A Homological–Based Description of Subdivided nD Objects

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Abstract. We present here a topo–geometrical description of a subdivided nD object called homological spanning forest representation. This representation is a convenient tool in order to completely control not only geometrical, but also advanced topological information of a given object. By codifying the underlying algebraic topological machinery in terms of coordinate–based graphs, we progress in the task to "combinatorialize" homological information at two levels: local and global. Therefore, our method presents several advantages with respect to the existing Algebraic topological models, and techniques based in Discrete Morse Theory. A construction algorithm has been implemented, and some examples are shown.

1 Introduction

One way to guarantee a consistent description of an object is to base such description on topological principles. A topological representation of an object defines a finite topological space made up of regions, arcs, and points which encode a particular partitioning. Several structures have been proposed to encode such a partitioning, including cellular complexes [1,2], combinatorial maps [3], graphs [4], etc.

We deal here with the problem of finding an efficient and robust geometrical and topological representation of a subdivided nD object given in terms of a cell complex and exploiting the notion of homology or, more precisely, using chain homotopy equivalences connecting the object with its homology groups (see [5] for more details).

In principle, homology is a purely algebraic notion related to the degree of connectivity at the level of formal sum of cells (connected components, holes or tunnels, cavities,...) and most of the models based on these ideas are algebraic–topological models (AT–model [6], AM–model [7],....). Nevertheless, it is possible to "combinatorialize" these models (eliminate its algebraic part) by using a graph representation of the algebraic operators (boundary operator, coboundary

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operator,...) and simplify its connectivity information by using hierarchical tree–like structures. Forman [8,9] used this idea in order to develop Discrete Morse Theory (DMT, for short), which has become a powerful tool in its applications to computational topology, computer graphics, image processing and geometric modeling.

With the philosophy of representing the object in terms of a finite number of topologically inessential "threads" as a goal, we translate DMT to a suitable algebraic homological setting, that of integral–chain complexes. We can progress in this way in the task to "combinatorialize" homological information at two level: local (DMT) and global (in terms of coordinated–based hierarchical forests based on chain homotopies [10]). The algebraic nature of the global approach can be combinatorialized if we place these chain homotopies and critical cells (homology generators) in a graph–based ambiance. Based on that, we develop here a non–unique combinatorial topo–geometric representation of a nD subdivided object, called homological spanning forest (or HSF, for short). We design and implement an algorithm for computing this model for a nD object embedded in $\mathbb{R}^n$ and we do some experiments in the three dimensional case.

2 Preliminaries

In this section, we first establish a notion of (combinatorial) *cell complex* in a finite–dimensional Euclidean space with the cell boundary information described in algebraic terms.

A *cell complex* $K = \{K_i\}_{i=0}^{\ell}$ embedded in $\mathbb{E}^\ell$ is a finite collection of cells $\{\sigma_i^{(r)i=1,\dots n} \in K_r\}$ of different dimensions $0 \geq r \geq \ell$ such that (see [11] for a formal definition of cell):

(i) $|K| = \bigcup_{i=1}^{n} \sigma_i = |K_0| \cup |K_1| \cup \ldots \cup |K_\ell|$. The set K_r consists of all the r–cells of K, for $0 \leq r \leq \ell$. It is possible that $K_i = \emptyset$ for some $0 < i \leq \ell$.
(ii) $\sigma_i \cap \sigma_j = \emptyset$ $(i \neq j)$;
(iii) If $dim(\sigma_i) = p$ (with $0 \geq p \geq \ell$), then $\partial\sigma_i \subset \bigcup_{i=1}^{p-1} K_i$,

The *p–skeleton* $K^{(p)}$ for K is the set of all k–cells with $0 \leq k \leq p$. The dimension of the cell complex is the smallest non–negative natural number r such that the condition $K^{(r)} = K^{(r+1)}$ is satisfied. If all the cells of K are convex sets of $\mathbb{E}^\ell$, then K is called *convex cell complex*. Simplicial, Cubical and some polyhedral complexes are special cases of convex cell complexes.

Roughly speaking, the idea of homology is to analyze the degree of connectivity of cell complexes using formal sums of cells. A *differential operator* for a cell complex K with coefficients in Λ is a linear map $d : \Lambda[K] \to \Lambda[K]$, such that the image of a p–cell σ is a linear combination of some $(p-1)$–cells of the boundary $\partial(\sigma)$ and $d \circ d = 0$. Taking into account that our cell complex K is embedded in $\mathbb{E}^\ell$, its geometric realization $|K|$ is a regular triangulable cell complex and there can be always defined a differential operator ∂, called *boundary operator*, with coefficients in the field Λ.

