics is necessarily Zeno-universal under the positive-frequency acceptance condition, but the automaton depicted in Fig. 2(c) shows that the converse does not hold.

## 3 Set of frequencies of runs in one-clock timed automata

In this section, we give a precise description of the set of frequencies of runs in single-clock timed automata. To this aim, we use the *corner-point abstraction* [8], a refinement of the region abstraction, and exploit the links between frequencies in the timed automaton and ratios in its corner-point abstraction. We fix a single-clock timed automaton $\mathcal{A} = (L, L_0, F, \Sigma, \{x\}, E)$.

### 3.1 The corner-point abstraction

Even though the corner-point abstraction can be defined for general timed automata [8], we focus on the case of single-clock timed automata.

If $M$ is the largest constant appearing in the guards of $\mathcal{A}$, the usual *region abstraction* of $\mathcal{A}$ is the partition $Reg_{\mathcal{A}}$ of the set of valuations $\mathbb{R}_+$ made of the singletons $\{i\}$ for $0 \leq i \leq M$, the open intervals $(i, i+1)$ with $0 \leq i \leq M-1$ and the unbounded interval $(M, \infty)$ represented by $\perp$. A piece of this partition is called a *region*. The corner-point abstraction refines the region abstraction by associating *corner-points* with regions. The singleton regions have a single corner-point represented by $\bullet$ whereas the open intervals $(i, i+1)$ have two corner-points $\bullet\!-$ (the left end-point of the interval) and $-\!\bullet$ (the right end-point of the interval). Finally, the region $\perp$ has a single corner-point denoted $\alpha_\perp$. We write $(R, \alpha)$ for the region $R$ pointed by the corner $\alpha$ and $(R, \alpha) + 1$ denotes its *direct time successor* defined by:

$$(R,\alpha)+1 = \begin{cases} ((i,i+1), \bullet\!-) & \text{if } (R,\alpha) = (\{i\}, \bullet) \text{ with } i < M, \\ ((i,i+1), -\!\bullet) & \text{if } (R,\alpha) = ((i,i+1), \bullet\!-), \\ (\{i+1\}, \bullet) & \text{if } (R,\alpha) = ((i,i+1), -\!\bullet), \\ (\perp, \alpha_\perp) & \text{if } (R,\alpha) = (\{M\}, \bullet) \text{ or } (\perp, \alpha_\perp). \end{cases}$$

Using these notions, we define the corner-point abstraction as follows.

**Definition 4.** *The* (unweighted) corner-point abstraction *of $\mathcal{A}$ is the finite automaton $\mathcal{A}_{cp} = (L_{cp}, L_{0,cp}, F_{cp}, \Sigma_{cp}, E_{cp})$ where $L_{cp} = L \times Reg_{\mathcal{A}} \times \{\bullet, \bullet\!-, -\!\bullet, \alpha_\perp\}$ is the set of states, $L_{0,cp} = L_0 \times \{0\} \times \{\bullet\}$ is the set of initial states, $F_{cp} = F \times Reg_{\mathcal{A}} \times \{\bullet, \bullet\!-, -\!\bullet, \alpha_\perp\}$ is the set of accepting states, $\Sigma_{cp} = \Sigma \cup \{\varepsilon\}$, and $E_{cp} \subseteq L_{cp} \times \Sigma_{cp} \times L_{cp}$ is the finite set of edges defined as the union of discrete transitions and idling transitions:*

- discrete transitions: $(\ell, R, \alpha) \xrightarrow{a} (\ell', R', \alpha')$ *if* $\alpha$ *is a corner-point of* $R$ *and there exists a transition* $\ell \xrightarrow{g,a,X'} \ell'$ *in* $\mathcal{A}$*, such that* $R \subseteq g$ *and* $(R', \alpha') = (R, \alpha)$ *if* $X' = \emptyset$*, otherwise* $(R', \alpha') = (\{0\}, \bullet)$*,*
- idling transitions: $(\ell, R, \alpha) \xrightarrow{\varepsilon} (\ell, R', \alpha')$ *if* $\alpha$ *(resp.* $\alpha'$*) is a corner-point of* $R$ *(resp.* $R'$*) and* $(R', \alpha') = (R, \alpha) + 1$.

We decorate this finite automaton with two weights for representing frequencies, one which we call the cost, and the other which we call the reward (by analogy with double-priced timed automata in [8]). The *(weighted) corner-point abstraction* $\mathcal{A}^F_{cp}$ is obtained from $\mathcal{A}_{cp}$ by labeling idling transitions in $\mathcal{A}_{cp}$ as follows: transitions $(\ell, R, \alpha) \xrightarrow{\varepsilon} (\ell, R, \alpha')$ with $(R, \alpha') = (R, \alpha) + 1$ ($\alpha' = \alpha + 1$ for short) are assigned cost 1 (resp. cost 0) and reward 1 if $\ell \in F$ (resp. $\ell \notin F$), and all other transitions are assigned both cost and reward 0. To illustrate this definition, the corner-point abstraction of the timed automaton in Fig. 1 is represented in Fig. 3.

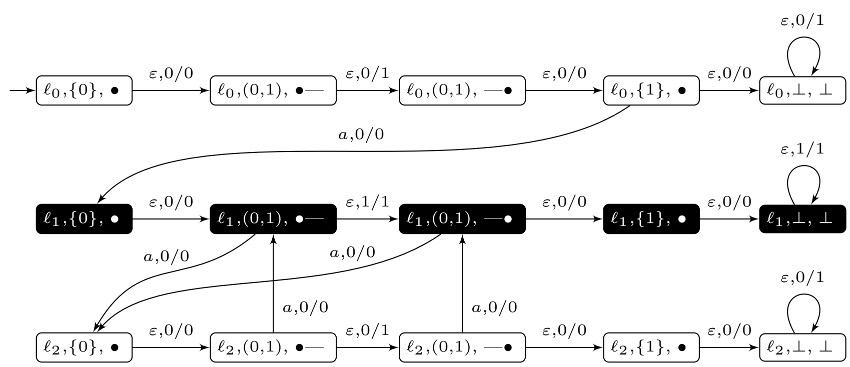



**Fig. 3.** The corner-point abstraction $\mathcal{A}^F_{cp}$ of $\mathcal{A}$ represented Fig. 1.

There will be a correspondence between runs in $\mathcal{A}$ and runs in $\mathcal{A}_{cp}$. As time is increasing in $\mathcal{A}$ we forbid runs in $\mathcal{A}_{cp}$ where two actions have to be made in 0-delay (this is easy to do as there should be no sequence $\ldots \xrightarrow{\sigma} (\ell, R, \alpha) \xrightarrow{\sigma'} \ldots$, where both $\sigma$ and $\sigma'$ are actions and $R$ is a punctual region).

Given $\pi$ a run in $\mathcal{A}^F_{cp}$ the *ratio* of $\pi$, denoted $\mathsf{Rat}(\pi)$, is defined, provided it exists, as the lim sup of the ratio of accumulated costs divided by accumulated rewards for finite prefixes. Run $\pi$ is said *reward-converging* (resp. *reward-diverging*) if the accumulated reward along $\pi$ is bounded (resp. unbounded). Reward-converging runs in $\mathcal{A}^F_{cp}$ are meant to capture Zeno behaviours of $\mathcal{A}$.

Given $\varrho$ a run in $\mathcal{A}$ we denote by $\mathsf{Proj}_{cp}(\varrho)$ the set of all runs in $\mathcal{A}^F_{cp}$ compatible with $\varrho$ in the following sense. We assume $\varrho = (\ell_0, v_0) \xrightarrow{\tau_0, a_0} (\ell_1, v_1) \xrightarrow{\tau_1, a_1} \cdots$, where move $(\ell_i, v_i) \xrightarrow{\tau_i, a_i} (\ell_{i+1}, v_{i+1})$ comes from an edge $e_i$. A run[4] $\pi = (\ell_0, R_0^1, \alpha_0^1) \to (\ell_0, R_0^2, \alpha_0^2) \to \cdots \to (\ell_0, R_0^{k_0}, \alpha_0^{k_0}) \to (\ell_1, R_1^1, \alpha_1^1) \to \cdots \to (\ell_1, R_1^{k_1}, \alpha_1^{k_1}) \cdots$ of $\mathcal{A}_{cp}^F$ is in $\mathsf{Proj}_{cp}(\varrho)$ if for all indices $n \geq 0$:

- for all $i \leq k_n$, $\alpha_n^i$ is a corner-point of $R_n^i$,
- for all $i \leq k_n - 1$, $(R_n^{i+1}, \alpha_n^{i+1}) = (R_n^i, \alpha_n^i) + 1$,
- $(R_{n+1}^1, \alpha_{n+1}^1)$ is the successor pointed-region of $(R_n^{k_n}, \alpha_n^{k_n})$ by transition $e_n$ (that is $(R_{n+1}^1, \alpha_{n+1}^1) = (\{0\}, \bullet)$ if $e_n$ resets the clock $x$ and otherwise $(R_{n+1}^1, \alpha_{n+1}^1) = (R_n^{k_n}, \alpha_n^{k_n})$),
- $v_n \in R_n^1$ and if $R_n^{k_n} \neq \perp$, $v_n + \tau_n \in R_n^{k_n}$,
- if $R_n^{k_n} = \perp$, the sum $\mu_n$ of the rewards since region $\{0\}$ has been visited for the last time has to be equal to $\lfloor v_n + \tau_n \rfloor$ or $\lceil v_n + \tau_n \rceil$.[5] Note that $\mu_n$ can be seen as the abstraction of the valuation $v_n$.

*Remark 5.* As defined above, the size of $\mathcal{A}_{cp}^F$ is exponential in the size of $\mathcal{A}$ because the number of regions is $2M$ (which is exponential in the binary encoding of $M$). We could actually take a rougher version of the regions [12], where only constants appearing in $\mathcal{A}$ should take part in the region partition. This partition, specific to single-clock timed automata is only polynomial in the size of $\mathcal{A}$. We choose to simplify the presentation by considering the standard unit intervals.

We will now see that the corner-point abstraction is a useful tool to deduce properties of the set of frequencies of runs in the original timed automata.

### 3.2 From $\mathcal{A}$ to $\mathcal{A}_{cp}^F$, and *vice-versa*

We first show that given a run $\varrho$ of $\mathcal{A}$, there exists a run in $\mathsf{Proj}_{cp}(\varrho)$, whose ratio is smaller (resp. larger) than the frequency of $\varrho$.

**Lemma 6 (From $\mathcal{A}$ to $\mathcal{A}_{cp}^F$).** *For every run $\varrho$ in $\mathcal{A}$, there exist $\pi$ and $\pi'$ in $\mathcal{A}_{cp}^F$ that can effectively be built and belong to $\mathsf{Proj}_{cp}(\varrho)$ such that:*

$$\mathsf{Rat}(\pi) \leq \mathsf{freq}_{\mathcal{A}}(\varrho) \leq \mathsf{Rat}(\pi').$$

*Run $\pi$ (resp. $\pi'$) minimizes (resp. maximizes) the ratio among runs in $\mathsf{Proj}_{cp}(\varrho)$.*

Such two runs of $\mathcal{A}_{cp}^F$ can be effectively built from $\varrho$, through the so-called *contraction* (resp. *dilatation*) operations. Intuitively it consists in minimizing (resp. maximizing) the time elapsed in $F$-locations.

Note that the notion of contraction cannot be adapted to the case of timed automata with several clocks, as illustrated by the timed automaton in Fig. 4. Consider indeed the run alternating delays $(\frac{1}{2} + \frac{1}{n})$ and $1 - (\frac{1}{2} + \frac{1}{n})$ for $n \in \mathbb{N}$, and switching between the left-most cycle $(\ell_1 - \ell_2 - \ell_1)$ and the right-most cycle $(\ell_3 - \ell_4 - \ell_3)$ following the rules: in round $k$, take $2^{2k}$ times the cycle $\ell_1 - \ell_2 - \ell_1$, then switch to $\ell_3$ and take $2^{2k+1}$ times the cycle $\ell_3 - \ell_4 - \ell_3$ and return back to

[4] For simplicity, we omit here the transitions labels

[5] Roughly, in the unbounded region $\perp$, the number of times an idling transition is taken should reflect how 'big' the delay $\tau_n$ is.

$\ell_1$ and continue with round $k+1$. This run cannot have any contraction since its frequency is $\frac{1}{2}$, whereas all its projections in the corner-point abstraction have ratio $\frac{2}{3}$, the $\limsup$ of a non-converging sequence. This strange behavior is due to the fact that the delays in $\ell_1$ and $\ell_3$ need to be smaller and smaller, and this converging phenomenon requires at least two clocks.

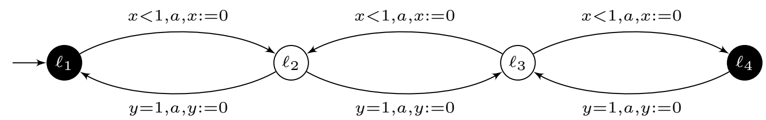


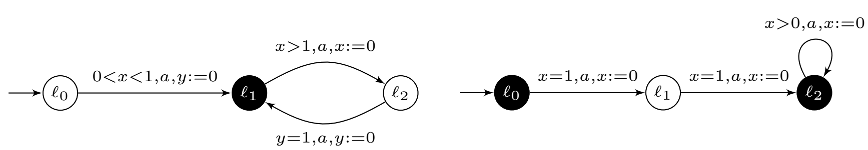

**Fig. 4.** A counterexample with two clocks for Lemma 6.

We now want to know when and how runs in $\mathcal{A}^F_{cp}$ can be lifted to $\mathcal{A}$. To that aim we distinguish between reward-diverging and reward-converging runs.

