

# Parsimonious Flooding in Geometric Random-Walks \*

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## Abstract

We study the information spreading yielded by the (*Parsimonious*) 1-*Flooding Protocol* in geometric Mobile Ad-Hoc Networks. We consider  $n$  agents on a convex plane region of diameter  $D$  performing independent random walks with move radius  $\rho$ . At any time step, every active agent  $v$  informs every non-informed agent which is within distance  $R$  from  $v$  ( $R > 0$  is the transmission radius). An agent is only active at the time step immediately after the one in which has been informed and, after that, she is removed. At the initial time step, a source agent is informed and we look at the *completion time* of the protocol, i.e., the first time step (if any) in which all agents are informed.

This random process is equivalent to the well-known *Susceptible-Infective-Removed (SIR)* infection process in Mathematical Epidemiology. No analytical results are available for this random process over any explicit mobility model.

The presence of removed agents makes this process much more complex than the (standard) flooding. We prove optimal bounds on the completion time depending on the parameters  $n$ ,  $D$ ,  $R$ , and  $\rho$ . The obtained bounds hold with high probability.

We remark that our method of analysis provides a clear picture of the dynamic shape of the information spreading (or infection wave) over the time.

**Keywords:** Geometric Random Walks, Information/Virus Spreading, Mobile Ad-Hoc Networks, Flooding Protocols.

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# 1 Introduction

In the *Geometric Random-Walk Model* [7, 10, 11],  $n$  agents perform independent random walks over a bounded plane region  $\mathcal{S}$  and we consider the following *infection* process. Every agent can be in three different states: non-informed (white) state, informed-active (red), informed-removed (black). During a time step, every agent performs one step of the random walk and every red agent informs (infects) all white agents lying within distance  $R$ . A white agent that has been informed (for the first time) at time step  $t$ , she becomes red at time step  $t + 1$ . Finally, when an agent becomes red she stays red for just one time step and, after that, she becomes black and stays that, forever.

At the initial time step, a source agent is in the red state. The completion time of the above infection process is the first time step in which every agent gets into the black state. If this time step does not exist then we say the infection process does not complete.

This random process is inspired by two main scenarios: The *Susceptible-Infected-Removed (SIR)* process which is widely studied in Mathematical Epidemiology [1, 2, 4] and the (*Parsimonious*) *1-Flooding Protocol* [3] in *geometric Mobile Ad-Hoc Networks (MANET)*. The (parsimonious)  $k$ -flooding protocol is the generalized version of the above infection process where a white agent, which is informed at time  $t$ , will remain red (so active) for the next  $k$  time steps.

While the standard flooding is extremely inefficient in terms of agent's energy consumption and message complexity, the  $k$ -flooding protocol (for small values of  $k$ ) strongly reduces the agent's energy consumption and the overall message complexity. However, as discussed later, the infection process yielded by the  $k$ -flooding (for small  $k$ ) over a dynamic network is much more complex than that yielded by the standard flooding.

The  $k$ -flooding has been studied in [3] on *Edge-Markovian Evolving Graphs (Edge-MEG)*. An Edge-MEG [6, 7] is a Markovian random process that generates an infinite sequence of graphs over the same set of  $n$  agents. If an edge exists at time  $t$  then, at time  $t + 1$ , it dies with probability  $q$ . If instead the edge does not exist at time  $t$ , then it will come into existence at time  $t + 1$  with probability  $p$ . The stationary distribution of an Edge-MEG with parameter  $p$  and  $q$  is the famous Erdős-Rényi random graph  $\mathcal{G}(n, \tilde{p})$  where  $\tilde{p} = \frac{p}{p+q}$ . The work [3] gives tight bounds on the  $k$ -flooding on stationary Edge-MEG for arbitrary values of  $p$ ,  $q$  and  $k$ . In particular, it derives the *reachability threshold* for the  $k$ -flooding, i.e., the smallest  $k = k(n, p, q)$  over which the protocol completes.

Edge-MEG is an analytical model of dynamic networks capturing time-dependencies, an important feature observed in real scenarios such as (faulty) wireless networks and P2P networks. However, it does not model important features of MANET. Indeed, in Edge-MEG, edges are independent Markov chains while, in MANET, there is a strong correlation among edges: agents use to act over a geometric space [7, 10, 11, 14].

**Our Contribution.** We study the 1-flooding protocol in the geometric random-walk model (the resulting MANET will be called *geometric-MANET*). The move radius  $\rho$  determines the maximal distance an agent can travel in one time step. Even though network modelling issues are out of the aims of this work, both the transmission radius and the move radius play a crucial role in our analysis and, thus, a small discussion about them is needed. Both parameters depend on several factors. In mathematical epidemiology, they depend on the agent's mobility, the kind of infection and on the agent's social behaviour that all together determine the average rate of "positive" contacts. In typical biological cases, the move radius can be significantly larger than the transmission radius. In MANET, the move radius depends on, besides the agent's mobility, the adopted protocol that tunes the transmission rate of the

agents: the larger is the time between two consecutive transmissions the larger is  $\rho$ . A larger  $\rho$  could yield a better message complexity at the cost of a larger completion time (the correct trade-offs is derived in [8] for the standard flooding).

It turns out that setting the “most convenient” value of the move radius is an issue that concerns both the network modelling and the protocol design. For these reasons, we investigate the 1-flooding for a wide range of values for parameters  $R$  and  $\rho$ .

Most of the known analytical studies concern geometric-MANET defined over squares or disks. We instead consider any *measurable* region  $\mathcal{S}$  of diameter  $D(\mathcal{S})$ . Informally speaking, a bounded region of the plane is said *measurable* if it is convex and it can be well approximated by a cover of square cells of suitable size. Furthermore, in a measurable region, we require the geometric random-walk model yields an almost-uniform stationary distribution of the agents. Square and disks are in fact the simplest examples of regions having all such properties.

We first show a negative result. Given a measurable region  $\mathcal{S}$ , if the transmission radius  $R$  is below the *connectivity threshold*<sup>1</sup> of  $\mathcal{S}$ , then, for any  $\rho \geq 0$ , w.h.p.<sup>2</sup> there are  $m = n^\alpha$  (for some constant  $\alpha > 0$ ) possible source agents for which the 1-flooding does not complete. This negative result also holds for the  $k$ -flooding for any  $k = O(1)$  and  $\rho \leq R$ .

We thus study the 1-flooding for  $R$  over the connectivity threshold of  $\mathcal{S}$ . We emphasize that, even in this case, nothing is known about the *connectivity* of the subset of red agents over the time and, more generally, about the impact of agent mobility on the 1-flooding process.

If  $\rho \leq R/(2\sqrt{2})$ , we prove that the information spreads at “optimal” speed  $\Theta(R)$ , i.e., the 1-flooding protocol w.h.p. completes within  $O(D(\mathcal{S})/R)$  time steps. Observe that, since  $\rho < R$ , this bound is asymptotically optimal.

Then, we consider move radii that can be up to any polynomial of  $R$ , i.e.  $\rho \leq \text{POLY}(R)$ . We prove that the information spreads at “optimal” speed that is  $\Theta(\rho)$ . So the 1-flooding w.h.p. completes in time  $O(D(\mathcal{S})/\rho)$  which is optimal for any  $\rho \geq R$ . Notice that this optimal information speed makes the 1-flooding time smaller than the static diameter  $O(D(\mathcal{S})/R)$  of the stationary graph: our bound is thus the first analytical evidence that agent’s mobility actually speeds-up this infection process. Finally, we observe that, in both cases, the energy-efficient 1-flooding protocol is as fast as the standard flooding [7, 8].

Can any upper bound for the 1-flooding time be directly extended to the  $k$ -flooding time, for any  $k > 1$ ? Rather surprisingly, the answer is in general not positive. Indeed, consider the two protocols running independently on the same experiment of the geometric-MANET. In the  $k$ -flooding process, an agent might be informed much earlier than what happens in the 1-flooding one: hence she might turn into the black state earlier and this might be fatal for some other white agents meet later. So a coupling between the two processes does not seem to exist. Another surprising fact is that, for the same reasons, increasing the transmission  $R$  does not necessarily imply the same or a better completion time of the  $k$ -flooding, for any  $k$ . In other words, it is not easy to establish whether the  $k$ -flooding time is a “monotone” function with respect to  $k$  or  $R$ . These facts provide a good insight about the crucial differences between the flooding process and the  $k$ -flooding one. Such differences make the analysis techniques adopted for the flooding almost useless. In particular, percolation theory [11, 10, 16], meeting and cover time of random walks on graphs [17], and the expansion/bootstrap arguments [7, 8] strongly rely on the fact that an informed agent will be active for all the flooding process. Furthermore, the analysis of  $k$ -flooding over the Edge-MEG model [3] strongly relies on the stochastic independence among the edges in  $\mathcal{G}(n, p)$ : a property that clearly does not hold in

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<sup>1</sup>The connectivity threshold refers to the uniform distribution of  $n$  (static) agents over  $\mathcal{S}$ .

