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Relation Liftings on Preorders and Posets *

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Abstract

The category $\mathsf{Rel}(\mathsf{Set})$ of sets and relations can be described as a category of spans and as the Kleisli category for the powerset monad. A set-functor can be lifted to a functor on $\mathsf{Rel}(\mathsf{Set})$ iff it preserves weak pullbacks. We show that these results extend to the enriched setting, if we replace sets by posets or preorders. Preservation of weak pullbacks becomes preservation of exact lax squares. As an application we present Moss's coalgebraic over posets.

Contents

*In terms of results and numbering, the material has appeared in our CALCO 2011 paper of the same title, but some typos were corrected and proofs and a small number of further comments were added.

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1 Introduction

Relation lifting [Ba, CKW, HeJ] plays a crucial role in coalgebraic logic, see eg [Mo, Bal, V].

On the one hand, it is used to explain bisimulation: If $T : \mathsf{Set} \longrightarrow \mathsf{Set}$ is a functor, then the largest bisimulation on a coalgebra $\xi : X \longrightarrow TX$ is the largest fixed point of the operator $(\xi \times \xi)^{-1} \circ \overline{T}$ on relations on X, where $\overline{T}$ is the lifting of T to $\mathsf{Rel}(\mathsf{Set}) \longrightarrow \mathsf{Rel}(\mathsf{Set})$. (The precise meaning of 'lifting' will be given in the Extension Theorem 5.3.)

On the other hand, Moss's coalgebraic logic [Mo] is given by adding to propositional logic a modal operator ∇, the semantics of which is given by applying $\overline{T}$ to the forcing relation $\Vdash \subseteq X \times \mathcal{L}$, where $\mathcal{L}$ is the set of formulas: If $\alpha \in T(\mathcal{L})$, then $x \Vdash \nabla\alpha \;\Leftrightarrow\; \xi(x)\; \overline{T}(\Vdash)\; \alpha$.

In much the same way as Set-coalgebras capture bisimulation, Pre-coalgebras and Pos-coalgebras capture simulation [R, Wo, HuJ, Kl, L, BK]. This suggests that, in analogy with the Set-based case, a coalgebraic understanding of logics for simulations should derive from the study of Pos-functors together with on the one hand their predicate liftings and on the other hand their ∇-operator. The study of predicate liftings of Pos-functors was begun in [KaKuV], whereas here we lay the foundations for the ∇-operator of a Pos-functor. In order to do this, we start with the notion of monotone relation for the following reason. Let $(X, \leq)$ and $(X', \leq')$ be the carriers of two coalgebras, with the preorders $\leq, \leq'$ encoding the simulation relations on X and X', respectively. Then a simulation between the two systems will be a relation $R \subseteq X \times X'$ such that $\geq ; R ; \geq' \subseteq R$, that is, R is a monotone relation. Similarly, $\Vdash$ will be a monotone relation. To summarise, the relations we are interested in are monotone, which enables us to use techniques of enriched category theory (of which no prior knowledge is assumed of the reader).

For the reasons outlined above, the purpose of the paper is to develop the basic theory of relation liftings over preorders and posets. That is, we replace the category Set of sets and functions by the category Pre of preorders or Pos of posets, both with monotone (i.e. order-preserving) functions. Section 2 introduces notation and shows that (monotone) relations can be presented by spans and by arrows in an appropriate Kleisli-category. Section 3 recalls the notion of exact squares. Section 4 characterises the inclusion of functions into relations $(-)_\diamond : \mathsf{Pre} \longrightarrow \mathsf{Rel}(\mathsf{Pre})$ by a universal property and shows that the relation lifting $\overline{T}$ exists iff T satisfies the Beck-Chevalley-Condition (BCC), which says that T preserves exact squares. The BCC replaces the familiar condition known from $\mathsf{Rel}(\mathsf{Set})$, namely that T preserves weak pullbacks. Section 5 lists examples of functors (not) satisfying the BCC and Section 6 gives the application to Moss's coalgebraic logic over posets.

Related work. The universal property of the embedding of a (regular) category to the category of relations is stated in Theorem 2.3 of [He]. Theorem 4.1 below generalizes this in passing from a category to a simple 2-category of (pre)orders.

Liftings of functors to categories of relations within the realm of regular categories have also been studied in [CKW].

2 Monotone relations

In this section we summarize briefly the notion of monotone relations on preorders and we show that their resulting 2-category can be perceived in two ways:

1. Monotone relations are certain *spans*, called *two-sided discrete fibrations*.

2. Monotone relations form a *Kleisli category* for a certain *KZ doctrine* on the category of preorders.

Definition 2.1. Given preorders $\mathscr{A}$ and $\mathscr{B}$, a *monotone relation R from $\mathscr{A}$ to $\mathscr{B}$*, denoted by

$$\mathscr{A} \xrightarrow{R} \mathrel{\mkern-14mu}\mapstochar\mkern14mu \mathscr{B}$$

is a monotone map $R : \mathscr{B}^{op} \times \mathscr{A} \longrightarrow 2$ where by 2 we denote the two-element poset on $\{0, 1\}$ with $0 \leq 1$.

Remark 2.2. Unravelling the definition: for a binary relation R, $R(b,a)=1$ means that a and b are related by R. Monotonicity of R then means that if $R(b,a)=1$ and $b_1 \leq b$ in $\mathscr{B}$ and $a \leq a_1$ in $\mathscr{A}$, then $R(b_1,a_1)=1$.

Relations compose in the obvious way. Two relations as on the left below

$$\mathscr{A} \xrightarrow{R} \mathscr{B} \qquad \mathscr{B} \xrightarrow{S} \mathscr{C} \qquad\qquad \mathscr{A} \xrightarrow{S \cdot R} \mathscr{C}$$

compose to the relation on the right above by the formula

$$S \cdot R(c,a) = \bigvee_b R(b,a) \wedge S(c,b) \tag{2.1}$$

hence the validity of $S \cdot R(c,a)$ is witnessed by at least one b such that both $R(b,a)$ and $S(c,b)$ hold.

Remark 2.3. The supremum in formula (2.1) is, in fact, exactly a coend in the sense of enriched category theory, see [Ke].

The above composition of relations is associative and it has monotone relations $\mathscr{A} \xrightarrow{\mathscr{A}} \mathscr{A}$ as units, where $\mathscr{A}(a,a')$ holds iff $a \leq a'$. Moreover, the relations can be ordered pointwise: $R \longrightarrow S$ means that $R(b,a)$ entails $S(b,a)$, for every a and b. Hence we have a 2-category of monotone relations $\mathsf{Rel}(\mathsf{Pre})$.

Remark 2.4. Observe that one can form analogously the 2-category $\mathsf{Rel}(\mathsf{Pos})$ of monotone relations on *posets*. In all what follows one can work either with preorders or posets. We will focus on preorders in the rest of the paper, the modifications for posets always being straightforward. Observe that both $\mathsf{Rel}(\mathsf{Pre})$ and $\mathsf{Rel}(\mathsf{Pos})$ have the crucial property: The only isomorphism 2-cells are identities.

Remark. The forgetful functor $V : \mathsf{Pre} \longrightarrow \mathsf{Set}$ extends to a faithful functor $\mathsf{Rel}(V) : \mathsf{Rel}(\mathsf{Pre}) \longrightarrow \mathsf{Rel}(\mathsf{Set})$ where $\mathsf{Rel}(\mathsf{Set})$ is the usual category of sets and relations.

2.A The functor $(-)_\diamond : \mathsf{Pre} \longrightarrow \mathsf{Rel}(\mathsf{Pre})$

We describe now the functor $(-)_\diamond : \mathsf{Pre} \longrightarrow \mathsf{Rel}(\mathsf{Pre})$ and show its main properties. The case of posets is completely analogous. For a monotone map $f : \mathscr{A} \longrightarrow \mathscr{B}$ define two relations

$$\mathscr{A} \xrightarrow{f_\diamond} \mathscr{B} \qquad\qquad \mathscr{B} \xrightarrow{f^\diamond} \mathscr{A}$$

by the formulas $f_\diamond(b,a) = \mathscr{B}(b,fa)$ and $f^\diamond(a,b) = \mathscr{B}(fa,b)$.

Lemma 2.5. *For every $f : \mathscr{A} \longrightarrow \mathscr{B}$ in* Pre *there is an adjunction in* $\mathsf{Rel}(\mathsf{Pre})$

$$f_\diamond \dashv f^\diamond : \mathscr{B} \longrightarrow \mathscr{A} \ .$$

Proof. This is easy: observe that if $\mathscr{A}(a,a')=1$, then

$$f^\diamond \cdot f_\diamond(a,a') = \bigvee_b f^\diamond(a',b) \wedge f_\diamond(b,a) = \bigvee_b \mathscr{B}(fa',b) \wedge \mathscr{B}(b,fa) = \mathscr{B}(fa,fa') = 1$$

since f is a monotone map. Hence $\eta^f : \mathscr{A} \longrightarrow f^\diamond \cdot f_\diamond$ holds.

For the comparison $f_\diamond \cdot f^\diamond \longrightarrow \mathscr{B}$, suppose that

$$f_\diamond \cdot f^\diamond(b,b') = \bigvee_a f_\diamond(b,a) \wedge f^\diamond(a,b') = \bigvee_a \mathscr{B}(b,fa) \wedge \mathscr{B}(fa,b') = 1$$

and use the transitivity of the order on $\mathscr{B}$ to conclude that $\mathscr{B}(b,b')=1$.

It is now easy to show that the triangle equalities

$$
\begin{tikzcd}
f_\diamond \arrow[r, "f_\diamond \eta^f"] \arrow[dr, equal] & f_\diamond \cdot f^\diamond \cdot f_\diamond \arrow[d, "\varepsilon^f f_\diamond"] \\
& f_\diamond
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
f^\diamond \arrow[r, "\eta^f f^\diamond"] \arrow[dr, equal] & f^\diamond \cdot f_\diamond \cdot f^\diamond \arrow[d, "f^\diamond \varepsilon^f"] \\
& f^\diamond
\end{tikzcd}
$$

hold and they witness the adjunction $f_\diamond \dashv f^\diamond$. ☐

Remark 2.6. Left adjoint morphisms in $\mathsf{Rel}(\mathsf{Pre})$ can be characterized as exactly those of the form $f_\diamond$ for some monotone map f. Therefore, if $L \dashv R : \mathscr{B} \relbar\joinrel\mapstochar\joinrel\rightarrow \mathscr{A}$ in $\mathsf{Rel}(\mathsf{Pre})$, then there exists a monotone map $f : \mathscr{A} \longrightarrow \mathscr{B}$ such that $f_\diamond = L$ and $f^\diamond = R$. Moreover, f is uniquely determined by L, R iff $\mathscr{B}$ is a poset.

To prove the claim, denote by $\eta : \mathscr{A} \longrightarrow R \cdot L$ the unit and by $\varepsilon : L \cdot R \longrightarrow \mathscr{B}$ the counit of $L \dashv R$. First we prove that for every a there is a b_0 such that

$$R(a, b_0) \wedge L(b_0, a) = 1$$

and that b_0 is unique up to isomorphism:

1. Due to η there is at least one b such that

   $$R(a, b) \wedge L(b, a) = 1$$

   holds: since $\mathscr{A}(a, a) = 1$, it is the case that $R \cdot L(a, a) = 1$.

2. Suppose that

   $$R(a, b_1) \wedge L(b_1, a) = 1 \quad \text{and} \quad R(a, b_2) \wedge L(b_2, a) = 1$$

   hold. Therefore the equalities

   $$R(a, b_1) \wedge L(b_2, a) = 1 \quad \text{and} \quad R(a, b_2) \wedge L(b_1, a) = 1$$

   hold as well. Then, due to ε, we have that $\mathscr{B}(b_1, b_2) = 1$ and $\mathscr{B}(b_2, b_1) = 1$. In other words, we have $b_1 \leq b_2$ and $b_2 \leq b_1$, that is, $b_1 \cong b_2$ and, if $\mathscr{B}$ is a poset then, using antisymmetry, we conclude that $b_1 = b_2$.

Define $fa = b_0$, which determines f uniquely iff $\mathscr{B}$ is a poset. That the assignment $a \mapsto fa$ is monotone, follows from the existence of η. Finally, we need to prove $L = f_\diamond$, that is, $L(b, a) = \mathscr{B}(b, fa)$ for all b, a. We know $L(fa, a)$ and $R(a, fa)$ by definition of f. Suppose $\mathscr{B}(b, fa)$, then $L(b, a)$ follows by monotonicity of L. Conversely, suppose $L(b, a)$. Using $\varepsilon : L \cdot R \longrightarrow \mathscr{B}$, we have $\bigvee_a L(b, a) \wedge R(a, b') \leq \mathscr{B}(b, b')$ and choosing $b' = fa$, we get $1 = L(b, a) \wedge R(a, fa) \leq \mathscr{B}(b, fa)$. ☐

Observe that if $f \longrightarrow g$, then $f_\diamond \longrightarrow g_\diamond$ holds. For if $\mathscr{B}(b, fa) = 1$ then $\mathscr{B}(b, ga) = 1$ holds by transitivity, since $fa \leq ga$ holds. Moreover, taking the lower diamond clearly maps an identity monotone map $id_{\mathscr{A}} : \mathscr{A} \longrightarrow \mathscr{A}$ to the identity monotone relation $\mathscr{A} \overset{\mathscr{A} = (id_{\mathscr{A}})_\diamond}{\relbar\joinrel\relbar\joinrel\mapstochar\joinrel\relbar\joinrel\rightarrow} \mathscr{A}$. Further, taking the lower diamond preserves composition:

$$(g \cdot f)_\diamond(c, a) = \mathscr{C}(c, gfa) = \bigvee_b \mathscr{C}(c, gb) \wedge \mathscr{B}(b, fa) = g_\diamond \cdot f_\diamond(c, a)$$

Hence we have a functor $(-)_\diamond : \mathsf{Pre} \longrightarrow \mathsf{Rel}(\mathsf{Pre})$ enriched in preorders. Moreover, $(-)_\diamond$ is *locally fully faithful*, i.e., $f_\diamond \longrightarrow g_\diamond$ holds iff $f \longrightarrow g$ holds.

