

# Computations in Finite Groups and Quantum Physics

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**Abstract.** Mathematical core of quantum mechanics is the theory of unitary representations of symmetries of physical systems. We argue that quantum behavior is a natural result of extraction of “observable” information about systems containing “unobservable” elements in their descriptions. Since our aim is physics where the choice between finite and infinite descriptions can not have any empirical consequences, we consider the problem in the finite background. Besides, there are many indications from observations — from the lepton mixing data, for example — that finite groups underly phenomena in particle physics at the deep level. The “finite” approach allows to reduce any quantum dynamics to the simple permutation dynamics and, thus, to express quantum observables in terms of permutation invariants of symmetry groups and their integer characteristics such as sizes of conjugate classes, sizes of group orbits, class coefficients, and dimensions of representations. Our study has been accompanied by computations with finite groups, their representations and invariants. We have used both our  $C$  implementation of algorithms for working with groups and computer algebra system *GAP*.

## 1 Introduction

Symmetry is the leading mathematical principle in quantum mechanics: only systems containing indistinguishable particles demonstrate quantum behavior — any violation of identity of particles destroys quantum interferences.

Mathematical description of any system uses arbitrarily chosen marks for registration and identification elements of the system. Elements of systems with symmetries are decomposed into “homogeneous” sets — group orbits. Only such relations and statements (they are called *invariants*) have objective meaning as are not dependent on relabeling elements lying on the same group orbit. An example of such invariant is the number of elements of a group orbit. To fix an element of a group orbit is possible only with respect to some additional system which appears as “*coordinate system*”, or “*observer*”, or “*measuring device*”. For example, no objective meaning can be attached to electric potentials  $\varphi$  and  $\psi$  or to points of space, denoted (marked) as vectors  $\mathbf{a}$  and  $\mathbf{b}$ . But the combinations

denoted as  $\psi - \varphi$  or  $\mathbf{b} - \mathbf{a}$  (in more general group notation  $\varphi^{-1}\psi$  and  $\mathbf{a}^{-1}\mathbf{b}$ ) are meaningful. These are examples of typical situations where observable objects or relations are group invariants depending on pairs of elements related to observed system and to observer.

The question of “whether the real world is discrete or continuous” or even “finite or infinite” is entirely *metaphysical*, since neither empirical observations nor logical arguments can validate one of the two adoptions — this is a matter of belief or taste. Since the choice between finite (discrete) and infinite (continuous) descriptions can not have any empirical consequences — “physics is independent of metaphysics” — we can boldly take advantage of “finite” consideration without any risk to destroy the physical content of a problem.

In this paper, we consider *finite quantum mechanics* from constructive, algorithmic point of view. Using the fact that *any representation* of finite group can be embedded into a permutation representation, we show that any quantum dynamics can be reduced to *permutations*, and quantum observables can be expressed in terms of *permutation invariants*. Note that the interpretational issues like “*wavefunction collapse*”, “*many-worlds*”, “*many-minds*” etc. disappear in the finite background. We discuss also experimental evidences of fundamental role of finite symmetry groups in particle physics.

## 2 Dynamical Systems and Quantum Evolution

Let us consider **dynamical system** with the finite set of (classical) **states**  $\Omega = \{\omega_1, \dots, \omega_N\}$  in the **discrete time**  $t \in \mathcal{T}$ , where  $\mathcal{T} = \mathbb{Z}$  or  $\mathcal{T} = [0, 1, \dots, T]$ . We assume that a finite **symmetry group**  $\mathbf{G} = \{\mathbf{g}_1, \dots, \mathbf{g}_M\} \leq \text{Sym}(\Omega)$  acts on the set of states.

**Classical evolution** (trajectory) of the dynamical system is a sequence of states evolving in time  $\dots, \omega_{t-1}, \omega_t, \omega_{t+1}, \dots \in \Omega^{\mathcal{T}}$ .

For reasons that will be clear later, we define **quantum evolution** as a sequence of permutations  $\dots p_{t-1}, p_t, p_{t+1} \dots \in \mathbf{G}^{\mathcal{T}}$ ,  $p_t \in \mathbf{G}$ .

In most physical problems, the whole set of states  $\Omega$  has a special structure of a set of functions  $\Omega = \Sigma^{\mathbf{X}}$  on some **space**  $\mathbf{X}$  with values in some set of **local states**  $\Sigma$ . In dynamical systems with such structure of the set of states nontrivial *gauge structures* — used in physical theories for description of forces — arise naturally. We assume that the space is a finite set  $\mathbf{X} = \{x_1, \dots, x_{|\mathbf{X}|}\}$  possessing nontrivial group of **space symmetries**  $\mathbf{F} = \{f_1, \dots, f_{|\mathbf{F}|}\} \leq \text{Sym}(\mathbf{X})$ . The local states form a finite set  $\Sigma = \{\sigma_1, \dots, \sigma_{|\Sigma|}\}$  provided with the group of **internal symmetries**  $\mathbf{I} = \{\gamma_1, \dots, \gamma_{|\mathbf{I}|}\} \leq \text{Sym}(\Sigma)$ . To combine the space  $\mathbf{F}$  and internal  $\mathbf{I}$  groups into the symmetry group  $\mathbf{G}$  of the whole set of states  $\Omega = \Sigma^{\mathbf{X}}$  we use the following equivalence class of *split extensions*

$$\mathbf{1} \rightarrow \mathbf{I}^{\mathbf{X}} \rightarrow \mathbf{G} \rightarrow \mathbf{F} \rightarrow \mathbf{1}, \quad (1)$$

where  $\mathbf{I}^{\mathbf{X}}$  is the group of  $\mathbf{I}$ -valued functions on the space  $\mathbf{X}$ . This is a natural generalization of constructions used in physical theories. Explicit formulas for

group operations in  $G$  expressed in terms of operations in  $F$  and  $I$  are given in [1,2] — we do not need them here.

