# Symbolic-manipulation constructions of Hilbert-space metrics in quantum mechanics 

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#### Abstract

${ }^{1}$ The problem of the determination of the Hilbert-space metric $\Theta$ which renders a given Hamiltonian $H$ self-adjoint is addressed from the point of view of applicability of computer-assisted algebraic manipulations. An exactly solvable example of the so called Gegenbauerian quantum-lattice oscillator is recalled for the purpose. Both the construction of suitable $\Theta=\Theta(H)$ (basically, the solution of the Dieudonne's operator equation) and the determination of its domain of positivity are shown facilitated by the symbolic algebraic manipulations and by MAPLE-supported numerics and graphics.


## 1 Introduction

In the series of papers [1] - 7] we interpreted the Dieudonné's quasi-Hermiticity relation [8]

$$
\begin{equation*}
H^{\dagger} \Theta=\Theta H \tag{1}
\end{equation*}
$$

as an operator equation which connects a given, "input" quantum Hamiltonian $H$ with an unknown, "output" operator $\Theta$ called the Hilbert-space metric of the quantum system in question ([9] cf. also a more detailed explanation of the underlying physics in Appendix A below).

In all of our papers we felt addressed by the underlying physics (i.e., by quantum mechanics in its form described, say, in Refs. [10, 11]) and did not pay too much attention to the underlying constructive mathematics. In our present paper we intend to fill this gap by redirecting our attention to the computer-assisted symbolic-manipulation background of our results.

We shall assume that both of the operators $H$ and $\Theta$ in Eq. (1) are defined in a vector space $\mathcal{V}$ with the Dirac-ket elements $|\psi\rangle \in \mathcal{V}$. For the sake of simplicity we shall further assume that $\operatorname{dim} \mathcal{V}=N<\infty$. In such a setting one may select various toy-model matrices $H^{(N)}$ and construct the eligible metrics $\Theta$. For such a purpose we shall choose the Gegenbauerian quantum

[^0]$N$-site lattices and study $H=H^{(N)}(a)$ and $\Theta=\Theta^{(N)}(a)$ of Ref. 2] (cf. section (2). The reasons of a facilitated algebraic tractability of Eq. (1) will be then clarified in section 3, with some complementary numerical aspects mentioned in section 4.

## 2 Gegenbauerian quantum $N$-site lattices

The general Hermitian conjugation prescribed by Eq. (15) of Appendix A must be compatible with the principles of Quantum Mechanics. This means that, formally speaking [11], our choice of the metric $\Theta$ must guarantee the Hermiticity of the observables (i.e., in our present paper, just of the Hamiltonian $H$ ) with respect to this conjugation [9. Such a requirement implies the necessity of the validity of the above-mentioned relation (11).

The latter relation will be called here, for the sake of brevity, Dieudonné equation. As long as this is the matrix equation, it seems to be an overdeterminate constraint. Its $N^{2}$ items have to be satisfied by the mere $N(N+1) / 2$ independent matrix elements of the general $N$ by $N$ real and symmetric matrix $\Theta$ with positive eigenvalues. Via a deeper study of an illustrative example taken from Ref. [2] we intend to demonstrate that the situation is much more user friendly.

First of all, the Hermitian conjugation may be perceived as a symmetry of Eq. (1) so that just its upper-triangular nontrivial subset remains relevant. This implies that the whole set of equations is in fact underdeterminate. $A$ priori, the independent solutions $\Theta$ will form an $N$-parametric family. This observation is compatible with the explicit constructive results published in Ref. [2].

Secondly, the message delivered by Ref. [2] was aimed at the physics audience. Our present study will complement these results by their more systematic derivation and by the more explicit explanation of their formal structure. Keeping this aim in mind we shall consider the $N$-dimensional matrix Schrödinger equation

$$
\begin{equation*}
H^{(N)}\left|\psi_{n}^{(N)}\right\rangle=E_{n}^{(N)}\left|\psi_{n}^{(N)}\right\rangle \tag{2}
\end{equation*}
$$

with the prescribed bound-state eigenvectors

$$
\left|\psi_{n}^{(N)}\right\rangle=\left(\begin{array}{c}
\left\langle 0 \mid \psi_{n}^{(N)}\right\rangle=G\left(0, a, E_{n}\right)  \tag{3}\\
\left\langle 1 \mid \psi_{n}^{(N)}\right\rangle=G\left(1, a, E_{n}\right) \\
\vdots \\
\left\langle N-1 \mid \psi_{n}^{(N)}\right\rangle=G\left(N-1, a, E_{n}\right)
\end{array}\right)
$$