The chain complex canonically associated to the cell complex K is the graded differential vector space $(C_*(K), \partial)$, where $C_p(K) = \Lambda[K_p]$, for all $p = 0, 1, \dots r$, and $\partial : C_*(K) \to C_{*-1}(K)$ is the previous boundary operator for the cell complex K. For instance, to find a boundary operator ∂ for a simplicial complex is straightforward, but it is not, in general, an easy task for others cell complexes. The following is one of the fundamental results in the theory of CW–complexes.

Theorem 1 (See [8]). *Let K a finite cell complex. There are algebraic boundary maps $\partial_p : C_p(K, \Lambda) \to C_{p-1}(K, \Lambda)$, for each p, so that $\partial_{p-1} \circ \partial_p = 0$ and such that the resulting differential complex $\{C_p(K, \Lambda), \partial_p\}_{p=0}^{r}$ calculates the homology of K. That is, if we define $H_p(C, \partial) = \text{Ker}\,(\partial_p)/\partial_{p+1}(C)$. In other words, $H_p(C, \partial) \cong H_p(|K|, \Lambda)$.*

From now on, a finite cell complex K is denoted by (C, ∂), where $\partial : C_*(K) \to C_{*-1}(K)$ is the boundary operator for $C_*(K)$ with coefficients in the finite field $\mathbb{Z}/\mathbb{Z}2 = \{0, 1\}$.

3 Integral–Chain Complexes

In [12], we recover the algebraic machinery underlying in Discrete Morse Theory, establishing a new framework for dealing with special chain complexes, that is the integral–chain complexes associated to finite cell complexes. In this section, we recall the main notions and results of this homological algebra work in order to understand our approach.

Definition 1. *[12] An* integral chain complex (C, ∂, ϕ) *is a graded vector space $C = \{C_p\}_{p=0}^{n}$ endowed with two linear maps: a differential operator $\partial : C_* \to C_{*+1}$, and an integral operator (also called* algebraic gradient vector field *[8] or* chain homotopy operator *[5]) $\phi : C_* \to C_{*+1}$, satisfying the global nil potency properties $\partial \circ \partial = 0$ and $\phi \circ \phi = 0$. An integral chain complex (C, ∂, ϕ) is* ∂–pure *(resp. ϕ–*pure*) if the condition $\partial = \partial \circ \phi \circ \partial$, called homology condition (resp. the condition $\phi = \phi \circ \partial \circ \phi$, called* strong deformation retract *condition) is satisfied.*

Examples of the application of integral operators is shown in Figure 1.

The computation of the homology of a chain complex (C, ∂) can be specified in terms of finding an integral operator $\phi : C_* \to C_{*+1}$, satisfying the Strong Deformation Retract (SDR for short) and homology conditions with regards to the differential operator ∂ ([13]).In spite of its simplicity, the following result is essential.

Fig. 1. A cell complex and the resulting cell complex after applying the integral operator $\phi(\langle 1 \rangle) = \langle 1, 2 \rangle$ (on the left) and a cell complex and the result after applying $\phi(\langle 1, 2 \rangle) = \langle 1, 2, 3 \rangle$ (on the right)

Lemma 1. *[Integral–Chain Lemma] An integral chain complex* (C, ∂, ϕ) *is integral–chain equivalent to its harmonic complex* $\pi(C, \partial, \phi)$*, meaning that its homological information can be extracted from that of the harmonic (see [12]). This last harmonic complex* $\pi(C, \partial, \phi)$ *is of the form* $(\pi(C), \partial_\pi, \phi_\pi)$ *where* $\pi(x) = (id + \phi \circ \partial + \partial \circ \phi)(x)$*,* $\partial_\pi(\pi(x)) = (\partial - \partial \circ \phi \circ \partial)(x)$ *and* $\phi_\pi(\pi(x)) = (\phi - \phi \circ \partial \circ \phi)(x)$*.* $\forall x \in C$*.*

We now define an algebraic constructor of a new integral–chain complex:

Definition 2. *Given a* ∂*–pure (resp. two* ϕ*–pure) integral–chain complexes* (C, ∂, ϕ) *and a differential operator* ∂' *satisfying the homology condition (resp. an integral operator* ϕ' *satisfying the strong deformation retract condition) for* $\pi_{(\partial,\phi)}(C)$*, a new* ∂*–pure (resp.* ϕ*–pure) integral chain complex* $(C, \partial + \partial' \circ \pi_{(\partial,\phi)}, \phi)$ *(resp.* $(C, \partial, \phi + \phi' \circ \pi_{(\partial,\phi)})$*) can be constructed. This new integral chain complex is called* composition *of* (C, ∂, ϕ) *by* ϕ'*.*

From now on, all the integral chain complexes we consider in the paper will be ϕ–pure integral–chain complexes.