<sup>2</sup>As usual, an event is said to hold *with high probability* (w.h.p.) if its probability is at least  $1 - 1/n$ .

geometric-MANET.

Our method of analysis significantly departs from all those mentioned above. Besides the optimal bounds on the completion time, our analyses provide a clear characterization of the geometric evolution of the infection process. We make use of a grid partition of  $\mathcal{S}$  into cells of size  $\Theta(R)$  and define a set of possible states a cell can assume over time depending on the number of red, white and black agents inside it. We then derive the local state-evolution law of any cell. Thanks to the regularity of this law, we can characterize the evolution of the geometric wave formed by the *red cells* (i.e. cells containing some red agent). A crucial property we prove is that, at any time step, white cells (i.e. cells containing white agents only) will never be adjacent to black cells, so there is always a red wave working between the black region and the white one. Furthermore, we show that the red wave eventually spans the entire region before all agents become black.

The generality of our method of analysis has further consequences. Thanks to the regularity of the red-wave shape, we are able to bound the time by which a given subregion will be infected for the first time. This bound is a function of the distance between the subregion and the initial position of the source agent. Actually, it is a function of the distance from the *closest* red agent at the starting time. So, our technique also works in the presence of an arbitrary set of source agents that aim to spread the same infection (or message). Under the same assumptions made for the single-source case, we can prove the completion time is w.h.p.  $\Theta(\text{ECC}(A, \mathcal{S})/R)$  (or  $\Theta(\text{ECC}(A, \mathcal{S})/\rho)$ ) where  $A$  is the set of the positions of the source agents at starting time and  $\text{ECC}(A, \mathcal{S})$  is the *geometric eccentricity* of  $A$  in  $\mathcal{S}$ . Finally, the specific arguments of our proofs can be easily adapted to get the same upper bounds for the  $k$ -flooding as well.

**Related Works.** As mentioned above, there are no analytical study of the parsimonious flooding process over any geometric mobility model. In what follows, we briefly discuss some analytical results concerning the flooding over some models of MANET. In [13], the flooding time is studied over a restricted geometric-MANET. Approximately, it corresponds to the case  $\rho = D$ . Notice that, under this restriction, the stochastic dependence between two consecutive agent positions is negligible. In [12], the speed of data communication between two agents is studied over a class of *Random-Direction* models yielding uniform stationary agent's distributions (including the geometric-MANET model). They provide an upper bound on this speed that can be interpreted as a *lower* bound on flooding time when the mobile network is very sparse and disconnected (i.e.  $R, \rho = o(1)$ ). Their technique, based on Laplacian transform of independent journeys, cannot be extended to provide any upper bound on the time of any version of the flooding. We observe that, differently from our model, both [13] and [12] assume that when an agent gets informed then all agents of her *current* connected component (no matter how large it is) will get informed in one time step. The same unrealistic assumption is adopted in [16], where bounds on flooding time (and some other tasks) are obtained in the Poisson approximation of the geometric-MANET model. In [7, 8], the first almost tight bounds for the flooding time over geometric-MANET have been given. As mentioned before, their proofs strongly rely on the fact that informed agents stay always active. Flooding and gossip time for random walks over the grid graph (so, the agent's space is a graph) have been studied in [17]. Here, an agent informs all agents lying in the same node. Besides the differences between 1-flooding and flooding discussed before, it is not clear whether their results could be extended to the random walk model over geometric spaces. Especially, in their model, there is no way to consider arbitrary values of the move radius.

**Roadmap of the paper.** The rest of the paper is organized as follows. After giving some

preliminary definitions in Section 2, we analyze the case  $\rho \leq R/(2\sqrt{2})$  in Section 3. In Section 4, we present the results for the case  $\rho \leq R^2/\sqrt{\log n}$ . In Section 5, we introduce a slight different geometric random-walk model and a more powerful technique to extend the bound  $O(D(\mathcal{S})/\rho)$  to any  $\rho \leq \text{POLY}(R)$ . In Section 6, we first briefly describe the extension to multi-source case and, then, derive the completion threshold of  $R$  for the 1-flooding. Finally, some open questions are discussed in Section 7. Most of the proofs are given in the Appendix.

## 2 Preliminaries

We first introduce the class of regions that can be considered feasible support spaces for our mobile agents.

**Definition 1** *For any  $l, \gamma > 0$ , a bounded connected plane region  $\mathcal{S}$  is  $(\ell, \gamma)$ -measurable if it is convex and there exists a grid-partition of the plane into squared cells of side length  $\ell$  in which a cell cover  $\mathcal{C}(\mathcal{S})$  of  $\mathcal{S}$  exists such that, for any cell  $c \in \mathcal{C}(\mathcal{S})$ ,  $\text{AREA}(c \cap \mathcal{S}) \geq \gamma \ell^2$ .*

We say that two cells are adjacent if they touch each other by side or by corner. The *cell-diameter*  $D_{\square}(\mathcal{S})$  of an  $(\ell, \gamma)$ -measurable  $\mathcal{S}$  is defined as follows. Given two cells  $c, c' \in \mathcal{C}(\mathcal{S})$ , define their *cell-distance*  $d_{\square}(c, c')$  as the length of the shortest cell-path  $p = \langle c = c_0, c_1, \dots, c_s = c' \rangle$  such that for every  $i$ ,  $c_i \in \mathcal{C}(\mathcal{S})$  and  $c_i$  is adjacent to  $c_{i+1}$ . Then, we define

$$D_{\square}(\mathcal{S}) = \max\{ d_{\square}(c, c') \mid c, c' \in \mathcal{C}(\mathcal{S}) \}$$

Similarly, we can define the *cell-distance* between a cell and any cell subset  $\mathcal{C}$ , i.e.,

$$d_{\square}(c, \mathcal{C}) = \min\{ d_{\square}(c, c') \mid c' \in \mathcal{C} \}$$

Observe that, since  $\mathcal{S}$  is convex, the (Euclidean) diameter  $D(\mathcal{S})$  and the *cell-diameter*  $D_{\square}(\mathcal{S})$  are tightly related:  $D_{\square}(\mathcal{S}) = \Theta(D(\mathcal{S})/\ell)$ . For instance, if  $\mathcal{S}$  is the  $L \times L$ -square ( $L > 0$ ) and  $\ell = o(L)$ , then its cell diameter is  $\Theta(L/\ell)$ .

According to the geometric random-walk model, there are  $n$  agents that perform independent random walks over  $\mathcal{S}$ . At any time step, an agent in position  $x \in \mathcal{S}$ , can move uniformly at random to any position in  $\mathcal{B}(x, \rho) \cap \mathcal{S}$ , where  $\mathcal{B}(x, \rho)$  is the disk of center  $x$  and radius  $\rho$ . This is a special case of the *Random Trip Model* introduced in [14] where it is proved that it admits a unique stationary agents distribution. In the sequel, we always assume that at time  $t = 0$  agents' positions are random w.r.t. the stationary distribution.

For the sake of simplicity, we assume that the average density of agents in  $\mathcal{S}$  (i.e. the average number of agents per area unit) is 1. It is straightforward to scale all the definitions and results to an arbitrary average density of agents.

Let us consider  $n$  mobile agents acting over an  $(\ell, \gamma)$ -measurable region  $\mathcal{S}$ . We say that the resulting geometric-MANET satisfies the *density condition* if, for every time step  $t = 0, 1, \dots, n$  and for every cell  $c \in \mathcal{C}(\mathcal{S})$ , the number  $\#_c$  of agents in  $c$  at time step  $t$  satisfies, with probability at least  $1 - (1/n)^4$ , the following inequalities

$$\eta_1 \ell^2 \leq \#_c \leq \eta_2 \ell^2, \text{ where } \eta_1, \eta_2 \text{ are positive constants.} \quad (1)$$

We will consider geometric-MANET that satisfy the density condition. For instance, if the region is the  $\sqrt{n} \times \sqrt{n}$ -square, this assumption implies  $R \geq \beta \sqrt{\log n}$ , for a suitable constant  $\beta > 0$ . More generally, the geometric random-walk model with  $n$  agents over an arbitrary

$(\ell, \Theta(1))$ -measurable region with  $\text{AREA}(\mathcal{S})/\ell^2 \leq \beta n / \log n$  for a suitable constant  $\beta > 0$  and move radius  $\rho = O(\ell)$  satisfies the density condition. The proof of this fact can be obtained by exploiting the same arguments used in [7, 8] for the discrete approximation of the geometric random-walk model on the square.

According to the 1-flooding protocol, at any time step, every agent can be in three different states: white (non-informed) state, red (informed-active), and black (informed-inactive). The configuration  $\text{CONF}(t)$  is defined as the set of the positions of the  $n$  agents together with their respective states at the end of time  $t$ .