2.B $\mathsf{Rel}(\mathsf{Pre})$ as a Kleisli category

The 2-functor $(-)_\diamond : \mathsf{Pre} \longrightarrow \mathsf{Rel}(\mathsf{Pre})$ is a *proarrow equipment with power objects* in the sense of Section 2.5 [MRW]. This means that $(-)_\diamond$ has a right adjoint $(-)^\dagger$ such that the resulting 2-monad on Pre is a KZ doctrine and $\mathsf{Rel}(\mathsf{Pre})$ is (up to equivalence) the corresponding Kleisli 2-category. All of the following results are proved in the paper [MRW], we summarize it here for further reference.

The 2-functor $(-)^\dagger$ works as follows:

1. On objects, $\mathscr{A}^\dagger = [\mathscr{A}^{op}, \mathbb{2}]$, the lowersets on $\mathscr{A}$, ordered by inclusion.
2. For a relation R from $\mathscr{A}$ to $\mathscr{B}$, the functor $R^\dagger : [\mathscr{A}^{op}, \mathbb{2}] \longrightarrow [\mathscr{B}^{op}, \mathbb{2}]$ is defined as the left Kan extension of $a \mapsto R(-, a)$ along the Yoneda embedding $\mathbb{y}_{\mathscr{A}} : \mathscr{A} \longrightarrow [\mathscr{A}^{op}, \mathbb{2}]$. This can be expressed by the formula:
$$R^\dagger(W) = b \mapsto \bigvee_a Wa \wedge R(b, a)$$
i.e., b is in the lowerset $R^\dagger(W)$ iff there exists a in W such that $R(b, a)$ holds.

It is easy to prove that $(-)^\dagger$ is a 2-functor and that $(-)^\dagger \dashv (-)_\diamond$ is a 2-adjunction of a KZ type. The latter means that if we denote by
$$(\mathbb{L}, \mathbb{y}, \mathbb{m}) \tag{2.2}$$
the resulting 2-monad on Pre, then we obtain the string of adjunctions $\mathbb{L}(\mathbb{y}_{\mathscr{A}}) \dashv \mathbb{m}_{\mathscr{A}} \dashv \mathbb{y}_{\mathbb{L}\mathscr{A}}$, see [M$_1$], [M$_2$], for more details.

The unit of the above KZ doctrine is the Yoneda embedding $\mathbb{y}_{\mathscr{A}} : \mathscr{A} \longrightarrow [\mathscr{A}^{op}, \mathbb{2}]$ and the multiplication $\mathbb{m}_A : [[\mathscr{A}^{op}, \mathbb{2}]^{op}, \mathbb{2}] \longrightarrow [\mathscr{A}^{op}, \mathbb{2}]$ is the left Kan extension of identity on $[\mathscr{A}^{op}, \mathbb{2}]$ along $\mathbb{y}_{[\mathscr{A}^{op}, \mathbb{2}]}$. In more detail:
$$\mathbb{m}_{\mathscr{A}}(\mathscr{W}) = a \mapsto \bigvee_W \mathscr{W}(W) \wedge W(a)$$
where $\mathscr{W}$ is in $[[\mathscr{A}^{op}, \mathbb{2}]^{op}, \mathbb{2}]$ and W is in $[\mathscr{A}^{op}, \mathbb{2}]$. Hence a is in the lowerset $\mathbb{m}_{\mathscr{A}}(\mathscr{W})$ iff there exists a lowerset W in $\mathscr{W}$ such that a is in W. The following result is proved in Section 2.5 of [MRW]:

Proposition 2.7. *The 2-functor* $(-)_\diamond : \mathsf{Pre} \longrightarrow \mathsf{Rel}(\mathsf{Pre})$ *exhibits* $\mathsf{Rel}(\mathsf{Pre})$ *as a Kleisli category for the KZ doctrine* $(\mathbb{L}, \mathbb{y}, \mathbb{m})$.

2.C Relations as spans

Monotone relations are going to be exactly certain spans, called *two-sided discrete fibrations* [S$_4$].

Definition 2.8. A *span* $(d_0, \mathscr{E}, d_1) : \mathscr{B} \longrightarrow \mathscr{A}$ from $\mathscr{B}$ to $\mathscr{A}$ is a diagram
$$\begin{tikzcd} & \mathscr{E} \arrow[dl, "d_0"'] \arrow[dr, "d_1"] & \\ \mathscr{A} & & \mathscr{B} \end{tikzcd}$$
of monotone maps. The preorder $\mathscr{E}$ is called the *vertex* of the span $(d_0, \mathscr{E}, d_1)$.

Remark 2.9. Given a span $(d_0, \mathscr{E}, d_1) : \mathscr{B} \longrightarrow \mathscr{A}$, the following intuitive notation might prove useful: a typical element of $\mathscr{E}$ will be denoted by a wiggly arrow
$$d_0(e) \overset{e}{\rightsquigarrow} d_1(e)$$
and $d_0(e)$ will be the *domain* of e and $d_1(e)$ the *codomain* of e.

Definition 2.10. A span $(d_0, \mathscr{E}, d_1) : \mathscr{B} \longrightarrow \mathscr{A}$ in Pre is a *two-sided discrete fibration* (we will say just *fibration* in what follows), if the following three conditions are satisfied. For every situation below on the left, there is a unique fill in on the right, denoted by $(d_0)_*(e')$, respectively $(d_1)_*(e)$:

$$
\begin{tikzcd}
a \arrow[d] & \\
a' \arrow[r, rightsquigarrow, "e'"'] & b'
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
a \arrow[r, rightsquigarrow, "(d_0)_*(e')"] \arrow[d] & b' \arrow[d, equal] \\
a' \arrow[r, rightsquigarrow, "e'"'] & b'
\end{tikzcd}
$$

$$
\begin{tikzcd}
a \arrow[r, rightsquigarrow, "e"] & b \arrow[d] \\
 & b'
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
a \arrow[r, rightsquigarrow, "e"] \arrow[d, equal] & b \arrow[d] \\
a \arrow[r, rightsquigarrow, "(d_1)_*(e)"'] & b'
\end{tikzcd}
$$

Every situation on the left can be written as depicted on the right:

$$
\begin{tikzcd}
a \arrow[r, rightsquigarrow, "e"] \arrow[d] & b \arrow[d] \\
a' \arrow[r, rightsquigarrow, "e'"'] & b'
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
a \arrow[r, rightsquigarrow, "e"] \arrow[d, equal] & b \arrow[d] \\
a \arrow[r, rightsquigarrow] \arrow[d] & b' \arrow[d, equal] \\
a' \arrow[r, rightsquigarrow, "e'"'] & b'
\end{tikzcd}
$$

Remark. Fibrations are jointly mono. In particular, if $\mathscr{B}, \mathscr{A}$ are discrete then $(d_0, \mathscr{E}, d_1) : \mathscr{B} \longrightarrow \mathscr{A}$ is a fibration iff it is a jointly mono.

Definition 2.11. A *comma object* of monotone maps $f : \mathscr{A} \longrightarrow \mathscr{C}$, $g : \mathscr{B} \longrightarrow \mathscr{C}$ is a diagram

$$
\begin{tikzcd}
f/g \arrow[r, "p_1"] \arrow[d, "p_0"'] \arrow[dr, phantom, "\nearrow"] & \mathscr{B} \arrow[d, "g"] \\
\mathscr{A} \arrow[r, "f"'] & \mathscr{C}
\end{tikzcd}
$$

where elements of the preorder f/g are pairs (a, b) with $f(a) \leq g(b)$ in $\mathscr{C}$, the preorder on f/g is defined pointwise and p_0 and p_1 are the projections. The whole "lax commutative square" as above will be called a *comma square*.

Example 2.12. Every span $(p_0, f/g, p_1) : \mathscr{A} \longrightarrow \mathscr{B}$ arising from a comma object of $f : \mathscr{A} \longrightarrow \mathscr{C}$, $g : \mathscr{B} \longrightarrow \mathscr{C}$ is a fibration.

A monotone relation $\mathscr{B} \stackrel{R}{\relbar\joinrel\mapstochar\joinrel\rightarrow} \mathscr{A}$ induces a fibration $(d_0, \mathscr{E}, d_1) : \mathscr{B} \longrightarrow \mathscr{A}$ with $\mathscr{E} = \{(a, b) \mid R(a, b) = 1\}$ ordered by $(a, b) \leq (a', b')$, if $a \leq a'$ and $b \leq b'$; and $(d_0, \mathscr{E}, d_1)$ induces the relation $R(a, b) = 1 \;\Leftrightarrow\; \exists e \in \mathscr{E} \,.\, d_0(e) = a, d_1(e) = b$.

Proposition 2.13. *Fibrations in* Pre *correspond exactly to monotone relations. Moreover, if* $(d_0, \mathscr{E}, d_1) : \mathscr{B} \longrightarrow \mathscr{A}$ *is the fibration corresponding to a relation* $R : \mathscr{B} \relbar\joinrel\mapstochar\joinrel\rightarrow \mathscr{A}$, *then* $R = (d_0)_\diamond \cdot (d_1)^\diamond$.

Proof. This is seen by the following *Grothendieck construction*:

1. Given a relation $R : \mathscr{A}^{op} \times \mathscr{B} \longrightarrow 2$, define the span $(d_0, \mathscr{E}, d_1) : \mathscr{B} \longrightarrow \mathscr{A}$ as follows:

(a) Objects of $\mathscr{E}$ are pairs (a,b), where a and b are objects of $\mathscr{A}$ and $\mathscr{B}$, respectively, with $R(a,b)=1$. A typical object is going to be denoted by

$$\begin{tikzcd} a \arrow[r, squiggly, "{(a,b)}"] & b \end{tikzcd}$$

(b) The preorder relation on $\mathscr{E}$: we put $(a,b) \leq (a',b')$, if $a \leq a'$, $b \leq b'$ in $\mathscr{A}$, $\mathscr{B}$, respectively. Diagrammatically:

$$\begin{tikzcd} a \arrow[r, squiggly, "{(a,b)}"] \arrow[d] & b \arrow[d] \\ a' \arrow[r, squiggly, "{(a',b')}"'] & b' \end{tikzcd}$$

(where we write, e.g., $a \longrightarrow a'$ to denote $a \leq a'$).

(c) The monotone maps $d_0 : \mathscr{E} \longrightarrow \mathscr{A}$ and $d_1 : \mathscr{E} \longrightarrow \mathscr{B}$ are then the obvious domain and codomain projections.

We verify now that $(d_0, \mathscr{E}, d_1)$ is a fibration.

(a) Suppose

$$\begin{tikzcd} a \arrow[d] & \\ a' \arrow[r, squiggly, "{(a',b')}"'] & b' \end{tikzcd}$$

is given. We define the cartesian lift as follows:

$$\begin{tikzcd} a \arrow[r, squiggly, "{(a,b')}"] \arrow[d] & b' \arrow[d, equal] \\ a' \arrow[r, squiggly, "{(a',b')}"'] & b' \end{tikzcd}$$

Here we have used the fact that R is monotone.

(b) Given

$$\begin{tikzcd} a \arrow[r, squiggly, "{(a,b)}"] & b \arrow[d] \\ & b' \end{tikzcd}$$

and $g : b \longrightarrow b'$, proceed analogously to the above: define the unique opcartesian lift as follows

$$\begin{tikzcd} a \arrow[r, squiggly, "{(a,b)}"] \arrow[d, equal] & b \arrow[d] \\ a \arrow[r, squiggly, "{(a,b')}"'] & b' \end{tikzcd}$$

(c) Suppose we are given a morphism

$$\begin{tikzcd} a \arrow[r, squiggly, "{(a,b)}"] \arrow[d] & b \arrow[d] \\ a' \arrow[r, squiggly, "{(a',b')}"'] & b' \end{tikzcd}$$

in $\mathscr{E}$. Then it is straightforward to see that it is equal to the composite

$$\begin{tikzcd}
a \arrow[r, squiggly, "{(a,b)}"] \arrow[d, equal] \arrow[dd, bend right=60] & b \arrow[d] \arrow[dd, bend left=60] \\
a \arrow[r, squiggly, "{(a,b')}"] \arrow[d] & b' \arrow[d, equal] \\
a' \arrow[r, squiggly, "{(a',b')}"'] & b'
\end{tikzcd}$$

2. Given a fibration $(d_0, \mathscr{E}, d_1) : \mathscr{B} \longrightarrow \mathscr{A}$, consider the following definition

   $$R(a, b) = 1 \quad \text{iff} \quad \text{there is } e \text{ in } \mathscr{E} \text{ with } d_0(e) = a \text{ and } d_1(e) = b$$

   That the assignment $(a, b) \mapsto R(a, b)$ gives a monotone map

   $$R : \mathscr{A}^{op} \times \mathscr{B} \longrightarrow 2$$

   is taken care of by the three conditions of Definition 2.10. In other words, we have obtained a relation from $\mathscr{B}$ to $\mathscr{A}$.

$\square$

Corollary. If $(d_0, \mathscr{E}, d_1) : \mathscr{B} \longrightarrow \mathscr{A}$ is the fibration corresponding $R : \mathscr{B} \longrightarrow\!\!\!\!\!\!\!\!\mapsto \mathscr{A}$, then $\mathsf{Rel}(V)R = \mathsf{Rel}(V)((d_0)_\diamond \cdot (d_1)^\diamond) = (V d_0)_\diamond \cdot (V d_1)^\diamond$.