**Lemma 7 (From $\mathcal{A}^F_{cp}$ to $\mathcal{A}$, reward-diverging case).** *For every reward-diverging run $\pi$ in $\mathcal{A}^F_{cp}$, there exists a non-Zeno run $\varrho$ in $\mathcal{A}$ such that $\pi \in \mathsf{Proj}_{cp}(\varrho)$ and $\mathsf{freq}_{\mathcal{A}}(\varrho) = \mathsf{Rat}(\pi)$.*

*Proof (Sketch).* The key ingredient is that given a reward-diverging run $\pi$ in $\mathcal{A}^F_{cp}$, for every $\varepsilon > 0$, one can build a non-Zeno run $\varrho_\varepsilon$ of $\mathcal{A}$ with the following strong property: for all $n \in \mathbb{N}$, the valuation of the $n$-th state along $\varrho_\epsilon$ is $\frac{\epsilon}{2^n}$-close to the abstract valuation in the corresponding state in $\pi$. The accumulated reward along $\pi$ diverges, hence $\mathsf{freq}_{\mathcal{A}}(\varrho_\varepsilon)$ is equal to $\mathsf{Rat}(\pi)$. $\square$

The restriction to single-clock timed automata is crucial in Lemma 7. Indeed, consider the two-clocks timed automaton depicted in Fig. 5(a). In its corner-point abstraction there exists a reward-diverging run $\pi$ with $\mathsf{Rat}(\pi) = 0$, however every run $\varrho$ satisfies $\mathsf{freq}_{\mathcal{A}}(\varrho) > 0$.




(a) A counterexample with two clocks. (b) Zeno case.

**Fig. 5.** Counterexamples to extensions of Lemma 7.

**Lemma 8 (From $\mathcal{A}^F_{cp}$ to $\mathcal{A}$, reward-converging case).** *For every reward-converging run $\pi$ in $\mathcal{A}^F_{cp}$, if $\mathsf{Rat}(\pi) > 0$, then for every $\varepsilon > 0$, there exists a Zeno run $\varrho_\varepsilon$ in $\mathcal{A}$ such that $\pi \in \mathsf{Proj}_{cp}(\varrho_\varepsilon)$ and $|\mathsf{freq}_{\mathcal{A}}(\varrho_\varepsilon) - \mathsf{Rat}(\pi)| < \varepsilon$.*

*Proof (Sketch).* A construction similar to the one used in the proof of Lemma 7 is performed. Note however that the result is slightly weaker, since in the reward-converging case, one cannot neglect imprecisions (even the smallest) forced *e.g.*, by the prohibition of the zero delays. □

Note that Lemma 8 does not hold in case $\mathsf{Rat}(\pi) = 0$, where we can only derive that the set of frequencies of runs $\varrho$ such that $\pi \in \mathsf{Proj}_{cp}(\varrho)$ is either $\{0\}$ or $\{1\}$ or included in $(0, 1)$. Also an equivalent to Lemma 7 for Zeno runs (even in the single-clock case!) is hopeless. The timed automaton $\mathcal{A}$ depicted in Fig. 5(b), where $F = \{\ell_0, \ell_2\}$ is a counterexample. Indeed, in $\mathcal{A}^F_{cp}$ there is a reward-converging run $\pi$ with $\mathsf{Rat}(\pi) = \frac{1}{2}$, whereas all Zeno runs in $\mathcal{A}$ have frequency larger than $\frac{1}{2}$.

### 3.3 Set of frequencies of runs in $\mathcal{A}$

We use the strong relation between frequencies in $\mathcal{A}$ and ratios in $\mathcal{A}^F_{cp}$ proven in the previous subsection to establish key properties of the set of frequencies.

**Theorem 9.** *Let $\mathcal{F}_\mathcal{A} = \{\mathsf{freq}_\mathcal{A}(\varrho) \mid \varrho$ run of $\mathcal{A}\}$ be the set of frequencies of runs in $\mathcal{A}$. We can compute* $\inf \mathcal{F}_\mathcal{A}$ *and* $\sup \mathcal{F}_\mathcal{A}$*. Moreover we can decide whether these bounds are reached or not. Everything can be done in NLOGSPACE.*

The above theorem is based on the two following lemmas dealing respectively with the set of non-Zeno and Zeno runs in $\mathcal{A}$.

**Lemma 10 (non-Zeno case).** *Let $\{C_1, \cdots, C_k\}$ be the set of reachable SCCs of $\mathcal{A}^F_{cp}$. The set of frequencies of non-Zeno runs of $\mathcal{A}$ is then $\cup_{1 \le i \le k}[m_i, M_i]$ where $m_i$ (resp. $M_i$) is the minimal (resp. maximal) ratio for a reward-diverging cycle in $C_i$.*

*Proof (Sketch).* First, the set of ratios of reward-diverging runs in $\mathcal{A}^F_{cp}$ is exactly $\cup_{1 \le i \le k}[m_i, M_i]$. Indeed, given two extremal cycles $c_m$ and $c_M$ of ratios $m$ and $M$ in an SCC $C$ of $\mathcal{A}^F_{cp}$, we show that every ratio $m \le r \le M$ can be obtained as the ratio of a run ending in $C$ by combining in a proper manner $c_m$ and $c_M$. Then, using Lemmas 6 and 7 we derive that the set of frequencies of non-Zeno runs in $\mathcal{A}$ coincides with the set of ratios of reward-diverging runs in $\mathcal{A}^F_{cp}$. □

**Lemma 11 (Zeno case).** *Given $\pi$ a reward-converging run in $\mathcal{A}^F_{cp}$, it is decidable whether there exists a Zeno run $\varrho$ such that $\pi$ is the contraction of $\varrho$ and $\mathsf{freq}_\mathcal{A}(\varrho) = \mathsf{Rat}(\pi)$.*

*Proof (Sketch).* Observe that every fragment of $\pi$ between reset transitions can be considered independently, since compensations cannot occur in Zeno runs: even the smallest deviation (such as a delay $\varepsilon$ in $\mathcal{A}$ instead of a cost 0 in $\pi$) will introduce a difference between the ratio and the frequency. A careful inspection of cases allows one to establish the result stated in the lemma. □

Using Lemmas 10 and 11, let us briefly explain how we derive Theorem 9. For each SCC $C$ of the corner-point abstraction $\mathcal{A}_{cp}^F$, the bounds of the set of frequencies of runs whose contraction ends up in $C$ can be computed thanks to the above lemmas. We can also furthermore decide whether these bounds can be obtained by a real run in $\mathcal{A}$. The result for the global automaton follows.

*Remark 12.* The link between $\mathcal{A}$ and $\mathcal{A}_{cp}^F$ differs in several aspects from [8]. First, a result similar to Lemma 6 was proven, but the runs $\pi$ and $\pi'$ were not in $\mathsf{Proj}_{cp}(\varrho)$, and more importantly it heavily relied on the reward-diverging hypothesis. Then the counter-part of Theorem 9 was weaker in [8] as there was no way to decide whether the bounds were reachable or not.

## 4 Emptiness and Universality Problems

*The emptiness problem.* In our context, the emptiness problem asks, given a timed automaton $\mathcal{A}$ whether there is a timed word which is accepted by $\mathcal{A}$ with positive frequency. We also consider variants where we focus on non-Zeno or Zeno timed words. As a consequence of Theorem 9, we get the following result.

**Theorem 13.** *The emptiness problem for infinite (resp. non-Zeno, Zeno) timed words in single-clock timed automata is decidable. It is furthermore NLOGSPACE-Complete.*

Note that the problem is open for timed automata with 2 clocks or more.

*The universality problem.* We now focus on the universality problem, which asks, whether all timed words are accepted with positive frequency in a given timed automaton. We also consider variants thereof which distinguish between Zeno and non-Zeno timed words. Note that these variants are incomparable: there are timed automata that, with positive frequency, recognize all Zeno timed words but not all non-Zeno timed words, and *vice-versa*.

A first obvious result concerns deterministic timed automata. One can first check syntactically whether all infinite timed words can be read (just locally check that the automaton is complete). Then we notice that considering all timed words exactly amounts to considering all runs. Thanks to Theorem 9, one can decide, in this case, whether there is or not a run of frequency 0. If not, the automaton is universal, otherwise it is not universal.

**Theorem 14.** *The universality problem for infinite (resp. non-Zeno, Zeno) timed words in deterministic single-clock timed automata is decidable. It is furthermore NLOGSPACE-Complete.*

*Remark 15.* Note that results similar to Theorems 13 and 14 hold when considering languages defined with a threshold $\lambda$ on the frequency.

If we relax the determinism assumption this becomes much harder!

**Theorem 16.** *The universality problem for infinite (resp. non-Zeno, Zeno) timed words in a single-clock timed automaton is non-primitive recursive. If two clocks are allowed, this problem is undecidable.*

*Proof (Sketch).* The proof is done by reduction to the universality problem for finite words in timed automata (which is known to be undecidable for timed automata with two clocks or more [2] and non-primitive recursive for one-clock timed automata [13]). Given a timed automaton $\mathcal{A}$ that accepts finite timed words, we construct a timed automaton $\mathcal{B}$ with an extra letter $c$ which will be interpreted with positive frequency. From all accepting locations of $\mathcal{A}$, we allow $\mathcal{B}$ to read $c$ and then accept everything (with positive frequency). The construction is illustrated on Fig. 6. It is easy to check that $\mathcal{A}$ is universal over $\Sigma$ iff $\mathcal{B}$ is universal over $\Sigma \cup \{c\}$. □

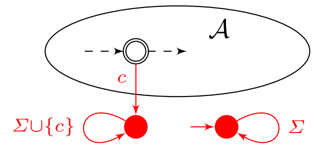



**Fig. 6.**

**Theorem 17.** *The universality problem for Zeno timed words with positive frequency in a one-clock timed automaton is decidable.*

*Proof (Sketch).* This decidability result is rather involved and requires some technical developments for which there is no room here. It is based on the idea that for a Zeno timed word to be accepted with positive frequency it is (necessary and) sufficient to visit an accepting location once. Furthermore the sequence of timestamps associated with a Zeno timed word is converging, and we can prove that from some point on, in the automaton, all guards will be trivially either verified or denied: for instance if the value of the clock is 1.4 after having read a prefix of the word, and if the word then converges in no more than 0.3 time units, then only the constraint $1 < x < 2$ will be satisfied while reading the suffix of the word, unless the clock is reset, in which case only the constraint $0 < x < 1$ will be satisfied. Hence the algorithm is composed of two phases: first we read the prefix of the word (and we use a now standard abstract transition system to do so, see [13]), and then for the tail of the Zeno words, the behaviour of the automaton can be reduced to that of a finite automaton (using the above argument on tails of Zeno words). □

## 5 Conclusion

In this paper we introduced a notion of (positive-)frequency acceptance for timed automata and studied the related emptiness and universality problems. This semantics is not comparable to the classical Büchi semantics. For deterministic single-clock timed automata, emptiness and universality are decidable by investigating the set of possible frequencies based on the corner-point abstraction. For (non-deterministic) single-clock timed automata, the universality problem restricted to Zeno timed words is decidable but non-primitive recursive. The restriction to single-clock timed automata is justified on the one hand by the undecidability of the universality problem in the general case. On the other hand, the techniques we employ to study the set of possible frequencies do not extend to timed automata with several clocks. A remaining open question is the decidability status of the universality problem for non-Zeno timed words, which is only known to be non-primitive recursive. Further investigations include a deeper study of frequencies in timed automata with multiple clocks, and also the extension of this work to languages accepted with some frequency larger than a given threshold.

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## Technical appendix

In this appendix, we present all proofs omitted in the core of the paper.

### Proofs for Section 3.2

*Definition of $F$-dilatation.* Let $\mathcal{A}$ be a timed automaton, $F \subseteq L$ a set of locations, and $\varrho = (\ell_0, 0) \xrightarrow{\tau_0, a_0} (\ell_1, v_1) \cdots$ an initial run in $\mathcal{A}$. We note $e_0, e_1, \cdots$ the edges fired along $\varrho$. We define the $F$-dilatation (or simply dilatation) of $\varrho$ as the run $\pi = (\ell_0, R_0^1 = \{0\}, \alpha_0^1 = \bullet) \to (\ell_0, R_0^2, \alpha_0^2) \to \cdots \to (\ell_0, R_0^{k_0}, \alpha_0^{k_0}) \to (\ell_1, R_1^1, \alpha_1^1) \to \cdots \to (\ell_1, R_1^{k_1}, \alpha_1^{k_1}) \cdots \in \mathsf{Proj}_{cp}(\varrho)$ in $\mathcal{A}_{cp}^F$ defined inductively as follows. Assume $n$ transitions of $\varrho$ are reflected in $\pi$: $\pi$ starts with $(\ell_0, R_0^1, \alpha_0^1) \to \cdots \to (\ell_0, R_0^{k_0}, \alpha_0^{k_0}) \to (\ell_1, R_1^1, \alpha_1^1) \to \cdots \to (\ell_n, R_n^1, \alpha_n^1)$ with $v_n \in R_n^1$ and $\alpha_n^1$ corner-point of $R_n^1$.