4 Homological Spanning Forest Representation

Discrete Morse Theory (DMT, for short) gives a positive answer to the problem of finding combinatorial chain homotopy operators ϕ for chain complexes $(C_*(K), \partial)$ of finite cell complexes, such that the integral homology of $(C_*(K), \partial, \phi)$ is a "good" approximation (measured in terms of critical cells, determined by the Betti numbers) to its differential homology. These combinatorial integral operators are seen in DMT as cell pairings (cell collapses) (see Figure 2). In [14], given a 2–manifold, an heuristic for computing optimal Morse pairings is developed. A pairing is considered optimal when the discrete gradient vector field has few critical cells (cells that are not paired). This heuristic computes optimal gradient vector fields in terms of hyper-forests. However, for general cell complexes this problem has not been solved yet.

Now, we progress in DMT with some slightly modifications with regards the classical theory, and without using, in principle, discrete Morse functions. In this way, we are able to obtain an optimal pairing for any finite cell complex without restriction.

Definition 3. *A combinatorial integral operator* $\mathcal{V}$ *defined on a cell complex* K *is a collection of disjoint pairs of (non–necessarily incident) cells* $\{\alpha^{(p)} \prec \beta^{(p+1)}\}$*. If the pairs are constituted by incident cells then,* $\mathcal{V}$ *is called combinatorial vector field ([8]). A cell* α *is a critical cell of* $\mathcal{V}$ *if it is not paired with any other cell in* $\mathcal{V}$*.*

In the sequel, we prefer to describe some important notions in terms of the barycentric subdivision of a cell complex rather than using its Hasse diagram.

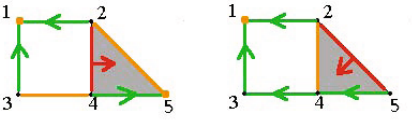

Fig. 2. A cell pairing on the left ($\langle 1\rangle$,$\langle 5\rangle$, $\langle 3,4\rangle$ and $\langle 2,5\rangle$ are critical), and an optimal one on the right ($\langle 1\rangle$ and $\langle 2,4\rangle$ are critical). The pairing is represented with an arrow from the cell of lower dimension to its paired cell of higher dimension.

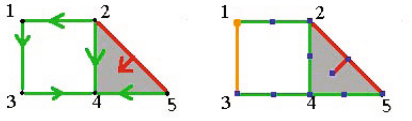

Fig. 3. A combinatorial vector field (on the left). On the right a gradient set of forest where cells $\langle 1\rangle$ and $\langle 1,3\rangle$ do not belong to the forest, $\langle 2,5\rangle$ and $\langle 2,4,5\rangle$ belong to F_1 and the rest of cells belong to F_0.

Definition 4. *Let (K,∂) be a finite convex cell complex of dimension m embedded in $\mathbb{R}^n$ and let $(BCS(K),\partial^{bcs})$ be the simplicial barycentric subdivision of K [15]. Let us consider a hierarchy of simplicial forests $\mathcal{F}=\{F_0,F_1,\ldots,F_{m-1}\}$ all contained in the 1–skeleton of $BCS(K)$, such that the nodes of F_p are vertex cells of dimension p and $p+1$ of K and its leaves are cells of dimension p, for all $0\leq p\leq m$. Such set of forests is called gradient set of forests for the cell complex (K,∂).*

An example of a combinatorial vector field and a gradient set of forest is shown in Figure 3.

A gradient set of forests $\mathcal{F}$ for (K,∂) can be expressed in combinatorial terms by means of a combinatorial vector field $\mathcal{V}_{\mathcal{F}}$, called *$\mathcal{F}$–gradient vector field.* In fact, $\mathcal{V}_{\mathcal{F}}$ is defined by choosing a root in each tree T of $\mathcal{F}$ and pairing incident cells (of different dimension) of T excepting the root. In this way, the roots of the trees become critical cells of $\mathcal{V}_{\mathcal{F}}$ as well as the rest of cells in K which do not appear in $\mathcal{F}$.