where, in the notation of MAPLE [12], the symbol $G(n, a, x)$ denotes the $n$-th Gegenbauer polynomial $G(n, a, x)$ equal to polynomial $C_{n}^{a}(x)$ in the notation of Ref. [13] or to $C_{n}^{(a)}(x)$ according to Ref. [14]. Under such an assumption, naturally, the explicit form of the related Hamiltonian is the
tridiagonal array

$$
\left[\begin{array}{cccccc}
0 & 1 / 2 a^{-1} & 0 & 0 & \cdots & 0  \tag{4}\\
2 \frac{a}{2 a+2} & 0 & (2 a+2)^{-1} & 0 & \cdots & \vdots \\
0 & \frac{2 a+1}{2 a+4} & 0 & (2 a+4)^{-1} & \ddots & 0 \\
0 & 0 & \frac{2 a+2}{2 a+6} & \ddots & \ddots & 0 \\
\vdots & \ddots & \ddots & \ddots & 0 & (2 a+2 N-4)^{-1} \\
0 & \cdots & 0 & 0 & \frac{2 a+N-1}{2 a+2 N-2} & 0
\end{array}\right]
$$

which defines the manifestly asymmetric $N$ by $N$ matrix $H^{(N)}$.
The validity of such an assignment is equivalent to the standard threeterm recurrences for the Gegenbauer polynomials while the $N$ by $N$ matrix truncation is equivalent to the implicit-equation identification of the real and non-degenerate spectrum $\sigma\left(H^{(N)}\right) \equiv\left\{E_{n}\right\}$ of the bound-state energies with the roots of the $N$-th Gegenbauer polynomial,

$$
\begin{equation*}
G\left(N, a, E_{n}\right)=0 . \tag{5}
\end{equation*}
$$

Naturally, such a secular equation may be considered solvable with an arbitrary numerical precision. The only nontrivial task represented by the complete description of the model will lie in the choice of a metric $\Theta$ compatible with Dieudonné Eq. (1). The method has not been described in Ref. [2] because it just consisted in the brute-force insertion of a general real and symmetric ansatz for $\Theta^{(N)}$ and in the subsequent trial and error analysis of Eq. (1) after its insertion.

In the metric-construction problem the key difficulties are twofold. Firstly, the ansatz for $\Theta^{(N)}$ contains too many (i.e., $N(N+1) / 2$ ) unknowns and there are no criteria for the clarification which ones of them should be selected as the "optimal" independent set. Even at the very small integers $N$, preliminary MAPLE-based brute-force algebraic symbolic-manipulation-solution experiments starting from Eq. (1) and from a few randomly selected $N$-plets of the tentative independent matrix elements of $\Theta$ generated just the obscure many-page results for all of the $N-$ and $a$-dependent matrix elements of $\Theta(H)$. The second difficulty emerged with the necessity of the parameter-range-specifying guarantee of the obligatorily positive-definite nature of any resulting $N$ by $N$ matrix candidate $\Theta_{\alpha}(H)$ for the metric (characterized or distinguished, in general, by a suitable multiindex $\alpha$ ).

The core of the success (i.e., of the resolution to both of these parallel algebro-numerical difficulties) has been revealed to lie in an interactive and iterative approach to both of the problems. In more detail this approach is to be described in what follows.

## 3 The Dieudonné's equation

In the light of the old Dieudonné's idea [8] it seems interesting to replace the current textbook Hermiticity property $H=H^{\dagger}$ of the current selfadjoint

Hamiltonians in quantum mechanics by the weaker assumption containing a nontrivial "metric" $\Theta \neq I$ [9]. In this context relation (11) guarantees the reality of the energies provided only that we require that the operator $\Theta=\Theta^{\dagger}$ is, roughly speaking [15], positive and invertible, i.e., tractable as a metric in the Hilbert space of states $\mathcal{H}^{(S)}$ where the superscript means "standard".