The most popular and intuitive approach to quantization — particularly well suited for dynamical systems with space — is Feynman’s path integral: the amplitude of quantum transition from initial to final state is computed by summing up the amplitudes along all possible classical trajectories connecting these states. As is well known, Feynman’s approach is equivalent to the traditional matrix formulation of quantum mechanics where the evolution of a system from an initial to a final state is described by an **evolution matrix**  $U$ :  $|\psi_0\rangle \rightarrow |\psi_T\rangle = U|\psi_0\rangle$ . The evolution matrix of a quantum dynamical system can be represented as the product of matrices corresponding to elementary time steps:  $U = U_{T\leftarrow T-1} \cdots U_{t\leftarrow t-1} \cdots U_{1\leftarrow 0}$ . In fact, it can be shown by straightforward examination that Feynman’s quantization rules — “multiply subsequent events” and “sum up alternative histories” — is simply a rephrasing of the matrix multiplication rule. For the sake of uniformity of consideration we adopt the evolution matrix approach throughout this paper.

Quantum mechanical evolution matrices are unitary operators acting in Hilbert spaces of (quantum) *state vectors* (called also “*wave functions*”, “*amplitudes*” etc.). *Quantum mechanical particles* are associated with unitary representations of certain groups. These representations are called “*singlets*”, “*doublets*”, and so on, in accordance with their dimensions. Multidimensional representations describe the *spin*. A *quantum mechanical experiment* is reduced to comparison of the system state vector  $\psi$  with some sample state vector  $\phi$  provided by a “*measuring apparatus*”. According to the Born rule, the probability to observe the coincidence of the states is equal to  $|\langle\phi|\psi\rangle|^2 / (\langle\phi|\phi\rangle\langle\psi|\psi\rangle)$ .

### 3 Groups, Numbers and Representations

All transitive actions of a finite group  $G = \{g_1, \dots, g_M\}$  on finite sets  $\Omega = \{\omega_1, \dots, \omega_N\}$  can easily be described [3]. Any such set is in one-to-one correspondence with *right*  $H \backslash G$  (or *left*  $G/H$ ) *cosets* of some subgroup  $H \leq G$ . The set  $\Omega$  is called a *homogeneous space* of the group  $G$  (*G-space*). Action of  $G$  on  $\Omega$  is *faithful*, if the subgroup  $H$  does not contain normal subgroups of  $G$ . We can write action in the form of permutations

$$\pi(g) = \begin{pmatrix} \omega_i \\ \omega_i g \end{pmatrix} \sim \begin{pmatrix} Ha \\ Hag \end{pmatrix}, \quad g, a \in G, \quad i = 1, \dots, N.$$

Maximal transitive set  $\Omega$  is the set of all elements of the group  $G$  itself, i.e., the set of cosets of the trivial subgroup  $H = \{1\}$ . The corresponding action is called *regular* and can be represented by the permutations

$$\Pi(g) = \begin{pmatrix} g_i \\ g_i g \end{pmatrix}, \quad i = 1, \dots, M. \tag{2}$$

To introduce a “quantitative” (“statistical”) description, let us assign to the elements of the set  $\Omega$  numerical “weights” from some suitable *number system*

$\mathcal{N}$  containing at least *zero* and *unity*. This allows to rewrite permutations by matrices — this is called *permutation representation*:

$$\pi(g) \rightarrow \rho(g) = (\rho(g)_{ij}), \quad \text{where } \rho(g)_{ij} = \delta_{\omega_i g, \omega_j}; \quad i, j = 1, \dots, \mathbb{N}. \quad (3)$$

Here  $\delta_{\alpha, \beta}$  is the Kronecker delta on  $\Omega$ .

The *cycle type* of a permutation is array of multiplicities of lengths of cycles in decomposition of the permutation into disjoint cycles. The cycle type is usually denoted by  $1^{k_1} 2^{k_2} \dots n^{k_n}$ , where  $k_i$  is the number of cycles of the length  $i$  in the permutation. The *characteristic polynomial* of permutation matrix (3) can be written immediately from the cycle type of the corresponding permutation  $\pi(g)$ :

$$\chi_{\rho(g)}(\lambda) = \det(\rho(g) - \lambda \mathbf{I}) = (\lambda - 1)^{k_1} (\lambda^2 - 1)^{k_2} \dots (\lambda^n - 1)^{k_n}. \quad (4)$$

The matrix form of permutations (2) representing the *regular* action

$$\Pi(g) \rightarrow \mathbf{P}(g) = (\mathbf{P}(g)_{ij}), \quad \mathbf{P}(g)_{ij} = \delta_{e_i g, e_j}, \quad i, j = 1, \dots, \mathbf{M} \quad (5)$$

is called the *regular representation* — this is a special case of (3).

For the sake of freedom of algebraic manipulations, one assumes usually that  $\mathcal{N}$  is an algebraically closed field — a standard choice is the field of complex numbers  $\mathbb{C}$ . If  $\mathcal{N}$  is a field, then the set  $\Omega$  can be treated as a basis of linear vector space  $\mathcal{H} = \text{Span}(\omega_1, \dots, \omega_{\mathbb{N}})$ .

The field  $\mathbb{C}$  is excessively large — most of its elements are non-constructive. What is really needed can be constructed as follows. As is clear from (4), all *eigenvalues* of permutation matrices are  $E$ th roots of unity, where  $E$  is the *exponent* of the group  $\mathbf{G}$  — the least common multiple of orders of the group elements. The  $E$ th roots of unity can be expressed in terms of  $\mathcal{P}$ th roots, where  $\mathcal{P}$  is some divisor of  $E$  called *conductor*. As a first step, we combine the roots of unity with *natural numbers*  $\mathbb{N} = \{0, 1, \dots\}$  to construct the set  $\mathcal{N}_{\mathcal{P}} = \mathbb{N}[r]$  of polynomials of the form  $n_1 + n_2 r + \dots + n_{\mathcal{P}} r^{\mathcal{P}-1}$ , where  $n_k \in \mathbb{N}$ ;  $r$  is *primitive*  $\mathcal{P}$ th root of unity, i.e. period of  $r$  is equal exactly to  $\mathcal{P}$ . For intuitive perception one could bear in mind the symbolics  $r = e^{2\pi i/\mathcal{P}}$  for the primitive root, but we will never use this representation. The following *algebraic* definitions are sufficient for all computations

1. *Multiplication*:  $r^k \times r^m = r^{k+m \bmod \mathcal{P}}$ ,
2. *Complex conjugation*:  $\overline{r^k} = r^{\mathcal{P}-k}$ .