Proof. On the left we have that $\mathsf{Rel}(V)((d_0)_\diamond \cdot (d_1)^\diamond)(b, a) = 1$ iff there is $w \in \mathscr{E}$ such that $b \le d_0(w)$ and $d_1(w) \le a$. On the right we have that $((V d_0)_\diamond \cdot (V d_1)^\diamond)(b, a) = 1$ iff there is $w \in \mathscr{E}$ such that $b = d_0(w)$ and $d_1(w) = a$. Since $(d_0, \mathscr{E}, d_1)$ is a fibration the two conditions are equivalent. $\square$

Remark 2.14. The proposition can be extended to any category enriched in Pre. The details are as follows. A span $(d_0, \mathscr{E}, d_1) : \mathscr{B} \longrightarrow \mathscr{A}$ in Pre is a *two-sided discrete fibration*, if the following three conditions are satisfied:

1. For each $m : \mathscr{K} \longrightarrow \mathscr{E}$, $a, a' : \mathscr{K} \longrightarrow \mathscr{A}$, $b : \mathscr{K} \longrightarrow \mathscr{B}$ and $\alpha : a' \longrightarrow a$ such that triangles

   $$\begin{tikzcd}
   \mathscr{K} \arrow[r, "m"] \arrow[dr, "a"'] & \mathscr{E} \arrow[d, "d_0"] \\
   & \mathscr{A}
   \end{tikzcd}
   \qquad\qquad
   \begin{tikzcd}
   \mathscr{K} \arrow[r, "m"] \arrow[dr, "b"'] & \mathscr{E} \arrow[d, "d_1"] \\
   & \mathscr{B}
   \end{tikzcd}$$

   commute, there is a unique $\bar{m} : \mathscr{K} \longrightarrow \mathscr{E}$ and a unique $d_0^*(\alpha) : \bar{m} \longrightarrow m$ such that

   $$\begin{tikzcd}
   \mathscr{K} \arrow[r, "\bar{m}"] \arrow[dr, "a'"'] & \mathscr{E} \arrow[d, "d_0"] \\
   & \mathscr{A}
   \end{tikzcd}
   \qquad\qquad
   \begin{tikzcd}
   \mathscr{K} \arrow[r, "\bar{m}"] \arrow[dr, "b"'] & \mathscr{E} \arrow[d, "d_1"] \\
   & \mathscr{B}
   \end{tikzcd}$$

   and

   $$\begin{tikzcd}
   \mathscr{K} \arrow[r, shift left=2, "\bar{m}"] \arrow[r, shift right=2, "m"'] \arrow[r, phantom, "\downarrow d_0^*(\alpha)"] & \mathscr{E} \arrow[r, "d_0"] & \mathscr{A}
   \end{tikzcd}
   \quad = \quad
   \begin{tikzcd}
   \mathscr{K} \arrow[r, shift left=2, "a'"] \arrow[r, shift right=2, "a"'] \arrow[r, phantom, "\downarrow\alpha"] & \mathscr{A}
   \end{tikzcd}$$

   $$\begin{tikzcd}
   \mathscr{K} \arrow[r, shift left=2, "\bar{m}"] \arrow[r, shift right=2, "m"'] \arrow[r, phantom, "\downarrow d_0^*(\alpha)"] & \mathscr{E} \arrow[r, "d_1"] & \mathscr{B}
   \end{tikzcd}
   \quad = \quad
   \begin{tikzcd}
   \mathscr{K} \arrow[r, "b"] & \mathscr{B}
   \end{tikzcd}$$

   commute. The 2-cell $d_0^*(\alpha)$ is called the *cartesian lift* of α.

2. For each $m : \mathscr{K} \longrightarrow \mathscr{E}$, $a : \mathscr{K} \longrightarrow \mathscr{A}$, $b, b' : \mathscr{K} \longrightarrow \mathscr{B}$ and $\beta : b \longrightarrow b'$ such that triangles

$$\begin{tikzcd} \mathscr{K} \arrow[r, "m"] \arrow[dr, "a"'] & \mathscr{E} \arrow[d, "d_0"] \\ & \mathscr{A} \end{tikzcd} \qquad \begin{tikzcd} \mathscr{K} \arrow[r, "m"] \arrow[dr, "b"'] & \mathscr{E} \arrow[d, "d_1"] \\ & \mathscr{B} \end{tikzcd}$$

commute, there is a unique $\bar{m} : \mathscr{K} \longrightarrow \mathscr{E}$ and a unique $d_1^*(\beta) : m \Rightarrow \bar{m}$ such that

$$\begin{tikzcd} \mathscr{K} \arrow[r, "\bar{m}"] \arrow[dr, "a"'] & \mathscr{E} \arrow[d, "d_0"] \\ & \mathscr{A} \end{tikzcd} \qquad \begin{tikzcd} \mathscr{K} \arrow[r, "\bar{m}"] \arrow[dr, "b'"'] & \mathscr{E} \arrow[d, "d_1"] \\ & \mathscr{B} \end{tikzcd}$$

and

$$\begin{tikzcd} \mathscr{K} \arrow[r, bend left, "m"] \arrow[r, bend right, "\bar{m}"'] \arrow[r, phantom, "\Downarrow d_1^*(\beta)"] & \mathscr{E} \arrow[r, "d_0"] & \mathscr{A} \end{tikzcd} \quad = \quad \begin{tikzcd} \mathscr{K} \arrow[r, "a"] & \mathscr{A} \end{tikzcd}$$

$$\begin{tikzcd} \mathscr{K} \arrow[r, bend left, "m"] \arrow[r, bend right, "\bar{m}"'] \arrow[r, phantom, "\Downarrow d_1^*(\beta)"] & \mathscr{E} \arrow[r, "d_1"] & \mathscr{B} \end{tikzcd} \quad = \quad \begin{tikzcd} \mathscr{K} \arrow[r, bend left, "b"] \arrow[r, bend right, "b'"'] \arrow[r, phantom, "\Downarrow \beta"] & \mathscr{B} \end{tikzcd}$$

commute. The 2-cell $d_1^*(\beta)$ is called the *opcartesian lift* of β.

3. Given any $\sigma : m \Rightarrow m' : K \longrightarrow E$, then the composite $d_0^*(d_0\sigma) \cdot d_1^*(d_1\sigma)$ is defined and it is equal to σ.

The easiest way of treating fibrations abstractly is that they are *algebras* for two (2-)monads simultaneously: they are *two-sided modules* in a certain precise sense. See [S_2] and [S_4].

Example 2.15. Suppose that $f : \mathscr{A} \longrightarrow \mathscr{B}$ is monotone. Recall the relations $f_\diamond : \mathscr{A} \nrightarrow \mathscr{B}$ and $f^\diamond : \mathscr{B} \nrightarrow \mathscr{A}$. Their corresponding fibrations are the spans

$$\begin{tikzcd} & id_{\mathscr{B}}/f \arrow[dl, "p_0"'] \arrow[dr, "p_1"] & \\ \mathscr{B} & & \mathscr{A} \end{tikzcd} \qquad \begin{tikzcd} & f/id_{\mathscr{B}} \arrow[dl, "p_0"'] \arrow[dr, "p_1"] & \\ \mathscr{A} & & \mathscr{B} \end{tikzcd}$$

arising from the respective comma squares.

Example 2.16. The relation $(\mathbb{y}_{\mathscr{A}})^\diamond$ from $\mathbb{L}\mathscr{A}$ to $\mathscr{A}$ will be called the *elementhood* relation and denoted by $\in_{\mathscr{A}}$, since $(\mathbb{y}_{\mathscr{A}})^\diamond(a, A) = \mathbb{L}\mathscr{A}(\mathbb{y}_{\mathscr{A}}a, A) = A(a)$ holds by the Yoneda Lemma.

2.D Composition of fibrations

Suppose that we have two fibrations as on the left below. We want to form their composite $\mathscr{E} \otimes \mathscr{F}$ *as a fibration*.

$$\begin{tikzcd} & \mathscr{E} \arrow[dl, "d_0^{\mathscr{E}}"'] \arrow[dr, "d_1^{\mathscr{F}}"] & \\ \mathscr{C} & & \mathscr{B} \end{tikzcd} \quad \begin{tikzcd} & \mathscr{F} \arrow[dl, "d_0^{\mathscr{F}}"'] \arrow[dr, "d_1^{\mathscr{F}}"] & \\ \mathscr{B} & & \mathscr{A} \end{tikzcd} \qquad \begin{tikzcd} & \mathscr{E} \otimes \mathscr{F} \arrow[dl, "d_0^{\mathscr{E}\otimes\mathscr{F}}"'] \arrow[dr, "d_1^{\mathscr{E}\otimes\mathscr{F}}"] & \\ \mathscr{C} & & \mathscr{A} \end{tikzcd}$$

The idea is similar to the ordinary relations: the composite is going to be a quotient of a pullback of spans, this time the quotient will be taken by a map that is surjective on objects, hence *absolutely dense*.

Remark 2.17. A monotone map $e : \mathscr{A} \longrightarrow \mathscr{B}$ is called *absolutely dense* (see [ABSV] and [BV]) iff

$$\mathscr{B}(b, b') = \bigvee_a \mathscr{B}(b, ea) \wedge \mathscr{B}(ea, b'),$$

that is, e is absolutely dense iff $e_\diamond \cdot e^\diamond = id$. Clearly, every monotone map surjective on objects is absolutely dense. The converse is true if $\mathscr{B}$ is a poset. If $\mathscr{B}$ is a preorder, then e is absolutely dense when each strongly connected component of $\mathscr{B}$ contains at least one element in the image of e.

In defining the composition of fibrations we proceed as follows: construct the pullback

$$\begin{tikzcd}
\mathscr{E} \circ \mathscr{F} \arrow[r, "q_1"] \arrow[d, "q_0"'] & \mathscr{F} \arrow[d, "d_0^{\mathscr{F}}"] \\
\mathscr{E} \arrow[r, "d_1^{\mathscr{E}}"'] & \mathscr{B}
\end{tikzcd}$$

and define $\mathscr{E} \otimes \mathscr{F}$ to be the following preorder:

1. Objects are wiggly arrows of the form $c \rightsquigarrow a$ such that there exists $b \in \mathscr{B}$ with $(c \rightsquigarrow b, b \rightsquigarrow a) \in \mathscr{E} \circ \mathscr{F}$.
2. Put $c \rightsquigarrow a$ to be less or equal to $c' \rightsquigarrow a'$ iff $c \leq c'$ and $a \leq a'$.

Define a monotone map $w : \mathscr{E} \circ \mathscr{F} \longrightarrow \mathscr{E} \otimes \mathscr{F}$ in the obvious way and observe that it is surjective on objects and, hence, absolutely dense.

We equip now $\mathscr{E} \otimes \mathscr{F}$ with the obvious projections $d_0^{\mathscr{E}\otimes\mathscr{F}} : \mathscr{E} \otimes \mathscr{F} \longrightarrow \mathscr{C}$ and $d_1^{\mathscr{E}\otimes\mathscr{F}} : \mathscr{E} \otimes \mathscr{F} \longrightarrow \mathscr{A}$. Then the following result is immediate.

Lemma 2.18. *The span $(d_0^{\mathscr{E}\otimes\mathscr{F}}, \mathscr{E} \otimes \mathscr{F}, d_1^{\mathscr{E}\otimes\mathscr{F}}) : \mathscr{A} \longrightarrow \mathscr{C}$ is a fibration.*

To summarize, we have

Proposition. Let S, R be monotone relations with associated fibrations $\mathscr{E}, \mathscr{F}$. Then the relation associated with $\mathscr{E} \otimes \mathscr{F}$ is $S \cdot R$, that is, we can write $\mathscr{E}^{S \cdot R} = \mathscr{E}^S \otimes \mathscr{E}^R$.

3 Exact squares

The notion of *exact squares* replaces the notion of weak pullbacks in the preorder setting and exact squares will play a central rôle in our extension theorem. Exact squares were introduced and studied by René Guitart in [Gu].

Definition 3.1. A lax square in Pre

$$\begin{tikzcd}
\mathscr{P} \arrow[r, "p_1"] \arrow[d, "p_0"'] \arrow[dr, phantom, "\nearrow"] & \mathscr{B} \arrow[d, "g"] \\
\mathscr{A} \arrow[r, "f"'] & \mathscr{C}
\end{tikzcd} \tag{3.3}$$

is *exact* iff the canonical comparison in $\mathsf{Rel}(\mathsf{Pre})$ below is an iso (identity).

$$\begin{tikzcd}
\mathscr{P} \arrow[d, "(p_0)_\diamond"', "|" marking] \arrow[dr, phantom, "\searrow"] & \mathscr{B} \arrow[l, "(p_1)^\diamond"', "|" marking] \arrow[d, "g_\diamond", "|" marking] \\
\mathscr{A} & \mathscr{C} \arrow[l, "f^\diamond", "|" marking]
\end{tikzcd} \tag{3.4}$$

Remark 3.2. In defining the canonical comparison, we use the adjunctions $(p_1)_\diamond \dashv (p_1)^\diamond$ and $f_\diamond \dashv f^\diamond$ guaranteed by Lemma 2.5.

Using the formula (2.1) we obtain an equivalent criterion for exactness namely that

$$\mathscr{C}(fa, gb) = \bigvee_w \mathscr{A}(a, p_0 w) \wedge \mathscr{B}(p_1 w, b) \tag{3.5}$$

Example 3.3. We give examples of exact squares in Pre. They all come from Guitart's paper [Gu], Example 1.14. The proofs follow immediately from the description (3.5) above.

