Jaime Gil Aluja and Ana Maria Gil Lafuente

Towards an Advanced Modelling of Complex Economic Phenomena

Studies in Fuzziness and Soft Computing, Volume 276

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Towards an Advanced Modelling of Complex Economic Phenomena

Pretopological and Topological Uncertainty Research Tools

Springer

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Introduction

The Paths towards Knowledge

The search for order and stability has traditionally been one of the fundamental objectives of **economic science**. We do not believe that there is any economist, whatever the school in which they have based their knowledge, who does not establish their decisions by thinking of obtaining an **equilibrium** or by thinking of breaking an existing equilibrium. But always with the idea of finding another, which may be more favourable for the interests which they wish to defend.

If we focus on the field of the **formalizing** of the phenomena in the sphere of knowledge a marked confusion can be observed in researchers, when a reality full of disarray which makes life **unstable** can be seen to be treated in the same way as in situations of equilibrium, enveloped in stability. It is this way because it results as difficult to accept that society, the economy and the activity of businesses such as those we have known up to now have no possibilities of survival in the immediate future in which many deep changes will be inevitable. When faced with this panorama there are no lack of studies which aim to find solutions by undertaking new paths in their economic studies, in which **fluctuations** and **instability** are taking an ever more fundamental position.

In effect, in a context of changes such as those which we are currently witnessing, who is capable of predicting the **evolution of events** with the necessary precision of a prophet? Maybe we should be satisfied with **less** and **better** employ that which is available to us. For this, a brief reflexion based on the possibilities which the answers offered by the laboratories where the new scientific discoveries are taught may result as useful. This confirms to us that research activity has reached a crossroads at which the future of humanity is at stake. On the one hand the **geometric conception** of knowledge and on the other the **Darwinian conception**. On the one side the sublime, monotonous and well-known reiterative songs, renewed only in their forms. The imposition of some pre-established beliefs from the blinding light of the Newtonian dawn, in which the reduction of the operation of the world to the predictability of a mechanism is dreamed of. And on the other side the emptiness of the unknown. The varying and at times dissonant whispers of notes which sound disjointed and incoherent. The attraction of adventure. The invitation to jump from the edge of a cliff where the distance to fall cannot be seen, only guided by the hope of opening new horizons. The response to the calling of **Ludwig Boltzmann**, of **Bertran Russell**, of **Lukasiewicz**,of **Zadeh**, of **Lorenz**, of **Prigogine** and of **Kaufmann**. The rejection of the yoke and burden of predestination and the proclamation of freedom of choice which time and again clashes against the wall of doubt.

Karl Popper[1] stated "that all events are caused by an event, in such a way that all events may be predicted or explained.... On the other hand, common sense attributes to sane adults the capacity to freely choose between various paths....". This kind of interior contradiction a greater problem which William James[2] designated "dilemma of determinism". When we move this dilemma to physics or economics then what is known as "the time paradox" appears, in which we gamble neither more nor less than **our relationship with the world** or **with society**. In effect, Is society already written or is it found to be in permanent construction? Over time economic science has fed on the knowledge supplied by physics and as from the Newtonian Dynamic to quantum physics temporal symmetry is accepted, without distinction between past and future, economic science has been seen impregnated by a nature of no time references. On the other hand, our **perception** of economic phenomena leads us to think that past and future play a different part.

The time paradox "was identified by the Viennese physicist **Ludwig Boltzmann**, who believed it possible to follow the example of **Charles Darwin** in biology and give an evolutionist description of physical phenomena. His attempt had the effect of demonstrating the contradictions between the laws of Newtonian Physics -based on the equivalence between past and future- and all attempts of evolutionist formulation which asserted an essential distinction between future and past"[3]. At present, however, this perception of reality and of time has changed, above all since the birth and development of **non-equilibrium physics** with concepts such as self-organisation and **dissipative structures**. We are constantly more conscious that the deepest roots of the new ways of looking with which researchers scrutinise and examine social systems, economics and management are to be found in the 19th century when the first essences of **evolutionism** were born.

In effect, in his fundamental work "On the Origin of Species" published in 1859, Darwin considered that the **fluctuations** in biological species, thanks to natural selection, give rise to an **irreversible** biological evolution. That a self-organisation of systems with a growing complexity takes place between the association of **fluctuations** (which is similar to the idea of chance, we would say uncertainty) and **irreversibility**.

The evolutionary description is found associated with the concept of **entropy**, which, in thermodynamics, allows us to distinguish between **reversible** and **irreversible** processes. In 1865, Clausius[4] associated entropy with the second principle of thermodynamics. His formulation of the two principles of thermodynamics is the following: "The energy of the universe is constant". "The entropy of the universe tends to a maximum". Faced with the energy which it continuously retains, entropy allows the establishment of a distinction between

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1 Popper, K.: L'univers irrésolu. Plaidoyer pour l'indéterminisme. Ed. Hermann. París, 1984, page. XV.

2 James, W.: The dilemma of determinism, in The Will to Believe. Ed. Dover. New York, 1956.

3 Prigogine, I.: *La fin des certitudes. Spanish translation with the title* «El fin de las certidumbres». Ed. Taurus. Buenos Aires, 1997, page 8.

4 Clausius, R.: *Ann. Phys*. CXXV, 1865, page 353.

reversible processes (constant entropy) and irreversible processes (entropy created). Therefore, in an isolated system the **entropy** increases when irreversible processes exist and remains constant in the presence of reversible processes. So, entropy reaches a maximum value when the system is nearing equilibrium and the irreversible process ends. It was **Ludwig Boltzmann** (1844-1906) who established a relationship between entropy and probability through the famous formula $H = K$. In P.

In 1872 Boltzmann published his famous "H Theorem". This theorem exposed how in the heart of a population of particles, the collisions between these modifies the distribution of the value of this function H at each moment until a minimum which corresponds with that which has come to be called the **Maxwell-Boltzmann equilibrium distribution** is reached. In this state the collisions no longer modify the distribution of velocities in the population and the magnitude H remains constant. In this way the collisions between particles lead to **equilibrium**.