The analysis of the information spreading will be performed in terms of the number of infected cells. Given a cell  $C \in \mathcal{C}(\mathcal{S})$ , the neighborhood  $N(C)$  is the set of cells formed by  $C$  and all its adjacent cells.

Let us observe that, the density condition implies that w.h.p. the (geometric) source eccentricity in  $\mathcal{S}$  is  $\Omega(D(\mathcal{S}))$ ; moreover the maximal speed of any message in the geometric-MANET is  $R + \rho$ . We thus easily get the following lower bound.

**Fact 2** *For any  $k$ , the  $k$ -flooding time is w.h.p.  $\Omega(D(\mathcal{S})/(R + \rho))$ .*

### 3 High Transmission-Rate or Low-Mobility

We warm-up with the case where the move radius is smaller than the transmission radius. More precisely, we assume  $\rho \leq R/(2\sqrt{2})$ . Under this assumption, an agent lying in a cell  $C$  cannot escape from  $N(C)$  in one step. This makes the analysis simpler than the cases with larger  $\rho$ . However, some of the crucial ideas and arguments adopted here will be exploited for the harder cases too.

We can now state the optimal bound on 1-flooding. Consider a network formed by  $n$  mobile agents with transmission radius  $R$  and move radius  $\rho \leq R/(2\sqrt{2})$ . The agents act over an  $(\ell, \gamma)$ -measurable region  $\mathcal{S}$  of diameter  $D(\mathcal{S})$  such that  $\ell = R/(2\sqrt{2})$ , and  $\gamma$  is any positive constant. Assume also that the resulting geometric-MANET satisfies the density condition.

**Theorem 3** *Under the above assumptions, the 1-flooding time is w.h.p.  $\Theta(D(\mathcal{S})/R)$ .*

The geometric-MANET defined over an  $L \times L$ -square  $\mathcal{Q}$  satisfies the above conditions provided that  $R \geq c_0 L \sqrt{\log n/n}$  for a sufficiently large constant  $c_0$  (see [7, 8]). We thus have the following

**Corollary 4** *Let  $R \leq L$  be such that  $R \geq c_0 L \sqrt{\log n/n}$  for a sufficiently large constant  $c_0$  and let  $\rho \leq R/(2\sqrt{2})$ . Then the 1-flooding time over  $\mathcal{Q}$  is w.h.p.  $\Theta(L/R)$ .*

#### 3.1 Proof of Theorem 3

For the sake of simplicity, we assume that the time step is divided into 2 consecutive phases: the *transmission phase* where every red agent transmits the information and the *move phase* where every agent performs one step of the random walk. We need to introduce the feasible states of a cell during the infection process.

**Definition 5** *At (the end of) any time step a cell  $C$  can be in 4 different states. It is white if it contains only white agents. It is red if it contains at least one red agent. It is black if it contains black agents only. It is grey if it is not in any of the previous 3 cases.*

We will show that the infection process, at any time step, has (w.h.p.) a well defined shape in which no white cell is adjacent to a black one and there is no grey cell. This shape is formalized in the following definitions. A subset of white cells is a *white component* if it is a connected component w.r.t. the subset of all the white cells.

**Definition 6** *A configuration  $\text{CONF}(t)$  is regular if the following properties hold*

- a) *No cell is grey.*
- b) *Every white component is adjacent to a red cell.*
- c) *No white cell is adjacent to a black cell.*

Observe that the starting configuration  $\text{CONF}(0)$  is regular.

**Definition 7** *A white cell is red-close if it is adjacent to a red cell.*

The next lemma determines the local state-evolution of any cell in a regular configuration (the proof is given in the Appendix).

**Lemma 8** *Consider any cell  $C$  at time step  $t$  and assume  $\text{CONF}(t)$  be regular. Then the following properties hold.*

- 1) *If  $C$  is red-close then it becomes red in  $\text{CONF}(t+1)$  w.h.p.*
- 2) *If  $C$  is (no red-close) white then it becomes white or red in  $\text{CONF}(t+1)$ .*
- 3) *If  $C$  is red then it becomes red or black in  $\text{CONF}(t+1)$ .*
- 4) *If  $C$  is black then it becomes red or black in  $\text{CONF}(t+1)$ .*

As an easy consequence of the above lemma, we get the following

**Lemma 9** *For any  $t \leq n$ , if  $\text{CONF}(t)$  is regular, then w.h.p.  $\text{CONF}(t+1)$  is regular as well.*

The above result on regular configurations provides a clear characterization of the shape of the infection process. We now analyze the speed of the process. Observe that we can now assume that all configurations are regular (w.h.p.). The proof of the next lemma is given in the Appendix.

**Lemma 10** *For any  $t < n$ , let  $w$  be any white cell in  $\text{CONF}(t+1)$  and let  $\text{Red}(t)$  be the set of red cells in  $\text{CONF}(t)$ . It holds w.h.p. that*

$$d_{\square}(w, \text{Red}(t+1)) \leq d_{\square}(w, \text{Red}(t)) - 1$$

Starting from the initial configuration  $\text{CONF}(0)$  (that has one red cell), Lemma 10 implies that *every* white cell in  $\mathcal{S}$  w.h.p will become red within  $O(D_{\square}(\mathcal{S})) = O(D(\mathcal{S})/R)$  time steps. Moreover, thanks to Lemmas 8 and 9, every red cell will become either red or black. Finally, when all cells are black or red, after the next time step there are no more white agent and the theorem follows.

## 4 Low Transmission-Rate or High-Mobility I

We consider the network formed by  $n$  mobile agents with transmission radius  $R \geq c_0 \sqrt{\log n}$  and move radius  $\rho$  such that  $R/2 \leq \rho \leq \alpha R^2 / \sqrt{\log n}$  for sufficiently small constant  $\alpha > 0$ . The agents act on a support region  $\mathcal{S}$  that satisfies two measurable conditions.

It is  $(\ell_0, \gamma)$ -measurable where  $\ell_0 = R/(4\sqrt{2})$  and  $\gamma$  is a positive constant. Moreover,  $\mathcal{S}$  is also  $(\ell, \gamma)$ -measurable where  $\ell = 2\ell_0$  and the grid of  $\ell$ -cells is a subgrid of the grid of cells of side length  $\ell_0$ . We assume the geometric-MANET over  $\mathcal{S}$  satisfies the density condition w.r.t. the  $\ell_0$ -cells.

**Theorem 11** *Under the above assumptions, the 1-flooding time is w.h.p. bounded by  $\Theta(D(\mathcal{S})/\rho)$ .*

The above theorem implies an optimal bound on 1-flooding time over an  $L \times L$  square  $\mathcal{Q}$  for any sufficiently large  $L$ .

**Corollary 12** *Let  $R \leq L$  be such that  $R \geq c_0 L \sqrt{\log n/n}$  for a sufficiently large constant  $c_0$  and let  $R/2 \leq \rho \leq \alpha R^2 / \sqrt{\log n}$ , for sufficiently small constant  $\alpha > 0$ . Then the 1-flooding time over  $\mathcal{Q}$  is w.h.p.  $\Theta(L/\rho)$ .*

### 4.1 Proof of Theorem 11

We adopt Def.s 5, 6, and 7 given in the previous section. Moreover, we need the further

**Definition 13** *Two cells are  $\rho$ -close if the (Euclidean) distance between their geometric centers is at most  $\rho$ .*

As in the previous section, we provide the local state-evolution law of any cell in a regular configuration (the proofs of all the next lemmas are given in the Appendix).

**Lemma 14** *Consider any cell  $C$  at time step  $t$  and assume  $\text{CONF}(t)$  be regular. Then the following properties hold w.h.p.*

- 1) *If  $C$  is red-close then it becomes red in  $\text{CONF}(t+1)$ .*
- 2) *If  $C$  is  $\rho$ -close to a red-close cell, then  $C$  becomes red in  $\text{CONF}(t+1)$ .*
- 3) *If  $C$  is white but it is not  $\rho$ -close to any red-close cell then it becomes white or red in  $\text{CONF}(t+1)$ .*
- 4) *If  $C$  is red or black but it is not  $\rho$ -close to any red-close cell then it becomes red or black in  $\text{CONF}(t+1)$ .*

As an easy consequence of the above lemma, we get the following

**Lemma 15** *For any  $t < n$ , if  $\text{CONF}(t)$  is regular, then w.h.p.  $\text{CONF}(t+1)$  is regular as well.*

In what follows, we assume that all configurations are regular (w.h.p.) and analyze the speed of the infection process. Differently from the previous section, we will show this “speed” is  $\Theta(\rho)$ . Notice that the cell diameters of  $\mathcal{S}$  w.r.t. the  $\ell_0$  and  $\ell$  respectively differs by a “small” constant factor only: for this reason both of them are denoted as  $D_{\square}(\mathcal{S})$ .