1. The square

$$\begin{tikzcd} \mathscr{A} \arrow[r, "f"] \arrow[d, "1_{\mathscr{A}}"'] \arrow[dr, phantom, "\nearrow"] & \mathscr{B} \arrow[d, "1_{\mathscr{B}}"] \\ \mathscr{A} \arrow[r, "f"'] & \mathscr{B} \end{tikzcd}$$

   where the comparison is identity, is always exact since

$$\mathscr{B}(fa, b) = \bigvee_w \mathscr{A}(a, w) \wedge \mathscr{B}(fw, b)$$

   holds by the Yoneda Lemma. Such a square is called a *Yoneda square* in [Gu].

2. The square

$$\begin{tikzcd} \mathscr{A} \arrow[r, "1_{\mathscr{A}}"] \arrow[d, "f"'] \arrow[dr, phantom, "\nearrow"] & \mathscr{A} \arrow[d, "f"] \\ \mathscr{B} \arrow[r, "1_{\mathscr{B}}"'] & \mathscr{B} \end{tikzcd}$$

   where the comparison is identity, is always exact since

$$\mathscr{B}(b, fa) = \bigvee_w \mathscr{B}(b, fw) \wedge \mathscr{A}(w, a)$$

   holds by the Yoneda Lemma. Again, squares of this form are called *Yoneda squares* in [Gu].

3. Every *comma square*

$$\begin{tikzcd} f/g \arrow[r, "d_1"] \arrow[d, "d_0"'] \arrow[dr, phantom, "\nearrow"] & \mathscr{B} \arrow[d, "g"] \\ \mathscr{A} \arrow[r, "f"'] & \mathscr{C} \end{tikzcd}$$

   is exact.

4. Every *op-comma square*

$$\begin{tikzcd} \mathscr{C} \arrow[r, "g"] \arrow[d, "f"'] \arrow[dr, phantom, "\nearrow"] & \mathscr{B} \arrow[d, "i_1"] \\ \mathscr{A} \arrow[r, "i_0"'] & f \triangleright g \end{tikzcd}$$

   is exact.

5. The square

$$\begin{tikzcd} \mathscr{A} \arrow[r, "1_{\mathscr{A}}"] \arrow[d, "1_{\mathscr{A}}"'] \arrow[dr, phantom, "\nearrow"] & \mathscr{A} \arrow[d, "f"] \\ \mathscr{A} \arrow[r, "f"'] & \mathscr{B} \end{tikzcd}$$

   (where the comparison is identity) is exact iff f is an *order-embedding*, i.e., iff the following holds: $fa \leq fa'$ iff $a \leq a'$.

   Such f's can also be called *fully faithful*.

6. The square

$$\begin{tikzcd} \mathscr{A} \arrow[r, "e"] \arrow[d, "e"'] \arrow[dr, phantom, "\nearrow"] & \mathscr{B} \arrow[d, "1_{\mathscr{B}}"] \\ \mathscr{B} \arrow[r, "1_{\mathscr{B}}"'] & \mathscr{B} \end{tikzcd}$$

(where the comparison is identity) is exact iff e is *absolutely dense*, i.e., iff

$$\mathscr{B}(b, b') = \bigvee_a \mathscr{B}(b, ea) \wedge \mathscr{B}(ea, b').$$

See, e.g., [ABSV] and [BV] for more details on absolutely dense maps.

7. The square

$$\begin{tikzcd} \mathscr{X} \arrow[r, "f"] \arrow[d, "1_{\mathscr{X}}"'] \arrow[dr, phantom, "\nearrow"] & \mathscr{A} \arrow[d, "u"] \\ \mathscr{X} \arrow[r, "1_{\mathscr{X}}"'] & \mathscr{X} \end{tikzcd}$$

is exact iff $f \dashv u : \mathscr{A} \longrightarrow \mathscr{X}$ holds. Moreover, the comparison in the above square is the unit of $f \dashv u$.

8. The square

$$\begin{tikzcd} \mathscr{A} \arrow[r, "1_{\mathscr{A}}"] \arrow[d, "u"'] \arrow[dr, phantom, "\nearrow"] & \mathscr{A} \arrow[d, "1_{\mathscr{A}}"] \\ \mathscr{X} \arrow[r, "f"'] & \mathscr{A} \end{tikzcd}$$

is exact iff $f \dashv u : \mathscr{A} \longrightarrow \mathscr{X}$ holds. Moreover, the comparison in the above square is the counit of $f \dashv u$.

9. The square

$$\begin{tikzcd} \mathscr{X}' \arrow[r, "f"] \arrow[d, "1_{\mathscr{X}'}"'] \arrow[dr, phantom, "\nearrow"] & \mathscr{A} \arrow[d, "u"] \\ \mathscr{X}' \arrow[r, "j"'] & \mathscr{X} \end{tikzcd}$$

is exact iff $f \dashv_j u : \mathscr{A} \longrightarrow \mathscr{X}$ holds, i.e., iff f is a left adjoint of u *relative to* j.

In general, relative adjointness means the existence of an isomorphism

$$\mathscr{X}(jx', ua) \cong \mathscr{A}(fx', a)$$

natural in x' and a, and due to

$$\mathscr{A}(fx', a) \cong \bigvee_w \mathscr{X}'(w, x') \wedge \mathscr{A}(fw, a)$$

this means precisely the exactness of the above square.

10. The square

$$\begin{tikzcd} \mathscr{A} \arrow[r, "j"] \arrow[d, "h"'] \arrow[dr, phantom, "\nearrow"] & \mathscr{B} \arrow[d, "l"] \\ \mathscr{X} \arrow[r, "1_{\mathscr{X}}"'] & \mathscr{X} \end{tikzcd}$$

is exact iff the comparison exhibits l as an *absolute* left Kan extension of h along j. In fact,

$$\mathscr{X}(x, lb) = \bigvee_a \mathscr{X}(x, ha) \wedge \mathscr{B}(ja, b)$$

asserts precisely that

(a) l is a left Kan extension of h along j.
For any $k : \mathscr{B} \longrightarrow \mathscr{X}$ we need to prove $l \longrightarrow k$ iff $h \longrightarrow k \cdot j$.
  i. Suppose $lb \leq kb$ for all b. Choose any a. Then $ha \leq lja$ by the square above. Since $lja \leq kja$ by assumption, hence $ha \leq kja$.
  ii. Suppose $ha \leq kja$ for all a. To prove $lb \leq kb$ for all b, it suffices to prove that $x \leq lb$ implies $x \leq kb$, for all x. Suppose $x \leq lb$, i.e., $\mathscr{X}(x, lb) = 1$. Hence $\bigvee_a \mathscr{X}(x, ha) \wedge \mathscr{B}(ja, b) = 1$. Choose a to witness $x \leq ha$ and $ja \leq b$. From our assumption we obtain $x \leq kja$, hence $x \leq kb$.

(b) l is an absolute left Kan extension of h along j.
We need to prove that for any $f : \mathscr{X} \longrightarrow \mathscr{X}'$, $f \cdot l$ is a left Kan extension of $f \cdot h$ along j. That is, for any $k : \mathscr{B} \longrightarrow \mathscr{X}'$ we need to prove $f \cdot l \longrightarrow k$ iff $f \cdot h \longrightarrow k \cdot j$.
This is proved in the same manner as above.

Observe that item 7 above is a special case of absolute Kan extensions by Bénabou's Theorem: $f \dashv u$ holds if the unit exhibits u as an absolute left Kan extension of identity along f.

Example 3.4. Every square (3.3) where f and p_1 are *left* adjoints, is exact iff $p_0 \cdot p_1^r = f^r \cdot g$, where we denote by f^r and p_1^r the respective right adjoints.

This is proved as follows. Firstly, the comparison $f \cdot p_0 \longrightarrow g \cdot p_1$ is equivalent to the comparison $p_0 \cdot p_1^r \longrightarrow f^r \cdot g$ due to adjunctions $f \dashv f^r$ and $p_1 \dashv p_1^r$. Further, we have

$$\bigvee_w \mathscr{A}(a, p_0 w) \wedge \mathscr{B}(p_1 w, b) = \bigvee_w \mathscr{A}(a, p_0 w) \wedge \mathscr{P}(w, p_1^r b) = \mathscr{A}(a, p_0 p_1^r b)$$

and

$$\mathscr{C}(fa, gb) = \mathscr{A}(a, f^r gb)$$

It follows that the square (3.3) is exact iff

$$\mathscr{A}(a, p_0 p_1^r b) = \mathscr{A}(a, f^r gb).$$

By the Yoneda Lemma, this is equivalent to $p_0 \cdot p_1^r = f^r \cdot g$.

Example 3.5. If the square on the left is exact, then so is the square on the right:

$$\begin{tikzcd} \mathscr{P} \arrow[r, "p_1"] \arrow[d, "p_0"'] \arrow[dr, phantom, "\nearrow"] & \mathscr{B} \arrow[d, "g"] \\ \mathscr{A} \arrow[r, "f"'] & \mathscr{C} \end{tikzcd} \qquad\qquad \begin{tikzcd} \mathscr{P}^{op} \arrow[r, "p_0^{op}"] \arrow[d, "p_1^{op}"'] \arrow[dr, phantom, "\nearrow"] & \mathscr{A}^{op} \arrow[d, "f^{op}"] \\ \mathscr{B}^{op} \arrow[r, "g^{op}"'] & \mathscr{C}^{op} \end{tikzcd}$$

To prove the claim, by (3.5), we need

$$\mathscr{C}^{op}(g^{op} b, f^{op} a) = \bigvee_w \mathscr{B}^{op}(b, p_1^{op} w) \wedge \mathscr{A}^{op}(p_0^{op} w, a)$$

But

$$\mathscr{C}^{op}(g^{op} b, f^{op} a) = \mathscr{C}(fa, gb)$$

and
$$\bigvee_w \mathscr{B}^{op}(b, p_1^{op} w) \wedge \mathscr{A}^{op}(p_0^{op} w, a) = \bigvee_w \mathscr{A}(a, p_0 w) \wedge \mathscr{B}(p_1 w, b)$$
and this finishes the proof.

Lemma 3.6. *Suppose that $(d_0^S, \mathscr{E}^S, d_1^S)$ and $(d_0^R, \mathscr{E}^R, d_1^R)$ are two-sided discrete fibrations. Then the pullback*

$$\begin{tikzcd}
\mathscr{E}^S \circ \mathscr{E}^R \arrow[r, "q_1"] \arrow[d, "q_0"'] & \mathscr{E}^R \arrow[d, "d_0^R"] \\
\mathscr{E}^S \arrow[r, "d_1^S"'] & \mathscr{B}
\end{tikzcd}$$

considered as a lax commutative square where the comparison is identity, is exact.

Proof. Suppose that $d_1^S(e) \leq d_0^R(f)$ holds. Then we have a situation
$$c \overset{e}{\rightsquigarrow} b \leq b' \overset{f}{\rightsquigarrow} a$$
and there exists w in $\mathscr{E}^S \circ \mathscr{E}^R$ of the form
$$c \overset{e'}{\rightsquigarrow} b' \overset{f}{\rightsquigarrow} a$$
that clearly satisfies $e \leq p_0(e', f)$ and $p_1(e', f) \leq f$. $\square$

Given monotone relations $\mathscr{A} \overset{R}{\nrightarrow} \mathscr{B}$ and $\mathscr{B} \overset{S}{\nrightarrow} \mathscr{C}$, the two-sided fibration corresponding to the composition $S \cdot R$ is the composition of the fibrations corresponding to S and R as described in Section 2.D. The properties described in the next Corollary are essential for the proof of Theorem 4.1.

Corollary 3.7. *Form, for a pair R, S, of monotone relations the following commutative diagram*

$$\begin{tikzcd}
& & \mathscr{E}^{S\cdot R} \arrow[dddll, bend right, "d_0^{S\cdot R}"'] \arrow[dddrr, bend left, "d_1^{S\cdot R}"] & & \\
& & \mathscr{E}^S \circ \mathscr{E}^R \arrow[u, "w"'] \arrow[dl, "q_0"'] \arrow[dr, "q_1"] & & \\
& \mathscr{E}^S \arrow[dl, "d_0^S"'] \arrow[dr, "d_1^S"] \arrow[rr, phantom, "\longrightarrow"] & & \mathscr{E}^R \arrow[dl, "d_0^R"'] \arrow[dr, "d_1^R"] & \\
\mathscr{C} & & \mathscr{B} & & \mathscr{A}
\end{tikzcd}$$

where the lax commutative square in the middle is a pullback square (hence the comparison is the identity), and w is a map, surjective on objects, coming from composing $\mathscr{E}^S$ and $\mathscr{E}^R$ as fibrations. Then the square is exact and w is an absolutely dense monotone map.

In the extension theorem we will demand that a certain functor $T : \mathsf{Pre} \longrightarrow \mathsf{Pre}$ preserves exact squares, whereas the proof of the theorem actually only needs the at first sight weaker requirement that T preserves strict exact squares and preserves the exactness of comma squares of the form $1_{\mathscr{A}}/1_{\mathscr{A}}$ (the former being needed for preservation of composition and the latter for preservation of identities). It therefore seems of interest to present the following result.

Proposition. For a locally monotone $T : \mathsf{Pre} \longrightarrow \mathsf{Pre}$, or $T : \mathsf{Pos} \longrightarrow \mathsf{Pos}$, the following are equivalent:

1. T preserves *lax* exact squares.

2. T preserves strict exact squares and exactness of comma squares of the form $1_{\mathscr{A}}/1_{\mathscr{A}}$, for all $\mathscr{A}$.

3. T preserves strict exact squares and exactness of comma squares of the form $f/1_{\mathscr{B}}$, $1_{\mathscr{A}}/f$, for all $f : \mathscr{A} \longrightarrow \mathscr{B}$.

4. T preserves strict exact squares and exactness of comma squares.

Proof. (1) implies (2): clear.