The recent growth of popularity and applicability of the quantum models requiring the unusual metrics $\Theta=\Theta^{(S)}>I$ may be found reviewed, e.g., in Refs. [10, 16, 17]. In our present paper we circumvent a number of technicalities by studying just the quantum models defined in finite-dimensional Hilbert spaces. Thus, we may identify $\mathcal{H}^{(F)} \equiv \mathbb{C}^{N}$. Moreover, for the sake of definiteness, we shall only pay attention to the models where the Hamiltonian matrices possess the tridiagonal real-matrix form.

### 3.1 An interactive algebraic-solution technique

As an illustrative example we shall use the Gegenbauerian quantum lattice model of Ref. [2] described in the preceding section. In fact, in Ref. [2] we described the results showing the feasibility of the brute-force construction of the complete family of the metrics $\Theta(H)$ admitted by the Dieudonnés linear algebraic constraints (11). In our present continuation of this effort we intend to provide a deeper insight in the problem explaining the reasons why our construction of the metrics appeared to be so successful.

We have to admit that with our very specific choice of the Gegenbauerian model in Ref. [2] we were unexpectedly fortunate. This fact may be demonstrated, say, by the recollection of the similar constructive attempts based on a different choice of the $N$ by $N$ "input" Hamiltonian as reported in Ref. [4]. After the construction of $\Theta(H)$ at the dimension as low as $N=4$ it has been argued there that the construction at the very next $N=5$ appeared almost prohibitively complicated. This is really in contrast with the results of Ref. [2] which proved valid at any integer $N$.

The core of the dimension-independent universality of the above-mentioned Gegenbauerian result may be seen in the combination of the extremely simple bidiagonal form of the Hamiltonians $H^{(N)}(a)$ with the comparably simple $a$-dependence of its matrix elements. The relevance of both of these ingredients becomes obvious when we recall the explicit $N=4$ sample of the Hamiltonian

$$
H^{(4)}(a)=\left[\begin{array}{cccc}
0 & (2 a)^{-1} & 0 & 0 \\
\frac{2 a}{2 a+2} & 0 & (2 a+2)^{-1} & 0 \\
0 & \frac{2 a+1}{2 a+4} & 0 & (2 a+4)^{-1} \\
0 & 0 & \frac{2 a+2}{2 a+6} & 0
\end{array}\right]
$$

together with the general ansatz for the metric

$$
\Theta^{(4)}(a)=\left[\begin{array}{llll}
k & b & c & d \\
b & f & g & h \\
c & g & m & n \\
d & h & n & j
\end{array}\right]
$$

In such a setting we may study the 16 -plet of the resulting relations, out of which Nr. 1, Nr. 6, Nr. 11 and Nr. 16 (i.e., diagonal items) remain trivial while the off-diagonal items form an antisymmetric matrix. Out of the remaining six independent items (say, Nr. 2, 3, 4, 7, 8, and 12) there is just one (viz., Nr. 4) which involves just two unknown quantities (viz., $h$ and $c$ ). This leads to the decision of taking $d, c, b$ and $k$ as independent parameters and of eliminating, in the first step, $h$ via item Nr. 4,

$$
h=\frac{c(a+1)}{2(a+2) a} .
$$

The inspection of the new set of items reveals that the simplest one is now just Nr. 3 which defines $g$ as a function of $b$ and $d$, with the next-step Nr. 8 defining $n$ as a function of $d$ and (newly known) $g$. We are left with the three items Nr. 2, 7 and 12 which couple $(k, f),(f, m)$ and $(m, j)$, respectively. As long as we decided to use $k$ as the fourth unconstrained parameter this means that in the same order we now define, step by step, the missing items $f, m$ and, ultimately, $j$. The result is complete yielding

$$
\begin{gathered}
h=1 / 2 \frac{c(a+1)}{(a+2) a} \\
g=1 / 2 \frac{b a+3 b+2 d a^{2}+4 d a+2 d}{(a+3) a} \\
n=1 / 2 \frac{-6 d a-10 d+b a+3 b}{(a+3) a(2 a+1)} \\
f=1 / 2 \frac{\left(2 c a^{2}+k a+c a+2 k\right)(a+1)}{(a+2) a^{2}} \\
m=1 / 2 \frac{2 c a^{3}+c a^{2}-7 c a+k a^{2}+5 k a+6 k}{(a+3) a^{2}(2 a+1)} \\
j=-1 / 4 \frac{6 c a^{2}+10 c a-k a^{2}-5 k a-6 k}{a^{2}(2 a+1)(a+2)(a+1)} .
\end{gathered}
$$