**Lemma 16** *For any  $t < n$ , let  $w$  be any white cell in  $\text{CONF}(t+1)$  and let  $\text{Red-close}(t)$  be the set of red-close cells in  $\text{CONF}(t)$ . It w.h.p. holds that*

$$d_{\square}(w, \text{Red-close}(t+1)) \leq \max\{d_{\square}(w, \text{Red-close}(t)) - \Theta(\rho/R), 0\}$$

The starting configuration  $\text{CONF}(0)$  is regular and contains one red cell. So, Lemma 16 implies that *every* white cell in  $\mathcal{S}$  w.h.p will become red within

$$O\left(\frac{D_{\square}(\mathcal{S})\ell}{\rho}\right) = O\left(\frac{D(\mathcal{S})}{\rho}\right) \text{ time steps}$$

Moreover, thanks to Lemma 14, every red cell will become either red or black. Finally, when all cells are black or red, in the next time step there are no more white agent and the theorem follows.

## 5 Low Transmission-Rate or High-Mobility II

In this section, we study the case when the move radius can be much larger than the transmission radius. More precisely, the move radius can be an arbitrary polynomial of the transmission radius, i.e.  $\rho = O(\text{POLY}(R))$ , we also assume  $R \geq c_0\sqrt{\log n}$ , for a sufficiently large  $c_0$ , and  $\rho \geq 5R$ .

A slightly different version of the geometric random-walk model is here adopted, the *cellular random walk*: the region  $\mathcal{S}$  is partitioned in squared *supercells* of edge length  $\rho$ . Then an agent lying in any position of a supercell  $C$  selects her next position independently and uniformly at random over the neighborhood  $N(C)$ . Moreover, an agent can be informed by another (red) agent only if they both belong to the same supercell. So, in the cellular random walk model, the *influence* of an agent lying in a supercell  $C$ , in the next time step, is always restricted to the subregion  $N(C)$ . Observe that the cellular random walk model preserves all significant features of the standard one while, on the other hand, it avoids several geometric technicalities. The latter would yield a much more elaborate analysis without increasing the relevance of the obtained results.

We consider a support region  $\mathcal{S}$  that satisfies two measurable conditions. It is  $(\ell, \gamma)$ -measurable where  $\ell = R/\sqrt{2}$  and  $\gamma$  is a positive constant. Moreover,  $\mathcal{S}$  is also  $(\rho, \gamma)$ -measurable where the grid of supercells is a subgrid of the grid of cells of side length  $\ell$ . In what follows, the cells of size length  $\rho$  are called *supercells* and those of size length  $\ell$  are called cells, simply. We also assume that the geometric-MANET over  $\mathcal{S}$  satisfies the density condition w.r.t. the cells. The density condition for the stationary phase of the cellular random-walk model for  $\rho$  over a region satisfying the two above measurability can be proved by similar arguments as in [7, 8]. Indeed, the measurability by supercells guarantees that the stationary distribution is almost uniform; this fact together with the measurability by cells implies the density condition.

**Theorem 17** *Under the above assumptions and  $5R \leq \rho \leq \text{POLY}(R)$ , the 1-flooding time is w.h.p.  $\Theta(D(\mathcal{S})/\rho)$ .*

The above theorem implies an optimal bound on 1-flooding time over an  $L \times L$  square  $\mathcal{Q}$ .

**Corollary 18** *Let  $R = o(L)$  be such that  $R \geq c_0L\sqrt{\log n/n}$  for a sufficiently large constant  $c_0$  and let  $5R \leq \rho \leq \text{POLY}(R)$ . Then the 1-flooding time over  $\mathcal{Q}$ , in the cellular random-walk model with move radius  $\rho$ , is  $\Theta(L/\rho)$  w.h.p.*

### 5.1 Proof of Theorem 17

The higher agent mobility forces us to analyze the infection process over the supercells (besides over the cells). During the information spreading, the state a supercell can assume is defined

by some bounds on the number of red and white agents inside it. Roughly speaking, the number of white agents must never be too small w.r.t. the number of red agents, moreover the latter must increase exponentially w.r.t.  $R^2$ . Since the infection process is rather complex, we need to consider a relative large number of possible supercell states. Our analysis will first show every supercell eventually evolves from the initial white state to the final black state according to a monotone process over a set of intermediate states. Then, we will show that the speed of this process is asymptotically optimal, i.e., proportional to the move radius. For a supercell  $C$  and a time step  $t$ , let  $\#_r^t(C)$ ,  $\#_w^t(C)$ , and  $\#_b^t(C)$  be, respectively, the number of red, white, and black agents in  $C$ . We define

$$\hat{h} = \left\lceil \log_{R^2} \left( c_0 \frac{\rho^2}{R^2} \log n \right) \right\rceil \quad (2)$$

Observe that the assumption  $\rho = O(\text{POLY}(R))$  implies  $\hat{h} = \Theta(1)$ .

**Definition 19** For any time  $t$  and for any supercell  $C$ , we define some possible states of  $C$  at time  $t$ .

**State  $h = 0$  (White State):**  $\#_r^t(C) = 0$  AND  $\#_b^t(C) = 0$ .

**State  $h = 1, \dots, \hat{h} - 1$  (Intermediate States):** The values of  $\#_r^t(C)$  and  $\#_w^t(C)$  satisfy

$$a_h R^{2h} \leq \#_r^t(C) \leq b_h R^{2h} \quad \text{AND} \quad \#_w^t(C) \geq c_h \rho^2$$

where  $a_h, b_h, c_h$  are constants (they will be fixed later in the Appendix) that satisfy  $a_h < b_h$ ,  $a_1 \geq a_2 \geq \dots \geq a_{\hat{h}-1} > 0$ ,  $0 < b_1 \leq b_2 \leq \dots \leq b_{\hat{h}-1}$ , and  $c_1 \geq c_2 \geq \dots \geq c_{\hat{h}-1} > 0$ .

**State  $\hat{h}$  (Red State):**  $\#_r^t(C) \geq 90 \frac{\rho^2}{R^2} \log n$ .

**State  $\hat{h} + 1$  (Black State):**  $\#_w^t(C) = 0$ .

All the above states are mutually disjoint but the last three ones, i.e.,  $\hat{h} - 1, \hat{h}, \hat{h} + 1$ . For every supercell  $C$  and time step  $t$ , in order to indicate that  $C$  satisfies the condition of state  $h$ , we will write  $h^t(C) = h$ .

**Definition 20** The configuration  $\text{CONF}(t)$  is regular if for every supercell  $C$  (1) an  $h \in \{0, 1, \dots, \hat{h}, \hat{h} + 1\}$  exists s.t.  $h^t(C) = h$  and (2) if  $h^t(C) = \hat{h} + 1$ , it holds that,  $\forall C' \in N(C)$ ,  $h^t(C') = \hat{h} \vee h^t(C') = \hat{h} + 1$ .

In the sequel, we exchange the order of the phases in a time step: the move phase now comes before the transmission one. Clearly, this does not change our asymptotical results. The next technical lemmas allow to control the 1-step evolution of the state of any supercell in terms of the number of red and white agents and how such agents spread over its cells (the proofs are given in the Appendix).

**Lemma 21** Let  $C$  be a supercell such that  $\#_w^t(N(C)) \geq \lambda \rho^2$ , for some constant  $\lambda \geq 720/c_0^2$ . Then, immediately after the move phase of time  $t + 1$  (and before the transmission phase), w.h.p., for every cell  $C$  in  $C$ , it holds that the number of white agents in  $C$  is at least  $(\lambda/36)R^2$ .

**Lemma 22** Let  $C$  be a supercell such that  $\#_r^t(C) \geq \lambda R^k$ , for some constants  $\lambda \geq 1800/c_0^2$  and  $k \geq 2$ . Then, immediately after the move phase of time  $t + 1$ , w.h.p. in every supercell  $C' \in N(C)$ , the cells in  $C'$  hit by some red agent are at least  $\min\{(\lambda/30)R^k, \rho^2/(2R^2)\}$ .

**Lemma 23** Let  $C$  be any supercell such that  $h^t(C) = \hat{h}$ . Then, immediately after the move phase of time  $t + 1$ , w.h.p. for every supercell  $C' \in N(C)$  all the cells in  $C'$  are hit by some red agent.

**Lemma 24** For any time step  $t$  and any supercell  $C$ , let  $\widehat{\#_r^t(C)} = \max\{\#_r^t(C'), C' \in N(C)\}$ . If, for some  $M > 0$ , it holds that  $\widehat{\#_r^t(C)} \leq M$ , then it holds that  $\#_r^{t+1}(C) \leq 68\eta_2 MR^2$  w.h.p.

We are now able to provide the local state-evolution law of any supercell in a regular configuration (the proofs of all the next lemmas are given in the Appendix). Let us define  $m^t(C) = \max\{h \mid \exists C' \in N(C) : h^t(C') = h\}$ .