- if $v_n + \tau_n \leq M$:
  - if $v_n + \tau_n \in R_n^1 = (c, c+1)$, $\alpha_n^1 = \bullet\!-$ and $\ell_n \in F$, then we let time elapse as much as possible and choose in $\mathcal{A}_{cp}^F$ the portion of path $(\ell_n, R_n^1, \bullet\!-) \to (\ell_n, R_n^1, -\!\bullet) \to (\ell_{n+1}, R_{n+1}^1, \alpha_{n+1}^1)$ where $(R_{n+1}^1, \alpha_{n+1}^1)$ is the successor pointed region of $(R_n^1, -\!\bullet)$ by transition $e_n$.
  - if $v_n + \tau_n \in R_n^1 = (c, c+1)$, $\alpha_n^1 = \bullet\!-$ and $\ell_n \notin F$, we choose to fire $e_n$ as soon as possible by selecting the following portion of path: $(\ell_n, R_n^1, \bullet\!-) \to (\ell_{n+1}, R_{n+1}^1, \alpha_{n+1}^1)$ where $(R_{n+1}^1, \alpha_{n+1}^1)$ is the successor pointed region of $(R_n^1, \bullet\!-)$ by transition $e_n$.
  - if $v_n + \tau_n \in R_n^1 = (c, c+1)$ and $\alpha_n^1 = -\!\bullet$ is the last corner of $R_n^1$ (that is the second one), we need to immediately fire $e_n$ in $\mathcal{A}_{cp}^F$ and thus choose $(\ell_n, R_n^1, -\!\bullet) \to (\ell_{n+1}, R_{n+1}^1, \alpha_{n+1}^1)$ where $(R_{n+1}^1, \alpha_{n+1}^1)$ is the successor pointed region of $(R_n^1, -\!\bullet)$ by transition $e_n$.
  - if $v_n + \tau_n \notin R_n^1$ and $\ell_n \notin F$, we fire $e_n$ as soon as possible, that is, we let time elapse until region $R_n^{k_n}$ with $v_n + \tau_n \in R_n^{k_n}$ and its first corner-point $\alpha_n^{k_n}$, and then fire $e_n$: $(\ell_n, R_n^1, \alpha_n^1) \to \cdots \to (\ell_n, R_n^{k_n}, \alpha_n^{k_n}) \to (\ell_{n+1}, R_{n+1}^1, \alpha_{n+1}^1)$ where $(R_{n+1}^1, \alpha_{n+1}^1)$ is the successor pointed region of $(R_n^{k_n}, \alpha_n^{k_n})$ by transition $e_n$.
  - if $v_n + \tau_n \notin R_n^1$ and $\ell_n \in F$, we fire $e_n$ as late as possible, that is, we let time elapse until region $R_n^{k_n}$ with $v_n + \tau_n \in R_n^{k_n}$ and its last corner-point $\alpha_n^{k_n}$, and then fire $e_n$: $(\ell_n, R_n^1, \alpha_n^1) \to \cdots \to (\ell_n, R_n^{k_n}, \alpha_n^{k_n}) \to (\ell_{n+1}, R_{n+1}^1, \alpha_{n+1}^1)$ where $(R_{n+1}^1, \alpha_{n+1}^1)$ is the successor pointed region of $(R_n^{k_n}, \alpha_n^{k_n})$ by transition $e_n$.
- if $v_n + \tau_n > M$:
  - if $R_n^1 \neq \perp$, we let time elapse until region $\perp$ and add a delay to respect the definition of the projection in $\mathcal{A}_{cp}^F$ which depends on $\ell_n$ and then fire $e_n$: $(\ell_n, R_n^1, \alpha_n^1) \to \cdots \to (\ell_n, R_n^i, \alpha_n^i) \to (\ell_n, \perp, \perp)\big(\to (\ell_n, \perp, \perp)\big)^{\nu_n} \to (\ell_{n+1}, R_{n+1}^1, \alpha_{n+1}^1)$ where $\nu_n = \begin{cases} \lceil v_n + \tau_n \rceil - M & \text{if } \ell_n \in F \\ \lfloor v_n + \tau_n \rfloor - M & \text{if } \ell_n \notin F \end{cases}$ and $(R_{n+1}^1, \alpha_{n+1}^1)$ is the successor pointed region of $(\perp, \perp)$ by transition $e_n$.
  - if $R_n^1 = \perp$, respecting the definition of the projection give two possible delays, our choice depends on $\ell_n$, then we fire $e_n$: $(\ell_n, \perp, \perp)\big(\to (\ell_n, \perp, \perp)\big)^{\nu_n} \to (\ell_{n+1}, R_{n+1}^1, \alpha_{n+1}^1)$ where $\nu_n = \begin{cases} \lceil v_n + \tau_n \rceil - \nu_{n-1} & \text{if } \ell_n \in F \\ \lfloor v_n + \tau_n \rfloor - \nu_{n-1} & \text{if } \ell_n \notin F \end{cases}$ and $(R_{n+1}^1, \alpha_{n+1}^1)$ is the successor pointed region of $(\perp, \perp)$ by transition $e_n$.

Similarly, we define the $F$-contraction of $\varrho$ as its $\bar{F}$-dilatation, *i.e.* the run $\pi \in \mathsf{Proj}_{cp}(\varrho)$ of $\mathcal{A}_{cp}^F$ which fires transition $e_n$ as soon as possible when $\ell_n \in F$ and as late as possible when $\ell_n \notin F$.

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**Lemma 6 (From $\mathcal{A}$ to $\mathcal{A}_{cp}^F$).** *For every run $\varrho$ in $\mathcal{A}$, its contraction $\pi$ and dilatation $\pi'$ in $\mathcal{A}_{cp}^F$ can effectively be built, they are in $\mathsf{Proj}_{cp}(\varrho)$ and they satisfy:*

$$\mathsf{Rat}(\pi) \leq \mathsf{freq}_{\mathcal{A}}(\varrho) \leq \mathsf{Rat}(\pi').$$

*Run $\pi$ (resp. $\pi'$) minimizes (resp. maximizes) the ratio among runs in $\mathsf{Proj}_{cp}(\varrho)$.*

*Proof.* The proof is based on the following intuitive lemma, whose proof is tedious but not difficult. Given a run $\varrho = (\ell_0, v_0) \xrightarrow{\tau_0, a_0} (\ell_1, v_1) \cdots$, in the sequel we abusevely denote by $\mathsf{freq}_{\mathcal{A}}(\varrho_n)$ the quantity given by $(\sum_{i \leq n | \ell_i \in F} \tau_i)/(\sum_{i \leq n} \tau_i)$. In the same spirit, given $\pi$ a run in $\mathcal{A}_{cp}^F$, we abusively denote by $\mathsf{Rat}(\pi_n)$, the ratio of accumulated costs divided by accumulated rewards for the finite prefix of length $n$.

**Lemma A** *Let $\varrho$ be a run of $\mathcal{A}$, and $\pi_n$ be the dilatation of $\varrho_n$ (for $n \in \mathbb{N}$). For all $n \in \mathbb{N}$, if $\mathsf{freq}_{\mathcal{A}}(\varrho_n) = \frac{c_n}{r_n}$ and $\mathsf{Rat}(\pi_n) = \frac{C_n}{R_n}$ then $C_n \geq c_n$ and $(R_n - C_n) \leq (r_n - c_n)$.*

Assuming the latter lemma, it is easy to conclude that $\mathsf{freq}_{\mathcal{A}}(\varrho) \leq \mathsf{Rat}(\pi)$ for $\pi$ the dilatation of $\varrho$. Indeed, given $n \in \mathbb{N}$, $R_n - C_n \leq r_n - c_n$ and $c_n > 0$ (the case $c_n = 0$ is straightforward) imply $\frac{R_n - C_n}{c_n} \leq \frac{r_n - c_n}{c_n}$. Moreover, $C_n \geq c_n$. Hence $\frac{R_n - C_n}{C_n} \leq \frac{R_n - C_n}{c_n}$. All together, this yields $\frac{R_n}{C_n} - 1 \leq \frac{r_n}{c_n} - 1$ which is equivalent to $\frac{C_n}{R_n} \geq \frac{c_n}{r_n}$. When $n$ tends to infinity, we obtain $\mathsf{Rat}(\pi) \geq \mathsf{freq}_{\mathcal{A}}(\varrho)$.

Using the fact that the $F$-contraction of $\varrho$ is the $\bar{F}$-dilatation of $\varrho$, one obtains $\mathsf{Rat}(\pi') \leq \mathsf{freq}_{\mathcal{A}}(\varrho)$ for $\pi'$ the contraction of $\varrho$. □

*Proof (of Lemma A).* The proof is by induction on $n$. The base case, for $n = 0$ is trivial, since the $R_0$, $C_0$, $r_0$, $c_0$ are all set to 0 by convention. Note that an initialization at step $n = 1$ would also be possible using cases 1 to 3 in the following cases enumeration.

Assume now that the lemma holds for $n \in \mathbb{N}$, and let us prove it for $n + 1$. Consider the prefix of length $n + 1$ of $\varrho$: $\varrho_{n+1} = (\ell_0, 0) \xrightarrow{\tau_0, a_0} (\ell_1, v_1) \cdots (\ell_n, v_n) \xrightarrow{\tau_n, a_n} (\ell_{n+1}, v_{n+1})$. We note $e_0, e_1, ...$ the edges fired along $\varrho$. Let us detail a careful inspection of cases, depending on the value of $v_n + \tau_n$ and whether $\ell_n \in F$.

**Case 1** $\mathsf{frac}(v_n) = 0$[6], $\ell_n \in F$, and $v_n + \tau_n \notin \mathbb{N}$.

In this case, $\tau_n = \mathcal{T}_n - \tau'$ with $\mathcal{T}_n \in \mathbb{N}$ and $\tau' \in (0, 1)$. By definition of the dilatation, $\pi_{n+1}$ is built from $\pi_n$ by firing $\mathcal{T}_n$ idling transitions weighted $1/1$ in $\mathcal{A}_{cp}$ (possibly interleaved with idling transitions weighted $0/0$) followed by the discrete transition weighted $0/0$ that corresponds to $e_n$. Thus, $C_{n+1} = C_n + \mathcal{T}_n$, $R_{n+1} = R_n + \mathcal{T}_n$, whereas $c_{n+1} = c_n + \tau_n$ and $r_{n+1} = r_n + \tau_n$. In particular, $C_n \geq c_n$ (induction hypothesis) and $\tau_n < \mathcal{T}_n$ imply $C_{n+1} \geq c_{n+1}$. Moreover, $R_{n+1} - C_{n+1} = R_n - C_n$ and $r_{n+1} - c_{n+1} = r_n - c_n$, and by induction hypothesis $R_n - C_n \leq r_n - c_n$. Hence $R_{n+1} - C_{n+1} \leq r_{n+1} - c_{n+1}$.

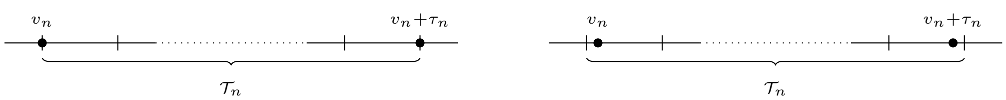
[6] $\mathsf{frac}(v)$ denotes its fractional part


(a) Case 1 ($\mathcal{T}_n > \tau_n$).     (b) Case 2 ($\mathcal{T}_n < \tau_n$).

**Fig. 7.** Cases 1 and 2.

**Case 2** $\mathsf{frac}(v_n) = 0$, $\ell_n \notin F$, and $v_n + \tau_n \notin \mathbb{N}$.
Here, $\tau_n = \mathcal{T}_n + \tau'$ with $\mathcal{T}_n \in \mathbb{N}$ and $\tau' \in (0, 1)$. In the dilatation, $\mathcal{T}_n$ transitions weighted $0/1$ will be fired before taking the transition corresponding to $e_n$. Thus $R_{n+1} = R_n + \mathcal{T}_n$, $C_{n+1} = C_n$, whereas $c_{n+1} = c_n$ and $r_{n+1} = r_n + \mathcal{T}_n + \tau'$. We immediately deduce that $C_{n+1} \geq c_{n+1}$. Morevoer $R_{n+1} - C_{n+1} = R_n + \mathcal{T}_n - C_n \leq r_n - c_n + \mathcal{T}_n < r_n + \mathcal{T}_n + \tau' - c_n = r_{n+1} - c_{n+1}$, where the second step uses the induction hypothesis.

**Case 3** $\mathsf{frac}(v_n) = 0$, and $v_n + \tau_n \in \mathbb{N}$.
In this case, $\tau_n = \mathcal{T}_n \in \mathbb{N}$ and exactly $\mathcal{T}_n$ transitions with reward 1 will be taken in $\mathcal{A}_{cp}$ before firing the transition that corresponds to $e_n$. In other words, the costs and rewards are exactly matched in the corner-point abstraction: $C_{n+1} - C_n = c_{n+1} - c_n$ and $R_{n+1} - R_n = r_{n+1} - r_n$. Notice that the last equalities hold regardless whether $\ell_n \in F$. Using the induction hypothesis ($C_n \geq c_n$ and $R_n - c_n \leq r_n - c_n$) we easily conclude: $R_{n+1} - C_{n+1} = R_n - C_n + r_{n+1} - c_{n+1} + c_n - r_n \leq r_{n+1} - c_{n+1}$, and $C_{n+1} = c_{n+1} + C_n - c_n \geq c_{n+1}$.




(a) Case 3 ($\mathcal{T}_n = \tau_n$).     (b) Case 4.1 ($\mathcal{T}_n > \tau_n$).

**Fig. 8.** Cases 3 and 4.1.

**Case 4** $\mathsf{frac}(v_n) \neq 0$ and $\ell_n \in F$

**Case 4.1** Assume first that the corner in the last state of $\pi_n$ is $\bullet\!-$. Then letting $\mathcal{T}_n = \lceil v_n + \tau_n - \lfloor v_n \rfloor \rceil$, in the dilatation, $\mathcal{T}_n$ idling transitions weighted $1/1$ will be fired in $\mathcal{A}_{cp}$ before firing the discrete transition corresponding to $e_n$. Thus $C_{n+1} = C_n + \mathcal{T}_n$, $R_{n+1} = R_n + \mathcal{T}_n$, whereas $c_{n+1} = c_n + \tau_n$ and $r_{n+1} = r_n + \tau_n$. We immediately obtain $C_{n+1} \geq c_{n+1}$ using the induction hypothesis and the fact that $\mathcal{T}_n > \tau_n$. Moreover, $R_{n+1} - C_{n+1} = R_n - C_n \leq r_n - c_n = r_{n+1} - c_{n+1}$.
Note that the picture on Fig. 8(b) represents the case $v_n + \tau_n \notin \mathbb{N}$, but the reasoning is valid for $v_n + \tau_n \in \mathbb{N}$ as well.