If  $\mathcal{P} = 1$ , then  $\mathcal{N}_1$  is the *semi-ring of natural numbers*  $\mathbb{N}$ .

If  $\mathcal{P} \geq 2$ , then *negative integer numbers* can be introduced via the definition

$$(-1) = \sum_{k=1}^{\mathcal{P}-1} r^{\frac{\mathcal{P}}{2}k}, \quad \text{where } p \text{ is any factor of } \mathcal{P}. \quad \text{So we obtain the } \textit{ring of integers } \mathbb{Z}.$$

If  $\mathcal{P} \geq 3$ , then the set  $\mathcal{N}_{\mathcal{P}}$  is a *commutative ring* embeddable into the field of complex numbers  $\mathbb{C}$ . This is the ring of *cyclotomic integers*:  $\mathcal{N}_{\mathcal{P}} = \mathbb{Z}[r] / \langle \Phi_{\mathcal{P}}(r) \rangle$ . Here  $\Phi_{\mathcal{P}}(r)$  is the  $\mathcal{P}$ th *cyclotomic polynomial* — the product of the binomials  $r - \zeta$ , where  $\zeta$  runs over *all primitive*  $\mathcal{P}$ th roots of unity.

The ring  $\mathcal{N}_{\mathcal{P}}$  is sufficient for almost all computations with finite quantum models. For simplicity of linear algebra we extend the ring  $\mathcal{N}_{\mathcal{P}}$  to the  $\mathcal{P}$ th *cyclotomic field*  $\mathbb{Q}_{\mathcal{P}} = \mathbb{Q}[r] / \langle \Phi_{\mathcal{P}}(r) \rangle$ . When computing matrices of *unitary* representations *square roots of dimensions* of representations arise as normalization factors. Since square roots of integers are always cyclotomic integers we can treat all irrationalities arising in computations — roots of unity and square roots of dimensions — as belonging to a ring of cyclotomic integers  $\mathcal{N}_n$  with some  $n$  (usually  $n > \mathcal{P}$ ). We can also construct a minimal *abelian number field*  $\mathcal{F}$  containing a given set of irrationalities. It is a subfield of the cyclotomic field  $\mathbb{Q}_n$ . The term *abelian* means here that  $\mathcal{F}$  is an extension with abelian Galois group. The command `Field(gens)` in the computer algebra system *GAP* [4] returns the *smallest* field that contains all elements from the list *gens*. As to the finite quantum systems discussed in this paper, the roots of unity and other irrationalities are only intermediate entities in description of quantum behavior — they disappear in the final “observables”.

Any linear representation of a finite group is equivalent to unitary, since one can always construct invariant inner product from an arbitrary one by “averaging over the group”. Starting from, e.g., the *standard inner product* in  $K$ -dimensional Hilbert space  $\mathcal{H}$

$$\langle \phi | \psi \rangle \equiv \sum_{i=1}^K \overline{\phi^i} \psi^i \quad (6)$$

we can come via the averaging to the *invariant inner product*:

$$\langle \phi | \psi \rangle \equiv \frac{1}{|G|} \sum_{g \in G} (U(g) \phi | U(g) \psi). \quad (7)$$

Here  $U$  is a representation of a group  $G$  in the space  $\mathcal{H}$ .

An important transformation of group elements — an analog of change of coordinates in physics — is the conjugation:  $a^{-1}ga \rightarrow g'$ ,  $g, g' \in G$ ,  $a \in \text{Aut}(G)$ . Conjugation by an element of the group itself, i.e., if  $a \in G$ , is called an *inner automorphism*. The equivalence classes with respect to the inner automorphisms are called *conjugacy classes*. The starting point in study of representations of a group is its decomposition into conjugacy classes

$$G = K_1 \sqcup K_2 \sqcup \dots \sqcup K_m.$$

The group multiplication induces *multiplication* on the classes. The product of  $K_i$  and  $K_j$  is the *multiset* of all possible products  $ab$ ,  $a \in K_i$ ,  $b \in K_j$ , decomposed into classes. This multiplication is obviously commutative, since  $ab$  and  $ba$  belong to the same class:  $ab \sim a^{-1}(ab)a = ba$ . Thus, the multiplication table for classes is given by

$$K_i K_j = K_j K_i = \sum_{k=1}^m c_{ijk} K_k. \quad (8)$$

The *natural integers*  $c_{ijk}$  — multiplicities of classes in the multisets — are called *class coefficients*.

This is a short list of main properties of linear representations of finite groups:

1. Any irreducible representation is contained in the regular representation. More specifically, there exists matrix  $T$  transforming simultaneously all matrices (5) to the form

$$T^{-1}P(g)T = \begin{pmatrix} D_1(g) & & & & \\ & d_2 \begin{pmatrix} D_2(g) & & & \\ & \ddots & & \\ & & D_2(g) & \\ & & & \ddots \end{pmatrix} & & & \\ & & & & & \\ & & & & & \\ & & & & d_m \begin{pmatrix} D_m(g) & & & \\ & \ddots & & \\ & & D_m(g) & \\ & & & \ddots \end{pmatrix} & & \\ & & & & & & & & & & \end{pmatrix}, \quad (9)$$

and any irreducible representation is one of  $D_j$ 's. The numbers of non-equivalent irreducible representation and conjugacy classes coincide. The number  $d_j$  is the dimension of the irreducible component  $D_j$  and simultaneously the multiplicity of its occurrence in the regular representation. It is clear from (9) that for the dimensions of irreducible representations the following relation holds:  $d_1^2 + d_2^2 + \dots + d_m^2 = |G| = M$ . The dimensions of irreducible representations divide the group order:  $d_j \mid M$ .