(2) implies (3): Suppose $f : \mathscr{A} \longrightarrow \mathscr{B}$ is a monotone map. We prove that T preserves exactness of the comma square

$$\begin{tikzcd}
& f/1_{\mathscr{B}} \arrow[dl, "\pi_0"'] \arrow[dr, "\pi_1"] & \\
\mathscr{A} \arrow[dr, "f"'] & \longrightarrow & \mathscr{B} \arrow[dl, "1_{\mathscr{B}}"] \\
& \mathscr{B} &
\end{tikzcd}$$

That T preserves exactness of comma squares of the form $1_{\mathscr{A}}/f$ is proved analogously.

Define $e : \mathscr{P} \longrightarrow f/1_{\mathscr{B}}$ by the universal property in

$$\begin{tikzcd}
& \mathscr{P} \arrow[d, "e"] & \\
& f/1_{\mathscr{B}} \arrow[dl, "\pi_0"'] \arrow[dr, "\pi_1"] & \\
\mathscr{A} \arrow[dr, "f"'] & \longrightarrow & \mathscr{B} \arrow[dl, "1_{\mathscr{B}}"] \\
& \mathscr{B} &
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
& & \mathscr{P} \arrow[dl, "s_1'"'] \arrow[dr, "s_0'"] \arrow[d, phantom, "(i)"] & & \\
& \mathscr{A} \arrow[dl, "1_{\mathscr{A}}"'] \arrow[dr, "f"] \arrow[d, phantom, "(ii)"] & {} & 1_{\mathscr{B}}/1_{\mathscr{B}} \arrow[dl, "s_1"'] \arrow[dr, "p_1'"] \arrow[d, phantom, "\overset{(iii)}{\longrightarrow}"] & \\
\mathscr{A} \arrow[dr, "f"'] & {} & \mathscr{B} \arrow[dl, "1_{\mathscr{B}}"'] \arrow[dr, "1_{\mathscr{B}}"] \arrow[d, phantom, "(iv)"] & {} & \mathscr{B} \arrow[dl, "1_{\mathscr{B}}"] \\
& \mathscr{B} \arrow[dr, "1_{\mathscr{B}}"'] & {} & \mathscr{B} \arrow[dl, "1_{\mathscr{B}}"] & \\
& & \mathscr{B} & &
\end{tikzcd}
\tag{3.6}$$

where (i), (ii), (iv) are pullbacks, and (iii) is a comma square.

Clearly, $e : \mathscr{P} \longrightarrow f/1_{\mathscr{B}}$ maps (a, b', b) in $\mathscr{P}$ to (a, b) in $f/1_{\mathscr{B}}$ and e is a monotone surjection.

The image under T of the diagram on the right of (3.6) is exact by assumptions. Hence the image under T of the diagram on the left of (3.6) is exact. Since e is a surjection, $e_\diamond \cdot e^\diamond = 1_{f/1_{\mathscr{B}}}$. Hence $(Te)_\diamond \cdot (Te)^\diamond = 1_{T(f/g)}$ holds since T preserves surjections (express surjectivity as a strict exact square). Thus

$$\begin{aligned}
(T\pi_0)_\diamond \cdot (T\pi_1)^\diamond &= (T\pi_0)_\diamond \cdot (Te)_\diamond \cdot (Te)^\diamond \cdot (T\pi_1)^\diamond \\
&= (Tf)^\diamond \cdot (Tg)_\diamond
\end{aligned}$$

proving exactness of

$$\begin{tikzcd}
& T(f/1_{\mathscr{B}}) \arrow[dl, "T\pi_0"'] \arrow[dr, "T\pi_1"] & \\
T\mathscr{A} \arrow[dr, "Tf"'] & \longrightarrow & T\mathscr{B} \arrow[dl, "1_{T\mathscr{B}}"] \\
& T\mathscr{B} &
\end{tikzcd}$$

(3) implies (4): Suppose

$$\begin{tikzcd}
& f/g \arrow[dl, "\pi_0"'] \arrow[dr, "\pi_1"] & \\
\mathscr{A} \arrow[rr, phantom, "\longrightarrow"] \arrow[dr, "f"'] & & \mathscr{B} \arrow[dl, "g"] \\
& \mathscr{C} &
\end{tikzcd}$$

is a comma square and define $e : \mathscr{P} \longrightarrow f/g$ by the universal property in

$$\begin{tikzcd}
& \mathscr{P} \arrow[d, "e"] & \\
& f/g \arrow[dl, "\pi_0"'] \arrow[dr, "\pi_1"] & \\
\mathscr{A} \arrow[rr, phantom, "\longrightarrow"] \arrow[dr, "f"'] & & \mathscr{B} \arrow[dl, "g"] \\
& \mathscr{C} &
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
& & \mathscr{P} \arrow[dl, "s'_1"'] \arrow[dr, "s'_0"] \arrow[dd, phantom, "(i)"] & & \\
& f/1_{\mathscr{C}} \arrow[dl, "p'_0"'] \arrow[dr, "s_0"] & & 1_{\mathscr{C}}/g \arrow[dl, "s_1"'] \arrow[dr, "p'_1"] & \\
\mathscr{A} \arrow[rr, phantom, "\overset{(ii)}{\longrightarrow}"] \arrow[dr, "f"'] & & \mathscr{C} \arrow[rr, phantom, "\overset{(iii)}{\longrightarrow}"] \arrow[dl, "1_{\mathscr{C}}"'] \arrow[dr, "1_{\mathscr{C}}"] \arrow[dd, phantom, "(iv)"] & & \mathscr{B} \arrow[dl, "g"] \\
& \mathscr{C} \arrow[dr, "1_{\mathscr{C}}"'] & & \mathscr{C} \arrow[dl, "1_{\mathscr{C}}"] & \\
& & \mathscr{C} & &
\end{tikzcd} \tag{3.7}$$

where (i) and (iv) are pullbacks, (ii) and (iii) are comma squares.

Clearly, $e : \mathscr{P} \longrightarrow f/g$ maps (a, c, b) in $\mathscr{P}$ to (a, b) in f/g and e is a monotone surjection.

The image under T of the diagram on the right of (3.7) is exact by assumptions. Hence the image under T of the diagram on the left of (3.7) is exact. Since e is a surjection, $e_\diamond \cdot e^\diamond = 1_{f/g}$. Hence $(Te)_\diamond \cdot (Te)^\diamond = 1_{T(f/g)}$ holds since T preserves surjections (express surjectivity as a strict exact square). Thus

$$\begin{array}{rcl}
(T\pi_0)_\diamond \cdot (T\pi_1)^\diamond & = & (T\pi_0)_\diamond \cdot (Te)_\diamond \cdot (Te)^\diamond \cdot (T\pi_1)^\diamond \\
& = & (Tf)^\diamond \cdot (Tg)_\diamond
\end{array}$$

proving exactness of

$$\begin{tikzcd}
& T(f/g) \arrow[dl, "T\pi_0"'] \arrow[dr, "T\pi_1"] & \\
T\mathscr{A} \arrow[rr, phantom, "\longrightarrow"] \arrow[dr, "Tf"'] & & T\mathscr{B} \arrow[dl, "Tg"] \\
& T\mathscr{C} &
\end{tikzcd}$$

(4) implies (1): Suppose that the lax square

$$\begin{tikzcd}
\mathscr{P} \arrow[r, "p_1"] \arrow[d, "p_0"'] \arrow[dr, phantom, "\nearrow"] & \mathscr{B} \arrow[d, "g"] \\
\mathscr{A} \arrow[r, "f"'] & \mathscr{C}
\end{tikzcd}$$

is exact.

Observe that there is an equality

$$
\begin{tikzcd}
 & \mathscr{S} \arrow[d, "e"] & \\
 & f/g \arrow[dl, "\pi_0"'] \arrow[dr, "\pi_1"] & \\
\mathscr{A} \arrow[dr, "f"'] \arrow[rr, phantom, "\longrightarrow"] & & \mathscr{B} \arrow[dl, "g"] \\
 & \mathscr{C} &
\end{tikzcd}
=
\begin{tikzcd}
 & & \mathscr{S} \arrow[dl, "s_1'"'] \arrow[dr, "s_0'"] \arrow[dd, phantom, "(i)"] & & \\
 & 1_{\mathscr{A}}/p_0 \arrow[dl, "p_0'"'] \arrow[dr, "s_0"] & & p_1/1_{\mathscr{B}} \arrow[dl, "s_1"'] \arrow[dr, "p_1'"] & \\
\mathscr{A} \arrow[dr, "1_{\mathscr{A}}"'] \arrow[rr, phantom, "\overset{(ii)}{\longrightarrow}"] & & \mathscr{P} \arrow[dl, "p_0"'] \arrow[dr, "p_1"] \arrow[rr, phantom, "\overset{(iii)}{\longrightarrow}"] & & \mathscr{B} \arrow[dl, "1_{\mathscr{B}}"] \\
 & \mathscr{A} \arrow[dr, "f"'] \arrow[rr, phantom, "\overset{(iv)}{\longrightarrow}"] & & \mathscr{B} \arrow[dl, "g"] & \\
 & & \mathscr{C} & &
\end{tikzcd}
\tag{3.8}
$$

where the diagrams on the right are: (i) is a pullback, (ii) and (iii) are comma objects, and (iv) is the original lax exact square. On the left, the morphism $e : \mathscr{S} \longrightarrow f/g$ is induced by the universal property of comma squares. Observe that e is a monotone surjection: e maps (a, w, b) in $\mathscr{S}$ to (a, b) in f/g, and for (a, b) in f/g there is (a, w, b) in $\mathscr{S}$ by exactness.

Therefore, the diagram

$$
\begin{tikzcd}
 & \mathscr{S} \arrow[d, "e"] & \\
 & f/g \arrow[dl, "\pi_0"'] \arrow[dr, "\pi_1"] & \\
\mathscr{A} \arrow[dr, "f"'] \arrow[rr, phantom, "\longrightarrow"] & & \mathscr{B} \arrow[dl, "g"] \\
 & \mathscr{C} &
\end{tikzcd}
=
\begin{tikzcd}
 & \mathscr{S} \arrow[dl, "\pi_0 \cdot e"'] \arrow[dr, "\pi_1 \cdot e"] & \\
\mathscr{A} \arrow[dr, "f"'] \arrow[rr, phantom, "\longrightarrow"] & & \mathscr{B} \arrow[dl, "g"] \\
 & \mathscr{C} &
\end{tikzcd}
$$

is exact, i.e., the equality

$$(\pi_0)_\diamond \cdot e_\diamond \cdot e^\diamond \cdot (\pi_1)^\diamond = f^\diamond \cdot g_\diamond$$

holds. This follows from $e_\diamond \cdot e^\diamond = 1_{f/g}$, since e is surjective and from the fact that comma squares are exact.

By assumption, in the diagram

$$
\begin{tikzcd}
 & & T(f/g) \arrow[dl, "Ts_1'"'] \arrow[dr, "Ts_0'"] \arrow[dd, phantom, "T(i)"] & & \\
 & T(1_{\mathscr{A}}/p_0) \arrow[dl, "Tp_0'"'] \arrow[dr, "Ts_0"] & & T(p_1/1_{\mathscr{B}}) \arrow[dl, "Ts_1"'] \arrow[dr, "Tp_1'"] & \\
T\mathscr{A} \arrow[dr, "1_{T\mathscr{A}}"'] \arrow[rr, phantom, "\overset{T(ii)}{\longrightarrow}"] & & T\mathscr{P} \arrow[dl, "Tp_0"'] \arrow[dr, "Tp_1"] \arrow[rr, phantom, "\overset{T(iii)}{\longrightarrow}"] & & T\mathscr{B} \arrow[dl, "1_{T\mathscr{B}}"] \\
 & T\mathscr{A} \arrow[dr, "Tf"'] \arrow[rr, phantom, "\overset{T(iv)}{\longrightarrow}"] & & T\mathscr{B} \arrow[dl, "Tg"] & \\
 & & T\mathscr{C} & &
\end{tikzcd}
$$

the square $T(i)$ is strict exact, and $T(ii)$, $T(iii)$ are lax exact squares. Also, the whole diagram is exact, being the image of the diagram

$$\begin{tikzcd}
 & \mathscr{S} \arrow[dl, "e"'] \arrow[dr, "e"] & \\
f/g \arrow[dr, "1_{f/g}"'] & & f/g \arrow[dl, "1_{f/g}"] \\
 & f/g \arrow[dl, "\pi_0"'] \arrow[dr, "\pi_1"] & \\
\mathscr{A} \arrow[dr, "f"'] \arrow[rr, phantom, "\longrightarrow"] & & \mathscr{B} \arrow[dl, "g"] \\
 & \mathscr{C} &
\end{tikzcd}$$

under T (use assumptions: the upper square is strict exact, and the lower square is a comma object).

We prove that $T(iv)$ is exact. Indeed:

$$\begin{aligned}
(Tf)^\diamond \cdot (Tg)_\diamond &= (1_{T\mathscr{A}})^\diamond \cdot (Tf)^\diamond \cdot (Tg)_\diamond \cdot (1_{T\mathscr{B}})_\diamond \\
&= (Tp_0')_\diamond \cdot (Ts_1')_\diamond \cdot (Ts_0')^\diamond \cdot (Tp_1')^\diamond \\
&= (Tp_0')_\diamond \cdot (Ts_0)^\diamond \cdot (Ts_1)_\diamond \cdot (Tp_1')^\diamond \\
&= (1_{T\mathscr{A}})^\diamond \cdot (Tp_0)_\diamond \cdot (Tp_1)^\diamond \cdot (1_{T\mathscr{B}})_\diamond \\
&= (Tp_0)_\diamond \cdot (Tp_1)^\diamond
\end{aligned}$$

□

4 The universal property of $(-)_\diamond : \mathsf{Pre} \longrightarrow \mathsf{Rel}(\mathsf{Pre})$

We prove now that the 2-functor $(-)_\diamond : \mathsf{Pre} \longrightarrow \mathsf{Rel}(\mathsf{Pre})$ has an analogous universal property to the case of sets. From that, the result on a unique lifting of T to $\overline{T}$ will immediately follow, see Theorem 5.3 below.