As much in the case of **Darwin** as in that of **Boltzmann**, **chance** (or, if prefered, uncertainty) and **evolution** are intimately related, but the results of their respective investigations lead to contrasting conclusions. For **Boltzmann**, the probability reaches its maximum when **uniformity** is being reached, while for **Darwin** evolution leads to new **self-organised structures**.

In comparison with these approaches, and **following** the **traditional physics** prototype in which complete and certain knowledge are linked and in which from certain initial conditions the predictability of the future and the possibility of returning to the past are guaranteed, marginal economic theory is supported by the **mechanics of movement**, which describe processes of a **reversible** and **determinist** character, where the direction of time plays no part whatsoever and in which there is no place for either uncertainty or irreversibility. To conclude, in classic studies, economic and management systems constitute **great automata**. But the incorporation of **instability** is causing a substantial change in which the concept of "economic law" acquires a new **meaning**.

It is true that some phenomena arising from the life of states, institutions and companies can be described through the use of **determinist equations**. But, on the other hand, others entail uncertain processes or, in any case, **stochastic processes**. Not only do they possess **laws** but also **facts** which do not result as a consequence of the laws and instead redefine their **possibilities**. It may occur that our own existence, with all its complexity, is also engraved in the primordial event baptised with the name of the Big-Bang. Ilya Prigogine[5] asked if time made its first appearance with the Big-Bang of if time previously existed in our universe. In this way the frontier of our knowledge, reasoning and speculation are difficult to define. The Big-Bang may be conceived as an event associated with instability, which implies that it is the starting point of our universe, but not of time. Therefore, time does not have a beginning and it is possible that it does not have an end. The reality is that economic science, which has searched so much for that which is **permanent**, **symmetry** and **laws**, has instead found that which is **mutable**, **irreversible** and **complex**.

[5] Prigogine, I.: La fin des certitudes. Spanish version, Ed. Taurus. Buenos Aires, 1997, pages. 11-12.

In this search, the scholars of economics and management are discovering processes in which the **transition of chaos to order** takes place, that is to say sequences which lead towards **self-organisation**. The question posed is **how** this creation of new structures takes place, which is to say this **self- organisation**. So, given the entropy of a system, if it is disturbed in such a way that a state remains sufficiently **near to equilibrium** the system itself responds by reestablishing the initial situation. This is a **stable system**. But, if a state is taken **far enough from equilibrium**, it enters a situation of **instability** in relation with the disturbance. This point is habitually named the bifurcation point[6]. From this, new phenomena take place which may correspond to behaviour far from the original. In this context determinist processes have no usefulness for predicting **which path** will be chosen between those existing and bifurcation. In many of these bifurcations **symmetry rupture** is produced.

The notions of time and of determinism have been present in western thought since pre-Socratic times, causing deeply felt tension when attempting to give an impulse to **objective knowledge** at the same time as promoting the humanist ideal of **freedom**. Science would fall into a contradiction if it were to opt for a determinist concept when we find ourselves involved in the task of developing a free society. Neither science and certitude nor ignorance and possibility can be identified. A scientific activity is coming to light in which investigations are not limited to the study of **simplified** and **ideal** phenomena but which are determined to unravel the **secrets** of societies written in a **real, essentially complex** world.

Irruption of Mathematical Spaces on Management Studies

Over many decades, economists have lived facing away from these ideas, closing the door to renewal and blocking the arrival of this breath of fresh air. Because of this, economic thought remained deeply-rooted as it had initially been, between 1880 and 1914, in **mechanistic mathematics**. The classic mechanics of Lagrange was used, which gave an impression of severity, compared with that which Perroux called the "laxity of the economic discourse". But on the other hand, the thoughts of researchers remain trapped by certain **economic laws**, parallel to the **laws of nature**, which prevent them from exercising one of their most prized treasures: **imagination**. The inevitable consequence is that the automatisms of the mathematics of determinism have exerted great prestige and still prevail today in many spheres of scientific activity in economics and business management.

But the search for new formal structures has not disappeared from the restless spirits of many researchers. This is the case of Lotfi A. Zadeh. The developments of physics and chaos and instability mathematics have provided, we believe, the important finding of Zadeh[7], who, with his fuzzy sets has created a fundamental change in the panorama of research within the spheres of social sciences. And in the evolutionary onrush of new proposals, introduced with thanks to him, concepts

[6] In a boolean environment the term bifurcation makes sense, but in multivalent logic trifurcation, pentafurcation..., or endecafurcation, can be produced, amongst others.

[7] Zadeh, L.: «Fuzzy Sets». Information and Control, 8th June 1965, pages 338-353.

as deeply-rooted as **profitability**, **economy**, **productivity**... expressed by cardinal functions, are losing their particular attraction in favour of other notions such as **relation**, **grouping**, **assignment** and **ordering**, which now acquire a new sense. This displacement is fundamental, because it means the transfer of **non-arithmetical** elements, considered contemporary in traditional studies, to the privileged position they currently occupy.

Little by little we are being provided with an arsenal of operative instruments of a non-numerical nature, in the shape of models and algorithms, capable of providing answers to the "aggressions" which our economics and management systems must withstand, coming from an environment full of turmoil. Despite this, it seems that the agglutinating element that constitutes the basic support on which the findings accomplished and those in the immediate future may settle has not yet been found. Can **the notions of pretopology and topology** help to achieve this objective?

In the work which we are presenting, we dare to propose a set of elements from which we hope arise focuses capable of renewing those structures of **economic thought** which are upheld by the **geometrical idea**, so deeply-rooted in the worshipping circles of the orthodox which, monopolising the means of power, assign privileges and deny beauty.