**Lemma 25** if  $\text{CONF}(t)$  is regular then the following implications hold w.h.p., for every supercell  $C$ :

- |  |               |                            |
|--|---------------|----------------------------|
| (a) $m^t(C) = 0$   | $\Rightarrow$ | $h^{t+1}(C) = 0$           |
| (b) $1 \leq m^t(C) \leq \hat{h} - 1$                                 | $\Rightarrow$ | $h^{t+1}(C) = m^t(C) + 1$  |
| (c) $m^t(C) = \hat{h} \wedge (h^t(C) = h \text{ with } h < \hat{h})$ | $\Rightarrow$ | $h^{t+1}(C) = \hat{h}$     |
| (d) $m^t(C) = \hat{h} \wedge h^t(C) = \hat{h}$                       | $\Rightarrow$ | $h^{t+1}(C) = \hat{h} + 1$ |
| (e) $m^t(C) = \hat{h} + 1$   | $\Rightarrow$ | $h^{t+1}(C) = \hat{h} + 1$ |

As a consequence of the above lemma, we get

**Lemma 26** For any  $t < n$ , if  $\text{CONF}(t)$  is regular, then w.h.p.  $\text{CONF}(t + 1)$  is regular as well.

**Lemma 27** With high probability the initial configuration  $\text{CONF}(0)$  is regular and a supercell  $C$  exists such that  $h^0(C) = 1$ .

For any time  $t$ , let  $\text{Red}(t)$  be the set of supercells whose state at time  $t$  is at least 1. For any supercell  $C$ , denote by  $d_{\square}^t(C)$  the distance w.r.t. supercells between  $C$  and  $\text{Red}(t)$ . Clearly, if  $C \in \text{Red}(t)$  then  $d_{\square}^t(C) = 0$ .

**Lemma 28** For any  $t \leq n$ , if  $\text{CONF}(t)$  is regular, then w.h.p., let for any supercell  $W$  such that  $h^t(W) = 0$ , it holds that

$$d_{\square}^{t+1}(W) \leq d_{\square}^t(W) - 1$$

Theorem 17 is an easy consequence of Lemmas 26, 27 and 28.

## 6 The Multi-Source Case and The Completion Threshold

**The multi-source 1-flooding.** Consider the 1-flooding process with  $n$  agents over the region  $\mathcal{S}$  whose starting configuration contains an arbitrary subset of source agents. Every source agent has the same message (infection) and, again, the goal is to bound the completion time. Let  $A$  be the set of positions of the source agents at starting time.

For any point  $x \in \mathcal{S}$ , define

$$d(x, A) = \min\{d(x, a) \mid a \in A\}$$

$$\text{ECC}(A, \mathcal{S}) = \max\{d(x, A) \mid x \in \mathcal{S}\}$$

The parameter  $\text{ECC}(A, \mathcal{S})$  is the geometric eccentricity of  $A$  in  $\mathcal{S}$ .

**Theorem 29** *Under the same assumptions of the single-source case, for any choice of the source positions  $A$ , the 1-flooding time is w.h.p.*

- i)  $\Theta(\text{ECC}(A, \mathcal{S})/R)$ , for any  $\rho \leq R/(2\sqrt{2})$ ;
- ii)  $\Theta(\text{ECC}(A, \mathcal{S})/\rho)$ , for any  $R/2 \leq \rho \leq \text{POLY}(R)$ .

*Idea of the proof.* The crucial observation is that in all cases (i.e. those of Sect.s 3, 4, and 5), the obtained local state-evolution law of the cells works for *any* starting configuration provided it is a regular one. It is easy to verify that, for any choice of the source subset, the starting configuration is w.h.p. regular. Moreover, our analysis of the speed of the information spreading does not change at all: the initial distance between every white cell (supercell) and its closest red cell (red-close supercell) decreases w.h.p. by 1 at any time step. In the case of more source agents, such initial distance is bounded by  $\text{ECC}(A, \mathcal{S})$ . □

**The completion threshold for  $R$ .** We show that the 1-flooding may not complete when  $R < D(\mathcal{S})\sqrt{\gamma \log n/n}$  for a sufficiently small constant  $\gamma$ . By using basic probability arguments it is easy to prove that, under the above assumption, the probability that *some* isolated agents exist in the initial configuration is *positive*. In the next lemma we state a much stronger result: we prove that, w.h.p., the number of such isolated agent is  $n^{\Theta(1)}$ . We don't know whether this result has been explicitly proved somewhere, however, in the appendix we give a proof based on the Method of Bounded Differences [15].

**Lemma 30** *A constant  $\gamma > 0$  exists such that, for sufficiently large  $n$ , for any  $R < D(\mathcal{S})\sqrt{\gamma \log n/n}$  and  $\rho > 0$ , w.h.p. there are  $m = n^\alpha$  (for some constant  $\alpha > 0$ ) isolated agents in the initial configuration.*

**Corollary 31** *A constant  $\gamma > 0$  exists such that, for sufficiently large  $n$ , for any  $R < D(\mathcal{S})\sqrt{\gamma \log n/n}$  and  $\rho > 0$ , w.h.p., there are source agents for which 1-flooding does not complete.*

By using arguments similar to those in the proof of Lemma 30 (and a smaller  $\gamma$ ), the above negative result can be extended to the  $k$ -flooding for any fixed constant  $k$  when  $\rho \leq R$ .

## 7 Conclusions

The probabilistic analysis of Information spreading in mobile/dynamic networks is a challenging issue that is the subject of several current research projects and some relevant advances have been obtained in the last 5 years by using approaches based on time/space discrete approximation of the evolving systems and discrete probability [3, 5, 6, 10, 12, 16, 17]. We believe that this *discrete* approach is promising to address several important related questions which are still far to be solved.

The more related open question to our work is the analysis of the  $k$ -flooding with  $k > 1$  under the completion threshold of the 1-flooding process. For instance, is the  $\Theta(\log n)$ -flooding able to complete even much *under* the above threshold? Which is the role of the move radius  $\rho$ ? A more general challenging issue is to extend the analysis of the parsimonious flooding to other explicit models of MANET such as the random way-point model [14, 9].

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## A Proofs of Section 3

### Proof of Lemma 8.

- 1) After the transmission phase of time step  $t + 1$ , every agent in the red-close cell  $C$  becomes red. Then, thanks to the density condition, w.h.p. at least a fraction of these red agents remains in  $C$  after the move phase as well. Thus, at time  $t + 1$ , w.h.p.  $C$  becomes red.
- 2) Observe that if a white cell  $w$  is not red-close then, since  $\text{CONF}(t)$  is regular,  $w$  is surrounded by white cells only, then at time  $t + 1$  no black agent can reach  $w$ . Notice that  $w$  may become red at time  $t + 1$ . Indeed, if there is a red-close cell in  $N(w)$ , after the move phase of time step  $t + 1$ , some red agent can reach  $w$ .
- 3) After the transmission phase of step  $t + 1$ , since  $R = 2\sqrt{2}\ell$ , all agents in  $N(C)$  will be either red or black. So, after the move phase, since  $\rho < \ell$  no white agent can reach  $C$ .
- 4) Since  $\text{CONF}(t)$  is regular, a black cell  $B$  is initially surrounded by black or red cells only. After the transmission phase of step  $t + 1$ , all agents in  $N(B)$  will be either red or black. So, after the move phase, since  $\rho < \ell$  no white agent can reach  $B$ .

□

**Proof of Lemma 10.** From Lemma 8, a white cell  $w$  in  $\text{CONF}(t + 1)$  was white in  $\text{CONF}(t)$  as well. Now, consider one shortest cell path  $p$  from  $w$  to a red cell  $R$  in  $\text{Red}(t)$ . Then, thanks to Lemma 8, the (red-close) white cell  $w'$  in  $p$  and adjacent to  $R$  will w.h.p. become red in  $\text{CONF}(t + 1)$ . Notice that some other white cell in  $p$  may become red in  $\text{CONF}(t + 1)$ . In any case, the lemma holds.

□

## B Proofs of Section 4

For the sake of simplicity, we here assume the parameter  $\gamma$  of the measurability of region  $\mathcal{S}$  be set to 1. If  $\gamma < 1$ , only the values of the constants in the proofs would change.

### Proof of Lemma 8.

*Claim 1).* Immediately after the transmission phase of time step  $t$ , all agents in  $C$  become red. Thanks to the density condition and the bound  $\rho \leq \alpha(R^2/\sqrt{\log n})$ , it is possible to prove that w.h.p at least one of such red agents remains in  $C$  after the move phase.

*Claim 2).* We first observe that, after the transmission phase of  $t + 1$ , all agents in  $C'$  becomes red. Define  $X$  be the indicator r.v. that equals 1 iff  $C$  is hit by some red agent immediately after the move phase. From the density condition, it holds that

$$\mathbf{P}(X = 0) \leq \left(1 - \frac{\ell_0^2}{\pi\rho^2}\right)^{\eta_1\ell_0^2} \leq \exp\left(-\frac{\eta_1\ell_0^4}{\pi\rho^2}\right)$$

The upper bound  $\rho \leq (R^2/\sqrt{\log n})$  implies that

$$\mathbf{P}(X = 0) \leq \exp(-\Theta(\log n))$$

So, w.h.p. cell  $C$  becomes red.