### 3.2 The case of general $N$

After the above-explained heuristic exercise we are prepared to consider the real and symmetric general ansatz for the metric

$$
\Theta^{(N)}(a)=\left[\begin{array}{cccc}
\theta_{11} & \theta_{12} & \ldots & \theta_{1, N}  \tag{6}\\
\theta_{12} & \theta_{22} & \ddots & \vdots \\
\vdots & \ddots & \ddots & \theta_{N-1, N} \\
\theta_{1, N} & \ldots & \theta_{N-1, N} & \theta_{N N}
\end{array}\right]
$$

and prove the general result.
Definition 1. At any $N \geq 2$ the insertion of the $N$ by $N$ Hamiltonian $H=H^{(N)}(a)$ given by Eq. (4) and of the general real and symmetric matrix ansatz (6) for the metric $\Theta=\Theta^{(N)}(a)$ defines the $N$ by $N$ matrix array (1)
of the linear Dieudonné equations $\mathcal{M}_{i, j}=0$. Its ordered version has the form $r_{\alpha}=0$ with

$$
\begin{align*}
& r_{1}=\mathcal{M}_{1, N}, \\
& r_{2}=\mathcal{M}_{1, N-1}, r_{3}=\mathcal{M}_{2, N}, \\
& r_{4}=\mathcal{M}_{1, N-2}, r_{5}=\mathcal{M}_{2, N-1}, r_{6}=\mathcal{M}_{3, N}, \\
& \ldots,  \tag{7}\\
& r_{(N-1)(N-2)) / 2+1}=\mathcal{M}_{1,2}, \ldots, r_{N(N-1) / 2}=\mathcal{M}_{N-1, N}
\end{align*}
$$

Theorem 1. In terms of the freely variable $N$-plet of the real initial parameters $\Theta_{1 j}, j=1,2, \ldots, N$ the Dieudonné equation in its ordered version (7) defines, step by step, the respective "missing" matrix elements

$$
\begin{array}{r}
\Theta_{2, N}, \\
\Theta_{2, N-1}, \Theta_{3, N}, \\
\ldots,  \tag{8}\\
\Theta_{2,2}, \Theta_{3,3}, \Theta_{4,4}, \ldots, \Theta_{N, N}
\end{array}
$$

in recurrent manner.
Proof. Once we revealed the diagonal-wise-arranged recurrent pattern it is easy and entirely straightforward to verify its validity by the corresponding trivial rearrangement of the two matrix multiplications in Eq. (11).

Remark 1. The diagonal-wise recurrent nature of Eq. (1) given by Theorem 1 has only been revealed by the post factum inspection of the results of Ref. [2].

## 4 The positive definiteness of the metric

During the recent years we are witnessing the remarkable growth of popularity of the building of quantum models which combine the "false" nonHermiticity $H \neq H^{\dagger}$ of the comparatively elementary Hamiltonian acting in a "friendly" Hilbert space $\mathcal{H}^{(F)}$ with the simultaneous "sophisticated" Hermiticity $H=H^{\ddagger}$ of the same Hamiltonian in another, less usual, amended, "standard" Hilbert space $\mathcal{H}^{(S)}$ endowed with a nontrivial metric $\Theta=\Theta^{(S)} \neq$ $I$. The essence of such an innovation becomes entirely transparent and obvious when one eliminates the partial confusion caused by the traditional terminology. The key point is that one never leaves the abstract theoretical framework of quantum theory. Just a few new mathematical tricks (like, typically, an unusual, non-unitary generalization of the most common Fourier transformation) are being added to the traditional textbook recipes.

### 4.1 The formulation of quantum theory using an ad hoc triplet of complementary Hilbert spaces

The Dieudonné-equation constraint imposed on a Hamiltonian $H$ is in fact equivalent to the manifest Hermiticity of its isospectral image

$$
\begin{equation*}
\mathfrak{h}=\Omega H \Omega^{-1}=\mathfrak{h}^{\dagger} \tag{9}
\end{equation*}
$$

[11]. In principle (though not always in practice) the latter operator is defined as acting in the physical Hilbert space $\mathcal{H}^{(P)}$ in which the traditional, trivial metric $\Theta^{(P)}=I$ is being used.