The concepts of **pretopology** and **topology**, habitually marginalized in economics and management studies, have centred our interest in recent times. We consider that it is not possible to conceive formal structures capable of representing **the Darwinism concept of economic behaviour** today without recurring to this fundamental generalisation of metric spaces in one way or another.

In our attempts to find a solid base to the structures proposed for the **treatment of economic phenomena**, we have frequently resorted to the **theory of clans** and the theory of affinities with results which we believe to be satisfactory. We would like to go further, establishing, if possible, the connection between their axiomatics at the same time as developing some **uncertain pretopologies** and **topologies** capable of linking previously unconnected theories, at the same time easing the creation of other new theories. Our aspirations are as ambitious as our enthusiasm is unflagging. We think that even though the flight does not reach so high, we will be capable of reaching levels sufficient enough to capture the attention of those searching for new paths towards a knowledge closer to the complex realities of modern times.

Now seems the right time to remember that in a "crude way", **topology** can be conceived as a branch of science which studies **space**. It therefore analyzes the idea of space and searches for properties **common to all spaces**. For this, it begins from the most general concept of space and studies the properties which belong as much to three-dimensional euclidean space R^3 and the n- dimensionals R^n as to the infinite-dimensional space H of Hilbert, to non-euclidean space and to the geometrics of Riemann, to only quote the most well-known. Therefore, topology does not act directly in either the linking operations between real numbers or in their generalisations.

Brief Summary of This Work

We believe that to arrive at such a high generalisation demands a certain trajectory to, in this way, imbue ourselves with the meaning and possibilities of these structures full of so much abstraction. The immersion in pretopological studies could be a good entry point towards the objectives which we pursue. For this reason, the axioms of the most general of pretopologies has been the starting point for our work.

In the first part of this study we have made a brief reference to ordinary pretopology to later move on to fuzzification. To better assume the meaning of pretopologies, we have begun with a principle example based on very singular financial products which have been passing all "tests" to the extent that new axioms were added to those of the most general pretopology until Moore's uncertain pretopology was reached. The proposed pretopologies do not usually always pass the obstacles which new axioms provide. We have wanted to develop a scheme of this nature, in an attempt to span the range of phenomena that economic and management reality may pose wider. Most of the time, fortunately, it will not be necessary to make use of Moore's uncertain pretopology. In many cases isotone uncertain pretopology or distributive uncertain pretopology will be enough.

Being able to arrive at Moore's uncertain pretopology has been a basic element for our algorithms as it has allowed us, with all of the necessary adaptations, to reach Moore closings and thanks to this we have been able to use, in the sphere which concerns us, the theory of affinities.

The task undertaken has obliged the following of a certain trajectory. Therefore, after a brief description of the fundamental concepts of uncertain ordinary pretopology, the conditions necessary for the existence of isotone uncertain pretopologies have been established. From here the existing relationships in a system are expressed by using a fuzzy graph. We have considered that this was susceptible to treatment until the "closes" were to be found on one side and the "opens" on another.

The traditional study of the fuzzy graph through its ∝-cuts has provided a **set of boolean graphs** with their range of Moore closing and their corresponding closes. In this first approximation uncertainty appears as a consequence of the possibility of accepting distinct levels. Simultaneously it is noted that **another kind of uncertainty** is born from the subjectivity in the assignment of valuations for each of the elements of the fuzzy subset. Moving into this field may lead, we hope, towards revealing proposals. Based on this reasoning we have proposed to deal with this approach in its **integrity**.

At the beginning of this task a crossroads appeared at which an important decision for the later development of our work needed to be taken. We refer to the **sense which should be given** to the referential from which the set of base elements among which the functional application will be performed should arise. There may be two interpretations. **The first** considers a "crisp" referential while the second is developed from **a referential of fuzzy subsets**.

During this work, as much in reference to pretopologies as topologies dealt with in the second part, we have been able to show that both focuses result as useful for **decision making** in the economics and management sphere. The choice of one or the other depends on the information available and the objectives to achieve.

We have started the second part of our work expounding the axioms which allow the definition of a topology in the most general way, which later allows an interesting game when passing to uncertainty. The notions of **filter base** and **base** on the one hand and the concept of **neighbourhood** on the other have helped us in this task. Immediately afterwards we have assumed the task of showing certain relationships of a singular importance which may exist between two or more topological spaces. In this way we will develop the concept of topological continuity and, leaning on the notion of **inverse of an application**, that of **homeomorphism**. Upon ending this block of knowledge in this way we hope to have brought the basic elements to achieve one of our main objectives to the surface: **uncertain topology**.

We will dedicate the second block of this part to uncertain topology. The **two focuses** noted in the study of pretopologies in uncertainty now become evident under the same axioms. The transformation of deterministic structures into uncertain structures has captured all of our attention. But uncertainty does not always appear for the same reason and choosing one or another, consigning the others to oblivion, would be showing a lack of scientific sensitivity.

We are able to isolate three ways in which to incorporate uncertainty. We have tested two of them in the pretopology of uncertainty, using a fuzzy subset as a descriptor of a physical or mental object and starting from a **referential set formed by the referentials of the fuzzy subset** in the first form and constructing the **referential set with fuzzy subsets** in the second form. The third and last form of incorporating uncertainty consists in converting the fuzzy elements into booleans by means of a breaking down in levels by $\propto$-cuts. We wish to express our conviction that we cannot expect that the paths of access to uncertainty are finished with that which we expound. On the contrary, we consider that these open doors should still provide interesting possibilities for future development.

The conglomeration of elements put forward in the first and second part of this work constitutes a formal body which, we believe, possesses a high level of homogeneity. The **axioms** established for the **pretopologies** and **topologies**, from which it has been able to establish a great number of properties, should lead us to a **third and final part** closer to the formal objectives and material of business economics and management. In this part we have proposed the connection of some operative instruments already used by ourselves with elaborated formal structures. In this way, we have dedicated each one of the blocks of which this part consists to two interesting but incomplete theories: the **theory of clans** and the **theory of affinities**. We consider that the connection of the previously existing with that achieved can provide unquestionable advantages at the time of their use for economic and management problem solving.