Theorem 4.1. *The 2-functor $(-)_\diamond : \mathsf{Pre} \longrightarrow \mathsf{Rel}(\mathsf{Pre})$ has the following three properties:*

1. *Every $f_\diamond$ is a left adjoint.*
2. *For every exact square (3.3) the equality $f^\diamond \cdot g_\diamond = (p_0)_\diamond \cdot (p_1)^\diamond$ holds.*
3. *For every absolutely dense monotone map e, the relation $e_\diamond$ is a split epimorphism with the splitting given by $e^\diamond$.*

Moreover, the functor $(-)_\diamond$ is universal w.r.t. these three properties in the following sense: if K is any 2-category where the isomorphism 2-cells are identities, to give a 2-functor $H : \mathsf{Rel}(\mathsf{Pre}) \longrightarrow \mathsf{K}$ is the same thing as to give a 2-functor $F : \mathsf{Pre} \longrightarrow \mathsf{K}$ with the following three properties:

1. *Every Ff has a right adjoint, denoted by $(Ff)^r$.*
2. *For every exact square (3.3) the equality $Ff^r \cdot Fg = Fp_0 \cdot (Fp_1)^r$ holds.*
3. *For every absolutely dense monotone map e, Fe is a split epimorphism, with the splitting given by $(Fe)^r$.*

Proof. It is trivial to see that $(-)_\diamond$ has the above three properties.

Given a 2-functor $H : \mathsf{Rel(Pre)} \longrightarrow \mathsf{K}$, define F to be the composite $H \cdot (-)_\diamond$. Such F clearly has the above three properties, since 2-functors preserve adjunctions.

Conversely, given $F : \mathsf{Pre} \longrightarrow \mathsf{K}$, define $H\mathscr{A} = F\mathscr{A}$ on objects, and on a relation $R = (d_0^R)_\diamond \cdot (d_1^R)^\diamond$ define $H(R) = Fd_0^R \cdot (Fd_1^R)^r$, where $(Fd_1^R)^r$ is the right adjoint of Fd_1^R in K.

It is easy to verify that H so defined preserves identities: the identity relation $id_\mathscr{A}$ on $\mathscr{A}$ is represented as a fibration

$$\begin{tikzcd} & 1_\mathscr{A}/1_\mathscr{A} \arrow[dl, "p_0"'] \arrow[dr, "p_1"] & \\ \mathscr{A} & & \mathscr{A} \end{tikzcd}$$

coming from the exact comma square

$$\begin{tikzcd} 1_\mathscr{A}/1_\mathscr{A} \arrow[r, "p_1"] \arrow[d, "p_0"'] \arrow[dr, phantom, "\nearrow"] & \mathscr{A} \arrow[d, "1_\mathscr{A}"] \\ \mathscr{A} \arrow[r, "1_\mathscr{A}"'] & \mathscr{A} \end{tikzcd} \tag{4.9}$$

Hence $H(id_\mathscr{A}) = Fp_0 \cdot (Fp_1)^r = F(1_\mathscr{A}) = 1_{F\mathscr{A}} = 1_{H\mathscr{A}}$ holds by our assumptions on F.

For preservation of composition use Corollary 3.7: first

$$H(S) \cdot H(R) = Fd_0^S \cdot (Fd_1^S)^r \cdot Fd_0^R \cdot (Fd_1^R)^r$$

by definition. Further, by exactness of the pullback from Corollary 3.7 and our assumption on F, we have

$$Fd_0^S \cdot (Fd_1^S)^r \cdot Fd_0^R \cdot (Fd_1^R)^r = Fd_0^S \cdot Fq_0 \cdot (Fq_1)^r \cdot (Fd_1^R)^r$$

and, finally, since Fw is split epi by Corollary 3.7 and our assumption on F, we obtain

$$Fd_0^S \cdot Fq_0 \cdot Fw \cdot (Fw)^r \cdot (Fq_1)^r \cdot (Fd_1^R)^r = Fd_0^{R\cdot S} \cdot (Fd_1^{R\cdot S})^r = H(R \cdot S)$$

and the proof is complete. $\square$

Remark. There is an analogous theorem with "Pos" replacing "Pre" and "surjective" replacing "absolutely dense".

5 The extension theorem

Definition 5.1. We say that a locally monotone functor $T : \mathsf{Pre} \longrightarrow \mathsf{Pre}$ satisfies the *Beck-Chevalley Condition* (BCC) if it preserves exact squares.

Remark 5.2. A functor satisfying the BCC has to preserve order-embeddings, absolutely dense monotone maps and absolute left Kan extensions. This follows from Example 3.3. Examples of functors (not) satisfying the BCC can be found in Section 6.

Theorem 5.3. *For a 2-functor $T : \mathsf{Pre} \longrightarrow \mathsf{Pre}$ the following are equivalent:*

1. *There is a 2-functor $\overline{T} : \mathsf{Rel(Pre)} \longrightarrow \mathsf{Rel(Pre)}$ such that*

$$\begin{tikzcd} \mathsf{Rel(Pre)} \arrow[r, "\overline{T}"] & \mathsf{Rel(Pre)} \\ \mathsf{Pre} \arrow[u, "(-)_\diamond"] \arrow[r, "T"'] & \mathsf{Pre} \arrow[u, "(-)_\diamond"'] \end{tikzcd} \tag{5.10}$$

2. *The functor T satisfies the BCC.*

3. *There is a distributive law $T \cdot \mathbb{L} \longrightarrow \mathbb{L} \cdot T$ of T over the KZ doctrine $(\mathbb{L}, \mathbb{y}, \mathbb{m})$ described in (2.2) above.*

Proof. The equivalence of 1. and 3. follows from general facts about distributive laws, using Proposition 2.7 above. See, e.g., [S_1]. For the equivalence of 1. and 2., observe that T satisfies the BCC iff

$$\mathsf{Pre} \xrightarrow{T} \mathsf{Pre} \xrightarrow{(-)_\diamond} \mathsf{Rel}(\mathsf{Pre})$$

satisfies the three properties of Theorem 4.1 above. $\square$ $\square$

Remark. There is an analogous theorem with "Pos" replacing "Pre".

Corollary 5.4. *If T is a locally monotone functor, the lifting $\overline{T}$ is computed as*

$$\overline{T}(R) = (Td_0)_\diamond \cdot (Td_1)^\diamond$$

where $(d_0, \mathscr{E}, d_1)$ is the two-sided discrete fibration corresponding to R.

Corollary. Let $T : \mathsf{Pre} \longrightarrow \mathsf{Pre}$ and $T_0 : \mathsf{Set} \longrightarrow \mathsf{Set}$ such that $TD = DT_0$ and $VT = T_0 V$ where $V : \mathsf{Pre} \longrightarrow \mathsf{Set}$ is the forgetful functor and D is its left-adjoint. Then T satisfies the BCC iff T_0 preserves weak pullbacks.

Proof. We show that $\overline{T}$ preserves composition of relations if $\overline{T_0}$ does. By Corollary 5.4 and the corollary after Proposition 2.13, we have $\mathsf{Rel}(V)\overline{T} = \overline{T_0}\mathsf{Rel}(V)$. Let S, R be two monotone relations. We have $\mathsf{Rel}(V)\overline{T}(S \cdot R) = \overline{T_0}\mathsf{Rel}(V)(S \cdot R) = \overline{T_0}(\mathsf{Rel}(V)S \cdot \mathsf{Rel}(V)R) = \overline{T_0}(\mathsf{Rel}(V)S) \cdot \overline{T_0}(\mathsf{Rel}(V)R) = \mathsf{Rel}(V)\overline{T}(S) \cdot \mathsf{Rel}(V)\overline{T}(R) = \mathsf{Rel}(V)(\overline{T}S \cdot \overline{T}R)$, hence $\overline{T}(S \cdot R) = \overline{T}S \cdot \overline{T}R$ by $\mathsf{Rel}(V)$ being faithful.

Conversely, any pullback in Set is mapped by D to a pullback in Pre and then to an exact square by T. Now from $TD = DT_0$ and the fact that any exact square of sets is a weak pullback it follows that T_0 preserves weak pullbacks. $\square$

6 Examples

Example 6.1. All the "Kripke-polynomial" functors satisfy the Beck-Chevalley Condition. This means the functors defined by the following grammar:

$$T ::= const_{\mathscr{X}} \mid Id \mid T^\partial \mid T + T \mid T \times T \mid \mathbb{L}T$$

where $const_{\mathscr{X}}$ is the constant-at-$\mathscr{X}$, T^∂ is the *dual* of T, defined by putting

$$T^\partial \mathscr{A} = (T\mathscr{A}^{op})^{op}$$

and $\mathbb{L}\mathscr{X} = [\mathscr{X}^{op}, 2]$ (the lowersets on $\mathscr{X}$, ordered by inclusion). Observe that $\mathbb{L}^\partial \mathscr{X} = [\mathscr{X}, 2]^{op}$, hence $\mathbb{L}^\partial \mathscr{X} = \mathbb{U}\mathscr{X}$ (the uppersets on $\mathscr{X}$, ordered by reversed inclusion).

To check that BCC is satisfied, suppose that the square

$$\begin{tikzcd} \mathscr{P} \arrow[r, "p_1"] \arrow[d, "p_0"'] \arrow[dr, phantom, "\nearrow"] & \mathscr{B} \arrow[d, "g"] \\ \mathscr{A} \arrow[r, "f"'] & \mathscr{C} \end{tikzcd} \tag{6.11}$$

is exact.

1. The functor $const_{\mathscr{X}}$.

   The image of square (6.11) under $const_{\mathscr{X}}$ is the square

$$\begin{tikzcd}
\mathscr{X} \arrow[r, "1_{\mathscr{X}}"] \arrow[d, "1_{\mathscr{X}}"'] \arrow[dr, phantom, "\nearrow"] & \mathscr{X} \arrow[d, "1_{\mathscr{X}}"] \\
\mathscr{X} \arrow[r, "1_{\mathscr{X}}"'] & \mathscr{X}
\end{tikzcd}$$

   where the comparison is the identity. This is an exact square (it is a Yoneda square).

2. The functor Id.

   This functor obviously satisfies the Beck-Chevalley Condition.

3. Suppose T satisfies the Beck-Chevalley Condition.

   The square

$$\begin{tikzcd}
\mathscr{P}^{op} \arrow[r, "p_0^{op}"] \arrow[d, "p_1^{op}"'] \arrow[dr, phantom, "\nearrow"] & \mathscr{A}^{op} \arrow[d, "f^{op}"] \\
\mathscr{B}^{op} \arrow[r, "g^{op}"'] & \mathscr{C}^{op}
\end{tikzcd}$$

   is exact by Example 3.5 and, by assumption, so is the square

$$\begin{tikzcd}
T(\mathscr{P}^{op}) \arrow[r, "T(p_0^{op})"] \arrow[d, "T(p_1^{op})"'] \arrow[dr, phantom, "\nearrow"] & T\mathscr{A}^{op} \arrow[d, "T(f^{op})"] \\
T(\mathscr{B}^{op}) \arrow[r, "T(g^{op})"'] & T(\mathscr{C}^{op})
\end{tikzcd}$$

   Finally, the square

$$\begin{tikzcd}
(T(\mathscr{P}^{op}))^{op} \arrow[r, "(T(p_1^{op}))^{op}"] \arrow[d, "(T(p_0^{op}))^{op}"'] \arrow[dr, phantom, "\nearrow"] & (T(\mathscr{B}^{op}))^{op} \arrow[d, "(T(g^{op}))^{op}"] \\
(T(\mathscr{A}^{op}))^{op} \arrow[r, "(T(f^{op}))^{op}"'] & (T(\mathscr{C}^{op}))^{op}
\end{tikzcd}$$

   is exact by Example 3.5 and this is what we were supposed to prove.

4. Suppose both T_1 and T_2 satisfy the Beck-Chevalley Condition. We prove that $T_1 + T_2$ does satisfy it.

   The image of (6.11) under $T_1 + T_2$ is

$$\begin{tikzcd}
T_1\mathscr{P} + T_2\mathscr{P} \arrow[r, "T_1p_1+T_2p_1"] \arrow[d, "T_1p_0+T_2p_0"'] \arrow[dr, phantom, "\nearrow"] & T_1\mathscr{B} + T_2\mathscr{B} \arrow[d, "T_1g+T_2g"] \\
T_1\mathscr{A} + T_2\mathscr{A} \arrow[r, "T_1f+T_2f"'] & T_1\mathscr{C} + T_2\mathscr{C}
\end{tikzcd}$$

   The assertion follows from the fact that coproducts are disjoint in Pre.

5. Suppose both T_1 and T_2 satisfy the Beck-Chevalley Condition. We prove that $T_1 \times T_2$ does satisfy it.

The image of (6.11) under $T_1 \times T_2$ is

$$\begin{tikzcd} T_1\mathscr{P} \times T_2\mathscr{P} \arrow[r, "T_1p_1 \times T_2p_1"] \arrow[d, "T_1p_0 \times T_2p_0"'] \arrow[dr, phantom, "\nearrow"] & T_1\mathscr{B} \times T_2\mathscr{B} \arrow[d, "T_1g \times T_2g"] \\ T_1\mathscr{A} \times T_2\mathscr{A} \arrow[r, "T_1f \times T_2f"'] & T_1\mathscr{C} \times T_2\mathscr{C} \end{tikzcd}$$

The assertion follows from how products are formed in Pre.