2. Any irreducible representation  $D_j$  is determined uniquely by its *character*  $\chi_j$  defined as the trace of the representation matrix:  $\chi_j(g) = \text{Tr}D_j(g)$ . This is a function on the conjugacy classes since  $\chi_j(g) = \chi_j(a^{-1}ga)$ . Obviously,  $\chi_j(\mathbf{1}) = d_j$ .
3. A compact form of recording all irreducible representations is the *character table*. The columns of this table are numbered by the conjugacy classes, while its rows contain values of characters of non-equivalent representation:

	$K_1$	$K_2$	$\dots$	$K_m$
$\chi_1$	1	1	$\dots$	1
$\chi_2$	$\chi_2(K_1) = d_2$	$\chi_2(K_2)$	$\dots$	$\chi_2(K_m)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\chi_m$	$\chi_m(K_1) = d_m$	$\chi_m(K_2)$	$\dots$	$\chi_m(K_m)$

By convention, the 1st column corresponds to the identity class, and the 1st row contains the *trivial* representation.

## 4 Finite Quantum Systems

In quantum mechanics all possible states of every physical system are represented by vectors  $\psi$  in a Hilbert space  $\mathcal{H}$ . It is assumed that vectors  $\psi$  and  $\psi'$  describe identical states if they are proportional through a complex factor:

$\psi' = \lambda\psi$ ,  $\lambda \in \mathbb{C}$ . Evolution of the system from any initial state  $\psi_0$  into the corresponding final state  $\psi_T$  is described by an *unitary* operator  $U$ :  $|\psi_T\rangle = U|\psi_0\rangle$ . The unitarity means that  $U$  belongs to the automorphism group of the Hilbert space:  $U \in \text{Aut}(\mathcal{H})$ . One may regard  $\text{Aut}(\mathcal{H})$  as a faithful representation of respective abstract group  $G$ . In the continuous time the dynamics can be expressed by the Schrödinger equation

$$i \frac{d}{dt} |\psi\rangle = H |\psi\rangle$$

in terms of the local *Hermitian* operator  $H$  called the *Hamiltonian* or *energy operator*. If  $H$  is independent of time, then the relation  $U = e^{-iHT}$  holds.

A finite quantum system is formulated in exactly the same way. The only difference is that now the group  $G$  is a finite group of order  $M$  having unitary representation  $U$  in  $K$ -dimensional Hilbert space  $\mathcal{H}_K$  over some abelian number field  $\mathcal{F}$  instead of  $\mathbb{C}$ . All possible evolution operators form the finite set  $\{U_1, \dots, U_M\}$  of unitary matrices from  $U$ .

Since the matrices  $U_j$  are non-singular, one can always introduce Hamiltonians by the formula  $H_j = i \ln U_j \equiv \sum_{k=0}^{p-1} \lambda_k U_j^k$ , where  $p$  is period of  $U_j$ ,  $\lambda_k$ 's are some coefficients<sup>1</sup>; but there is no need to do so.

More generally, *hermitian operators*  $A$  describing *observables* in quantum formalism can be written as elements of the group algebra representation:

$$A = \sum_{k=1}^M \alpha_k U_k.$$

Finite groups — unless they are many-component direct products — can be often generated by a small number of elements. For example, all simple and all symmetric groups are generated by two elements. The algorithm restoring the whole group from  $n_g$  generators is very simple. It is reduced to  $n_g(M - n_g - 1)$  group multiplications. So the finite quantum models are well suited for computer algebra methods.

#### 4.1 Reducing Quantum Dynamics to Permutations

It follows from decomposition (9) that any  $K$ -dimensional representation  $U$  can be extended to an  $N$ -dimensional representation  $\tilde{U}$  in a Hilbert space  $\mathcal{H}_N$ , in such a way that the representation  $\tilde{U}$  corresponds to the *permutation action* of the group  $G$  on some  $N$ -element set of entities  $\Omega = \{\omega_1, \dots, \omega_N\}$ . It is clear that  $N \geq K$ .

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<sup>1</sup> Note that the logarithmic function being essentially a construction from continuous mathematics introduces into the  $\lambda_k$ 's a *non-algebraic* element — namely,  $\pi$  — expressed by *infinite* sum of elements from  $\mathcal{F}$ . In other words, the  $\lambda_k$ 's are elements of a *transcendental extension* of  $\mathcal{F}$ .

The case when  $N$  is strictly greater than  $K$  is most interesting. Clearly, the additional “hidden parameters” — appearing in this case due to increase of the number of states (dimension of space) — in no way can affect the data relating to the space  $\mathcal{H}_K$  since both  $\mathcal{H}_K$  and its complement in  $\mathcal{H}_N$  are invariant subspaces of the extended space  $\mathcal{H}_N$ . Thus, *any quantum problem* in  $K$ -dimensional Hilbert space can be reformulated in terms of permutations of  $N$  things.

From the algorithmic point of view, manipulations with permutations are much more efficient than the linear algebra operations with matrices. Of course, degrees of permutations  $N$  might be much larger than dimensions of matrices  $K$ . However, the very possibility to *reduce quantum dynamics to permutations* is much more important conceptually than the algorithmic issues.