A ϕ–pure integral–chain complex $(C(K),\partial,\tilde{\mathcal{V}_{\mathcal{F}}})$ can be derived from $(C(K),\partial,\mathcal{V}_{\mathcal{F}})$. If $\sigma^{(p)}$ is a node of $\mathcal{F}$ which is not a root, $\tilde{\mathcal{V}_{\mathcal{F}}}(\sigma^{(p)})$ is defined as the sum of the $(p+1)$–cells of K existing in the unique path within the forest F_p, joining σ and the corresponding root. In other case, its value is zero. This *ϕ–pure $\mathcal{F}$–gradient integral operator* $\tilde{\mathcal{V}_{\mathcal{F}}}$ satisfies the SDR–condition $\mathcal{V}_{\mathcal{F}}\partial\mathcal{V}_{\mathcal{F}}=\mathcal{V}_{\mathcal{F}}$.

The main algorithm of this paper is designed using as main piece the following proposition which is already proved in [12].

Proposition 1. *[12] Let (K,∂) be a finite convex cell complex and let $\mathcal{F}$ a gradient set of forest for K. Let $(C(K),\partial,\phi)$ be the integral–chain complex, being ϕ the pure gradient integral operator derived from $\mathcal{F}$. Then, the harmonic complex*

of $(C(K), \partial, \tilde{\mathcal{V}_{\mathcal{F}}})$ *is isomorphic to* $(\text{ Ker } \mathcal{V}_{\mathcal{F}} \setminus \mathcal{V}_{\mathcal{F}}(C(K)), \partial_\pi, 0)$. *This last integral–chain complex, called Morse cell complex* $\mathcal{M}(C(K), \mathcal{F})$ *is constituted by finite linear combinations of the different critical cells of* $\mathcal{V}_{\mathcal{F}}$ *and* ∂_π *can be seen as the boundary operator of the corresponding cell complex determined by the critical cells. Given a critical* p*–cell* $\sigma^{(p)}$*, then* $\partial_\pi(\sigma^{(p)}) = (\partial - \partial \circ \phi \circ \partial)(\sigma^p)|_{criticalcells}$*, that is, the linear combination of the critical* $(p-1)$*–cells appearing in* $(\partial - \partial \circ \phi \circ \partial)(\sigma^{(p)})$.

Let us note that we can repeatedly apply Prop. 1 to the successive Morse complexes, previously describing a corresponding gradient forest for each of them. We can express this task in the following way:

$$(C(K), \partial) \Rightarrow \mathcal{M}(C(K), \mathcal{F}^0) \Rightarrow \mathcal{M}(\mathcal{M}(C(K), \mathcal{F}^0), \mathcal{F}^1) \Rightarrow$$
$$M(\mathcal{M}(\mathcal{M}(C(K), \mathcal{F}^0), \mathcal{F}^1), \mathcal{F}^2) \Rightarrow \ldots \Rightarrow H_*(C(K), 0),$$

where $(C, \partial) \Rightarrow (C', \partial')$ means that there is a chain homotopy equivalence [15] between the chain complexes (C, ∂) and (C', ∂').

On the other hand, Prop. 2 provides us the integral–chain complex $(C(K), \partial, h)$ which is composition of $(C(K), \partial, 0)$ by the successive gradient forests. If we obtain in this way a Morse complex, with all the possible gradient forest being trivial (constituted of only one node), the process stops and h is the "key" operator for getting the homology groups and corresponding homology generators of K as well as a topological interpretation of K in terms of trees in the 1–skeleton of the barycentric subdivision of K. This geo–topological (coordinate–based) representation is called *Homological Spanning Forest* (or HSF, for short) representation for K (see Figure 4).

Theorem 2. *In the previous conditions, the integral operator* $h : C_*(K) \rightarrow C_{*+1}(K)$ *specifies a set of forest* $\mathcal{G} = \{G_0, G_1, \ldots, G_m\}$ *in* $BCS(K)$ *called HSF–representation for* K.

5 Implementation and Experiments

In [6] the authors present an algorithm to reduce a initial chain complex up to its minimal homological expression. The advantage of this method is that the obtained integral operator encodes the homological information of the initial complex (homology groups, cohomology, homology generators, relations between them, etc.). The complexity in time of this method is $O(n^3)$.