**Case 4.2** Assume now that the corner in the last state of $\pi_n$ is $-\!\bullet$. In this situation, we cannot conclude immediately, since $C_{n+1} = C_n + \mathcal{T}_n$ and $c_{n+1} = c_n + \tau_n$, with $\mathcal{T}_n = \lceil v_n + \tau_n - \lceil v_n \rceil \rceil$ is incomparable to $\tau_n$ in general. Instead, we need to reason in a more global way, taking into account some previous steps in $\varrho$ and $\pi$. Let us

consider the least index $i$ such that the corner of the last state in $\pi_{n-i}$ is not $-\bullet$. For this index, $\pi_{n-i}$ ends either with the pointed region $((\lfloor v_{n-i} \rfloor, \lceil v_{n-i} \rceil), \bullet-)$ or with $(\{v_{n-i}\}, \bullet)$. We then consider the suffix of path $\pi_{n+1}$ after $\pi_{n-i}$. Notice that the clock $x$ was not reset along this suffix (since no pointed region of the form $(\cdot, \cdot, \bullet)$ was reached). For this part, the accumulated reward is $\mathcal{T}_{n,i} = \lceil v_n + \tau_n - \lfloor v_{n-i} \rfloor \rceil$. The corresponding part in $\varrho$ has an accumulated delay $\tau_{n,i} = v_n + \tau_n - v_{n-i} = \sum_{j=n-i}^{n} \tau_j$ (since the clock has not been reset). Note that $\mathcal{T}_{n,i} \geq \tau_{n,i}$.
Let us now discuss the cost accumulated along the suffix of path $\pi_{n+1}$ after $\pi_{n-i}$. By definition of the dilatation, no idling transition can be fired from a state with an $F$-location along this suffix, else the last state of $\pi_{n-i+1}$ has not the corner $-\bullet$. Thus, the accumulated cost along the suffix of path $\pi_{n+1}$ after $\pi_{n-i}$ is equal to $\mathcal{T}_{n,i}$. However, the corresponding part in $\varrho$ has an accumulated cost $c_{n,i}$ smaller than $\tau_{n,i}$ (due to the potential time spent in locations not in $F$).
The above discussion can be summarised as follows: $R_{n+1} = R_{n-i} + \mathcal{T}_{n,i}$, $C_{n+1} = C_{n-i} + \mathcal{T}_{n,i}$, $r_{n+1} = r_{n-i} + \tau_{n,i}$ and $c_{n+1} = c_{n-i} + c_{n,i}$ with $\mathcal{T}_{n,i} \geq \tau_{n,i} \geq c_{n,i}$. We can thus derive that $C_{n+1} \geq c_{n+1}$, using both the induction hypothesis stating that $C_{n-i} \geq c_{n-i}$ and the fact that $\mathcal{T}_{n,i} \geq c_{n,i}$. It remains to prove that $R_{n+1} - C_{n+1} \leq r_{n+1} - c_{n+1}$. By the above equalities, this is equivalent to prove that $R_{n+i} - C_{n+i} \leq (r_{n+i} - c_{n+i}) + (\tau_{n,i} - c_{n,i})$ which is true by the induction hypothesis stating that $(R_{n+i} - C_{n+i}) \leq (r_{n+i} - c_{n+i})$ and the fact that $\tau_{n,i} \geq c_{n,i}$.

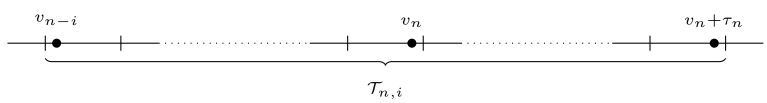



**Fig. 9.** Case 4.2

**Case 5** $\mathsf{frac}(v_n) \neq 0$ and $\ell_n \notin F$

**Case 5.1** Symmetrically to what precedes, the easy case is when the corner in the last state of $\pi_n$ is $-\bullet$. Then, letting $\mathcal{T}_n = \lfloor v_n + \tau_n \rfloor - \lceil v_n \rceil < \tau_n$, we can write $R_{n+1} = R_n + \mathcal{T}_n$ and $C_{n+1} = C_n$. Since $c_{n+1} = c_n$ and $r_{n+1} = r_n + \tau_n$, we deduce the desired inequalities.

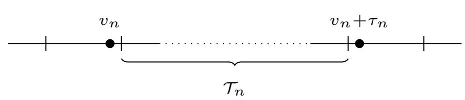



**Fig. 10.** Case 5.1 ($\mathcal{T}_n < \tau_n$).

**Case 5.2** Assume now that the corner in the last state of $\pi_n$ is $\bullet-$. Therefore, the last pointed region in $\pi_n$ is $((\lfloor v_n \rfloor, \lceil v_n \rceil), \bullet-)$, and we let $\mathcal{T}_n = \lfloor v_n + \tau_n - \lfloor v_n \rfloor \rfloor$. By definition of the dilatation, this can only happen if $\ell_{n-1} \notin F$. We then consider the least index $i$ such that the last corner in $\pi_{n-i}$ is not $\bullet-$. For this index, the last pointed region in $\pi_{n-i}$ is either $((\lfloor v_{n-i}\rfloor, \lceil v_{n-i}\rceil), \!-\!\bullet)$ or $(\{v_{n-i}\}, \bullet)$. Moreover, all locations $\ell_j$ for $n-i \le j \le n$ are not in $F$. We define $\mathcal{T}_{n,i} = \lfloor v_n + \tau_n \rfloor - \lceil v_{n-i} \rceil$. Note that $\mathcal{T}_n \le \sum_{j=n-i}^{n} \tau_j$. Using this notation, $R_{n+1} = R_{n-i} + \mathcal{T}_{n,i}$, and $C_{n+1} = C_{n-j}$. In $\mathcal{A}$, $r_{n+1} = r_{n-j} + \sum_{j=n-i}^{n} \tau_j$ and $c_{n+1} = c_{n-i}$. We trivially derive $C_{n+1} \ge c_{n+1}$ using the analogous induction hypothesis at rank $n-i$. Moreover, $R_{n+1} - C_{n+1} = R_{n-i} + \mathcal{T}_{n,i} - C_{n-i} < R_{n_i} + \sum_{j=n-i}^{n} \tau_j - C_{n-i} \le r_{n-i} + \sum_{j=n-i}^{n} \tau_j - c_{n-i} = r_{n+1} - c_{n+1}$, using the induction hypothesis at rank $n-i$ in the next to last step.

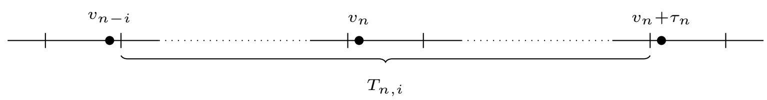



**Fig. 11.** Case 5.2.

Let us notice that we ignored the unbounded region through the all proof. However it can be treated exactly in the same way. Indeed, we can consider the accumulated reward since the last reset and its difference with the valuation in $\varrho$ instead of the corner-point.

Note that in cases 4.2 and 5.2, the induction relies on other cases (4.1, 5.1, and 1, 2, 3). However, the induction is well-founded since those cases are treated independently. $\square$

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**Lemma 7 (From $\mathcal{A}_{cp}^{F}$ to $\mathcal{A}$, reward-diverging case).** *For every reward-diverging run $\pi$ in $\mathcal{A}_{cp}^{F}$, there exists a non-Zeno run $\varrho$ in $\mathcal{A}$ such that $\pi \in \mathsf{Proj}_{cp}(\varrho)$ and $\mathsf{freq}_{\mathcal{A}}(\varrho) = \mathsf{Rat}(\pi)$.*

*Proof.* Given $\varrho$ a run and $n \in \mathbb{N}$, we denote by $\varrho[n]$ the valuation of the $n$-th state along the run. Similarly, if $\pi$ belongs to $\mathsf{Proj}_{cp}(\varrho)$, we consider the states of $\pi$ which correspond with a state of $\varrho$ (those which are just before a discrete transition) and we note $\pi[n]$ the valuation of the corner of the $n$-th state if the region is bounded. Otherwise, $\pi[n]$ is the sum of all the rewards since the last region $\{0\}$. Lemma 7 relies on the following lemma:

**Lemma B** *For every reward-diverging run $\pi$ in $\mathcal{A}_{cp}^{F}$, for all $\varepsilon > 0$, there exists a run $\varrho_\varepsilon$ of $\mathcal{A}$ such that, for all $n \in \mathbb{N}$, $|\pi[n] - \varrho_\epsilon[n]| \le \frac{\epsilon}{2^n}$.*

Let us assume the Lemma B and consider apart the cases where $\mathsf{Rat}(\pi) = 0$ and $\mathsf{Rat}(\pi) > 0$.

Assume first that $\pi$ is a reward-diverging run in $\mathcal{A}_{cp}^{F}$ with $\mathsf{Rat}(\pi) = 0$. Given $\varepsilon > 0$, let $\varrho$ be a run of $\mathcal{A}$ such that, for all $n \in \mathbb{N}$, $|\pi[n] - \varrho[n]| < \frac{\varepsilon}{2^n}$. If $C_n$ and $R_n$ are the accumulated costs and rewards in the first $n$ steps of $\pi$ in $\mathcal{A}_{cp}^{F}$, then $\mathsf{Rat}(\pi) = \limsup_{n\to\infty} \frac{C_n}{R_n}$ and $\mathsf{freq}_{\mathcal{A}}(\varrho) = \limsup_{n\to\infty} \frac{C_n + \sum_{i\le n} \alpha_i \varepsilon/2^i}{R_n + \sum_{i \le n} \beta_i \varepsilon / 2^i}$ where for every $i$, $\alpha_i \in \{-1, 0, 1\}$ and $\beta_i \in \{-1, 1\}$. Hence $\mathsf{freq}_{\mathcal{A}}(\varrho) \le \limsup_{n\to\infty} \frac{C_n+\varepsilon}{R_n-\varepsilon}$ (because $R_n > \varepsilon$ for $n$ large enough). Since $\lim_{n\to\infty} \frac{C_n}{R_n} = 0$ and $\lim_{n\to\infty} R_n = \infty$, we deduce $\limsup_{n\to\infty} \frac{C_n+\varepsilon}{R_n-\varepsilon} = 0$ which means $\mathsf{freq}_{\mathcal{A}}(\varrho) = 0 = \mathsf{Rat}(\pi)$.

Assume now that $\pi$ is a reward-diverging run in $\mathcal{A}_{cp}^F$ with $\mathsf{Rat}(\pi) > 0$. Using the same notations as in the previous case, $|\frac{C_n}{R_n} - \frac{C_n + \sum_{i \leq n} \alpha_i \varepsilon / 2^i}{R_n + \sum_{i \leq n} \beta_i \varepsilon / 2^i}| \leq \frac{C_n \varepsilon + R_n \varepsilon}{R_n (R_n - \varepsilon)}$. The latter term tends to 0 as $n$ tends to infinity. As a consequence $\mathsf{freq}_{\mathcal{A}}(\varrho) = \mathsf{Rat}(\pi)$. □

*Proof (of Lemma B).* We show that given a reward-diverging run $\pi$ in $\mathcal{A}_{cp}^F$, we can build a run $\varrho$ such that $\pi \in \mathsf{Proj}_{cp}(\varrho)$ and the $\varrho[i]$ are as close as we want of the $\pi[i]$. More precisely, we show that we can choose suitable delays. In the case where $\pi[i]$ is different than $\pi[i+1]$, the choice of the delay allows to be as close as wanted of $\pi[i+1]$. If $\pi[i]$ and $\pi[i+1]$ are equal but an upper bound of a region, we can move nearer to $\pi[i+1] = \pi[i]$ by the new delay. If the region is unbounded, and $\pi[i+1]$ larger than the maximal constant, it is again a good case. The only difficulty is the case where the new delay force us to move further than $\pi[i+1] = \pi[i]$. The solution is to consider globally the sequence of the delays in the same corner together with the delay leading to it. Thanks to the non-Zenoness, this sequence is necessarily finite. Therefore, we can effectively choose suitable delays to respect the condition at the end of the sequence and thus all along the sequence. Note this lemma is a simpler version of the Lemma 3 in [3]. □

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*Details on the counterexample Fig.5(a).* We explicit here a reward-diverging run $\pi$ in $\mathcal{A}_{cp}^F$ of zero ration and explain why every run $\varrho$ in $\mathcal{A}$ has a positive frequency. First, $\pi$ consists (omitting idling transitions weighted 0/0) in the following sequence of transitions :

$$(\ell_0, \{0\}^2, \bullet) \xrightarrow{\varepsilon, 0/1} (\ell_0, -, -) \xrightarrow{a, 0/0} \Big((\ell_1, -, -) \xrightarrow{a, 0/0} (\ell_2, -, -) \xrightarrow{\varepsilon, 0/1} (\ell_2, -, -) \xrightarrow{a, 0/0} \Big)^\omega.$$

The ratio of $\pi$ is thus zero because the accumulated cost of $\pi$ is zero whereas the reward diverges. On the other hand, let us consider a run $\varrho$ of $\mathcal{A}$ and prove that its frequency is positive. Indeed, $\varrho$ reads necessarily a word of the form $(1-\tau_0, a).\big((\tau_i, a).(1-\tau_i)\big)_{1 \leq i}$ where $\tau_0 \in (0,1)$ and $\tau_{i+1} > \tau_i$ for all $0 \leq i$. The frequency of $F = \{\ell_1\}$ in $\varrho$ is thus given by:

$$\mathsf{freq}_{\mathcal{A}}(\varrho) = \limsup_{n \to \infty} \frac{\sum_{i \leq n} \tau_i}{\sum_{i \leq n} 1} > \limsup_{n \to \infty} \frac{\sum_{i \leq n} \tau_0}{\sum_{i \leq n} 1}.$$

Hence, $\mathsf{freq}_{\mathcal{A}}(\varrho) > \tau_0 > 0$.