#### 4.2 Connection with Observation. The Born Rule

In quantum mechanics, the link between mathematical description and experiment is provided by the *Born rule*, stating that the *probability* to observe a quantum system being in the state  $\psi$  by apparatus tuned to the state  $\phi$  is expressed by the number

$$\mathbf{P}(\phi, \psi) = \frac{|\langle \phi | \psi \rangle|^2}{\langle \phi | \phi \rangle \langle \psi | \psi \rangle}. \quad (10)$$

This expression can be rewritten in a form including the pair “system–apparatus” in more symmetric way

$$\mathbf{P}(\phi, \psi) = \frac{|\langle \phi | \psi \rangle|^2}{|\langle \phi | \psi \rangle|^2 + \|\phi \wedge \psi\|^2}.$$

Here  $\phi \wedge \psi$  is exterior (Grassmann) product of the vectors  $\phi$  and  $\psi$ , which is the  $K(K-1)/2$ -dimensional vector with the components in the unitary basis  $(\phi \wedge \psi)^{ij} = \phi^i \psi^j - \phi^j \psi^i$  and with the square of norm

$$\|\phi \wedge \psi\|^2 = \sum_{i=1}^{K-1} \sum_{j=i}^K |\phi^i \psi^j - \phi^j \psi^i|^2.$$

There are many philosophical speculations concerning the concept of probability and its interpretation. However, what is really used in practice is the *frequency interpretation*: the probability is the ratio of the number of favorable cases to the total number of cases. In the case of finite sets there are no complications at all: the probability is the rational number — the ratio of the number of singled out elements of a set to the total number of elements of the set.

It can be shown that if data about states of a system and apparatus are represented in the permutation basis by *natural numbers*, then formula (10) gives *rational numbers* in the invariant subspaces of the permutation representation also, in spite of possible presence of cyclotomics and square roots in the intermediate computations.

Let us consider permutation action of the group  $G = \{g_1, \dots, g_M\}$  on the set entities  $\Omega = \{\omega_1, \dots, \omega_N\}$ . We will describe the (quantum) states of the system and apparatus in the permutation representation by the vectors

$$|n\rangle = \begin{pmatrix} n_1 \\ \vdots \\ n_N \end{pmatrix} \text{ and } |m\rangle = \begin{pmatrix} m_1 \\ \vdots \\ m_N \end{pmatrix}, \quad (11)$$

respectively. It is natural to assume that  $n_i$  and  $m_i$  are *natural numbers*, interpreting them as the “multiplicities of occurrences” of the element  $\omega_i$  in the system and apparatus states, respectively. In other words, the vectors  $|n\rangle$  and  $|m\rangle$  are elements of  $N$ -dimensional module  $H_N$  over the semi-ring  $N$ . Permutation action of  $G$  on  $\Omega$  is equivalent to matrix representation of  $G$  in the module  $H_N$ . We can turn the module  $H_N$  into the Hilbert space  $\mathcal{H}_N$  by extending the semi-ring  $N$  to an abelian number field  $\mathcal{F}$  compatible with the structure of  $G$ .

Of course, due to the symmetry the numbers  $n_i$  and  $m_i$  are not observable. Only their *invariant combinations* are observable. Since the standard inner product defined in (6) is invariant for the permutation representation, in accordance with the Born rule we have

$$\mathbf{P}(m, n) = \frac{(\sum_i m_i n_i)^2}{\sum_i m_i^2 \sum_i n_i^2}. \quad (12)$$

It is clear that for non-vanishing natural vectors  $|n\rangle$  and  $|m\rangle$  expression (12) is a rational number strictly greater than zero. This means, in particular, that it is impossible to observe destructive quantum interference here. However, the *destructive interference* of the vectors with natural components can be observed in the proper invariant subspaces of the permutation representation.

## 5 Example: Group of Permutations of Three Things

$S_3$  is the smallest non-commutative group providing a non-trivial quantum behavior. Nevertheless,  $S_3$  has important applications in the lepton sector of flavor physics. The group consists of six elements having the following representation by permutations

$$g_1 = (), \quad g_2 = (2, 3), \quad g_3 = (1, 3), \quad g_4 = (1, 2), \quad g_5 = (1, 2, 3), \quad g_6 = (1, 3, 2). \quad (13)$$

The group can be generated by many pairs of its elements. Let us choose, for instance,  $g_2$  and  $g_6$  as generators.  $S_3$  decomposes into three conjugacy classes

$$K_1 = \{g_1\}, \quad K_2 = \{g_2, g_3, g_4\}, \quad K_3 = \{g_5, g_6\} \quad (14)$$

with the following multiplication table

$$K_1 K_j = K_j, \quad K_2^2 = 3K_1 + 3K_3, \quad K_2 K_3 = 2K_2, \quad K_3^2 = 2K_1 + K_3.$$

The group  $S_3$  has the following character table

$$\begin{array}{c|ccc}
 & K_1 & K_2 & K_3 \\
 \chi_1 & 1 & 1 & 1 \\
 \chi_2 & 1 & -1 & 1 \\
 \chi_3 & 2 & 0 & -1
 \end{array} . \tag{15}$$

Matrices of permutation representation of generators are

$$P_2 = \begin{pmatrix} 1 & \cdot & \cdot \\ \cdot & \cdot & 1 \\ \cdot & 1 & \cdot \end{pmatrix} \text{ and } P_6 = \begin{pmatrix} \cdot & \cdot & 1 \\ 1 & \cdot & \cdot \\ \cdot & 1 & \cdot \end{pmatrix} . \tag{16}$$

The eigenvalues of  $P_2$  and  $P_6$  are  $(1, 1, -1)$  and  $(1, r, r^2)$ , respectively;  $r$  is a primitive third root of unity with cyclotomic polynomial  $\Phi_3(r) = 1 + r + r^2$ .

Since any permutation representation contains one-dimensional invariant subspace with the basis vector  $(1, \dots, 1)^T$ , the only possible structure of decomposition of permutation representation into irreducible parts is the following

$$\tilde{U}_j = \begin{pmatrix} 1 & 0 \\ 0 & U_j \end{pmatrix}, \quad j = 1, \dots, 6, \tag{17}$$

where the matrices  $1$  and  $U_j$  correspond to one-dimensional trivial (character  $\chi_1$ ) and two-dimensional faithful (character  $\chi_3$ ) representations, respectively.