With this intention we connect the **axioms** of the **theory of clans** with those established for the distinct topologies. We have been able to verify that certain concepts, basic in the sphere of **topology**, result as valid when they are transferred to the study of **clans**. For our own tranquility we perform a "test" using some problems form the area of economics management, which validate in practise as much as from a technical perspective.

We end this third part with a block destined to recall the most significant elements of the theory of affinities to, once adequately restructured, affect pretopological spaces. In this sense the comparison between the isotone pretopology and Moore closing has constituted a good starting point. From here it results as essential, to our understanding, to find the sources from which the closes may be found and subsequently Moore closings. The connection of these elements with pretopologies and topologies therefore results as immediate. With this the definition of affinity acquires its greatest meaning.

However, the practical needs, in the sense of capacity of use of this host of knowledge to deal with economics and management problems, demands the availability of **operative methods**. These would arrive in the form of algorithms of an alternative use: that of the **maximum inverse correspondence** and that of the **maximum complete sub-matrices.** We consider that with the presentation of these algorithms and with their use in a revealing supposition we can close our work, albeit in a provisional way.

With this work we aim to begin the task of finding the **bases** on which to **place the structure of economics and management systems**, taking into account that the deep mutations produced in these go beyond, on many occasions, the limits within which the own strengths of the system allows a return to positions of equilibrium. Going beyond these limits means the birth of a new context, **not connected to lineal processes**. Our proposal is aimed at demonstrating the wide possibilities which, in this sense, **uncertain pretopological and topological spaces** acquire.

A Return to the Origin?

It seems that Epicuro was the first to expose the problem of the inseparability of the **determinist** world of the atoms and human **freedom**. It is true that the formulation of the **laws of nature** contributes an important element in not denying evolution in the name of the truth of being, but on the contrary trying to describe the movements characterised by a speed which varies with the passing of time. Despite their formulation, these laws entail the **supremacy** of the **being** over **evolution**, as evident in Newton's law which linked force and acceleration: it is deterministic and reversible in time. But, in spite of the fact that **Newtonian physics** was relegated by the two great discoveries of the 20^{th} century, **quantum mechanics** and **relativity**, his **determinism** and **temporal symmetry** have survived. As it is known, quantum mechanics does not describe trajectories but wave functions, but its fundamental equation, the **Schrödinger equation**, is determinist and of reversible time.

But, if for a great quantity of physicists, amongst whom one can find **Einstein**, the problem of determinism and of time has been resolved ("time is just an illusion"), for philosophers is continues to be a question mark on which the sense of human existence depends. In this way **Henri Bergson**[8] asserts that "time postpones or, better said, is a postponement. Therefore it must be elaboration. Will it not be then the vehicle for creation and election? Does the existence of time not then prove that there is indetermination in things?" In this way, for **Bergson** realism and indeterminism walk hand in hand. **Karl Popper** also considers that "the determinism of Laplace -confirmed as it appears to be by the determinism of physical theories and his brilliant success- is the most solid and serious obstacle in the way of an explanation and an apology of human freedom, creativity and responsibility".[9]

With these precedents we have undertaken the task of developing this book which aims towards the objective of widening the perspective of economics and management studies, from a description of certain **spaces** capable of granting a place for the geometrical concepts of certainty and reversibility and also the innovation which uncertainty or irreversibility may mean. If in the most well-known treatise of economics and management idealised systems of economics and management are described which are stable and reversible, we have aimed to move closer to the world in which we live characterised by instability and evolution, which brings a certain complexity. From here the transition to **uncertain pretopologies** and **topologies**. We believe that even within chaotic systems an **order** may exist. This order is that which we are searching for, reaching far from the past into the future.

8 Bergson, H.: «Le possible et le réel», in: Oeuvres. Presses Universitaires de France. París, 1970, page 1333.

9 Popper, K. : L'univers irrésolu. Plaidoyer pour l'indéterminisme. Ed. Hermann. París, 1984, page 2.

Precedents: Intuitive and Axiomatic Aspects of Topology

Brief Historical Overview of Topology

Since several decades ago economic science has dived into many distinct fields of knowledge to find the necessary elements with the objective of a better understanding of the complexity inherent to the systems in which stable equilibriums overflow. Maybe the moment has arrived to return to a path already timidly undertaken on many occasions but then abandoned, we believe prematurely, without fully achieving the desired objectives. We are refering to **topology**.

The word **topology** comes from the Greek terms $\tau o\pi o\sigma$ (place) and $\lambda o\gamma o\sigma$ (study). It first appeared in 1847 when **Johann Benedikt Listing** used it in one of his works.

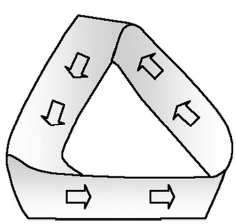
In its origins topology included work relative to the properties which physical or mental objects support/maintain when they are submitted to continuous transformations such as deformations, doubling, lengthening, etc... but not breakage (for example,a triangle is topologically equivalent to a sphere). In short, it could be said that it deals with the study of soft and gradual changes, of the analysis of the **non-broken**.

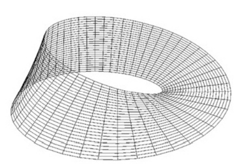
But its existence is previous to the word with which it is known today. Indeed, it is considered that the first to use it was **Gottfried Wilhelm Leibniz** (1646-1716). Among his discoveries is found the mathematical principle of continuity which he called "análysis situs" (position analysis) in the year 1679. It is also known as "rubber band geometry". When Leibniz refered to análysis situs he wanted to demonstrate a genuine geometrical analysis in the sense of expressing the "place", the "position", against the algebraic analysis which deals with magnitudes. In this way, while the cartesian coordinates were refered to as specific **quantities**, Leibniz aimed to deal with the geometry of sets independent to the quantities which the elements of these sets could define. **Topology** constitutes a kind of geometry in which longitudes, angles, faces and forms are infinitely changeable. All of the geometrical forms studied in our youth are the same for a topologist. **Topology** studies those properties of **forms which do not change** when **reversible continuous transformations** take place. In what concerns us, we emphasize his interest for the problems of groupings, limits, neighbourhood and morphology of **economics and management objects**.