*Claim 3).* Assume, by way of contradiction, that immediately after the transmission phase of  $t + 1$ , a black agent exists that is within Euclidean distance  $\rho$  from  $C$ . Let  $b$  the black agent closest to  $C$  and let  $B$  containing it. Let  $z$  be the cell which is the closest one to  $C$  among those in  $N(B)$ . Since the region is convex, by simple geometric argument, we have that the

Euclidean distance between the centers of  $C$  and  $z$  is equal to the Euclidean distance between  $B$  and  $C$ . This implies that  $C$  is  $\rho$ -close to  $z$ . Since  $\text{CONF}(t)$  is regular, it is not hard to prove that cell  $z$  can be neither black nor white at time step  $t$ . It turns out that  $z$  must be red: this would contradict the hypothesis that  $C$  is not  $\rho$ -close to a red-close cell at time step  $t$ .

*Claim 4).* Immediately after the transmission phase of  $t + 1$ , consider the white agent  $w$  closest to cell  $C$ . If  $d(w, C) > 0$ , we easily get the claim. Assume thus  $d(w, C) \leq \rho$  and let  $w$  be the cell containing  $w$ . Let  $z$  be the cell which is the closest one to  $C$  among those in  $N(w)$ . Since  $\text{CONF}(t)$  is regular, it is not hard to prove that cell  $z$  can be neither red nor black at time step  $t$ . So, cell  $z$  must be white at time  $t$ . After the transmission of time step  $t + 1$ , all agents in  $z$  must go into the red state since otherwise  $w$  would not be the closest white agent to  $C$ . Now, since  $z$  is  $\rho$ -close to  $C$ , by using the same argument of Claim 2, we have that after the move phase, at least a red agent hits cell  $C$  thus the latter becomes red as well.

□

**Proof of Lemma 16.** From Lemma 14, a white cell  $w$  in  $\text{CONF}(t + 1)$  was white in  $\text{CONF}(t)$  as well. Consider one shortest (white) cell path  $p(w, R)$  from  $w$  to Red-close( $t$ ) in  $\text{CONF}(t)$ . Then, thanks to Claim 2 of Lemma 14, every cell  $w'$  in  $p(w, R)$  which is  $\rho$ -close to  $R$  will w.h.p. become red in  $\text{CONF}(t + 1)$ . Notice that the number of such new red cells is at least  $\lfloor \rho / (\sqrt{2}\ell) \rfloor$ . Notice that some other white cell in  $p(w, R)$  may become red in  $\text{CONF}(t + 1)$ . In any case, the lemma holds.

□

## C Proofs of Section 5

For the sake of simplicity, we here assume the parameter  $\gamma$  of the measurability of region  $\mathcal{S}$  be set to 1. If  $\gamma < 1$ , only the values of the constants in the proofs would change.

**Proof of Lemma 21.** Let  $X$  be the number of white agents in cell  $C$  immediately after the move phase of time step  $t + 1$ . At time  $t$ , for every white agent  $j$  in  $N(C)$ , let  $Y_j$  be the indicator r.v. that is 1 iff agent  $j$  moves to cell  $C$ . Notice that  $\mathbf{P}(Y_j = 1) \geq R^2/(18\rho^2)$ . Clearly,  $X = \sum_j Y_j$  and

$$\mathbf{E}[X] \geq \#_w^t(N(C)) \frac{R^2}{18\rho^2} \geq \frac{\lambda}{18} R^2$$

Since the r.v.  $Y_j$  are independent, we can apply Chernoff's Bound and get that, w.h.p.  $X \geq \frac{\lambda}{36} R^2$ . The lemma follows by applying the union bound with respect to all cells in  $C$ . □

**Proof of Lemma 22.** We distinguish two cases. Firstly consider the case that

$$\rho \geq c_0 R \sqrt{\frac{\lambda}{18} \log n} \tag{3}$$

Let  $E$  be a subset of the red agents in  $C$  at time  $t$  of cardinality  $M = \min\{\lambda R^k, (18\rho^2)/R^2\}$ . Let  $Z$  be the number of cells in  $C'$  hit by some red agent in  $E$  immediately after the move phase at time  $t + 1$ . We first bound the expected value of  $Z$ . For any cell  $C$  in  $C'$ , let  $Y_C$  the indicator r.v. that is 1 iff  $C$  is hit. It holds that

$$\mathbf{P}(Y_C = 0) \leq \left(1 - \frac{R^2}{18\rho^2}\right)^M \leq \exp\left(-\frac{R^2}{18\rho^2} M\right)$$

Hence,

$$\mathbf{P}(Y_C = 1) \geq 1 - \exp\left(-\frac{R^2}{18\rho^2}M\right) \geq (1 - 1/e)\frac{R^2}{18\rho^2}M$$

where the second inequality derives from the inequality  $\exp(-x) \leq 1 - (1 - 1/e)x$ , for any  $0 \leq x \leq 1$ . We then get

$$\mathbf{E}[Z] = \sum_C \mathbf{E}[Y_C] \geq \frac{2\rho^2}{R^2}(1 - 1/e)\frac{R^2}{18\rho^2}M \geq \frac{M}{15}$$

Notice that r.v.  $Y_C$  are not independent. So, in order to prove a concentration result for  $Z$ , we use the *method of bounded differences* [15]. For every red agent  $j$  in  $E$ , let  $X_j$  be the r.v. that is equal to the cell  $C$  in  $C'$  hit by agent  $j$  immediately after the move phase of time  $t + 1$  and  $X_j = 0$  if  $j$  does not hit any cell in  $C'$ . Observe that r.v.  $X_j$  are independent. Now, consider function  $F(X_1, \dots, X_M)$  that returns the number of cells in  $C'$  hit by some red agent. It is easy to see that  $F$  has Lipschitz constant equal to 1. Moreover it holds that  $F(X_1, \dots, X_M) = Z$ . By applying Lemma 1.2 in [15], we get

$$\begin{aligned} \mathbf{P}\left(Z \leq \min\left\{\frac{\lambda R^k}{30}, \frac{\rho^2}{2R^2}\right\}\right) &\leq \mathbf{P}\left(Z \leq \frac{M}{30}\right) \leq \mathbf{P}\left(Z \leq \frac{1}{2}\mathbf{E}[Z]\right) \\ &\leq 2\exp\left(-\frac{M}{450}\right) \end{aligned}$$

From Inequality (3), the assumption  $R \geq c_0\sqrt{\log n}$ , and the lemma's hypothesis on  $\lambda$ , it is easy to see that we get high probability. The thesis follows by applying the union bound with respect to all the supercells in  $N(C)$ .

Now consider the case that

$$\rho < c_0R\sqrt{\frac{\lambda}{18}\log n} \quad (4)$$

Define

$$q = \left\lfloor \sqrt{\frac{\lambda}{18}}\frac{R^2}{\rho} \right\rfloor$$

Notice that  $q \geq 1$ , since Inequality (4) holds. We partition every cell into subcells of side length  $s = \ell/q$ . Let  $m = (9\rho^2)/s^2$ . Observe that  $m$  is an upper bound on the number of subcells in  $N(C)$ . It holds that

$$m = \frac{9\rho^2}{s^2} = \frac{9\rho^2}{\ell^2}q^2 \leq \frac{18\rho^2}{R^2} \left(\sqrt{\frac{\lambda}{18}}\frac{R^2}{\rho}\right)^2 = \lambda R^2$$

Thus,  $m \leq \lambda R^2$ . Let  $E$  be a subset of the red agents in  $C$  at time  $t$  of cardinality  $m$ . Let  $Z$  be the number of subcells in  $C'$  hit by some red agent in  $E$  immediately after the move phase at time  $t + 1$ . We first bound the expected value of  $Z$ . For any subcell  $s$  in  $C'$ , let  $Y_s$  the indicator r.v. that is 1 iff  $s$  is hit. It holds that

$$\mathbf{P}(Y_s = 0) \leq \left(1 - \frac{s^2}{9\rho^2}\right)^m \leq \exp\left(-\frac{s^2}{9\rho^2}m\right) = \frac{1}{e}$$

Hence,  $\mathbf{P}(Y_s = 1) \geq 1 - 1/e$ . We then get

$$\mathbf{E}[Z] = \sum_S \mathbf{E}[Y_S] \geq \frac{\rho^2}{s^2}(1 - 1/e) \geq \frac{m}{15}$$