Both the Hilbert spaces $\mathcal{H}^{(P)}$ and $\mathcal{H}^{(S)}$ may be perceived as unitary equivalent. We may deduce

$$
\begin{equation*}
\mathfrak{h}^{\dagger}=\left(\Omega^{-1}\right)^{\dagger} H^{\dagger} \Omega^{\dagger} \tag{10}
\end{equation*}
$$

After we abbreviate $\Omega^{\dagger} \Omega:=\Theta$ we end up with the Dieudonné's relation (11).

### 4.2 The Gegenbauerian illustrative example

For our real and finite-dimensional Gegenbauerian Hamiltonians $H=H^{(N)}(a)$ which are given in advance, the Dieudonné's relation (11) forms the set of $N^{2}$ constraints imposed upon the $[N(N+1) / 2]$-plet of the unknown real matrix elements of the metric matrix $\Theta=\Theta^{\dagger}>0$. In papers [1] - [7] we proposed the non-numerical, symbolic-manipulation approach to constructions of a complete solution of this linear algebraic system. What remains for us to construct is the appropriate domain $\mathcal{D}$ of free parameters for which these candidates for the metric remain positive definite, i.e., truly eligible in the appropriate definitions of the generalized Hermitian conjugation and/or of the appropriate Hilbert-space inner product.

In Ref. [2] we discussed a few specific examples of candidates $\Theta^{(N)}(a)$ for the Gegenbauerian metrics. We revealed that such a study leads to a purely numerical description supporting the hypothesis that the domains $\mathcal{D}^{(N)}(a)$ change "smoothly" with $N$ and lead to the non-empty and sufficiently large limiting domains $\lim _{N \rightarrow \infty} \mathcal{D}^{(N)}(a)=\mathcal{D}^{(\infty)}(a) \neq \emptyset$.

A comparatively weak $N$-dependence characterizes even the domains $\mathcal{D}^{(N)}(a)$ at small $N \gtrsim 5$. The direct evaluation of the eigenvalues of $\Theta^{(N)}(a)$ (i.e., the more precise determination of the boundaries $\partial \mathcal{D}^{(N)}(a)$ ) only suffers of the errors caused by the multiple-scale nature of these eigenvalues.

In Ref. [2] we were only able to provide a transparent graphical illustration of the free-parameter-dependence of the spectrum of selected $\Theta^{(N)}(a)$ s at the very first nontrivial dimension $N=3$. In the context of programming in MAPLE (offering an adaptable floating-point precision arithmetics) the remedy is easy. One may take, say, the $N=3$ toy metric of Ref. [2],

$$
\Theta_{g}^{(3)}(a)=\left[\begin{array}{ccc}
2 a^{2} & 2 g a & 0 \\
2 g a & a+1 & g \\
0 & g & \frac{a+2}{2 a+1}
\end{array}\right]
$$

and represent the triplet of eigenvalues $p_{j}(g), j=1,2,3$ (as sampled in Figure Nr. 3 of loc. cit. at $a=1$ ) in logarithmic scale yielding, say, the adapted $a=1$ secular equation

$$
\operatorname{det}\left[\begin{array}{ccc}
2-e^{-t t r-20} & 2 g & 0 \\
2 g & 2-e^{-t t r-20} & g \\
0 & g & 1-e^{-t t r-20}
\end{array}\right]=0
$$

i.e., the non-polynomial version of our eigenvalue problem,

$$
4-8 e^{-t t r-20}+5\left(e^{-t t r-20}\right)^{2}-6 g^{2}-\left(e^{-t t r-20}\right)^{3}+5 g^{2} e^{-t t r-20}=0
$$

In a test run the numerical analysis of this equation reproduced the results given in Table Nr. 1 of loc. cit..

On this basis one may expect that the key problem brought by the rescaling appears tractable by the MAPLE-based numerical software. The main gain came with the substantial extension of the feasibility of the graphical determinations of the parameter-dependence of the physical domains $\mathcal{D}_{g}^{(N)}(a)$ with the growth of $N$. The characteristic illustration is offered by Figures 1 and 2 which clearly demonstrate the emergence of an obvious pattern which was not accessible without rescaling.


Figure 1: Seven eigenvalues $p=p(g)$ of metric $\Theta_{g}^{(7)}(1)$.


Figure 2: Nine eigenvalues $p=p(g)$ of metric $\Theta_{g}^{(9)}(1)$.