---

**Lemma 8 (From $\mathcal{A}_{cp}^F$ to $\mathcal{A}$, reward-converging case).** *For every reward-converging run $\pi$ in $\mathcal{A}_{cp}^F$, if $\mathsf{Rat}(\pi) > 0$, then for every $\varepsilon > 0$, there exists a Zeno run $\varrho_\varepsilon$ in $\mathcal{A}$ such that $\pi \in \mathsf{Proj}_{cp}(\varrho_\varepsilon)$ and $|\mathsf{freq}_{\mathcal{A}}(\varrho_\varepsilon) - \mathsf{Rat}(\pi)| < \varepsilon$.*

*Proof.* Lemma 8 uses the following lemma:

**Lemma C** *For every reward-converging run $\pi$ in $\mathcal{A}_{cp}^F$, for all $\varepsilon > 0$, there exists a Zeno run $\varrho_\varepsilon$ in $\mathcal{A}$ such that $\pi \in \mathsf{proj}_{cp}(\varrho_\varepsilon)$ and for all $n \in \mathbb{N}$, $|\pi[n] - \varrho_\varepsilon[n]| < \varepsilon$.*

Assuming Lemma C and that $\mathsf{Rat}(\pi) > 0$, let $n_\pi$ be the length of the smallest prefix of $\pi$ such that there is no transition with non-Zero reward after. Thanks to the convergence of the reward of $\pi$, $n_\pi$ is necessarily finite. Given $\varepsilon > 0$, the run $\varrho_{\varepsilon'}$ given by the Lemma C with $\varepsilon' = \frac{\varepsilon}{n_\pi+1}$ satisfies the desired property. $\square$

**Remark D** *Note that if $\pi$ is a contraction, then $\pi$ is the contraction of $\varrho_\varepsilon$ defined in the proof of Lemma C.*

Remark that, if $\pi$ is of ratio 0, three cases are possible:

- only $F$-locations are along $\pi$ and the reward of $\pi$ is 0, then if $\pi$ is the contraction of $\varrho$, $\mathsf{freq}_{\mathcal{A}}(\varrho) = 1$,
- only $\bar{F}$-locations are along $\pi$, the frequency of each run $\varrho$ whose contraction is $\pi$, is 0
- otherwise, neither 1 nor 0 can be the frequency of an run $\varrho$ of contraction $\pi$.

These results follow immediately from the prohibition of zero-delays.

*Proof (of Lemma C).* Let $\pi$ be a reward-converging run in $\mathcal{A}_{cp}^F$, and $\varepsilon \in (0,1)$. As $\pi$ is reward-converging, it ends with transitions weighted $0/0$ and its longest prefix $\pi'$ not ending with a transition weighted $0/0$ exists. To prefix $\pi'$, one can associate a finite run $\varrho'$ of $\mathcal{A}$, as we did for reward-diverging runs (see proof of Lemma B): for all indices $i$ less than the length of $\pi'$, $|\pi'[i] - \varrho'[i]| < \frac{\varepsilon}{2^i}$. For the suffix of $\pi$, composed only of transitions weighted $0/0$, we define a corresponding run in $\mathcal{A}$ with total duration less than $\varepsilon$. This can, *e.g.*, be achieved by taking successive delays of $\frac{\varepsilon}{2^k}$ for $k \geq 1$. Concatenating $\varrho'$ and the run defined above yields a run in $\mathcal{A}$ always $\varepsilon$-close to $\pi$. $\square$

## Proofs for Section 3.3

**Lemma 10 (non-Zeno case).** *Let $\{C_1, \cdots, C_k\}$ the set of reachable SCCs of $\mathcal{A}_{cp}^F$. The set of frequencies of non-Zeno runs of $\mathcal{A}$ is then $\cup_{1\leq i\leq k}[m_i, M_i]$ where $m_i$ (resp. $M_i$) is the minimal (resp. maximal) ratio for a reward-diverging cycle in $C_i$.*

*Proof.* The lemma is based on the following lemma which expresses the set of ratios in $\mathcal{A}_{cp}^F$ for reward-diverging runs ending up in a given SCC.

**Lemma E** *Let $C_i$ be an SCC of $\mathcal{A}_{cp}^F$. If $\mathcal{R}_i$ denotes the set of ratios of reward-diverging simple cycles in $C_i$, then the set of ratios of reward-diverging runs of $\mathcal{A}_{cp}^F$ ending in $C_i$ is the interval $[m_i, M_i]$, where $m_i = \min(\mathcal{R}_i)$ and $M_i = \max(\mathcal{R}_i)$.*

Admitting the lemma for now, we conclude as follows. The set of ratios for the reward-diverging runs in $\mathcal{A}_{cp}^F$ is thus $\cup_{1\leq i\leq k}[m_i, M_i]$, where $m_i$ is the minimal frequency for a simple reward-diverging cycle in SCC $C_i$, and $M_i$ the maximal one. By the Lemma 7, we know that $\cup_{1\leq i\leq k}[m_i, M_i]$ is included in $\mathcal{F}_{nZ}$, the set of frequencies of non-Zeno runs in $\mathcal{A}$. Moreover, thanks to the Lemma 6 and the convexity of the intervals $[m_i, M_i]$, we can show the other inclusion $\mathcal{F}_{nZ} \subseteq \cup_{1\leq i\leq k}[m_i, M_i]$ as follows. Let $\varrho$ be a non-Zeno run in $\mathcal{A}$. We distinguish between two cases:

- if the contraction and the dilatation of $\varrho$ are both reward-diverging, then either the clock is reset infinitely often along $\varrho$ or from some point on, the value of the clock along $\varrho$ lies in the unbounded region forever. In the first case, there is some state of the form $(\ell, \{0\}, \bullet)$ in $\mathcal{A}_{cp}$ which is visited infinitely often by both the contraction and the dilatation. In the

second case, from some point on, they will follow the same transitions between states of the form $(\ell, \bot, \alpha_\bot)$ (within the unbounded region). In both cases, the contraction and the dilatation both end up in the same SCC, say $C_i$. Their frequencies, and that of $\varrho$ (thanks to Lemma 6) thus lie in the interval $[m_i, M_i]$.
- if the contraction (resp. dilatation) of $\varrho$ is reward-converging, the frequency of $\varrho$ is 1 (resp. 0), in this case, the dilatation (resp. contraction) is reward-diverging and of ratio 1 (resp. 0), therefore 1 (resp. 0) is in $\cup_{1\le i\le k}[m_i, M_i]$.

As a consequence, the set $\mathcal{F}_{nZ}$ of frequencies of non-Zeno runs of $\mathcal{A}$ is equal to the set $\cup_{1\le i\le k}[m_i, M_i]$ of frequencies of the reward-diverging runs of $\mathcal{A}^F_{cp}$. □

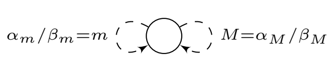
*Proof (of Lemma E).* Let $\pi$ be a reward-diverging run of $\mathcal{A}^F_{cp}$. To $\pi$ we associate the SCC $C_\pi$ of $\mathcal{A}^F_{cp}$ where $\pi$ ends up in. First observe that the influence of the prefix leading $C_\pi$ is negligible in the computation of the ratio because $\pi$ is reward-diverging. Precisely, the ratio of the prefix of length $n$ (for $n$ large enough) is:

$$\mathsf{Rat}(\pi_{|n}) = \frac{p_{pref} + P_n}{q_{pref} + Q_n}$$

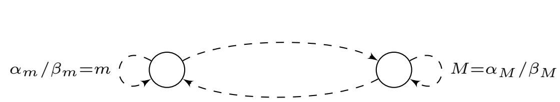
where $p_{pref}/q_{pref}$ is the ratio of the shortest prefix of $\pi$ leading to $C_\pi$. The sequence $Q_n$ diverges when $n$ tends to infinity because $\pi$ is reward-diverging. Hence $\lim_{n\to\infty} \mathsf{Rat}(\pi_{|n}) = \lim_{n\to\infty} \frac{P_n}{Q_n}$. As a consequence, without loss of generality, we assume that $\mathcal{A}^F_{cp}$ is restricted to $C_\pi$ and $\pi$ starts in some state of $C_\pi$.

Observe now that reward-converging cycles in $\mathcal{A}^F_{cp}$ necessarily have reward (and hence cost) 0, and thus do not contribute to the ratio $\mathsf{Rat}(\pi)$. Hence we can assume w.l.o.g. that $\pi$ does not pass through reward-converging cycles. Following the proof of [3, Prop. 4], we can decompose $\pi$ into (reward-converging) cycles and prove that $\mathsf{Rat}(\pi)$ lies between $m = \min(\mathcal{R}_{C_\pi})$ and $M = \max(\mathcal{R}_{C_\pi})$. Note that the extremal values ($m$ and $M$) are obtained by a run reaching a cycle with extremal ratio, and iterating it forever.

Let us now show that any value in the interval $[m, M]$ is the ratio of some run in $\mathcal{A}^F_{cp}$ which ends up in the SCC $C_\pi$. The arguments are inspired by [4]. Given $\lambda \in (0, 1)$, we explain how to build a run with ratio $r_\lambda = (1-\lambda)m + \lambda M$. To do so, for $(a_n) \in (\mathbb{Q} \cap (m, M))^{\mathbb{N}}$ a sequence of rational numbers converging to $\lambda$, we build an run $\pi$ such that $|\mathsf{Rat}(\pi_{|f(n)}) - a_n| < \frac{1}{n}$ for some increasing function $f \in \mathbb{N}^{\mathbb{N}}$.




(a) Case 1




(b) Case 2

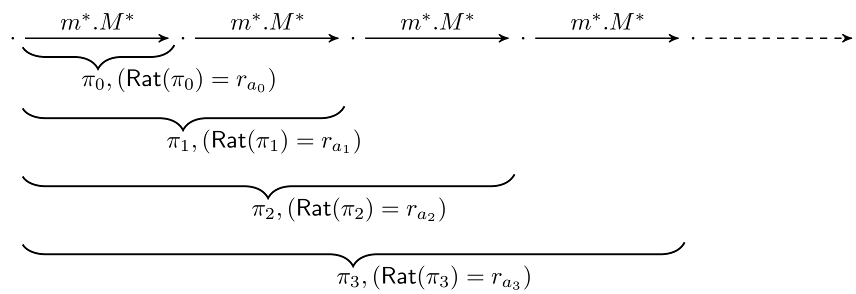
**Fig. 12.** The two possible cases.

*Case 1.* We first assume for simplicity that in $C_\pi$ two cycles of respective ratio $m$ and $M$ share a state, as depicted in Fig. 12(a), and prove a stronger result: we build a run $\pi$ such that $\mathsf{Rat}(\pi_{|f(n)}) = a_n$. Since two cycles, one of minimal ratio, and the other of maximal ratio share a common state, it suffices to explain how to combine these two cycles to obtain ratio $r_\lambda$.

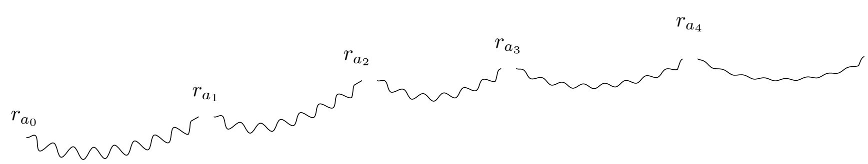
Assume $a_0 = p_0/q_0$ with $(p_0, q_0) \in \mathbb{N}^2$. Let us show how to build a finite run $\pi$ of ratio $r_{a_0} = (1 - p_0/q_0)m + (p_0/q_0)M$. Assume $m = \alpha_m/\beta_m$ where $\alpha_m$ is the cost of the cycle, and $\beta_m$ its reward, and similarly $M = \alpha_M/\beta_M$. Taking $(q_0 - p_0)\beta_M$ times the cycle of ratio $m$ and then $p_0\beta_m$ times the cycle of ratio $M$ yields an finite run $\pi_0$ with the desired property (this will be $\pi_{|f(0)}$). Indeed:

$$\frac{\big((q_0 - p_0)\beta_M\big)\alpha_m + \big(p_0\beta_m\big)\alpha_M}{\big((q_0 - p_0)\beta_M\big)\beta_m + \big(p_0\beta_m\big)\beta_M} = \frac{(q_0 - p_0)\beta_M\alpha_m + p_0\beta_m\alpha_M}{q_0\beta_M\beta_m} = \frac{q_0 - p_0}{q_0}m + \frac{p_0}{q_0}M = r_{a_0}.$$

To build an infinite run with ratio $r_\lambda$, we incrementally build prefixes $\pi_n$ (which will be $\pi_{|f(n)}$) of ratio $r_{a_n}$, starting with $\pi_0$, as depicted in the picture below.




Run $\pi_{n+1}$ has $\pi_n$ as prefix, then iterates the cycle of minimal ratio, and finally iterates the cycle of maximal ratio in order to compensate $r_{a_n}$ and reach ratio $r_{a_{n+1}}$. We assume $a_n = p_n/q_n$ with $(p_n, q_n) \in \mathbb{N}^2$. In $\pi_{n+1}$ the number of iterations of the cycle of ratio $m$ (resp. the cycle of ratio $M$) is globally $b_{n+1}(q_{n+1} - p_{n+1})\beta_M$ (resp. $b_{n+1}p_{n+1}\beta_m$) for some $b_{n+1} \in \mathbb{N}_{>0}$. This construction ensures that $r_\lambda$ is an accumulation point of the set of ratios for the prefixes $\pi_n$. Moreover, since each path fragment starts with iterations of the cycle of minimal ratio first, $r_\lambda$ is the largest accumulation point of the sequence of the ratios of prefixes after each cycle. The sequence of the prefixes' ratios is schematized below. The oscillations during a cycle become negligible when the length of the run increases. In the picture below, they are represented by shorter and shorter dashes.