For every red agent  $j$  in  $E$ , let  $X_j$  be the r.v. that is equal to the subcell  $s$  in  $C'$  hit by agent  $j$  immediately after the move phase of time  $t + 1$  and  $X_j = 0$  if  $j$  does not hit any subcell in  $C'$ . Observe that r.v.  $X_j$  are independent. Now, consider function  $F(X_1, \dots, X_m)$  that returns the number of subcells in  $C'$  hit by some red agent. Clearly  $F$  has Lipschitz constant equal to 1 and  $F(X_1, \dots, X_m) = Z$ . By applying Lemma 1.2 in [15], we get

$$\mathbf{P} \left( Z \leq \frac{m}{30} \right) \leq \mathbf{P} \left( Z \leq \frac{1}{2} \mathbf{E}[Z] \right) \leq 2 \exp \left( -\frac{m}{450} \right)$$

It is not hard to see that  $m \geq (\lambda R^2)/4$ . From this and the assumed bound on  $\lambda$ , we get the high probability. Observe that the number of cells in  $C'$  hit by red agents is at least  $Z/q^2$ . This implies that, w.h.p., the number of cells in  $C'$  hit by red agents is at least

$$\frac{m}{30} \frac{1}{q^2} \geq \frac{9\rho^2}{30s^2q^2} = \frac{9\rho^2}{30\ell^2} = \frac{18\rho^2}{30R^2} \geq \frac{\rho^2}{2R^2} \geq \min \left\{ \frac{\lambda R^k}{30}, \frac{\rho^2}{2R^2} \right\}$$

The thesis follows by applying the union bound with respect to all the supercells in  $N(C)$ .  $\square$

**Proof of Lemma 23.** By hypothesis  $\#_r^t(C) \geq K$  where

$$K = 90 \frac{\rho^2}{R^2} \log n$$

Let  $C$  be any cell in  $N(C)$  and let  $Y$  be the indicator r.v. that is 1 iff cell  $C$  is hit by some red agent immediately after the move phase of time  $t + 1$ . It holds that

$$\mathbf{P}(Y = 0) \leq \left( 1 - \frac{\ell^2}{9\rho^2} \right)^K \leq \exp \left( -\frac{\ell^2}{9\rho^2} K \right) = \exp \left( -\frac{R^2}{18\rho^2} 90 \frac{\rho^2}{R^2} \log n \right) = e^{-5 \log n}$$

The thesis follows by applying the union bound with respect to all the cells in  $N(C)$ .  $\square$

**Proof of Lemma 24.** Assume that  $\widehat{\#}_r^t(C) \leq M$ , then  $\#_r^t(N(C)) \leq 9M$ . So, immediately after the move phase at time  $t + 1$ , the red agents in  $C$  are at most  $9M$ . Notice that any red agent can transmit over at most 15 cells. From the density condition, w.h.p. every cell contains at most  $\eta_2 \ell^2$  agents. It thus follows that w.h.p.

$$\#_r^{t+1}(C) \leq 9M \cdot 15\eta_2 \ell^2 \leq 68\eta_2 MR^2$$

$\square$

**Setting the constants in Def. 19.** We can now set constants  $a_h$ ,  $b_h$ , and  $c_h$  for  $h = 1, 2, \dots, \hat{h} - 1$ :

$$a_h = \frac{\eta_1^h}{2 \cdot 2160^{h-1} 20^{\frac{(h-1)(h-2)}{2}}} \quad (5)$$

$$b_h = 15 \cdot 68^{h-1} \eta_2^h \quad (6)$$

$$c_h = \frac{\eta_1}{2 \cdot 20^{h-1}} \quad (7)$$

**Proof of Lemma 25.**

*Implication (a).* It directly follows from the definitions of state 0 and of cellular random-walk model.

*Implication (b).* Let  $m = m^t(C)$  and let  $\bar{C} \in N(C)$  be a supercell such that  $h^t(\bar{C}) = m$ . Since  $1 \leq m \leq \hat{h} - 1$ , from the definition of intermediate states it derives that  $\#_w^t(N(C)) \geq c_m \rho^2$ . From Lemma 21 (it holds that  $c_m \geq 720/c_0^2$ , for sufficiently large  $c_0$ ) it follows that, immediately after the move phase of time  $t + 1$ , every cell in  $C$  contains at least  $(c_m/36)R^2$  white agents, w.h.p.

Since  $h^t(\bar{C}) = m$ , it holds that  $\#_r^t(\bar{C}) \geq a_m R^{2m}$ . In virtue of Lemma 22 applied to  $\bar{C}$  (it holds that  $a_m \geq 1800/c_0^2$ , for sufficiently large  $c_0$ ) we obtain that, immediately after the move phase of time  $t + 1$ , w.h.p. the cells in  $C$  hit by red agents are at least  $\min\{(a_m/30)R^{2m}, \rho^2/(2R^2)\}$ . To summarize, w.h.p., immediately before the transmission phase of time  $t + 1$ , every cell in  $C$  contains at least  $(c_m/36)R^2$  white agents and at least  $\min\{(a_m/30)R^{2m}, \rho^2/(2R^2)\}$  of those cells are hit by red agents. After the transmission, all the white agents in any hit cell get red, thus, w.h.p., it holds that

$$\#_r^{t+1}(C) \geq \frac{c_m}{72} \min\left\{\frac{a_m}{15} R^{2m+2}, \rho^2\right\} \quad (8)$$

We distinguish two cases. Firstly consider the case that  $m \leq \hat{h} - 2$ . Notice that from the definition of  $\hat{h}$  it holds that  $(a_m/15)R^{2m+2} \leq (a_m/15)R^{2\hat{h}-2} \leq \rho^2$ . Thus inequality (8) becomes

$$\#_r^{t+1}(C) \geq \frac{c_m a_m}{1080} R^{2m+2}$$

From the definitions (5) and (7) it holds that  $a_{h+1} \leq a_h c_h / 1080$  (for  $h = 1, \dots, \hat{h} - 2$ ). Thus, the lower bound on red agents of state  $m + 1$  is satisfied:

$$\#_r^{t+1}(C) \geq a_{m+1} R^{2(m+1)} \quad (9)$$

Now we prove the upper bound. Since  $\text{CONF}(t)$  is regular and for any  $C' \in N(C)$   $h^t(C') \leq m$ , it holds that

$$\#_r^t(C') \leq b_m R^{2m}$$

From Lemma 24, since  $\widehat{\#_r^t(C)} \leq b_m R^{2m}$ , it derives that, w.h.p.

$$\#_r^{t+1}(C) \leq 68\eta_2 b_m R^{2m+2}$$

From definition (6), it holds that  $b_{h+1} \geq 68\eta_2 b_h$  (for  $h = 1, \dots, \hat{h} - 2$ ). Thus, the upper bound on red agents of state  $m + 1$  is satisfied:

$$\#_r^{t+1}(C) \leq b_{m+1} R^{2(m+1)} \quad (10)$$

It remains to show the lower bound on white agents. Let  $W$  be the number of white agents in the supercell  $C$  immediately after the move phase of time  $t + 1$ . The white agents in  $C$  at time  $t + 1$  are those that were in  $C$  immediately after the move phase and that are not got red (after the transmission phase). It follows that

$$\#_w^{t+1}(C) \geq W - \#_r^{t+1}(C)$$

Recall that immediately after the move phase of time  $t + 1$ , every cell in  $C$  contains at least  $(c_m/36)R^2$  white agents, w.h.p. Thus, w.h.p., it holds that

$$W \geq \frac{2\rho^2}{R^2} \frac{c_m}{36} R^2 = \frac{c_m}{18} \rho^2$$

By combining this inequality with inequality (10), we have that (w.h.p.)

$$\#_w^{t+1}(C) \geq \frac{c_m}{18} \rho^2 - b_{m+1} R^{2(m+1)}$$

From the definition of  $\hat{h}$  and the assumption that  $m \leq \hat{h} - 2$ , it holds that

$$R^{2(m+1)} \leq R^{2\hat{h}-2} \leq c_0 \frac{\rho^2}{R^2} \log n \leq \frac{\rho^2}{c_0}$$

Hence,

$$\#_w^{t+1}(C) \geq \frac{c_m}{18} \rho^2 - b_{m+1} \frac{\rho^2}{c_0} = \left( \frac{c_m}{18} - \frac{b_{m+1}}{c_0} \right) \rho^2$$

From definitions (6) and (7), it holds that  $c_{h+1} \leq c_h/18 - b_{h+1}/c_0$  (for  $h = 1, \dots, \hat{h} - 2$ ), for sufficiently large  $c_0$ . Thus, the lower bound on white agents of state  $m + 1$  is satisfied:

$$\#_w^{t+1}(C) \geq c_{m+1} \rho^2 \quad (11)$$

Finally, from inequalities (9), (10), and (11), it derives that, w.h.p.,  $h^{t+1}(C) = m + 1$ .