There is no doubt, however, that the person who established the solid bases of what would be later known as topology was **Bernhard Riemann** (1826-1866), disciple of **Carl Friedrich Gauss** (1777-1855). In his doctoral thesis Riemann established fundamental topological concepts among which that of "extended magnitude several times" stands out, the origin of what would later become known as **topological space**. In this way the idea of **functions space** arose (all the possible forms of a function in a given dominion) and also that of **space of positions of a geometrical figure**. In relation to the study of surfaces, it associated whole numbers to spaces which had previously been defined. These are the **Betti numbers**, which have been the seed of **algebraic topology**. The development of Riemann's ideas was limited principally to the lack of an element which later arose with force. We are refering to the **set theory**. An important contribution appeared with the work of **Georg Cantor** (1845-1918) thanks to the systematization of the set theory and the notions of "accumulation point", "closed set", "open set" and "dimensions", amongst others. Later, once into the 20^{th} century, significant contributions were made by **David Hilbert** (1862-1943), **Maurice Frechet** (1878-1973) and **Frigyes Riest** (1880-1956), amongst others. Without doubt the key figure of the topology that we know today was **Felix Hausdorff** (1868-1942). His definition of topology, formulated in 1914, would serve as a base for that accepted at present times, formulated by **Alexandroff** in 1928. He made it evident that when mathematicians demonstrate theorems of analysis they always use similar methods and that the important thing was not metric properties but the relationships of proximity between the points and subsets. He dispensed with the notion of distance, although he accepted those **minimum properties** which allowed the consideration of **proximity between points and subsets.**

Some Classical Approaches of Topology

Over time some curious problems which, in some way, illustrate a part of the content of topology have been transmitted. Among those most well-known we may mention "the Möbius band", "the Seven Bridges of Königsberg", "the Four Colour Problem" and "the Three Bodies in Space".





1. In 1858, the German mathematician A. F. Möbius demonstrated that if you take a strip of paper long enough, you twist one of its extremes 180 degrees and then stick the two extremes together to create a ring, you obtain an object with a single face which, apparently, is to perform the impossible. From every perspective and no matter how many times you turn the surface you always find one continuous face.

If we cut the strip in half longitudinally until we reach the starting point of the cut then we are not left with two closed bands but only with one. This is what is known as “cutting into ribbons”.

Topologists are used to taking simple figures as a base to create more complicated surfaces, generalising the results in three-dimensions, with a view to include figures of four, five..., *n* dimensions. In this way a flat surface such as a sheet of paper is considered as if it were a perforated sphere which has been stretched and squashed until the aforementioned sheet of paper is created.

It is difficult to recognise such figures when they have been stretched in a fanciful way. To be able to identify them it is normal to characterise each topological type by means of **simple invariant properties**. One of these, relative to a surface, is the **number of edges**. A second is the **number of faces**. A Möbius band is an example of **a face**. The third characteristic invariant is the aforementioned **Betti number**, defined by the **maximum number of transversal cuts** which may be made in a surface without dividing it, taking into account that a transversal cut begins and ends at the edge. The ribbon cuts also allow the finding of the Betti number of a surface and are performed in a way that starts and ends at one point on the surface, always avoiding the edge. The maximum number of ribbon cuts possible without dividing a surface with an edge leads to the Betti number. The Betti number of a Möbius band is one. Each ribbon cut intercepts only one transversal which shows that named the **fundamental relationship of duality**. **S**. **Lefschetz** generalised this relationship to n dimensions and in 1927 formulated, the “duality theorem” which has been considered one of the milestones of topological development of the 20th century. As a final point, the numbers corresponding to the three topological invariants are always the same for an object, although its form may be changed with stretching but always without breakage or rejoining.

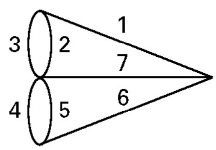
If a band formed after performing three 180 degree twists in one of its extremes before fastening the two extremes is considered, it can be observed that this band also has only one edge and a single face, a transversal cut is the only possibility. The Betti number is therefore one. Essentially, we find ourselves before an ordinary Möbius band (a single twist of 180 degrees), although there is a difference in the way it is situated in relation with three-dimensional space. Indeed, while here the edge of the band shows a knot, in the ordinary Möbius band there is a simple curve without knots.

2. The problem of the **Bridges of Königsberg** is found immersed in net theory and graph theory. It is one of the oldest of topology and it is normally presented by means of a graph formed by four vertices and seven non-orientated arcs.

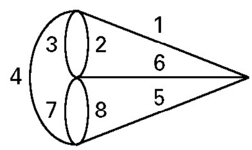
In the 17th century the Prussian city of Königsberg had been built around an island named Kneiphof, joined to each bank of the river Pregel by two bridges (and therefore, four bridges in total) and by another bridge to a neighbouring island which, in turn, was connected to the riverbanks by another two bridges.

The problem was finding the way of crossing each of the seven bridges without crossing any of them more than once. The local people saw this problem as a riddle without an answer. It was in 1736 when the Swiss mathematician **Leonhard Euler**, interested in the problem, found an ingenious answer to the impossible by using a mathematical demonstration of the impossibility of such a route, which then created a number n of bridges. His explanation consisted of a revealing example of the deceptive simplicity of topological approaches. The problem of Königsberg is linked with the well-known exercise of creating a specific figure on a sheet of paper without lifting the pencil nor passing more than once over the same line.