In this section we present a new algorithm, based in Prop. 1, where the resulting HSF representation encodes exactly the same information that the previous mentioned method, and besides the advantages of providing such a representation of the object, by using graph techniques, the time complexity is reduced. The heart of the proposed algorithm runs in linear time, and the question of how many times the loop should be executed, crucially depends on the particular complex.

Given an initial cell complex $C_*(K)$, Algorithm 1 computes its HSF–representation. The algorithm consists of an iterative process, where at each

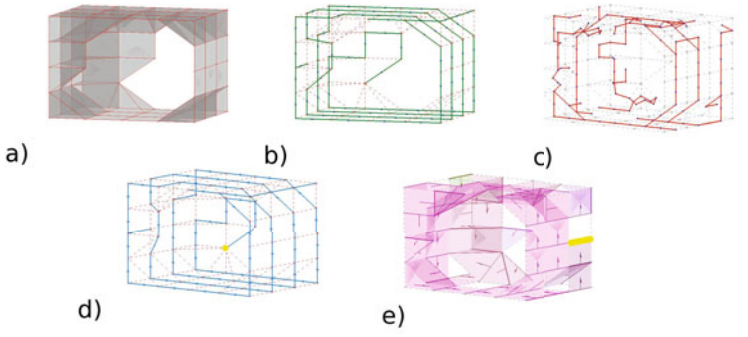

Fig. 4. In Figure 4 a) we can see a cell complex. Part of its homological spanning forest representation (G_0 and G_1) is shown in Figures 4 b) and 4 c). Figures 4 d) and 4 e) represent the optimal combinatorial pairing. The resulting critical cells are colored in yellow in Figures 4 b) and 4 c).

step i, a gradient set of forests $\{F_0 \dots F_p\}$ is computed over $C_*(K)^i$. The function $\mathcal{M}$ returns a Morse complex, constituted by finite linear combinations of the different critical cells in $(C_*(K)^i, \mathcal{F}^i)$. Once the computed $\mathcal{F}^i$ is trivial, the process stops. The guarantee that the minimal number of critic cells is obtained at the end of the algorithm arises in the fact that the algorithm only stops when $\partial = 0$ for every cell in K.

Algorithm 1. HSF$(C_*(K), \partial)$

$i = 0$
while ! $(Trivial\ (\mathcal{F}^i))$ **do**
  $F_0 = SpanningTree_c\ (C_{0,1}(K)^i)$
  $F_1 = SpanningTree_c\ (C_{1,2}(K)^i \backslash F_0)$
  $\dots$
  $F_p = SpanningTree_c\ (C_{p,p+1}(K)^i \backslash F_{p-1})$
  $\mathcal{F}^i = F_0 \cup F_1 \cup \dots \cup F_p$
  $\mathcal{G}^i = \mathcal{G}^{i-1} \bigcup \mathcal{F}^i$
  $C_*(K)^{i+1} = \mathcal{M}(C_*(K)^i, \mathcal{F}^i)$
  $i = i + 1$
end while
return $(\mathcal{G}^i)$

The union operation $\bigcup$ in Algorithm 1 consists of the integration of the information residing in the forests $\{F_0 \dots F_p\}$ to the global forest $\mathcal{G}$. This itegration is done by using the algebraic composition operation of Prop. 2.

The computation of the gradient set of forests is performed using the Algorithm $SpanningTree_c$. Algorithm $SpanningTree_c$ is a basic spanning tree algorithm, where some extra conditions need to be considered. The basic idea