## 5 Summary

From the point of view of Quantum Mechanics it is rather unfortunate that for a given Hamiltonian $H$ the specification of the metric $\Theta(H)$ prescribed by Dieudonné Eq. (1) is ambiguous [9. In this context, our series of papers [1] [7] has been devoted to the constructive study of the one-to-many mappings $H \rightarrow \Theta(H)$. In essence, we offered there a new methodical recipe of a systematic suppression of the ambiguity of the menu of eligible $\Theta(H) \mathrm{s}$.

In our present continuation and extension of this paper we decided to explain the symbolic-manipulation aspects of such a recipe in more detail. Emphasizing that such a problem would be hardly tractable and/or solvable
without an essential interaction between the abstract quantum theory and the symbolic-manipulation techniques and algebraic constructionis assisted by contemporary computers.

One of byproducts of such an interaction between methods has been shown to lie in the possible amendments of the numerical aspects of the necessary simultaneous analysis and study of physical domains and of the other properties of both of the physics-representing operators $H$ and $\Theta$. On the basis of these results one can conclude that the confirmation of feasibility of a methodical symbiosis between algebra and analysis seems to be able to contribute to the contemporary quick growth of popularity of the practical phenomenological applications of quantum models with nontrivial metrics, e.g., in optics [18].

## Appendix A. A compact review of Quantum Mechanics using nontrivial metrics $\Theta \neq I$

In the spirit of certain textbooks on Quantum Mechanics [19] the Dirac's ket vectors $|\psi\rangle \in \mathcal{V}$ may be interpreted as describing different quantum systems the nature of which may vary with the other characteristics of the system in question. Thus, one can start from the most common single-particle (or, in some applications, single quasi-particle) quantum models where, roughly speaking, there exists an operator $\hat{Q}$ of the particle position with eigenvalues $q \in \mathbb{R}$ and eigenkets $|q\rangle$. One then introduces the standard (often called Dirac's) Hermitian conjugation (i.e., transposition plus complex conjugation)

$$
\begin{equation*}
\mathcal{T}^{\text {Dirac })}:|\psi\rangle \rightarrow\langle\psi| . \tag{11}
\end{equation*}
$$

In this manner one represents the kets of states $|\psi\rangle$ by the concrete squareintegrable functions $\psi(q)=\langle q \mid \psi\rangle$ (of the coordinate $q$ ) called wave functions. In this context the discrete, finite-dimensional models exemplified by Eq. (3) may be perceived as the most convenient testing ground of the methods.

In all of the similar classes of examples the vector space of states $\mathcal{V}$ becomes endowed with the most common inner product

$$
\begin{equation*}
\left\langle\psi_{a} \mid \psi_{b}\right\rangle=\int_{-\infty}^{\infty} \psi_{a}^{*}(q) \psi_{b}(q) d q \tag{12}
\end{equation*}
$$

with, possibly, the integration replaced by the infinite or finite summation. Thus, we may (and usually do) set $\mathcal{V} \equiv L^{2}(\mathbb{R})$, etc.

Without any real danger of misunderstanding we may speak here about the "friendly" Hilbert space of states $\mathcal{H}^{(F)} \equiv \mathcal{V}$, calling the variable $q$ in Eq. (12) "the coordinate". In parallel we usually perform a maximally convenient choice of the Hamiltonian $H$ based on the so called principle of correspondence which "dictates" us to split $H=T+V$ where the general interaction operator $V$ is represented, say, by a kernel $V\left(q, q^{\prime}\right)$ when acting upon the wave functions. In practice, this kernel is most often chosen as proportional to the Dirac's delta-function so that $V$ becomes an elementary multiplicative operator $V=V_{\text {local }}=V(q)$. Similarly, the most popular and preferred form of the "kinetic energy" $T$ is a differential operator, say, $T=T_{\text {local }}=-d^{2} / d q^{2}$ in single dimension and suitable units.

A word of warning emerges when we perform a Fourier transformation in $\mathcal{H}^{(F)}$ so that the variable $q$ becomes replaced by $p$ (= momentum). One should rather denote the latter, "Fourier-image" space by the slightly different symbol $\mathcal{H}^{(P)}$, therefore (with the superscript still abbreviating "physical" [11]). Paradoxically, after this change of frame the kinetic operator $T_{\text {local }}$ becomes multiplicative while $V_{\text {local }}$ becomes strongly non-local in momenta.