The reasoning of Euler is very simple. He began with the representation of the problem by means of a **graph** in which the **vertices** substitute the parts of solid ground and the **arcs or edges** substitute the bridges. He named those vertices which result in an even number of trajectories **even vertices** and those which result in an odd number **odd vertices**. The number of journeys necessary to traverse a connecting graph is equal to half the number of odd vertices. As it is not possible to construct a graph with an odd number of odd vertices (each arc has two vertices) the problem of the bridges of Königsberg does not have a solution. It would be necessary to add another arc, which is to say, another bridge.



Impossible
(Königsberg Bridges)



Possible

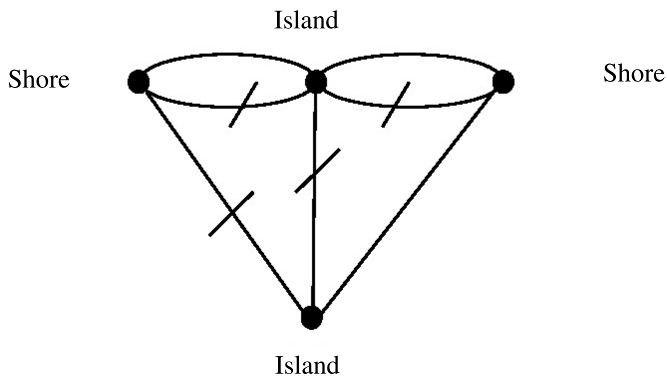
The idea of Euler consisted of creating a **connected graph in which circuits do not exist**. In a graph of this nature the number of vertices is equal to the number of arcs plus one. For this one begins with a graph in which those arcs which produce **circuits** are eliminated without deleting any vertex. The maximum number of arcs destroyed in a way in which all the vertices remain connected is the Betti number of the graph.

We will look at this analytically. Let us assume a graph with initial V vertices and A arcs of which B is eliminated. Therefore, we have:

$$V = 1 + (A - B)$$

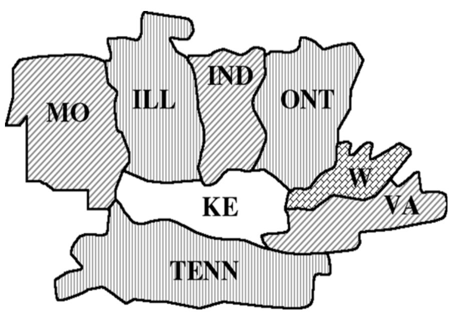
$$B = 1 + (A - V)$$

It can be graphically proven that the Betti number of the graph of the bridges of Königsberg is four, as it is necessary to eliminate 4 arcs for circuits not to exist in the graph. Let us take a look at this:



A = seven arcs (seven bridges)
V = four vertices (four areas of separated land)
$B = 1 + 7 - 4 = 4.$

Although the idea which supports the naming of the "Betti number" had already been used prior to that by **G. R. Kirchhoff in 1847** and by **James Clerk Maxwell in 1873**, it was **Enri Poincaré** who settled on the name in 1895 in honour of the mathematical physicist **Enrico Betti** (1823-1892) who, in 1871, had created the Riemann connection numbers.



3. The well-known four- colour problem is also placed in the field of graphs. It is as simple as it is unsolvable and as unsolvable as it is difficult to demonstrate its unsolvability. In short it is stated that, given a geographical map each country should be coloured in a different colour from those adjacent to it, using the least number of colours. It is necessary to take into account that a single point of contact does not presuppose the existence of a border. So, it must be demonstrated that four colours are enough to colour any map of a single plan.

For many countries only three colours were necessary, and for some well-known cases such as the American State of Kentucky and its surrounding states four colours are necessary.

In 1946 the Belgian mathematician **S. M. de Backer** tested that any map with a number of countries equal or lower to 35 could be coloured, in the stated circumstances, with no more than four colours. Later other researchers managed to increase the number of countries significantly, although a general demonstration for any quantity of countries is still not known.

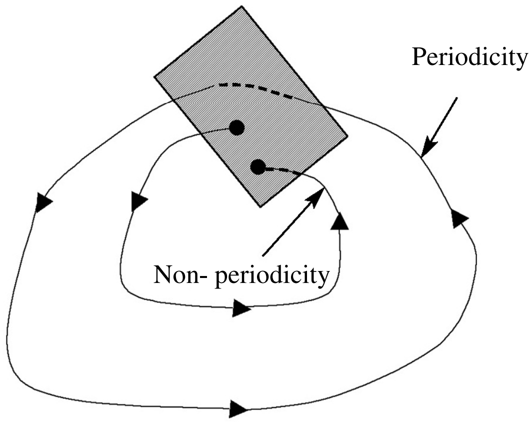
It is evident that the **four colour theorem** was enunciated only for the cases of flat or spherical surfaces, as other distinct problems arise for other geometrical figures.

4. In 1887 King Oscar of Sweden offered a prize of 2,500 Krona to whoever gave the answer to the question of "if the solar system is stable". In celestial mechanics the interaction of two bodies does not pose a problem, but that of three presents great difficulties. **Poincaré** won the prize in 1890 with the memoir "on the Three-body Problem and the Equations of dynamics".

The movement of two bodies (the Sun and the Earth, for example) is periodical. The period is of one year. This shows that these bodies cannot crash into one another, nor can they infinitely move away from one another. They have not previously done this and therefore they can never do this. **Periodicity** constitutes a very useful element for controlling stability.

In the third chapter of his memoir, **Poincaré** strived to explain **the existence** of **periodical solutions** for differential equations. Assuming that at a determined moment the system is found in a specific state and that at a later time it once again returns to the same state. All positions and velocities are the same as before. Therefore, the **movement** which has driven a state to itself once again should be repeated, once and again: it is a periodical movement. Given a point in a many dimensional space, as time passes the point will move giving rise to a curve. When will the curve form a closed loop? As the question does not affect either the **form**, the **size** or the **position** of the loop, the answer corresponds to topology. In this way the existence of periodical solutions depends on the topological properties of the **relationship** between the position of a point now and its position in a later period.