To construct decomposition (17) we should determine matrices  $U_j$  and  $T$  such that  $\tilde{U}_j = T^{-1}P_jT$ . In addition we impose unitarity on all the matrices. Clearly, it suffices to perform the procedure only for matrices of generators. There are different ways to construct decomposition (17).

If we start with the diagonalization of  $P_6$ , we come to the following<sup>2</sup>

$$\begin{aligned}
 U_1 &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad U_2 = \begin{pmatrix} 0 & r^2 \\ r & 0 \end{pmatrix}, \quad U_3 = \begin{pmatrix} 0 & r \\ r^2 & 0 \end{pmatrix}, \\
 U_4 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad U_5 = \begin{pmatrix} r^2 & 0 \\ 0 & r \end{pmatrix}, \quad U_6 = \begin{pmatrix} r & 0 \\ 0 & r^2 \end{pmatrix}.
 \end{aligned} \tag{18}$$

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<sup>2</sup> Note the peculiarity of representation (18) — its matrices are very similar to matrices of permutations: there is exactly one non-zero entry in each column and in each row. But in contrast to permutation matrices in which any non-zero entry is *unity*, non-zeros in (18) are *roots of unity*. This is because  $S_3$  is one of the so-called *monomial groups* [5] for which all irreducible representations can be constructed as induced from one-dimensional representations of their subgroups — choosing diagonal form for  $U_6$  is just equivalent to inducing (18) from representation of cyclic subgroup  $\mathbb{Z}_3 \leq S_3$ . Most groups, at least of small orders, are just monomial. For example, it can be checked with the help of *GAP* that the total number of all non-isomorphic groups of order  $< 384$  is equal to 67424, but only 249 of them are *non-monomial*. The minimal non-monomial group is the 24-element group  $SL(2, 3)$  of  $2 \times 2$  matrices in the characteristic 3 with unit determinants.

The transformation matrix (up to inessential degrees of freedom for its entries) takes the following form

$$\mathbb{T} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & r^2 \\ 1 & r^2 & 1 \\ 1 & r & r \end{pmatrix}, \quad \mathbb{T}^{-1} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & r & r^2 \\ r & 1 & r^2 \end{pmatrix}. \quad (19)$$

Otherwise, the diagonalization of  $P_2$  leads to another second component of decomposition (17) (we present here only the generator matrices)

$$U'_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad U'_6 = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}.$$

The transformation matrix in this case takes the form

$$\mathbb{T}' = \begin{pmatrix} \frac{1}{\sqrt{3}} & \sqrt{\frac{2}{3}} & 0 \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \end{pmatrix}, \quad \mathbb{T}'^{-1} = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \sqrt{\frac{2}{3}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}. \quad (20)$$

The matrix  $\mathbb{T}'$  is known in particle physics under the names *Harrison-Perkins-Scott* or *tribimaximal* mixing matrix. It is used to description of neutrino oscillation data.

The information about “quantum behavior” is encoded, in fact, in transformation matrices like (19) or (20).

Let  $|n\rangle = \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}$  and  $|m\rangle = \begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix}$  be system and apparatus state vectors in the “permutation” basis. Transformation of these vectors from the permutation to “quantum” basis with the help of, say, (19) leads to

$$\begin{aligned} |\tilde{\psi}\rangle &= \mathbb{T}^{-1} |n\rangle = \frac{1}{\sqrt{3}} \begin{pmatrix} n_1 + n_2 + n_3 \\ n_1 + n_2 r + n_3 r^2 \\ n_1 r + n_2 + n_3 r^2 \end{pmatrix}, \\ |\tilde{\phi}\rangle &= \mathbb{T}^{-1} |m\rangle = \frac{1}{\sqrt{3}} \begin{pmatrix} m_1 + m_2 + m_3 \\ m_1 + m_2 r + m_3 r^2 \\ m_1 r + m_2 + m_3 r^2 \end{pmatrix}. \end{aligned}$$

Projections of the vectors onto two-dimensional invariant subspace are:

$$|\psi\rangle = \frac{1}{\sqrt{3}} \begin{pmatrix} n_1 + n_2 r + n_3 r^2 \\ n_1 r + n_2 + n_3 r^2 \end{pmatrix}, \quad |\phi\rangle = \frac{1}{\sqrt{3}} \begin{pmatrix} m_1 + m_2 r + m_3 r^2 \\ m_1 r + m_2 + m_3 r^2 \end{pmatrix}. \quad (21)$$

The same manipulation with matrix (20) leads to

$$|\psi'\rangle = \begin{pmatrix} n_1 \sqrt{\frac{2}{3}} - n_2 \frac{1}{\sqrt{6}} - n_3 \frac{1}{\sqrt{6}} \\ -n_2 \frac{1}{\sqrt{2}} + n_3 \frac{1}{\sqrt{2}} \end{pmatrix}, \quad |\phi'\rangle = \begin{pmatrix} m_1 \sqrt{\frac{2}{3}} - m_2 \frac{1}{\sqrt{6}} - m_3 \frac{1}{\sqrt{6}} \\ -m_2 \frac{1}{\sqrt{2}} + m_3 \frac{1}{\sqrt{2}} \end{pmatrix}. \quad (22)$$

Constituents of Born's probability (10) for the two-dimensional subsystem — clearly, the same in both cases (21) and (22) — are

$$\langle \psi | \psi \rangle = Q_3(n, n) - \frac{1}{3} L_3(n)^2, \quad (23)$$

$$\langle \phi | \phi \rangle = Q_3(m, m) - \frac{1}{3} L_3(m)^2, \quad (24)$$

$$|\langle \phi | \psi \rangle|^2 = \left( Q_3(m, n) - \frac{1}{3} L_3(m) L_3(n) \right)^2, \quad (25)$$

where  $L_N(n) = \sum_{i=1}^N n_i$  and  $Q_N(m, n) = \sum_{i=1}^N m_i n_i$  are linear and quadratic permutation invariants, respectively.