We now consider the case  $m = \hat{h} - 1$ . From inequality (8) it holds that (w.h.p.)

$$\begin{aligned} \#_r^{t+1}(C) &\geq \frac{c_{\hat{h}-1}}{72} \min \left\{ \frac{a_{\hat{h}-1}}{15} R^{2\hat{h}}, \rho^2 \right\} \\ &\geq \frac{c_{\hat{h}-1}}{72} \min \left\{ \frac{a_{\hat{h}-1}}{15} c_0 \frac{\rho^2}{R^2} \log n, \rho^2 \right\} \\ &= c_0 \frac{c_{\hat{h}-1} a_{\hat{h}-1}}{15 \cdot 72} \frac{\rho^2}{R^2} \log n \quad (\text{since } \frac{a_{\hat{h}-1} c_0 \log n}{15 R^2} \leq \frac{a_{\hat{h}-1}}{15 c_0} \leq 1) \\ &\geq 90 \frac{\rho^2}{R^2} \log n \quad (\text{for sufficiently large } c_0) \end{aligned}$$

and thus  $h^{t+1}(C) = \hat{h} = m + 1$ .

*Implication (c).* Since  $m^t(C) = \hat{h}$ , there is a supercell  $\bar{C}$  adjacent to  $C$  such that  $h^t(\bar{C}) = \hat{h}$ . From Lemma 23, immediately after the move phase of time  $t + 1$ , w.h.p. all the cells in  $C$  are hit by some red agent. Since  $h^t(C) < \hat{h}$  and  $\text{CONF}(t)$  is regular, it holds that  $\#_w^t(C) \geq c_{\hat{h}-1} \rho^2$ . It follows that immediately after the transmission phase, w.h.p. the number of red agents in  $C$  is at least

$$c_{\hat{h}-1} \rho^2 \geq 90 \frac{\rho^2}{R^2} \log n$$

for sufficiently large  $c_0$ . This implies that  $h^{t+1}(C) = \hat{h}$ .

*Implication (d).* Since  $h^t(C) = \hat{h}$ , from Lemma 23, immediately after the move phase of time  $t + 1$ , w.h.p. all the cells in  $C$  are hit by some red agent. It follows that immediately after the transmission phase there are no white agents in  $C$ . Thus, w.h.p.  $\#_w^{t+1}(C) = 0$  and  $h^{t+1}(C) = \hat{h} + 1$ .

*Implication (e).* Let  $\bar{C} \in N(C)$  be a supercell such that  $h^t(\bar{C}) = \hat{h} + 1$ . Since  $\text{CONF}(t)$  is regular, it must be the case that  $h^t(C) \geq \hat{h}$  (given that  $C \in N(\bar{C})$ ). This implies that either  $h^t(C) = \hat{h}$  or  $h^t(C) = \hat{h} + 1$ . We distinguish the two cases. Firstly, if  $h^t(C) = \hat{h}$ , from Lemma 23, immediately after the move phase of time  $t + 1$ , w.h.p. all the cells in  $C$  are hit by some red agent. It follows that immediately after the transmission phase there are no white agents in  $C$ . Thus  $h^{t+1}(C) = \hat{h} + 1$ .

Consider now the case  $h^t(C) = \hat{h} + 1$ . Since  $\text{CONF}(t)$  is regular,  $\forall C' \in N(C)$  we have  $h^t(C') = \hat{h} \vee h^t(C') = \hat{h} + 1$ . We distinguish two cases.

- 1)  $\exists C' \in N(C) : h^t(C') = \hat{h}$ : From Lemma 23, it follows that  $h^{t+1}(C) = \hat{h} + 1$ .
- 2)  $\forall C' \in N(C) h^t(C') = \hat{h} + 1$ : It holds that  $\#_w^t(N(C)) = 0$  and thus no white agent can move into supercell  $C$  during time step  $t + 1$ . It follows that  $h^{t+1}(C) = \hat{h} + 1$ .

□

**Proof of Lemma 26.** From Lemma 25 it is easy to verify that if  $\text{CONF}(t)$  is regular then, w.h.p. for every supercell  $C$ , it holds that  $h^{t+1}(C) \in \{0, 1, \dots, \hat{h}, \hat{h} + 1\}$ .

Let  $C$  be a supercell such that  $h^{t+1}(C) = \hat{h} + 1$ . From Lemma 25 it must be the case that either  $h^t(C) = \hat{h}$  or  $h^t(C) = \hat{h} + 1$ . Distinguish the two cases.

- 1)  $h^t(C) = \hat{h}$ : It holds that  $\forall C' \in N(C) m^t(C') \geq \hat{h}$ . From Lemma 25 (c), (d), and (e), it follows that  $h^{t+1}(C') = \hat{h} \vee h^{t+1}(C') = \hat{h} + 1$
- 2)  $h^t(C) = \hat{h} + 1$ : It holds that  $\forall C' \in N(C) m^t(C') \geq \hat{h} + 1$ . From Lemma 25 (e) it follows that  $h^{t+1}(C') = \hat{h} + 1$ .

□

**Proof of Lemma 27.** At the very beginning of time step 0, all the agents are white except the source one which is red. Immediately after the move phase the source agent moves to a cell  $c$ . Immediately after the transmission phase, w.h.p. all the agents (except one) in the cell  $c$  get red. Let  $C$  be the supercell containing cell  $c$ . From the density condition, it holds that (w.h.p.)

$$\eta_1 \ell^2 \leq \#_r^0(C) \leq 15\eta_2 \ell^2$$

since at most 15 cells are affected by the transmission of the source agent. Thus, the bounds on red agents of state 1 are satisfied:  $a_1 R^2 \leq \#_r^0(C) \leq b_1 R^2$ , since  $a_1 = \eta_1/2$  and  $b_1 = 15\eta_2$ .

Consider now the number of white agents. In the supercell  $C$  there are at least  $\rho^2/\ell^2 - 15$  cells that are not affected by the transmission. From the density condition, those cells contain at least  $\eta_1 \ell^2 (\rho^2/\ell^2 - 15)$  white agents. It follows that, w.h.p.

$$\#_w^0(C) \geq \eta_1 \ell^2 \left( \frac{\rho^2}{\ell^2} - 15 \right) = \eta_1 \rho^2 - 15\eta_1 \ell^2 \geq \frac{\eta_1}{2} \rho^2$$

where the last inequality holds because  $\rho \geq 5R$ . Hence  $\#_w^0(C) \geq c_1 \rho^2$  and  $h^0(C) = 1$ . Clearly all the other supercells are in the state 0. □

**Proof of Lemma 28.** Let  $P$  be a shortest path of supercells between  $W$  and  $\text{Red}(t)$ . Path  $P$  contains a supercell  $W'$  such that  $h^t(W') = 0$  and  $W'$  is adjacent to a supercell in  $\text{Red}(t)$ . From Lemma 25 it follows that the state of  $W'$  at time  $t + 1$  is at least 1 and thus  $W' \in \text{Red}(t + 1)$ . Hence,  $d_{\square}^{t+1}(W) \leq d_{\square}^t(W)$ . □

## D Proof of Section 6

**Proof of Lemma 30.** For the sake of simplicity, we consider  $\mathcal{S}$  as the square of edge length  $\sqrt{n}$  and set  $R \leq \sqrt{\gamma \log n}$  where  $\gamma$  is a suitable positive constant. At time  $t = 0$ , consider the random disk graph  $G(n, R)$  formed by the  $n$  agents. Define the indicator r.v.  $X_i$  which equals 1 iff agent  $i$  is isolated in  $G(n, R)$ . We have that

$$\mathbf{P}(X_i = 1) \geq \left(1 - \frac{\pi R^2}{n}\right)^{n-1} \geq \exp(-2\pi\gamma \log n)$$

This implies that

$$\mathbf{E} \left[ X = \sum_i X_i \right] \geq n^{1-2\pi\gamma}$$

Unfortunately, r.v.  $X_i$ 's are not independent so we cannot apply Chernoff's bound: we instead apply the Method of Bounded Differences [15]. Consider the r.v.  $Z_i$  that returns the position (i.e. coordinates) of agent  $i$  in  $\mathcal{S}$ . Observe that such r.v. are independent. Then we define the function  $F(Z_1, Z_2, \dots, Z_n)$  that returns the number of isolated agents. Observe that  $Z = X$ . We now need the following useful Lemma 1.2 in [15].

In order to apply this lemma to our function  $F$ , we need to find out the correct values for coefficients Lipschitz's constants  $c_k$ . We claim that, in our geometric case,  $c_k \leq 7$ : indeed, changing the position of any agent in the square can affect the  $R$ -disks of at most 6 isolated nodes. So, by setting this bound on the  $c_k$  and  $t = \beta \mathbf{E}[F(Z_1, \dots, Z_n)]$  for a suitable small constant  $\beta > 0$ , we can choose  $\gamma > 0$  such that

$$\mathbf{P}(X \leq n^{1/2}) = \mathbf{P}(F(Z_1, \dots, Z_n) \leq n^{1/2}) \leq \exp(-n^{\Theta(1)})$$

□