A simple example can be revealing. If we would like to know if an artificial satellite has a periodical orbit, instead of following all of its trajectory all around the Earth with a telescope we focus on it in a way that "sweeps" a plane which goes from north to south, from one point of the horizon to the other and which is in line with the centre of our planet. We take note of the place which it has passed the first time, its speed and its direction. We must then wait only **focusing on the plane**. Periodicity demands that it must once again pass by the same point, at the same speed and in the same direction. Therefore, instead of observing all states, just a few are enough. This surface is known as the **Poincaré section**



From this idea, **Poincaré** went deep into another approach, known as **Hill's reduced model**. Three bodies are considered, one of which possesses a mass so small that it does not affect the other two, of large mass. In exchange these do affect the first (for example, an interstellar particle and two planets). The two large bodies move forming elliptical paths around their mutual centre of gravity, but the tiny body moves oscillating from one side to another with nothing to change its direction. **Poincaré** used his **superficial section method** to try to find periodical movements of the tiny body, but what he found was a very complicated and counterintuitive behaviour: the system began activity in one state, followed a curve which took another state when it returned to the Poincaré section, then another and another successively. The system, in summary, passed through the Poincaré section with an uncertain sequence of points. Poincaré had found a chaotic panorama.

The main reason for that which we have just stated has been to illustrate the idea of topology from an anecdotal historical perspective, at the same time as allowing the introduction of some basic ideas which have survived over time.

In the last fifty years there has been an increase of interest in the use of topology to deal with a large number of problems by a great number of mathematicians and also of physicists and engineers. Due to this, important applications in the study of **flow nets** and **closed circuits** (Kirchhoff), in **magnetic fields** (Maxwell) and also in studies of the disposition of colours and design of printed electronic circuits, amongst others have taken place. We believe that it is time that **economists** and **management specialists** are also interested by this important branch of mathematics, as maybe it would be possible to represent **economics and management systems** through **topological spaces**, finding the "laws", if they exist, of the mutations which take place when the limits capable of producing a return to equilibrium are passed, but the new situations do not give rise to a traumatic rupture with the past. The work which we are currently presenting follows this path.

We know of **the use of topology** to represent problems expressed through **"non-lineal" differential equations**, which is to say equations with derivatives **without** effects proportional to the causes. The simplification which linearity brings has historically come imposed on many occasions by the difficulty, when not the impossibility, of its calculation and solution. **Economics and management phenomena** rarely present an adopted lineal form in reality which is why there is **an urgent need** for the recourse to complex non-lineal schemes.

That which we have just stated should not lead us to think that topology is an insurmountably difficult part of mathematics. The opposite is in fact the case, above all in the basic ideas which they report. In effect, many topological concepts are habitually employed. From here we arrive at the notions of **interior** and **exterior**, **connection** and **non-connection**, known and fostered in schools, even from the youngest infants.

In the texts which are dedicated to explaining topology there exists the habit of presenting some **theorems** which are naturally susceptible to draw the attention of readers and arouse their curiosity. Here we have five of these:

a) **The wind cannot simultaneously blow in all areas of the Earth**. There must exist an area without wind at all times. This place may be, for example, the South Pole.
b) If the wind is blowing in all areas of the northern hemisphere at a determined moment, then at this moment it must blow in all directions at the equator. For example, a place must exist where the wind blows north-easterly.
c) Under the same circumstances at least **two diametrically opposite points** must exist at the equator where the wind blows in radically opposite directions.
d) At a determined moment there exists at least one point **on the Earth and its antipode** which have the same temperature and the same humidity.
e) If the Earth were divided into three single great powers then at least **one of them would never witness a sunset**, given that in at least one of the three cases it would be a given fact that of two of their points one should be an antipode of the other.

Each one of these theorems is a consequence of mathematical discoveries of a most general nature. In this way the first two are particular cases of the **general "fixed point" theorem**, which was formulated in 1904 by the Dutch mathematician **L. E. J. Brouwer**[1]. The theorems c) and d) are a direct result of the general "**antipodal point**" theorem formulated for n dimensions in 1933 by the Polish researchers **K. Borsuk** and **S. M. Ulam**. The last of these theorems was deduced from another **more general for n dimensions**, elaborated in 1930 by the Soviet mathematicians **L. Lusternick and L. Schnirelmann**.

The content of chapter I of the great work published with the name of the author **Nicolas Bourbaki**[2] results of great interest. **Eléments de Mathématique**,

[1] Shinbrot, M.: «Teoremas del punto fijo», in *Matemáticas en el mundo moderno* (various authors). Ed. Blume. Madrid, 1974, pages. 165-171.

[2] Bourbaki, N.: *Eléments de Mathématique. Livre III Topologie Générale.* Hermann & Cie. Editeurs. París, 1940. As it is known, N. Bourbaki has been used as the pseudonym of an important group of mainly French mathematicians (between 10 and 20 depending on the moment in time).

whose third book is integrally dedicated to topological structures. Its clarity has given us a a lot in our attempt to develop some algorithms based on uncertain topology.

Considerations on the Subject of the Idea of Topology

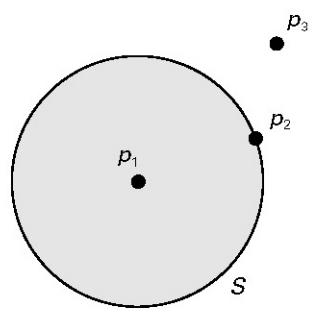
The aim of everything that we have just shown is to "create" the necessary conditions in order to stimulate the study of topology. If we have achieved to open scientific curiosity then it is possible that the need appears for an explanation of why **interest exists** for **topological spaces**. We will do this through some brief ideas.