Note that:

1. Expressions (23)–(25) consist of the *invariants of permutation representation*. This is a manifestation of fundamental role of permutations in quantum description.
2. Expressions (23) and (24) are always positive rational numbers for  $|n\rangle$  and  $|m\rangle$  with different components.
3. Conditions for *destructive quantum interference* — vanishing Born's probability — are determined by the equation

$$3(m_1 n_1 + m_2 n_2 + m_3 n_3) - (m_1 + m_2 + m_3)(n_1 + n_2 + n_3) = 0.$$

This equation has infinitely many solutions in natural numbers. An example

$$\text{of such a solution is: } |n\rangle = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \quad |m\rangle = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}.$$

Thus, we have obtained essential features of quantum behavior from “permutation dynamics” and “natural” interpretation (11) of quantum amplitude by a simple transition to invariant subspaces.

Recall once more that any permutation representation contains the trivial one-dimensional subrepresentation and, hence, has  $(N - 1)$ -dimensional invariant subspace. The inner product in this subspace can be expressed in terms of the permutation invariants by the formula

$$\langle \phi | \psi \rangle = Q_N(m, n) - \frac{1}{N} L_N(m) L_N(n).$$

The identity  $Q_N(n, n) - \frac{1}{N} L_N(n)^2 \equiv \frac{1}{N^2} \sum_{i=1}^N (L_N(n) - N n_i)^2$  shows explicitly that  $\langle \psi | \psi \rangle > 0$  for  $|n\rangle$  with different components  $n_i$ . This inner product does not contain irrationalities for natural  $|n\rangle$  and  $|m\rangle$ . This is not the case for other invariant subspaces. Nevertheless irrationalities disappear in the squared modulus of the inner product  $|\langle \phi | \psi \rangle|^2$ . To give a simple illustration let us consider

the cyclic group  $\mathbb{Z}_3$ . Its three-dimensional permutation representation decomposes into three one-dimensional irreducible components. E.g., for the generator  $g = (1, 2, 3)$  of  $\mathbb{Z}_3$  we have

$$P = \begin{pmatrix} \cdot & 1 & \cdot \\ \cdot & \cdot & 1 \\ 1 & \cdot & \cdot \end{pmatrix} \longrightarrow \tilde{U} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & r^2 \end{pmatrix}, \quad r \text{ is a primitive third root of unity.}$$

The inner product in one-dimensional subspace corresponding to the eigenvalue, say  $r$ , contains irrationalities:  $\langle \phi | \psi \rangle = \frac{1}{3} (\mathbb{Q}_3(m, n) + rC(m, n) + r^2C'(m, n))$ , but  $|\langle \phi | \psi \rangle|^2 = \frac{1}{9} (\mathbb{Q}_3(m, m) - C(m, m)) (\mathbb{Q}_3(n, n) - C(n, n))$  is free of them. The invariants  $C(m, n) = m_1n_3 + m_2n_1 + m_3n_2$  and  $C'(m, n) = m_1n_2 + m_2n_3 + m_3n_1$  are specific for the group  $\mathbb{Z}_3$  in contrast to  $L_N(n)$  and  $Q_N(m, n)$  that are common to all permutation groups.

## 6 Finite Symmetry Groups in Particle Physics

At present, all observations concerning fundamental particles [6] are compatible with the Standard Model (SM). The SM is a gauge theory with the group of internal (gauge) symmetries  $\Gamma = \text{SU}(3) \times \text{SU}(2) \times \text{U}(1)$ . In the context of Grand Unified Theory (GUT)  $\Gamma$  is assumed to be a subgroup of some larger (simple) group. With respect to space-time symmetries, the elementary particles are divided into two classes: *bosons*, responsible for physical forces (roughly speaking, they are elements of the gauge group) and *fermions*, usually treated as particles of matter. The fermions of the SM are divided into three *generations* of *quarks* and *leptons* as follows (antiparticles are omitted for brevity):

	Generations					
	1		2		3	
Up-quarks	Up	$u$	Charm	$c$	Top	$t$
Down-quarks	Down	$d$	Strange	$s$	Bottom	$b$
Charged leptons	Electron	$e^-$	Muon	$\mu^-$	Tau	$\tau^-$
Neutrinos	Electron neutrino	$\nu_e$	Muon neutrino	$\nu_\mu$	Tau neutrino	$\nu_\tau$

Between generations particles differ only by their mass and quantum property called *flavor*. The flavor changing transitions — taking place in such phenomena as weak decays of quarks and neutrino oscillations — are described by  $3 \times 3$  unitary *mixing matrices*. The outputs of experiments allow to calculate magnitudes of elements of these matrices.

In the case of quarks (“*in the quark sector*”), the mixing matrix describing transitions between up- and down-type quarks is the *Cabibbo–Kobayashi–Maskawa* (CKM) matrix

$$V_{\text{CKM}} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix},$$

where  $|V_{\alpha\beta}|^2$  represents the probability that the quark (of flavor)  $\beta$  decays into a quark  $\alpha$ . The current experimental data rounded to three significant digits are:

$$\begin{pmatrix} |V_{ud}| & |V_{us}| & |V_{ub}| \\ |V_{cd}| & |V_{cs}| & |V_{cb}| \\ |V_{td}| & |V_{ts}| & |V_{tb}| \end{pmatrix} = \begin{pmatrix} 0.974 & 0.225 & 0.004 \\ 0.225 & 0.974 & 0.041 \\ 0.009 & 0.040 & 0.999 \end{pmatrix}.$$

More precise values can be found in [6].