6. Suppose that T satisfies the Beck-Chevalley Condition. We prove that $\mathbb{L}T$ does satisfy it again.

   It suffices to prove that $\mathbb{L}$ satisfies the Beck-Chevalley Condition. The image of square (6.11) under $\mathbb{L}$ is the square

$$\begin{tikzcd} \mathbb{L}\mathscr{P} \arrow[r, "\mathbb{L}p_1"] \arrow[d, "\mathbb{L}p_0"'] \arrow[dr, phantom, "\nearrow"] & \mathbb{L}\mathscr{B} \arrow[d, "\mathbb{L}g"] \\ \mathbb{L}\mathscr{A} \arrow[r, "\mathbb{L}f"'] & \mathbb{L}\mathscr{C} \end{tikzcd}$$

   First recall how $\mathbb{L}$ is defined on monotone maps: for example, $\mathbb{L}f : \mathbb{L}\mathscr{A} \longrightarrow \mathbb{L}\mathscr{C}$ is defined as a left Kan extension along $f^{op} : \mathscr{A}^{op} \longrightarrow \mathscr{C}^{op}$. This means that, for every lowerset $W : \mathscr{A}^{op} \longrightarrow \mathbb{2}$,

$$(\mathbb{L}f)(W) = \bigvee_a \mathscr{C}^{op}(f^{op}a, -) \wedge Wa$$

   or, in a more readable fashion,

$$(\mathbb{L}f)(W) : c \mapsto \bigvee_a \mathscr{C}(c, fa) \wedge Wa$$

   Hence c is in the lowerset $(\mathbb{L}f)(W)$ iff there is a in W such that $c \leq fa$. Observe that $\mathbb{L}$ is indeed a functor: it clearly preserves identities and composition (for that, see Theorem 4.47 of [Ke]) up to isomorphisms. But these canonical isomorphisms are identities, since $[\mathscr{X}^{op}, \mathbb{2}]$ is always a *poset*.

   We employ Example 3.4: both $\mathbb{L}f$ and $\mathbb{L}p_1$ are left adjoints with $(\mathbb{L}f)^r = [f^{op}, \mathbb{2}]$ and $(\mathbb{L}p_1)^r = [p_1^{op}, \mathbb{2}]$. Hence it suffices to prove that

$$\mathbb{L}p_0 \cdot [p_1^{op}, \mathbb{2}] = [f^{op}, \mathbb{2}] \cdot \mathbb{L}g$$

   Moreover, by the density of principal lowersets of the form $\mathscr{B}(-, b_0)$ in $\mathbb{L}\mathscr{B}$ and the fact that all the monotone maps $\mathbb{L}p_0$, $[p_1^{op}, \mathbb{2}]$, $[f^{op}, \mathbb{2}]$, $\mathbb{L}g$ preserve suprema (since they all are left adjoints), it suffices to prove that

$$(\mathbb{L}p_0 \cdot [p_1^{op}, \mathbb{2}])(\mathscr{B}(-, b_0)) = ([f^{op}, \mathbb{2}] \cdot \mathbb{L}g)(\mathscr{B}(-, b_0)) \tag{6.12}$$

   holds for all b_0.

   The left-hand side is isomorphic to

$$\mathbb{L}p_0(\mathscr{B}(p_1-, b_0)) = a \mapsto \bigvee_w \mathscr{A}(a, p_0w) \wedge \mathscr{B}(p_1w, b_0)$$

   By exactness of (6.11), this means that

$$\mathbb{L}p_0(\mathscr{B}(p_1-, b_0)) = a \mapsto \mathscr{C}(fa, gb_0)$$

   Observe further that

$$(\mathbb{L}g)(\mathscr{B}(-, b_0)) = c \mapsto \bigvee_b \mathscr{C}(c, gb) \wedge \mathscr{B}(b, b_0)$$

hence

$$(\mathbb{L}g)(\mathscr{B}(-,b_0)) = c \mapsto \mathscr{C}(c, gb_0)$$

by the Yoneda Lemma.

The right hand side of (6.12) is therefore isomorphic to

$$([f^{op}, 2] \cdot \mathbb{L}g)(\mathscr{B}(-,b_0)) = [f^{op}, 2](c \mapsto \mathscr{C}(c, gb_0)) = a \mapsto \mathscr{C}(fa, gb_0)$$

Example 6.2. Recall the adjunction $Q \dashv I : \mathsf{Pos} \longrightarrow \mathsf{Pre}$, where I is the inclusion functor and $Q(\mathscr{A})$ is the quotient of $\mathscr{A}$ obtained by identifying a and b whenever $a \leq b$ and $b \leq a$. The functors Q and I are locally monotone and map exact squares to exact squares. Hence, if $T : \mathsf{Pre} \longrightarrow \mathsf{Pre}$ satisfies the BCC, so does $QTI : \mathsf{Pos} \longrightarrow \mathsf{Pos}$.

Example 6.3. The *powerset functor* $\mathbb{P} : \mathsf{Pre} \longrightarrow \mathsf{Pre}$ is defined as follows. The order on $\mathbb{P}\mathscr{A}$ is the Egli-Milner preorder, that is, $\mathbb{P}(A, B) = 1$ if and only if

$$\forall a \in A\ \exists b \in B\ a \leq b \text{ and } \forall b \in B\ \exists a \in A\ a \leq b \tag{6.13}$$

$\mathbb{P}f(A)$ is the direct image of A. The functor $\mathbb{P}$ is locally monotone and satisfies the BCC.

The *finitary powerset functor* $\mathbb{P}_\omega$ is defined similarly: $\mathbb{P}_\omega\mathscr{A}$ consists of the finite subsets of $\mathscr{A}$ equipped with the Egli-Milner preorder. $\mathbb{P}_\omega$ is locally monotone and satisfies the BCC.

The powerset functor $\mathbb{P}$ is locally monotone and satisfies the BCC. This follows from the unnumbered corollary of Section 5. For a direct argument consider an exact square:

$$\begin{tikzcd} \mathscr{P} \arrow[r, "p_1"] \arrow[d, "p_0"'] \arrow[dr, phantom, "\nearrow"] & \mathscr{B} \arrow[d, "g"] \\ \mathscr{A} \arrow[r, "f"'] & \mathscr{C} \end{tikzcd} \tag{6.14}$$

By (3.5) we have to show that for $A \in \mathbb{P}\mathscr{A}$ and $B \in \mathbb{P}\mathscr{B}$

$$\mathbb{P}\mathscr{C}(\mathbb{P}f(A), \mathbb{P}g(B)) = \bigvee_W \mathbb{P}\mathscr{A}(A, \mathbb{P}p_0(W)) \wedge \mathbb{P}\mathscr{B}(\mathbb{P}p_1(W), B) \tag{6.15}$$

Assume $\mathbb{P}\mathscr{C}(\mathbb{P}f(A), \mathbb{P}g(B)) = 1$. Then

$$\forall a \in A\ \exists b \in B\ fa \leq gb \text{ and } \forall b \in B\ \exists a \in A\ fa \leq gb \tag{6.16}$$

We have to find $W \in \mathbb{P}\mathscr{P}$ such that $\mathbb{P}\mathscr{A}(A, \mathbb{P}p_0(W))$ and $\mathbb{P}\mathscr{B}(\mathbb{P}p_1(W), B)$. Let $W = \{w \in \mathscr{P} \mid \exists a \in A\ a \leq p_0 w \text{ and } \exists b \in B\ p_1 w \leq b\}$. It is easy to see that W satisfies $\forall w \in W\ \exists a \in A\ \mathscr{A}(a, p_0 w)$ and $\forall w \in W\ \exists b \in B\ \mathscr{B}(p_1 w, b)$. Consider $a \in A$. By (6.16) there exists $b \in B$ such that $\mathscr{C}(fa, gb)$. By (3.5) there exists $w \in W$ such that $\mathscr{A}(a, p_0 w)$. So $\mathbb{P}\mathscr{A}(A, \mathbb{P}p_0(W)) = 1$. Similarly, we can show that for all $b \in B$ exists $w \in W$ with $\mathscr{B}(p_1 w, b)$. This shows that $\mathbb{P}$ preserves exact squares, hence it satisfies the BCC.

The proof that $\mathbb{P}_\omega$ satisfies the BCC goes along the same lines.

Example 6.4. Given a preorder $\mathscr{A}$, a subset $A \subseteq \mathscr{A}$ is called *convex* if $x \leq y \leq z$ and $x, z \in A$ imply $y \in A$.

The *convex powerset functor* $\mathbb{P}^c : \mathsf{Pos} \longrightarrow \mathsf{Pos}$ is defined as follows. $\mathbb{P}^c\mathscr{A}$ is the set of convex subsets of $\mathscr{A}$ endowed with the Egli-Milner order. $\mathbb{P}^c f(A)$ is the direct image of A. This is a well defined locally monotone functor. Notice that $\mathbb{P}^c \cong Q\mathbb{P}I$. This follows from the fact that if $\mathscr{A}$ is a poset and $A, B \in \mathbb{P}I\mathscr{A}$, then $\mathbb{P}I\mathscr{A}(A, B) = 1$ and $\mathbb{P}I\mathscr{A}(B, A) = 1$ if and only if A and B have the same convex hull. Hence, by Example 6.2, $\mathbb{P}^c$ satisfies the BCC.

The *finitely-generated convex powerset* $\mathbb{P}^c_\omega$ is defined similarly to $\mathbb{P}^c$. The only difference is that the convex sets appearing in $\mathbb{P}^c_\omega\mathscr{A}$ are convex hulls of finitely many elements of $\mathscr{A}$. Then $\mathbb{P}^c_\omega$ is locally monotone and is isomorphic to $Q\mathbb{P}_\omega I$, thus it also satisfies the BCC. Again, we have that $\mathbb{P}^c_\omega = Q\mathbb{P}_\omega I$ and $\mathbb{P}^c_\omega$ satisfies the BCC.

Observe that both functors are self-dual: $(\mathbb{P}^c)^\partial = \mathbb{P}^c$ and $(\mathbb{P}^c_\omega)^\partial = \mathbb{P}^c_\omega$.

Example 6.5. Since the lowerset functor $\mathbb{L} : \mathsf{Pre} \longrightarrow \mathsf{Pre}$ satisfies the Beck-Chevalley Condition by Example 6.1, we can compute its lifting $\overline{\mathbb{L}} : \mathsf{Rel}(\mathsf{Pre}) \longrightarrow \mathsf{Rel}(\mathsf{Pre})$. We show how $\overline{\mathbb{L}}$ works on the relation $\mathscr{A} \overset{R}{\longrightarrow\!\!\!\!\!\!\!\!\!\!\mapsto} \mathscr{B}$. The value $\overline{\mathbb{L}}(R)$ is, by Theorems 4.1 and 5.3, given by $(\mathbb{L}d_0)_\diamond \cdot (\mathbb{L}d_1)^\diamond$ where $(d_0, \mathscr{E}^R, d_1) : \mathscr{A} \longrightarrow \mathscr{B}$ is the two-sided discrete fibration corresponding to R. Using the formula (2.1) for relation composition, we can write

$$\overline{\mathbb{L}}(R)(B, A) = \bigvee_W \mathbb{L}\mathscr{B}(B, \mathbb{L}d_0(W)) \wedge \mathbb{L}\mathscr{A}(\mathbb{L}d_1(W), A) \tag{6.17}$$

where $B : \mathscr{B}^{op} \longrightarrow 2$ and $A : \mathscr{A}^{op} \longrightarrow 2$ are arbitrary lowersets. Since $\mathbb{L}d_1$ is a left adjoint to restriction along $d_1^{op} : (\mathscr{E}^R)^{op} \longrightarrow \mathscr{A}^{op}$, we can rewrite (6.17) to

$$\overline{\mathbb{L}}(R)(B, A) = \bigvee_W \mathbb{L}\mathscr{B}(B, \mathbb{L}d_0(W)) \wedge \mathbb{L}\mathscr{E}^R(W, A \cdot d_1^{op})$$

and, by the Yoneda Lemma, to

$$\overline{\mathbb{L}}(R)(B, A) = \mathbb{L}\mathscr{B}(B, \mathbb{L}d_0(A \cdot d_1^{op}))$$

Hence the lowersets B and A are related by $\overline{\mathbb{L}}(R)$ if and only if the inclusion

$$B \subseteq \mathbb{L}d_0(A \cdot d_1^{op})$$

holds in $[\mathscr{B}^{op}, 2]$. Recall that

$$\mathbb{L}d_0(A \cdot d_1^{op}) = b \mapsto \bigvee_w \mathscr{B}(b, d_0 w) \wedge (A \cdot d_1^{op})(w)$$

Therefore the inclusion $B \subseteq \mathbb{L}d_0(A \cdot d_1^{op})$ is equivalent to the statement: For all b in B there is (b_1, a_1) such that $R(b_1, a_1)$ and $b \leq b_1$ and a_1 in A.

Observe that the above condition is reminiscent of one half of the Egli-Milner-style of the relation lifting of a powerset functor. This is because $\mathbb{L}$ is the "lower half" of two possible "powerpreorder functors". The "upper half" is given by $\mathbb{U} : \mathsf{Pre} \longrightarrow \mathsf{Pre}$ where $\mathbb{U} = \mathbb{L}^\partial$.