The truly deep change of the paradigm comes with the models where the necessity of the observability of the coordinate $q$ is abandoned completely. One may still start from the vector space of kets $\mathcal{V}$ and endow it with the Hilbert-space structure via the similarly looking inner product

$$
\begin{equation*}
\left\langle\psi_{a} \mid \psi_{b}\right\rangle=\int_{\mathcal{C}} \psi_{a}^{*}(s) \psi_{b}(s) d s \tag{13}
\end{equation*}
$$

The non-existence of the position operator $\hat{Q}$ changes the physics of course. The key point is that we lose the one-to-one correspondence between the integration path $\mathcal{C}$ and the spectrum $\mathbb{R}$ of any coordinate-mimicking operator. The physics-independent optional variable $s$ remains purely formal.

In such a setting our choice of the physical observables remains unrestricted and is entirely arbitrary. It need not be related to any usual classical system, either. For illustration one might recall the pedagogically motivated paper [20]. In a slightly provocative demonstration of the abstract nature of quantum theory the variable $s$ in Eq. (13) has been chosen there as an observable "time" of a hypothetical "quantum clock" system.

Definition (13) of the inner product in $\mathcal{H}^{(F)}$ or in $\mathcal{H}^{(P)}$ is the theoretical framework within which the traditional quantum mechanics works, reducing the full power of the theory to something acceptable via analogies with classical physics. Still, one need not move too far. Even in Ref. [1], for example, we did not treat the variable $s$ in Eq. (13) as a purely formal quantity, having preserved at least a part of its relation to the position $q$. In particular, we stayed in the middle of the path towards abstraction and worked still with the usual one-dimensional ordinary differential Schrödinger equation

$$
\begin{equation*}
-\frac{d^{2}}{d s^{2}} \psi_{n}(s)+V(s) \psi_{n}(s)=E_{n} \psi_{n}(s), \quad n=0,1, \ldots \tag{14}
\end{equation*}
$$

where the key nonstandard features can be seen in

- the admissibility of the complex potentials sampled by the power-lawanharmonic family $V(s)=-(\text { is })^{2+\delta}$ of Ref. [21] giving the "standard" real and discrete bound-state spectra at any $\delta>0$ (cf. the proofs in [22]);
- the admissibility of the replacement of the usual real line of $s$ by a suitable complex curve $\mathcal{C}=\mathbb{C}(\mathbb{R})$ which may even be, in principle, living on a complicated multisheeted Riemann surface [23];
- in the possibility of a systematic study of its discrete analogues and simplifications.

Naturally, one partially leaves the more or less safe guidance offered by the principle of correspondence. Just a partial revitalization of this guidance is
possible (cf., e.g., a nice example-based discussion of this point in Ref. [24]). A partial reward for this loss can be sought in a new tractability of some traditionally contradictory quantization problems (say, when working, say, with the relativistic Klein-Gordon equations [25]).

The main theoretical difficulty consists in the vast ambiguity of the necessary appropriate generalization of Eq. (11). The general recipe (explained already in [9] or, more explicitly, in [11]) is metric-dependent and reads

$$
\begin{equation*}
\mathcal{T}_{\Theta}^{(\text {general })}:|\psi\rangle \rightarrow\langle\langle\psi|:=\langle\psi| \Theta . \tag{15}
\end{equation*}
$$

This means that using the language of wave functions $\psi(s)$ with $s \in \mathcal{C}$ we must replace the most common single-integral definition (13) of the inner product in the original "friendly" Hilbert space $\mathcal{H}^{(F)}$ by the more sophisticated double-integral formula

$$
\begin{equation*}
\left\langle\left\langle\psi_{a} \mid \psi_{b}\right\rangle=\int_{\mathcal{C}} \int_{\mathcal{C}} \psi_{a}^{*}(s) \Theta\left(s, s^{\prime}\right) \psi_{b}\left(s^{\prime}\right) d s d s^{\prime}\right. \tag{16}
\end{equation*}
$$

In terms of an integral-operator-kernel representation $\Theta\left(s, s^{\prime}\right)$ of our abstract metric operator $\Theta=\Theta^{\dagger}>0$ this recipe defines a more sophisticated inner product which converts the ket-vector space $\mathcal{V}$ into the metric-dependent (and physics-representing, "standard" [11) Hilbert space $\mathcal{H}^{(S)}$.

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