In the lepton sector weak interaction processes are described by the *Ponte-corvo–Maki–Nakagawa–Sakata* (PMNS) mixing matrix

$$U_{\text{PMNS}} = \begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu1} & U_{\mu2} & U_{\mu3} \\ U_{\tau1} & U_{\tau2} & U_{\tau3} \end{pmatrix}.$$

Here indices  $e, \mu, \tau$  correspond to neutrino flavors — this means that the neutrinos  $\nu_e, \nu_\mu, \nu_\tau$  are produced with  $e^+, \mu^+, \tau^+$  (or produce  $e^-, \mu^-, \tau^-$ ), respectively, in weak processes. The indices 1, 2, 3 correspond to the *mass eigenstates*, i.e., neutrinos  $\nu_1, \nu_2, \nu_3$  with definite masses  $m_1, m_2, m_3$ . Numerous experiments with solar, atmospheric, reactor, and accelerator neutrinos indicate the existence of discrete symmetries that can not be deduced from the SM. The phenomenological pattern is the following [7]:

1.  $\nu_\mu$  and  $\nu_\tau$  flavors are presented with equal weights in all three mass eigenstates  $\nu_1, \nu_2, \nu_3$  (this is called “*bi-maximal mixing*”):  
 $|U_{\mu i}|^2 = |U_{\tau i}|^2, \quad i = 1, 2, 3;$
2. all three flavors are presented equally in  $\nu_2$  (“*trimaximal mixing*”):  
 $|U_{e2}|^2 = |U_{\mu2}|^2 = |U_{\tau2}|^2;$
3.  $\nu_e$  is absent in  $\nu_3$ :  $|U_{\mu3}|^2 = 0$ .

These relations together with the normalization condition for probabilities allow to determine moduli-squared of all matrix elements:

$$(|U_{li}|^2) = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} & 0 \\ \frac{1}{6} & \frac{1}{3} & \frac{1}{2} \\ \frac{1}{6} & \frac{1}{3} & \frac{1}{2} \end{pmatrix}. \quad (26)$$

A particular form of unitary matrix satisfying data (26) was suggested by Harrison, Perkins, and Scott in [8]:

$$U_{\text{TB}} = \begin{pmatrix} \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix}. \quad (27)$$

This so-called *tribimaximal* (TB) mixing matrix coincides — up to the trivial permutation of two columns corresponding to the renaming  $\nu_1 \leftrightarrow \nu_2$  of states

— with transformation matrix (20) decomposing the natural permutation representation of the group  $S_3$  into irreducible components. This means that we can identify the flavor basis with the representation basis of permutations of three things, and the mass basis is a basis of irreducible decomposition of this representation. In [9] Harrison and Scott study in detail connections of the neutrino mass matrix with the character table and class algebra of the group  $S_3$ . At present, much effort is devoted to the construction and study of models based on finite flavor symmetries (for recent reviews, see, for example, [10,11]). The most popular groups for constructing such models are:

- $T = A_4$  — the tetrahedral group;
- $T'$  — the double covering of  $A_4$ ;
- $O = S_4$  — the octahedral group;
- $I = A_5$  — the icosahedral group;
- $D_N$  — the dihedral groups ( $N$  even);
- $Q_N$  — the quaternionic groups (4 divides  $N$ );
- $\Sigma(2N^2)$  — the groups in this series have the structure  $(\mathbb{Z}_N \times \mathbb{Z}_N) \rtimes \mathbb{Z}_2$ ;
- $\Delta(3N^2)$  — the structure  $(\mathbb{Z}_N \times \mathbb{Z}_N) \rtimes \mathbb{Z}_3$ ;
- $\Sigma(3N^3)$  — the structure  $(\mathbb{Z}_N \times \mathbb{Z}_N \times \mathbb{Z}_N) \rtimes \mathbb{Z}_3$ ;
- $\Delta(6N^2)$  — the structure  $(\mathbb{Z}_N \times \mathbb{Z}_N) \rtimes S_3$ .

As to the quark sector, observations do not give such sharp picture as in the lepton case. In [12] the  $D_{14}$  symmetry was suggested for explanation of the value of the Cabibbo angle (one of the parameters of the CKM matrix), but without any connection with the leptonic symmetries. The natural attempts to find discrete symmetries unifying leptons and quarks still remain not very successful, though there are some encouraging observations, for example, the *quark-lepton complementarity* (QLC) — observation that the sum of quark and lepton mixing angles is equal approximately to  $\pi/4$ .

The origin of finite symmetries among fundamental particles is unclear. There are different attempts to explain — sometimes looking a bit complicated and artificial, for example, these symmetries are treated as symmetries of manifolds arising at compactification of a higher dimensional theory to four spacetime dimensions [13]. The idea that symmetries at the most fundamental level are *per se* finite looks more attractive in our opinion. In this approach, unitary groups used in physical theories can be treated simply as repositories of all finite groups having faithful representation of corresponding dimensions:  $U(n)$  contains all finite groups with faithful  $n$ -dimensional representations. Of course, due to redundancy of the field  $\mathbb{C}$ ,  $U(n)$  is not a minimal group with this property.

Such small groups as  $S_3$ ,  $A_4$ , etc. are most likely only remnants of large combinations of more fundamental finite symmetries that are expected to exist at the GUT scale. Unfortunately the GUT scale ( $10^{16}$  GeV) being close to the Planck scale ( $10^{19}$  GeV) is out of reach of experiments (the most powerful colliders to date can provide only about  $10^4$  GeV). Thus, the only practical way is to construct models, study them by the computational group theory methods, and compare consequences of these models with available experimental data.

## Conclusion

“Finite” analysis shows that quantum behavior is a manifestation of indistinguishability of objects, i.e., fundamental impossibility to trace the identity of homogeneous objects in the process of their evolution.

Only “statistical” statements about numbers of certain invariant combinations of elements may have objective significance. These statements can be expressed in terms of group invariants and natural numbers characterizing symmetry groups, such as dimensions of its representations, class coefficients etc.

Any quantum mechanical problem can be reduced to permutations since permutation representations contain all other representations. This — together with natural interpretation of quantum amplitudes as vectors of “multiplicities of occurrences” of underlying permuted entities — makes quantum mechanical problems constructive and particularly suitable for their study by computer algebra and computational group theory methods.

The models based on finite groups are now extensively studied in particle physics, since there are strong observational evidences of finite symmetries in fundamental physical processes.

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