Example 6.6. The relation liftings $\overline{\mathbb{P}}$, $\overline{\mathbb{P}^c}$, $\overline{\mathbb{P}_\omega}$, $\overline{\mathbb{P}^c_\omega}$ of the (convex) powerset functor and their finitary versions yield the "Egli-Milner" style of the relation lifting. More precisely, for a relation $\mathscr{B} \overset{R}{\longrightarrow\!\!\!\!\!\!\!\!\!\!\mapsto} \mathscr{A}$ we have $\overline{\mathbb{P}}(R)(B, A)$ (respectively $\overline{\mathbb{P}_\omega}(R)(B, A)$, $\overline{\mathbb{P}^c}(R)(B, A)$, $\overline{\mathbb{P}^c_\omega}(R)(B, A)$) if and only if

$$\forall a \in A \; \exists b \in B \; R(b, a) \text{ and } \forall b \in B \; \exists a \in A \; R(b, a).$$

To compute the lifting of $\mathbb{P}^c$, consider a monotone relation $\mathscr{A} \overset{R}{\longrightarrow\!\!\!\!\!\!\!\!\!\!\mapsto} \mathscr{B}$ and the induced fibration $(d_0, \mathscr{E}, d_1) : \mathscr{A} \longrightarrow \mathscr{B}$. We know that $\overline{\mathbb{P}^c}(R) = (\mathbb{P}^c d_0)_\diamond \cdot (\mathbb{P}^c d_1)^\diamond$, so

$$\overline{\mathbb{P}^c}(R)(B, A) = \bigvee_E \mathbb{P}^c\mathscr{B}(B, \mathbb{P}^c d_0(E)) \wedge \mathbb{P}\mathscr{A}(\mathbb{P}^c d_1(E), A) \tag{6.18}$$

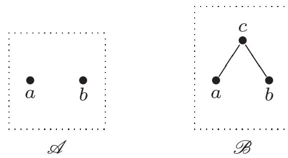
We prove that $\overline{\mathbb{P}^c}(R)(B, A) = 1$ implies $\forall a \in A \; \exists b \in B \; R(b, a)$ and $\forall b \in B \; \exists a \in A \; R(b, a)$. Consider a witness E and $a \in A$. Since $\mathbb{P}^c\mathscr{A}(\mathbb{P}^c d_1(E), A) = 1$, there exists $(b', a') \in E$ such that $\mathscr{A}(a', a)$. Since $\mathbb{P}^c\mathscr{B}(B, \mathbb{P}^c d_0(E)) = 1$, there exists $b \in B$ such that $\mathscr{B}(b, b')$. Since R is monotone and $R(b', a') = 1$ we obtain $R(b, a) = 1$. So $\forall a \in A \; \exists b \in B \; R(b, a)$. The second part is analogous.

Conversely, if $\forall a \in A \; \exists b \in B \; R(b, a)$ and $\forall b \in B \; \exists a \in A \; R(b, a)$, define the subset of $\mathscr{E}$ as follows:

$$E = \{ b \rightsquigarrow a \mid b \in B, \; a \in A \}$$

Then E is convex, since both B and A are convex. Both $\mathbb{P}^c\mathscr{B}(B, \mathbb{P}^c d_0(E)) = 1$ and $\mathbb{P}^c\mathscr{A}(\mathbb{P}^c d_1(E), A) = 1$ hold for obvious reasons. Hence $\overline{\mathbb{P}^c}(R)(B, A) = 1$ holds.

Example 6.7. To find a functor that does not satisfy the BCC, it suffices, by Remark 5.2, to find a locally monotone functor $T : \mathsf{Pre} \longrightarrow \mathsf{Pre}$ that does not preserve order-embeddings. For this, let T be the *connected components functor*, i.e., T takes a preorder $\mathscr{A}$ to the discretely ordered poset of connected components of $\mathscr{A}$. T does not preserve embedding $f : \mathscr{A} \longrightarrow \mathscr{B}$ indicated below.



7 An Application: Moss's Coalgebraic Logic over Posets

We show how to develop the basics of Moss's coalgebraic logic over posets. For reasons of space, this development will be terse and assume some familiarity with, e.g., Sections 2.2 and 3.1 of [KuL].

Since the logics will have propositional connectives but no negation (to capture the semantic order on the logical side) we will use the category DL of bounded distributive lattices. We write $F \dashv U : \mathsf{DL} \longrightarrow \mathsf{Pos}$ for the obvious adjunction; and $P : \mathsf{Pos}^{op} \longrightarrow \mathsf{DL}$ where $UP\mathscr{X} = [\mathscr{X}, \mathbb{2}]$ and $S : \mathsf{DL} \longrightarrow \mathsf{Pos}^{op}$ where $SA = \mathsf{DL}(A, \mathbb{2})$. Note that $UP = [-, \mathbb{2}]$ and recall $\mathbb{L} = [(-)^{op}, \mathbb{2}]$. Further, let $T : \mathsf{Pos} \longrightarrow \mathsf{Pos}$ be a locally monotone finitary functor that satisfies the BCC.

We define coalgebraic logic abstractly by a functor $L : \mathsf{DL} \longrightarrow \mathsf{DL}$ given as

$$L = FT^{\partial}U$$

where the functor $T^{\partial} : \mathsf{Pos} \longrightarrow \mathsf{Pos}$ is given by $T^{\partial}\mathscr{X} = (T(\mathscr{X}^{op}))^{op}$. By Example 6.1, T^{∂} satisfies the BCC. The formulas of the logic are the elements of the initial L-algebra $FT^{\partial}U(\mathcal{L}) \longrightarrow \mathcal{L}$. The formula given by some $\alpha \in T^{\partial}U(\mathcal{L})$ is written as $\nabla\alpha$. The semantics is given by a natural transformation

$$\delta : LP \longrightarrow PT^{op}$$

Before we define δ, we need for every preorder $\mathscr{A}$, the relation[1]

$$\begin{tikzcd} {[\mathscr{A}, \mathbb{2}]} \arrow[r, "\ni_{\mathscr{A}}", "/" marking] & \mathscr{A}^{op} \end{tikzcd}$$

given by the evaluation map $\mathsf{ev}_{\mathscr{A}} : \mathscr{A} \times [\mathscr{A}, \mathbb{2}] \longrightarrow \mathbb{2}$. Observe that

$$\ni_{\mathscr{A}} = (\mathbb{y}_{\mathscr{A}^{op}})^{\diamond} \tag{7.19}$$

since $(\mathbb{y}_{\mathscr{A}^{op}})^{\diamond}(a, V) = [\mathscr{A}, \mathbb{2}](\mathbb{y}_{\mathscr{A}^{op}}a, V) = Va$ holds by the Yoneda Lemma.

Lemma 7.1. *For every monotone map $f : \mathscr{A} \longrightarrow \mathscr{B}$ we have*

$$\begin{tikzcd} {[\mathscr{A}, \mathbb{2}]} \arrow[r, "\ni_{\mathscr{A}}", "/" marking] & \mathscr{A}^{op} \\ {[\mathscr{B}, \mathbb{2}]} \arrow[u, "{[f,\mathbb{2}]^{\diamond}}", "/" marking] \arrow[r, "\ni_{\mathscr{B}}"', "/" marking] & \mathscr{B}^{op} \arrow[u, "(f^{op})^{\diamond}"', "/" marking] \end{tikzcd}$$

Diagrammatic Proof. The square

$$\begin{tikzcd} \mathscr{A}^{op} \arrow[r, "\mathbb{y}_{\mathscr{A}^{op}}"] \arrow[d, "f^{op}"'] & \mathbb{L}(\mathscr{A}) \arrow[d, "\mathbb{L}(f)"] \\ \mathscr{B}^{op} \arrow[r, "\mathbb{y}_{\mathscr{B}^{op}}"'] & \mathbb{L}(\mathscr{B}) \end{tikzcd}$$

[1] The type of $\ni_{\mathscr{X}}$ conforms with the logical reading of $\ni$ as $\Vdash$. Indeed, $\ni(x, \varphi)$ & $\varphi \subseteq \psi \Rightarrow \ni(x, \psi)$ and $\ni(x, \varphi)$ & $x \leq y \Rightarrow \ni(y, \varphi)$, where φ, ψ are uppersets of $\mathscr{X}$.

commutes in Pre, since y is natural. Hence the square

$$\begin{tikzcd}
\mathscr{A}^{op} \arrow[r, "(\mathsf{y}_{\mathscr{A}^{op}})_\diamond", "\shortmid" marking] \arrow[d, "(f^{op})_\diamond"', "\shortmid" marking] & \mathbb{L}(\mathscr{A}) \arrow[d, "(\mathbb{L}(f))_\diamond", "\shortmid" marking] \\
\mathscr{B}^{op} \arrow[r, "(\mathsf{y}_{\mathscr{B}^{op}})_\diamond"', "\shortmid" marking] & \mathbb{L}(\mathscr{B})
\end{tikzcd}$$

commutes in $\mathsf{Rel}(\mathsf{Pre})$ since $(-)_\diamond$ is a 2-functor.

Now observe that $\mathbb{L}(f) \dashv [f, \mathbb{2}]$ holds by the definition of $\mathbb{L}$ on morphisms. Hence $(\mathbb{L}(f))^\diamond \dashv [f, \mathbb{2}]^\diamond$ holds. Since adjoints are determined uniquely up to isomorphisms, this shows that $(\mathbb{L}(f))_\diamond = [f, \mathbb{2}]^\diamond$ (we use that isomorphisms are identities in $\mathsf{Rel}(\mathsf{Pre})$).

Thus, taking right adjoints everywhere in the above square we obtain the square from the claim of the lemma. $\square$

Computational Proof. By definition

$$\begin{aligned}
\ni_{\mathscr{A}} \cdot [f, \mathbb{2}]^\diamond(a, V) &= \bigvee_W \ni_{\mathscr{A}}(a, W) \wedge [\mathscr{A}, \mathbb{2}](V \cdot f, W) \\
&= \ni_{\mathscr{A}}(a, V \cdot f) \\
&= (V \cdot f)(a)
\end{aligned}$$

where the second step is due to the Yoneda Lemma. Analogously:

$$\begin{aligned}
(f^{op})^\diamond \cdot \ni_{\mathscr{B}}(a, V) &= \bigvee_b \mathscr{B}^{op}(f^{op} a, b) \wedge \ni_{\mathscr{B}}(b, V) \\
&= \ni_{\mathscr{B}}(fa, V) \\
&= V(fa)
\end{aligned}$$

$\square$

Corollary 7.2. *For every locally monotone functor T that satisfies the Beck-Chevalley Condition and for every monotone map $f : \mathscr{A} \longrightarrow \mathscr{B}$, we have*

$$\begin{tikzcd}
\overline{T}[\mathscr{A}, \mathbb{2}] \arrow[r, "\overline{T}\ni_{\mathscr{A}}", "\shortmid" marking] & \overline{T}\mathscr{A}^{op} \\
\overline{T}[\mathscr{B}, \mathbb{2}] \arrow[r, "\overline{T}\ni_{\mathscr{B}}"', "\shortmid" marking] \arrow[u, "\overline{T}[f,\mathbb{2}]^\diamond", "\shortmid" marking] & \overline{T}\mathscr{B}^{op} \arrow[u, "\overline{T}(f^{op})^\diamond"', "\shortmid" marking]
\end{tikzcd}$$

Coming back to $\delta : LP \longrightarrow PT^{op}$. It suffices, due to $F \dashv U$, to give

$$\tau : T^\partial UP \longrightarrow UPT^{op}$$

Observe that, for every preorder $\mathscr{X}$, we have

$$UPT^{op}(\mathscr{X}) = [T^{op}\mathscr{X}, \mathbb{2}] = \mathbb{L}((T^{op}\mathscr{X})^{op})$$

By Proposition 2.7, to define $\tau_{\mathscr{X}}$ it suffices to give a relation from $T^\partial UP\mathscr{X}$ to $(T^{op}\mathscr{X})^{op}$, and we obtain it from Theorem 5.3 by applying $\overline{T^\partial}$ to the relation $\ni_{\mathscr{X}}$. That $\tau_{\mathscr{X}}$ so defined is natural, follows from Corollary 7.2. This follows [KKuV] with the exception that here now we need to use T^∂.

Example 7.3. Recall the functor $\mathbb{P}^c_\omega$ of Example 6.4 and consider a coalgebra $c : \mathscr{X} \longrightarrow \mathbb{P}^c_\omega \mathscr{X}$. On the logical side we allow ourselves to write $\nabla\alpha$ for any finite subset α of $U(\mathcal{L})$. Of course, we then have to be careful that the semantics of α agrees with the semantics of the convex closure of α. Interestingly, this is done automatically by the machinery set up in the previous section, since $\mathbb{P}^c_\omega = Q\mathbb{P}_\omega I$ and all these functors are self-dual. By Example 6.6, the semantics of $\nabla\alpha$ is given by

$$x \Vdash \nabla\alpha \quad \Leftrightarrow \quad \forall y \in c(x) \exists \varphi \in \alpha . y \Vdash \varphi \ \text{ and } \forall \varphi \in \alpha \exists y \in c(x) . y \Vdash \varphi.$$

8 Conclusions

We hope to have illustrated in the previous two sections that, after getting used to handle the $(-)_\diamond, (-)^\diamond$ and $(-)^{op}$, the techniques developed here work surprisingly smoothly and will be useful in many future developments. For example, an observation crucial for both [KKuV, KuL] is that composing the singleton map $X \longrightarrow \mathscr{P}X$, $x \mapsto \{x\}$, with the relation $\ni_X \colon \mathscr{P}X \longrightarrow\!\!\!\!\!\!\!+\;\; X$ is id_X. Referring back to (7.19), we find here the same relationship

$$\ni_{\mathscr{A}} \circ (\mathbb{y}_{\mathscr{A}^{op}})_\diamond = (\mathbb{y}_{\mathscr{A}^{op}})^\diamond \circ (\mathbb{y}_{\mathscr{A}^{op}})_\diamond = id_{\mathscr{A}^{op}}$$

The question whether the completeness proof of [KKuV] and the relationship between ∇ and predicate liftings of [KuL] can be carried over to our setting are a direction of future research.

Another direction is the generalisation to categories which are enriched over more general structures than 2, such as commutative quantales. Simulation, relation lifting and final coalgebras in this setting have been studied in [Wo].

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