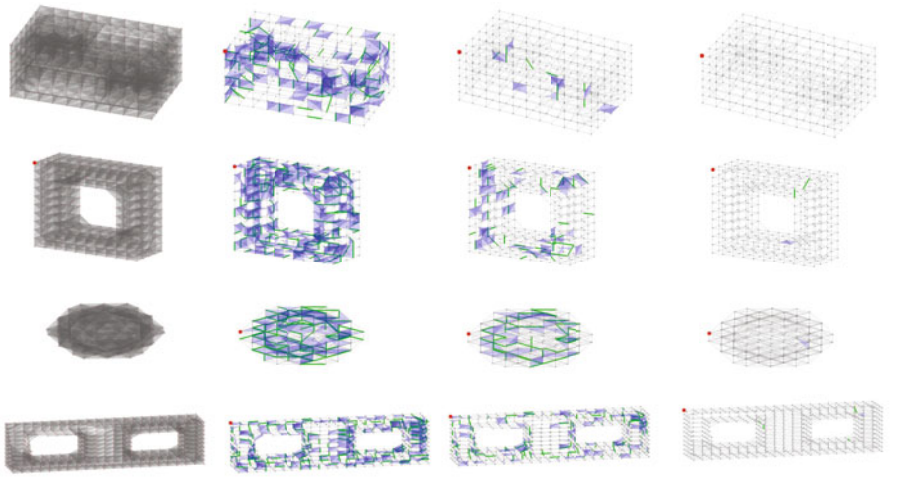

Fig. 5. A Bing's house, Torus, Sphere and Double Torus cell complexes, and their corresponding Morse complexes after some reductions. The final number of critical cells in the final complexes coincides with the Betti Numbers.

of this method, is, to construct valid trees (by joining p–cells and $(p+1)$–cells) that satisfy the global nil potency properties and the SDR condition. Therefore, we must asure that no cycles are created throughout the process.

We have used Depht First Search for the implementation, but any other spanning tree algorithm could be used instead. The implementation is written in C++, and it works either with simplicial or cell complexes. Several experiments have been performed (see Figure 5) using well known examples (Torus, Bing's house, Double Torus, Sphere, etc.). The software has provided valid HSF–representations and the minimum number of critical cells for each example.

6 Conclusions

In this paper we develop a non–unique combinatorial topo–geometric representation of a nD subdivided object, called homological spanning forest (or HSF, for short). This representation is a convenient tool in order to compute not only the minimum number of critical cells but also geometric (local curvature, normals to the boundary, Ricci curvature,....) and advanced topological information (reconstruction of the boundary, homological classification of cycles, relative homology with regards any sub-complex, skeletons, (co)homology operations, ...). Advantages with respect to existing Algebraic topological models, and DMT–based techniques have been shown. In a near future, we have the intention to deal with the "good" behaviour of the HSF representation for objects embedded in $\mathbb{R}^n$ with regards to combinatorial, geometric and topological changes, simplification, recognition, visualization, etc.

References

1. Kovalevsky, V.: Finite topology as applied to image analysis. Computer Vision, Graphics and Image Processing 46, 141–161 (1989)
2. Klette, R.: Cell complexes through time. Communication and Information Technology Research Technical Report 60 (2000)
3. Edmonds, J.: A combinatorial representation for polyhedral surfaces. Notices Amer. Math. Soc. 7 (1960)
4. Kropatsch, W.G., Haxhimusa, Y., Ion, A.: Multiresolution image segmentations in graph pyramids. In: Applied Graph Theory in Computer Vision and Pattern Recognition, pp. 3–41 (2007)
5. Eilenberg, S., Mac Lane, S.: On the groups $h(\pi, n)$, i, ii, iii. Annals of Math. 58, 60, 60, 55–106,48–139, 513–557 (1953,1954)
6. González-Díaz, R., Real, P.: On the cohomology of 3d digital images. Discrete Appl. Math. 147, 245–263 (2005)
7. González-Díaz, R., Jiménez, M.J., Medrano, B., Real, P.: Chain homotopies for object topological representations. Discrete Appl. Math. 157, 490–499 (2009)
8. Forman, R.: A Discrete Morse Theory for Cell Complexes. In: Yau, S.T. (ed.) Topology and Physics for Raoul Bott. International Press (1995)
9. Forman, R.: Morse theory for cell complexes. Adv. in Math. 134, 90–145 (1998)
10. Molina-Abril, H., Real, P.: Homological computation using spanning trees. In: Bayro-Corrochano, E., Eklundh, J.-O. (eds.) CIARP 2009. LNCS, vol. 5856, pp. 272–278. Springer, Heidelberg (2009)
11. Whitehead, J.: Combinatorial homotopy i. Bull. Amer. Math. Soc. 55, 213–245 (1949)
12. Molina-Abril, H., Real, P.: Homological optimality in discrete morse theory through chain homotopies. Submitted to Pattern Recognition Letters (2011)
13. Gugenheim, V.K.A.M., Lambe, L.A., Stasheff, J.D.: Perturbation theory in differential homological algebra. Illinois J. Math. 33, 357–373 (1989)
14. Lewiner, T., Lopes, H., Tavares, G., Matmídia, L.: Towards optimality in discrete Morse theory. Experimental Mathematics 12 (2003)
15. Munkres, J.: Elements of algebraic topology. Addison Wesley, Reading (1984)