*Case 2* In the general case, in the SCC $C_\pi$ of $\mathcal{A}^F_{cp}$ the cycles with minimal and maximal ratios do not necessarily share a common state: two finite runs connect the two cycles, as represented on Fig 12(b). We fix two cycles of minimal and maximal ratios, and two finite paths $\pi_{mM}$ and $\pi_{Mm}$ that connect those cycles in $C_\pi$. Similarly to the first case, we show how to build a sequence of finite runs $(\pi_n)$ with $|\mathsf{Rat}(\pi_n) - r_{a_n}| < \frac{1}{n}$, and prove that the influence of the finite paths linking the cycles is negligible when $n$ tends to infinity. The run $\pi_{n+1}$ is defined as the concatenation of $\pi_n$ with $\pi_{Mm}$ then iterations of the cycle of minimal

ratio $m$ then $\pi_{mM}$ and ending with iterations of the cycle with maximal ratio $M$. If $\tilde{p}$ and $\tilde{q}$ are respectively the cost and the reward of $\pi_{mM}$ and $\pi_{Mm}$ together, then the ratio of $\pi_{n+1}$ is:

$$\mathsf{Rat}(\pi_{n+1}) = \frac{b_{n+1}(q_{n+1} - p_{n+1})\beta_M \alpha_m + b_{n+1} p_{a_{n+1}} \beta_m \alpha_M + (n+1)\tilde{p}}{b_{n+1}(q_{n+1} - p_{n+1})\beta_M \beta_m + b_{n+1} p_{a_{n+1}} \beta_m \beta_M + (n+1)\tilde{q}}$$

Since this value tends to $r_{a_{n+1}}$ when $b_{n+1}$ tends to infinity, $b_{n+1}$ can be chosen such that $|\mathsf{Rat}(\pi_{n+1}) - r_{a_{n+1}}| < 1/(n+1)$. This way, $\lim_{n\to\infty} \mathsf{Rat}(\pi_n)$ agrees with $\lim_{n\to\infty} r_{a_n}$, that is $\lim_{n\to\infty} \mathsf{Rat}(\pi_n) = r_\lambda$. The function $f$ is defined by '$f(n)$ is the length of $\pi_n$'. □

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**Lemma 11 (Zeno case).** *Given $\pi$ a reward-converging run in $\mathcal{A}^F_{cp}$, it is decidable whether there exists a Zeno run $\varrho$ such that $\pi$ is the contraction of $\varrho$ and $\mathsf{freq}_\mathcal{A}(\varrho) = \mathsf{Rat}(\pi)$.*

*Proof.* This proof is composed of two parts. First, we study how to detect if the reward-converging execution $\pi$ in $\mathcal{A}^F_{cp}$ is a contraction, that is if there exists an execution in $\mathcal{A}$ whose contraction is $\pi$. If $\pi$ is a contraction, then by Lemma 8, we can construct a Zeno execution $\varrho$ in $\mathcal{A}$ whose contraction is $\pi$ (in particular $\mathsf{freq}_\mathcal{A}(\varrho) \geq \mathsf{Rat}(\pi)$) and such that $\mathsf{freq}_\mathcal{A}(\varrho)$ is as near as we want from $\mathsf{Rat}(\pi)$. The second step is to decide if we can construct an *optimal* $\varrho$ in $\mathcal{A}$, that is a Zeno execution $\varrho$ whose contraction is $\pi$ and such that $\mathsf{freq}_\mathcal{A}(\varrho) = \mathsf{Rat}(\pi)$. To do so, we see that we can study $\pi$ independently on each fragment between its resets. For each of these fragments of $\pi$, we provide necessary and sufficient conditions for them to be exactly reflected in $\mathcal{A}$. Thus, there exists an optimal $\varrho$ if and only if all of these fragments respect these conditions. Indeed, a tiny difference between the ratio of a fragment and the corresponding frequency on $\varrho$ is never neglected because on the one hand $\varrho$ is Zeno and on the other hand the contraction minimizes the frequency.

Let $\pi$ be a reward-converging run in $\mathcal{A}^F_{cp}$. By definition of contractions, we easily verify whether $\pi$ is the contraction of some run in $\mathcal{A}$. It is the case if and only if the two following conditions are satisfied by $\pi$:

$(i)$ from each state of $\pi$ of the form $(\ell, (i, i+1), \bullet\!-)$ where $\ell \notin F$, $\pi$ follows an idling transition to $(\ell, (i, i+1), -\!\bullet)$;

$(ii)$ after each move $(\ell, (i, i+1), \bullet\!-) \xrightarrow{1/1} (\ell, (i, i+1), -\!\bullet)$ where $\ell \in F$, $\pi$ goes to $(\ell, \{i+1\}, \bullet)$ by an idling transition.

We first consider two simple cases:

- If $\mathsf{Rat}(\pi) = 0$, then there exists a Zeno run $\varrho$ in $\mathcal{A}$ whose contraction is $\pi$, such that $\mathsf{freq}_\mathcal{A}(\varrho) = \mathsf{Rat}(\pi) = 0$ if and only if there are only (non $F$)-locations along $\pi$. Otherwise, because of the Zenoness of $\varrho$ and the positivity of all the delays, $\mathsf{freq}_\mathcal{A}(\varrho) > 0$.
- If $\mathsf{Rat}(\pi) = 1$ and $\pi$ is the contraction of some run $\varrho$, then $\mathsf{freq}_\mathcal{A}(\varrho) \leq \mathsf{Rat}(\pi)$ hence by definition of the contraction $\mathsf{freq}_\mathcal{A}(\varrho) = \mathsf{Rat}(\pi)$.

We now assume that $0 < \mathsf{Rat}(\pi) < 1$. For any run $\varrho$ such that $\pi$ is the contraction of $\varrho$, $\mathsf{Rat}(\pi) \leq \mathsf{freq}_\mathcal{A}(\varrho)$. Thus in order to have equality we will have to minimize delays spent in $F$-locations when building $\varrho$.

Note that resets of the clock are not reflected in the corner-point abstraction, but could easily be. Therefore, in the sequel, we abusively speak of resets in $\pi$. In the rest of the proof, we will work independently on the reset-free parts of $\pi$, let us shortly argue why this reasoning holds in this context. Let $\varrho$ be a path containing finitely many resets and such that $\varrho = \varrho^1 \xrightarrow{a_1} \varrho^2 \xrightarrow{a_2} \cdots \varrho^n$ where all the $\varrho^i$'s are reset-free. Let $\pi$ be the contraction of $\varrho$, let us notice that $\pi$ can be written as $\pi^1 \xrightarrow{a_1} \pi^2 \xrightarrow{a_2} \cdots \pi^n$ where $\pi^i$ corresponds to the contraction of $\varrho^i$ (for $1 \leq i \leq n$). By definition of the contraction, we know that $\mathsf{Rat}(\pi^i) \leq \mathsf{freq}_{\mathcal{A}}(\varrho^i)$ for each $1 \leq i \leq n$. In particular, if there exists $i$ such that $\mathsf{Rat}(\pi^i) \neq \mathsf{freq}_{\mathcal{A}}(\varrho^i)$, it is necessarily the case that $\mathsf{Rat}(\pi^i) < \mathsf{freq}_{\mathcal{A}}(\varrho^i)$. In this situation, it is clearly impossible to have that $\mathsf{Rat}(\pi) = \mathsf{freq}_{\mathcal{A}}(\varrho)$.

Let us now detail the different cases that can arise:

- Assume there is an unbounded number of resets along $\pi$, and assume $\varrho$ is a run such that $\pi$ is the contraction of $\varrho$ and that $\mathsf{Rat}(\pi) = \mathsf{freq}_{\mathcal{A}}(\varrho)$. Because $\pi$ is reward-converging, after some point, all rewards are 0. Hence, only the prefix of $\pi$ before this point contributes to the ratio. By construction of the contraction, the ratio of $\pi$ up to this point (say it is $a/b$, which is by assumption $< 1$) is smaller or equal to the frequency of $\varrho$ up to that point, there must be equality as $\mathsf{Rat}(\pi) = \mathsf{freq}_{\mathcal{A}}(\varrho)$. We will prove now that from this point on, $\pi$ only visits $F$-states. Due to condition $(i)$ above, if a state $(\ell, (i, i+1), \bullet\!-)$ is visited, then $\ell \in F$ (otherwise there should be a reward of 1 on the next edge). If $(\ell, \{i\}, \bullet)$ is visited, then this is because we have just seen a reset, thus $i = 0$, and the next move should be an idling move (time is strictly increasing), leading to $(\ell, (0, 1), \bullet\!-)$. For the same reason we also get that $\ell \in F$. Since zero-delays are forbidden in $\mathcal{A}$, the run $\varrho$ necessarily spends some positive delay in the $F$-locations. As a consequence the equality $\mathsf{Rat}(\pi) = \mathsf{freq}_{\mathcal{A}}(\varrho)$ cannot hold because $\mathsf{Rat}(\pi) = a/b < (a + c)/(b + c) = \mathsf{freq}_{\mathcal{A}}(\varrho)$ for some $c > 0$ (the duration of the tail of $\varrho$). There is no run $\varrho$ such that $\pi$ is the contraction of $\varrho$ and $\mathsf{Rat}(\pi) = \mathsf{freq}_{\mathcal{A}}(\varrho)$.
- Assume now that the number of resets along $\pi$ is finite. The run $\pi$ can be split into fragments between resets. As explained at the beginning of the proof, each fragment has to be fairly reflected in $\varrho$ to not hinder the optimality of the construction. There is finite fragments and an infinite one. The finite fragments of $\pi$ are treated as follows. The two conditions that they have to satisfy in order to not block the construction of an optimal $\varrho$ are:
  - a discrete transition of $\pi$ going from an $F$-location (resp. $\overline{F}$-location) to an $\overline{F}$-location (resp. $F$-location) has to be fired from a punctual region ($\{i\}$) or from the region $\perp$, moreover this discrete transition has to occur after a sub-fragment in $F$-location (resp. $\overline{F}$-location) whose reward is positive.
  - the same way, the end of the fragment has to go from a punctual region or from the region $\perp$.

  These conditions are necessary and sufficient, the proof is based on the disjunction of cases of the proof of Lemma A. In the disjunction, we seen that for every $F$-fragment (resp. $\bar{F}$-fragment) of $\pi$, the ratio is smaller or equal to the frequency of the corresponding fragment in $\varrho$. Furthermore, the equality holds only for the case 3 and the cases 4.2 and 5.2 whether $v_{n-i}$ and $v_n + \tau_n$ belong to $\mathbb{N}$. Then every sub-fragment ($F$-fragment or $\bar{F}$-fragment) of a finite fragment (separated by resets) has to correspond to one of these cases. This is clearly equivalent to the two above conditions.

  If no finite fragment hinders the minimization, the infinite suffix of $\pi$ without resets has to be considered. The conditions for this infinite fragment are:

- as above, a discrete transition of $\pi$ going from an $F$-location (resp. $\bar{F}$-location) to an $\bar{F}$-location (resp. $F$-location) has to be fired from a punctual region or from the region $\perp$ after a sub-fragment in $F$-location (resp. $\bar{F}$-location) whose reward is positive.
- moreover, the fragment has to end in $\bar{F}$-locations.

First, in this infinite fragment the reward is finite. Hence, if there is an unbounded alternation of $F$- and $\bar{F}$-locations, then there is (at least) an $F$-fragment of reward 0 and there is no optimal $\varrho$. Else, as above, the first condition is necessary and ensures the good behaviors before stabilization of $\pi$ in $F$ or $\bar{F}$. Moreover, if $\pi$ ends in $\bar{F}$, then $\varrho$ can be constructed choosing delays to minimize the frequency by tending to the reward corresponding in $\pi$. Otherwise, $\pi$ ends in $F$ with a pointed region $((i, i{+}1), \alpha)$ or $(\perp, \alpha_\perp)$:

- $((i, i+1), \bullet\!-)$: the sum of the delays can at best tend to $i + \varepsilon$,
- $((i, i+1), -\!\bullet)$: either the last fragment in $\bar{F}$ does not end in a suitable region, or $\pi$ is not a contraction,
- $(\perp, \alpha_\perp)$: the sum of the delays can at best tend to the reward of the fragment $+\varepsilon$ because this reward has to be always smaller or equal to the sum of the delay to $\pi$ be the contraction of the built $\varrho$.

To conclude, a careful inspection of the corner-point abstraction allows us to decide whether there exists a run $\varrho$ in $\mathcal{A}$ whose contraction is $\pi$ and such that $\mathsf{Rat}(\pi) = \mathsf{freq}_{\mathcal{A}}(\varrho)$. □

A similar lemma holds for dilatations:

**Lemma F** *Given $\pi$ a reward-converging run of $\mathcal{A}^F_{cp}$, it is decidable whether there exists a Zeno run $\varrho$ such that $\pi$ is the dilatation of $\varrho$ and $\mathsf{freq}_{\mathcal{A}}(\varrho) = \mathsf{Rat}(\pi)$.*

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**Theorem 9.** *Let $\mathcal{F}_{\mathcal{A}} = \{\mathsf{freq}_{\mathcal{A}}(\varrho) \mid \varrho \text{ run of } \mathcal{A}\}$ be the set of frequencies of runs in $\mathcal{A}$. We can compute $\inf \mathcal{F}_{\mathcal{A}}$ and $\sup \mathcal{F}_{\mathcal{A}}$. Moreover we can decide whether these bounds are reached or not. Everything can be done in NLOGSPACE.*

*Proof.* With each run $\pi$ of the corner-point abstraction $\mathcal{A}^F_{cp}$ is associated the SCC it ends up in. Let us argue that given $C$ an SCC of $\mathcal{A}^F_{cp}$:

1. the infimum of the frequencies of runs of $\mathcal{A}$ whose contraction ends up in $C$ can be computed, and
2. it is decidable whether the bound is reached.

First of all, if there is no reward-converging (simple) cycle in $C$, then all runs in $\mathcal{A}^F_{cp}$ ending up in $C$ are reward-diverging. For each such run $\pi$, there exists a non-Zeno run $\varrho$ in $\mathcal{A}$ with $\mathsf{Rat}(\pi) = \mathsf{freq}_{\mathcal{A}}(\varrho)$, thanks to Lemma 7. In this case, Lemma 10 allows us to conclude.

Assume now that $C$ contains a reward-converging cycle, and let $S_{rc}$ be the set of states in $\mathcal{A}^F_{cp}$ that belong to a reward-converging cycle. The set $\mathsf{S}$ of cycle-free finite runs in $\mathcal{A}^F_{cp}$ ending up in a state of $S_{rc}$ is finite and therefore contains a run $\pi_{\min}$ with minimal ratio $r_{\min}$. The infimum $r^*$ of the ratios of runs of $\mathcal{A}^F_{cp}$ ending up in $C$ is thus $\min(r_{\min}, m)$ where $m$ is the minimal ratio of reward-diverging simple cycles co-reachable from $C$. Moreover, it is also the infimum of the frequencies of runs of $\mathcal{A}$ whose contraction ends up staying in $C$. This bound is reached by a non-Zeno run of $\mathcal{A}$ whose contraction ends up staying in $C$ if and only if $m = r^*$ and there is a reward-diverging cycle of ratio $m$ in $C$ (Lemma 10). We do not need to consider non-Zeno runs whose contraction is reward-converging because their frequency is necessarily equal to 1. On the other hand, the infimum may be reached by a Zeno run whose contraction ends up staying in $C$. This contraction may be reward-converging or diverging. We distinguish three cases: $r_{\min} > m$, $r_{\min} < m$ and $r_{\min} = m$.

**Case** $r_{\min} > m$ The infimum of ratios of runs in $\mathcal{A}^F_{cp}$ ending up in $C$ is then $m$: iterating the cycle of minimal ratio $m$ many times and then going to a reward-converging cycle in $C$ yields a ratio which tends to this infimum. However, the infimum of the ratios is not reached by runs of $\mathcal{A}^F_{cp}$ ending up in $C$. A fortiori, frequency $m$ cannot be realized by a run in $\mathcal{A}$ whose contraction is reward-converging and ends up in $\mathcal{A}^F_{cp}$.

**Case** $r_{\min} < m$ The infimum of ratios of runs in $\mathcal{A}^F_{cp}$ ending up in $C$ is $r_{\min}$. Note that considering only runs whose contraction is reward-converging suffices. Indeed, a contraction reward-diverging of a Zeno run is necessarily of ratio 0, therefore its existence would imply that $m = 0$. Using the proof of Lemma 11, we can decide if there exists a reward-converging run $\pi$ of $\mathcal{A}^F_{cp}$ ending up in $C$ such that there exists a Zeno run $\varrho$ whose contraction is $\pi$ and such that $\mathsf{freq}_{\mathcal{A}}(\varrho) = \mathsf{Rat}(\pi) = r_{\min}$.

In this case, $\pi$ can be decomposed either as follows:

$$1):\ \pi = \pi_0.\pi_p.(\ell^{\bar{F}}, \{i\}, \bullet) \xrightarrow{0/0} (\ell^{\bar{F}}, (i, i+1), \bullet\!\!-\!\!-) \xrightarrow{0/1} (\ell^{\bar{F}}, (i, i+1), -\!\!-\!\!\bullet)\big(\xrightarrow{0/0} (\ell^{\bar{F}}_i, (0,1), -\!\!-\!\!\bullet)\big)_{i\in\mathbb{N}}$$

or as follows :

$$2):\ \pi = \pi_0.\pi_p.(\ell^{\bar{F}}, \bot, \alpha_\bot)\big(\xrightarrow{0/0} (\ell^{\bar{F}}_i, \bot, \alpha_\bot)\big)_{i\in\mathbb{N}}$$

where, first $\pi_0$ ends up resetting the clock and its fragments between resets satisfy the good conditions that is the ones to be a contraction and the ones of the proof of the Lemma 11 over finite fragments, second the suffix contains only $\bar{F}$-locations and no resets and finally the factor $\pi_p$ satisfies the good conditions over the beginning of a fragment. Note that $\pi_p$ can ends in an $F$-location or an $\bar{F}$-location. If there exists such a run $\pi$ taking several cycles in the suffix, the same run $\tilde{\pi}$ whose suffix simply consists in the infinite iteration of the first cycle taken by $\pi$ satisfies the required properties. Therefore, the set of states of $\mathcal{A}^F_{cp}$ from which a suitable suffix runs is computable. The set of cycle-free prefixes satisfying the conditions and ending up in this set is also computable. Moreover, adding cycle iterations to the prefix cannot help to meet the conditions. As a consequence, the existence of such a $\pi$ is decidable.

**Case** $r_{\min} = m$ The infimum of the ratios can be the frequency of some Zeno run whose contraction is reward-diverging only if $r_{\min} = m = 0$. In this case, the infimum is reached by a Zeno run whose contraction ending up in $C$ is reward-diverging if and only if there is a reward-diverging contraction ending up in $C$ visiting only locations of $\bar{F}$, resetting infinitely often the clock and visiting finitely often the region $\{1\}$. On the other hand, the Zeno runs with reward-converging contraction can be treated similarly to the case $r_{\min} < m$ since the introduction of a finite number of minimal or reward-converging cycles in the minimal prefix cannot help to reach the bound.

We can thus decide whether the infimum is reached for a run of $\mathcal{A}$ by considering the question in subsets of runs (sorted with respect to the SCC in which the contraction ends up) forming a partition of the set of the runs of $\mathcal{A}$.

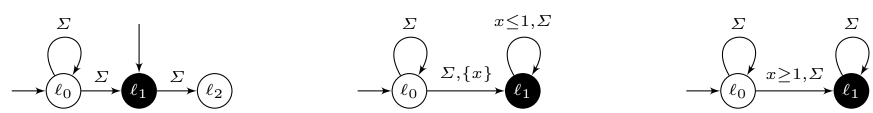
Similarly given $C$ an SCC of $\mathcal{A}_{cp}^F$:

1. the supremum of the frequencies of runs of $\mathcal{A}$ whose dilatation ends up in $C$ can be computed, and
2. it is decidable whether the bound is reached.

Let us now conclude the proof. If the infimum (resp. supremum) of the frequencies of runs in $\mathcal{A}$ is reached by some run, then the ratio of its contraction (resp. dilatation) is equal to this infimum (resp. supremum) frequency. By considering the bounds of the sets of frequencies for runs of $\mathcal{A}$ whose contraction (resp. dilatation) end up in each SCC, the bounds of the set of frequencies of runs of $\mathcal{A}$ are respectively the minimum and the maximum of these latter. Moreover, the two bounds are reached if and only if they are reached for some SCC. □

**Proofs for Section 4**

The different variants of the universality problems for timed automata under frequency-acceptance are incomparable, as illustrated by the following examples:




(a) $\mathcal{A}_1$. (b) $\mathcal{A}_2$. (c) $\mathcal{A}_3$.

**Fig. 13.** Counterexamples for the comparison between universality problems.

Let us explain why the universality problems with positive-frequency acceptance are not comparable when considering respectively finite timed words, Zeno timed words or non-Zeno timed words. The three timed automata of Fig. 13 illustrate this. The timed automaton $\mathcal{A}_1$ is universal for finite timed words but neither for the Zeno ones or the non-Zeno ones. In the same way, $\mathcal{A}_2$ and $\mathcal{A}_3$ are universal respectively for Zeno timed words and non-Zeno timed words but not for the other types of words.

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**Theorem 16.** *The universality problem for infinite (resp. non-Zeno, Zeno) timed words in a one-clock timed automaton is non-primitive recursive. If two clocks are allowed, this problem is undecidable.*

*Proof.* We want to check whether $\mathcal{A}$ is universal for Zeno timed words with positive frequency. We first check that every Zeno timed word can be read in $\mathcal{A}$: this is equivalent to checking that all finite timed word can be read in $\mathcal{A}$, and this can be done [8]. Thus, w.l.o.g.

we assume that $\mathcal{A}$ reads all Zeno timed words, and we now only need to take care of the accepting condition.

From $\mathcal{A}$ we build the timed automaton $\mathcal{B}$ composed of two copies of $\mathcal{A}$, one with a tag $\mathsf{w}$ and one with a tag $\mathsf{b}$. The symbol $\mathsf{w}$ (resp. $\mathsf{b}$) is for white (resp. black). If $\ell' \in F$, then for all transitions leading to $\ell'$, we will have transitions from the $\mathsf{w}$-copy to the $\mathsf{b}$-copy. All others are maintained. In $\mathcal{B}$ once the $\mathsf{b}$-copy is entered, it is never left, and for Zeno timed words, the Büchi condition can be reduced to "enter the $\mathsf{b}$-copy and read the rest of the word".

More formally, we define $\mathcal{B} = (L', L'_0, F', \Sigma, \{x\}, E')$ as follows:

– $L' = L \times \{\mathsf{w}, \mathsf{b}\}$, $L'_0 = L_0 \times \{\mathsf{w}\}$ and $F' = L \times \{\mathsf{b}\}$;
– if $\ell \xrightarrow{g,a,X'} \ell'$ is in $E$, then the following edges are in $E'$:
  - $(\ell, \mathsf{b}) \xrightarrow{g,a,X'} (\ell', \mathsf{b})$ and $(\ell, \mathsf{w}) \xrightarrow{g,a,X'} (\ell', \mathsf{w})$;
  - $(\ell, \mathsf{w}) \xrightarrow{g,a,X'} (\ell', \mathsf{b})$ if $\ell' \in F$;

The correctness of the transformation is stated in the following straightforward lemma.

**Lemma G** *$\mathcal{A}$ is universal with positive frequency for Zeno timed words iff $\mathcal{B}$ is universal with positive frequency for Zeno timed words.*

In the following we will write configurations of $\mathcal{B}$ as triples $(\ell, \mathsf{tag}, v)$ where $\ell$ is a location of $\mathcal{A}$, $\mathsf{tag} \in \{\mathsf{w}, \mathsf{b}\}$ and $v$ is a value for the clock. Furthermore we let $M$ be the maximal constant the clock is compared with in $\mathcal{B}$, and if $0 \leq c < M$, $I_c$ denotes the interval $(c; c+1)$ whereas $I_M$ denotes the interval $(M; +\infty)$.

An infinite execution $(\ell_0, \mathsf{tag}_0, v_0) \to (\ell_1, \mathsf{tag}_1, v_1) \cdots \to (\ell_p, \mathsf{tag}_p, v_p) \to \ldots$ in $\mathcal{B}$ *stabilizes* after $n_0$ steps whenever there exists some integer $c$ such that for every $n \geq n_0$, either $v_n \in I_c$, or $v_n = 0$, or $v_n \in I_0$. Note that every infinite run which reads a Zeno word stabilizes.

Given a Zeno timed word $w$, our aim is to analyze all runs that read $w$, so that we will be able to detect whether $w$ is accepted or not. Therefore we need to be able to compute a uniform bound after which all runs which read $w$ stabilizes. This is the aim of the next lemma.

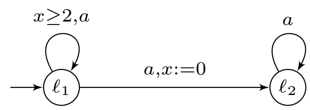
**Lemma H (Uniform stabilization)** *Let $w$ be a Zeno timed word (which can be read in $\mathcal{B}$ by assumption). Then, there exists some integer $n_0$ such that every execution that reads $w$ stabilizes after $n_0$ steps.*

To prove this lemma we need the notion of duration of a timed word. If $w = (t_0, a_0) \ldots (t_k, a_k)$ is a finite timed word, the duration of $w$ is $t_k$ and is denoted *duration*$(w)$. If $w$ is an infinite timed word, we let $w_{\leq n}$ be the $n$-th prefix of $w$, then the duration of $w$ is *duration*$(w) = \lim_{n\to\infty}$ *duration*$(w_{\leq n})$. Note that this is a finite value if and only if $w$ is a Zeno timed word.

*Proof.* Let $D$ be the duration of $w$, and for every $n$, $d_n$ be the duration of $w_{\leq n}$ (the prefix of length $n$ of $w$). We fix some integer $N$ such that $D - d_N < 1$, and we write $V_N$ for the set of possible valuations for clock $x$ after having read prefix $w_{\leq n}$ in $\mathcal{B}$. The set $V_N$ is finite.

Let $\varrho = (\ell_0, \mathsf{tag}_0, v_0) \to (\ell_1, \mathsf{tag}_1, v_1) \ldots$ be a run that reads $w$ in $\mathcal{B}$. Then $v_N \in V_N$, and either $\varrho$ stabilizes after $N$ steps, or $\varrho$ stabilizes after $N + k$ steps, where $k$ is the smallest integer such that $\lfloor v_N \rfloor + 1 = \lfloor v_N + d_{N+k} - d_N \rfloor$. This property does not depend on the choice of the execution, but only on the value $v_N$. Since there are finitely many $v_N$, as $V_N$ is finite, we can find a maximal $k$, denoted $k_0$ which will work for all the $v_N$. We choose $n_0$ either as $N$ (in case $\lfloor v_N \rfloor = \lfloor v_N + d_{N+k} - d_N \rfloor$ for every $k$), or as $N + k_0$ where $k_0$ is defined as above. By the previous analysis, we are done, every run $\varrho$ which reads $w$ stabilizes after $n_0$ steps. $\square$

*Example 17.* Take for instance the following timed automaton:




and the Zeno timed word $w = (a, 2)(a, 2 + 1/2)(1, 2 + 3/4)...$ whose duration is 3. There are infinitely many runs that read $w$, each one depends on the time where it takes the transition to $\ell_2$. We have that all runs that read $w$ stabilize after 2 steps.

Hence a Zeno word that is read in $\mathcal{B}$ will have a prefix (up to $n_0$ steps) and a Zeno tail that will satisfy clock constraints in a "straightforward" manner. We will take advantage of this structure to draw an algorithm for deciding whether there is a Zeno word that is not accepted by $\mathcal{B}$.

***Tail of Zeno words.*** We construct a finite automaton $\mathcal{B}_f$ that will somehow recognize the tail of Zeno behaviours. We build the automaton as follows: the set of states is $Q = L \times \{\mathsf{w}, \mathsf{b}\} \times (\{0, 1, \ldots, M-1, M\})$.

- There is a transition $(\ell, \mathsf{tag}, c) \xrightarrow{a} (\ell', \mathsf{tag}', 0)$ whenever there is a transition $(\ell, \mathsf{tag}) \xrightarrow{g,a,\{x\}} (\ell', \mathsf{tag}')$ in $\mathcal{B}$ with $[\![x \in I_c]\!] \subseteq [\![g]\!]$;
- There is a transition $(\ell, \mathsf{tag}, c) \xrightarrow{a} (\ell', \mathsf{tag}', c)$ whenever there is a transition $(\ell, \mathsf{tag}) \xrightarrow{g,a,\emptyset} (\ell', \mathsf{tag}')$ in $\mathcal{B}$ with $[\![x \in I_c]\!] \subseteq [\![g]\!]$

A state $(\ell, \mathsf{tag}, c)$ is accepting if $\mathsf{tag} = \mathsf{b}$, and we assume a Büchi condition. We parameterize $\mathcal{B}_f$ with the set of initial states $Q_0 \subseteq Q$, and we then write $\mathcal{B}_f^{Q_0}$.

This abstraction is a region abstraction for tails of Zeno behaviours in the following sense:

**Lemma I** *Let $q = (\ell, \mathsf{tag}, c)$ be a state of $\mathcal{B}_f$, and $u$ be an infinite (untimed) word. There is an equivalence between the two following properties:*

1. *$u$ can be read along some path $\varpi$ from $q$ in $\mathcal{B}_f$;*
2. *for every $v \in \{c\} \cup I_c$, for every increasing timestamps sequence $\tau$ which is convergent, and such that $v + duration(\tau) \in \overline{I_c} \cap (v; v+1)$, the timed word $w = (u, \tau)$ can be read along some path $\pi$ in $\mathcal{B}$.*

*In this equivalence we can furthermore assume the sequence of locations and tags encountered along $\varpi$ coincide with the sequence of locations and tags encountered along $\pi$.*

*Proof.* We first prove the implication 2 $\Rightarrow$ 1. Assume $w$ is a Zeno timed word read along the path $\pi = (\ell, \mathsf{tag}) \xrightarrow{g_0, a_0, Y_0} (\ell_1, \mathsf{tag}_1) \xrightarrow{g_1, a_1, Y_1} \ldots$ from configuration $(\ell, \mathsf{tag}, v)$. The corresponding run has the form $(\ell, \mathsf{tag}, v) \xrightarrow{\tau_0, a_0} (\ell_1, \mathsf{tag}_1, v_1) \xrightarrow{\tau_1, a_1} \ldots$ and by assumption for every $j \geq 0$, $v + \sum_{i=0}^{j} \tau_i \in I_c$.

- Assume the clock $x$ is never reset along $\pi$ (all $Y_k$'s are empty), then for every $k \geq 1$, $v_k = v + \sum_{0 \leq i < k} \tau_i$ and thus $v_k \in I_c$, which implies $[\![x \in I_c]\!] \subseteq [\![g_k]\!]$. In that case, by construction of $\mathcal{B}_f$, we get that there is path $(\ell, \mathsf{tag}, c) \xrightarrow{a_0} (\ell_1, \mathsf{tag}_1, c) \xrightarrow{a_1} \dots$ in $\mathcal{B}_f$.
- Assume the clock $x$ is reset along $\pi$, and that $Y_k = \{x\}$ is the first time $x$ is reset along $\pi$. Then the same argument as before applies to the prefix of $\pi$ up to $\ell_{k-1}$. Then we have that for every $j > k$, $v_j$ is either 0 (in case $Y_{j-1} = \{x\}$) or lies in $(0;1)$, and that $[\![x \in (0;1)]\!] \subseteq [\![g_j]\!]$ in any case (time is supposed to be strictly monotonic). Thus we can build a path $\varpi$ in the finite automaton which reads the untiming of $w$.

We now prove the implication $1 \Rightarrow 2$. Assume that $u = a_0 a_1 \dots$ is an infinite (untimed) word which is read along some path $\varpi = (\ell_0, \mathsf{tag}_0, c_0) \xrightarrow{a_0} (\ell_1, \mathsf{tag}_1, c_1) \xrightarrow{a_1} \dots$ in $\mathcal{B}_f$ with $(\ell_0, \mathsf{tag}_0, c_0) = (\ell, \mathsf{tag}, c)$. Take now a value $v \in \{c\} \cup I_c$ for clock $x$ and take an increasing timestamps sequence $(t_i)_{i \geq 0}$ that is convergent and such that $v + \sup_i t_i \in \overline{I_c} \cap (v; v+1)$.

By construction of $\mathcal{B}_f$ there is a path $\pi = (\ell_0, \mathsf{tag}_0, c_0) \xrightarrow{g_0, a_0, Y_0} (\ell_1, \mathsf{tag}_1, c_1) \xrightarrow{g_1, a_1, Y_1} \dots$ that corresponds to $\varpi$. In particular, if $Y_i = \emptyset$, then $c_{i+1} = c_i$, and if $Y_i = \{x\}$, then $c_{i+1} = 0$. We distinguish between two cases:

- the clock $x$ is never reset (for all $i \geq 0$, $Y_i = \emptyset$), in which case for all $i$, $c_i = c$. We define for every $i \geq 1$, $v_i = v + t_{i-1}$. By assumption on $(t_i)_i$, it holds that $v_i \in I_c = I_{c_i}$. Thus, the following run reads the timed word $w = ((t_i)_{i \geq 0}, u)$:
$$(\ell, \mathsf{tag}, v) = (\ell_0, \mathsf{tag}_0, v_0) \xrightarrow{t_0, a_0} (\ell_1, \mathsf{tag}_1, v_1) \xrightarrow{t_1 - t_0, a_1} \dots$$
- the clock $x$ is reset along $\pi$, and $i_0$ is the smallest index such that $Y_{i_0} = \{x\}$. We then define $I = \{i_0 < i_1 < \dots\}$ the set of index of sets $Y_i$'s where $Y_i = \{x\}$. We then define
$$v_{j+1} = \begin{cases} v + t_j & \text{if } j < i_0 \\ 0 & \text{if } j \in I \\ t_j - t_{i_k} & \text{if } i_k < j < i_{k+1} \end{cases}$$
Then, the following run reads the timed word $w = ((t_i)_{i \geq 0}, u)$:
$$(\ell, \mathsf{tag}, v) = (\ell_0, \mathsf{tag}_0, v_0) \xrightarrow{t_0, a_0} (\ell_1, \mathsf{tag}_1, v_1) \xrightarrow{t_1 - t_0} \dots$$
because the values of the clock never exceeds 1 after having reset the clock for the first time (and time is increasing, hence the constraint $x \in I_0$ is then always satisfied when firing a transition).

This conclues the proof of the second implication. □

**Lemma J** *Let $Q_0 \subseteq Q$ be a set of initial states for $\mathcal{B}_f$. Then we can decide in polynomial space whether $\mathcal{B}_f^{Q_0}$ is universal.*

*Proof.* $\mathcal{B}_f$ is a Büchi automaton, this result is thus standard, see [7]. □

***Handling the prefix.*** We use an abstraction which is now standard in the context of single-clock timed automata, see [1, 2, 8, 5, 9, 6, 10] This is a symbolic transition system associated with $\mathcal{B}$, which is denoted $\mathsf{Abst}_{\mathcal{B}}$ and defined as follows. We let $\Gamma$ be the finite set $2^{L \times \{\mathsf{b}, \mathsf{w}\} \times \{0, 1, \dots, M\}}$. The states of $\mathsf{Abst}_{\mathcal{B}}$ are tuples $(\gamma, h, \gamma')$ where $\gamma, \gamma' \in \Gamma$ and $h \in \Gamma^*$ is a finite word over alphabet $\Gamma$. Informally an abstract state $(\gamma, h, \gamma')$ represents a set of configurations of $\mathcal{B}$, where $\gamma$ are those states where the value of $x$ is an integer, $\gamma'$ are those states where the value of $x$ is larger than $M$, and $h$ encodes the order on the fractional part of the other states. More precisely, an abstract state $(\gamma, h = \gamma_1 \ldots \gamma_m, \gamma')$ represents a set of states of $\mathcal{B}$ $S = \{(\ell_j, \mathsf{tag}_j, v_j) \mid j \in J\}$ such that:

- $\gamma = \{(\ell_j, \mathsf{tag}_j, v_j) \mid v_j \in \{0, 1, \ldots, M\}\}$
- $\bigcup_{l=1}^{m} \gamma_l = \{(\ell_j, \mathsf{tag}_j, v_j) \mid j \in J,\ v_j \in I_{c_j},\ c_j < M\}$
- $\gamma' = \{(\ell_j, \mathsf{tag}_j, M) \mid v_j > M\}$
- if $(\ell_i, \mathsf{tag}_i, v_i) \in \gamma_i$ and $(\ell_j, \mathsf{tag}_j, v_j) \in \gamma_j$, $i \leq j$ iff $\mathsf{frac}(v_i) \leq \mathsf{frac}(v_j)$

We then write $abstr(S) = (\gamma, h, \gamma')$, and $S \in concr((\gamma, h, \gamma'))$.

We omit the definition of the abstract transitions in $\mathsf{Abst}_{\mathcal{B}}$, which is rather tedious and can be found for instance in [9].

This abstraction 'computes' the set of executions that can read finite timed words in $\mathcal{B}$ in the following sense.

**Lemma K** *Let $Conf = (\gamma, h, \gamma')$ be an abstract configuration. Then $Conf$ is reachable in $\mathsf{Abst}_{\mathcal{B}}$ iff there exists a finite timed word $w$ such that $Conf = abstr(S_w)$, where $S_w$ is the set of configurations that are reached after reading $w$ in $\mathcal{B}$.*

*Proof.* This is proven in close terms for instance in [9] (where the point-of-view of alternating timed automata is taken). ☐

***Gluing everything.*** We define $\mathcal{Z}$ the set of all sets of states $Q_0 \subseteq Q$ such that $\mathcal{A}_f^{Q_0}$ is universal. This set can be computed thanks to Lemma J. If $(\gamma, h, \gamma')$ is an abstract state of the above symbolic transition system, we write $set((\gamma, h, \gamma'))$ for the set $\gamma \cup \gamma' \cup \bigcup_{l=1}^{m} \gamma_l$, assuming $h = \gamma_1 \ldots \gamma_m$.

**Proposition L** *There is a Zeno timed word not accepted by $\mathcal{B}$ iff in $\mathsf{Abst}_{\mathcal{B}}$, it is possible to reach a configuration $Conf$ such that $set(Conf) \notin \mathcal{Z}$.*

*Proof.* Assume that $\mathcal{B}$ is not universal, and consider a Zeno timed word $w$ which is not accepted by $\mathcal{B}$. By assumption it can be read in $\mathcal{B}$. Also there exists some integer $n_0$ such that any run in $\mathcal{B}$ which reads $w$ stabilizes after $n_0$ steps (Lemma H). Assume $w_1$ is the $n_0$-th prefix of $w$ and that $w = w_1 \cdot w_2$. Let $Conf = abstr(S_{w_1})$. We will prove that writing $Q_0$ for $set(Conf)$, it is the case that $\mathcal{B}_f^{Q_0}$ does not accept $\mathsf{untime}(w_2)$. Towards a contradiction assume it is not the case. Then let $\pi_2$ be a path that accepts $\mathsf{untime}(w_2)$ in $\mathcal{B}_f^{Q_0}$. It starts from some $q_0 \in Q_0$. We have that $q_0 \in set(abstr(S_{w_1}))$, and thus there is some configuration $(\ell, \mathsf{tag}, v) \in S_{w_1}$ which corresponds to $q_0$, and there is some run in $\mathcal{B}$ which reads $w_1$ and reaches $(\ell, \mathsf{tag}, v)$. We can then apply Lemma I and lift $\pi_2$ into a run $\varpi_2$ that accepts $w_2$ (it visits the same locations and the same tags as $\pi_2$, which is accepting). This is the expected contradiction.

Assume now that in $\mathsf{Abst}_{\mathcal{B}}$, it is possible to reach a configuration $Conf$ such that $Q_0 = set(Conf) \notin \mathcal{Z}$. Let $w_1$ be a finite timed word such that $Conf = abstr(S_{w_1})$ (Lemma K). As $\mathcal{B}_f^{Q_0}$ is not universal, there is an infinite (untimed) word $u_2$ which is not accepted by $\mathcal{B}_f^{Q_0}$. Let $\alpha$ be the largest fractional part involved in $S_{w_1}$, and $T$ be the largest timestamp of $w_1$. We define the (Zeno) timed word $w = w_1 \cdot w_2$ where the untiming of $w_2$ is $u_2$ and the timestamps of $w_2$ are $T + \epsilon \cdot \left(\frac{1}{2}\right)$, $T + \epsilon \cdot \left(\frac{1}{2} + \frac{1}{4}\right)$, ..., $T + \epsilon \cdot \left(\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^k}\right)$, ... where $\epsilon$

is chosen such that $0 < \epsilon < 1 - \alpha$. It is then obvious that any infinite run in $\mathcal{B}$ that reads $w$ stabilizes after $n_0 = |u_1|$ steps (because any infinite run which reads $w$ starts with a prefix reading $w_1$, hence after $|u_1|$ steps, it is in a configuration of $S_{w_1}$).

Towards a contradiction, consider a run $\varrho$ in $\mathcal{B}$ which accepts $w$. The suffix $w_2$ of $w$ satisfies the second condition of Lemma I. Applying the equivalence stated in this lemma, the first condition is also satisfied, and this corresponds to the untimed word $u_2$. The corresponding path in $\mathcal{B}_f$, which starts from some $q \in Q_0$, is accepting because $\varrho$ is accepting, and tags and locations are preserved. Hence it is the case that $\mathcal{B}_f^{\{q\}}$ accepts $u_2$. It is then the case that $\mathcal{B}_f^{Q_0}$ accepts $u_2$, which contradicts the assumption. □

The set $\mathcal{Z}$ is upward-closed, hence we can decide the reachability problem w.r.t. $\mathcal{Z}$ in the well-structured transition system $\mathsf{Abst}_{\mathcal{B}}$ (see for instance [9]). Hence, as a consequence of Proposition L above, we get Theorem 17.

**Remark M** *The construction made in the proof of the above theorem can be adapted to prove that the universality problem for Zeno timed words in one-clock timed automata with a standard Büchi acceptance condition is decidable.*

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