We remind ourselves that in sets theory it is only possible to study the relationship between a point and a set. Therefore, this relationship remains limited to the possession of the set or not.

However, upon considering R^2 in the set S (which is a subset of E) formed by an open ball, the three points p_1, p_2, p_3 in the first figure on this page.

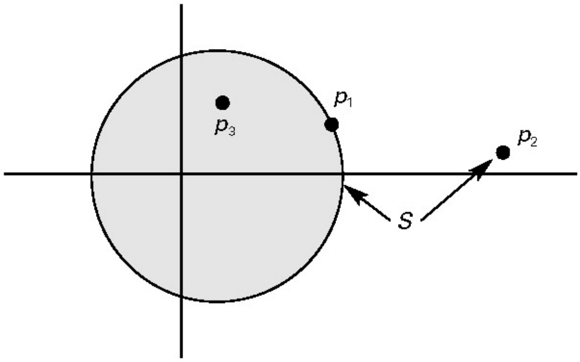
P_1 can be "seen" **inside** $\boldsymbol{S}$, p_2 **on the border of** $\boldsymbol{S}$ and p_3 **outside** $\boldsymbol{S}$. This initial interpretation has its foundation in the notion of the proximity of p_1 to S and also to its complement.

In this way the **adherence point** of set S is born, considered as that which **is not outside**. In other words, it is that which is as close to S as we would like.



From now on, we will designate a topological space by using *(E, T(E))*. So, if S is a subset of E, it would be said that a point $p \in E$ is adherent to S when any **range** U of p crosses S. This is equivalent to saying that **no** range of p **exists** which is totally included in $E - S$. Therefore, any point of the set S is an adherence point of S. The set of adherence points of S is its **adherent or adherence application**.

We will now move on to the concept of **accumulation**. If we consider in R^2 the set S formed by the union of the circle and point p_2, the points p_1, p_2, p_3 of the following figure are all **adherence points**.



However, the points p_1, p_3 have a distinct sense to p_2. In effect, "near" to p_1, p_3 other points of S can be found, which does not occur with p_2, which is **isolated**. From here the notions of **accumulation point** and **isolation point**.

Given a topological space $(E, T(E))$ and also given a subset of E, S, we will say that $p \in E$ is an **accumulation point of *S***, if any range U of p contains a point of S distinct to p. The set of all of the accumulation points of S are called **accumulation** or **set derived from *S***. The points which do not fulfil this condition are **isolated points**.

We will now separate the adherence points according to proximity as much of the set S as the complement of this set. Those points which are "near" to both the set S and its complement are named **border points**. In other words, when all the range U of p crosses S and its complement $E - S$. The **set of border points** is called **border**.

Finally, among the adherence points it fits to consider those found "within" the set S and, therefore which may "separate" from the complementary set $E - S$. In this way, therefore, a point $p \in S$ is **interior to** S, if S is a range of p. The set of points inside S are named **interior**. It is therefore possible to formulate the following theorem: Given a topological space $(E, T(E))$ and a subset S of E , one would say that $p \in S$ is **interior to** S if and only if it is not an adherence point of its complement $E - S$.

It does not result difficult to accept that the **interior** has the **dual properties** of **adherence**.

We have made reference to a concept that of range of a point, for which some remarks are convenient. We will do this through its most elemental properties:

1 The point p belongs to all range of p.
2 If U is a range of p, any set $T \supset U$ is a range of p.
3 If U_1 and U_2 are ranges of p, the same occurs with the intersection $U_1 \cap U_2$. This is generalizable to a finite number of ranges of p.
4 A range U of p is also a range of all points x of a suitable range F of p.

The axioms 1) and 4) have constituted decisive axioms for a **generalisation** of topological spaces. In this way it has been able to define a topological space $T(E)$, from a set E when a system $U(p)$ of subsets of E corresponds to each element p of E, named ranges of U of p, which verify the following axioms:

1 $p \in U$, for all of the range $U \in U(p)$.
2 If $U \in U(p)$ and $F \supset U$, then $F \in U(p)$.
3 If U_1, $U_2 \in U(p)$, then $U_1 \cap U_2 \in U(p)$; $E \in U(p)$
4 For each $U \in U(p)$, there exists an $F \in U(p)$, so that $U \in U(q)$ for all $q \in F$.

In this way, a set E together with a topology $T(E)$ is named topological space $(E, T(E))$. The elements of E take the name of **points** of topological space.

The axioms 1) and 4) have their correspondence with the axioms of the **Hausdorff range**, introduced by F. Hausdorff in his fundamental work of 1914 "Grundzüge der Mengenlehre". In effect, a topological space $(E, T(E))$ is said to be of Hausdorff when it fulfils one of the following axioms, equivalent to each other.

a) If $p \neq q$, with p, q being two points of E, the ranges $U \in U(p)$ and $F \in U(q)$ exist, with $U \cap F = \emptyset$ (Hausdorff separation axiom).
b) The intersection of all the closed ranges of a point p only contains p.

The first of these axioms is named of separation as the points p and q remain separated by the ranges U and F.

Elemental Notions in Topological Spaces

Everything that we have just stated has the purpose of bringing to the surface sufficient elements in order to establish some definitions which we will summarise.

If S is a subset of the referential E:

1 A point $p \in E$ is named **interior** of S when a range $U \in U(p)$ exists which belongs entirely to S. In classic nomenclature, the set of all the interiors is known with the denomination of **opening** of S.
2 A point $p \in E$ is named **exterior** of S when a range $U \in U(p)$ exists which belongs entirely to the **compliment** of S. The set of points exterior to S are known as **exterior** of S.
3 A point $p \in E$ is a **border** point of S, or with respect to S, when in each range of p there exists points which belong to S and to the compliment of S. The set of all these points p of S is named the **border** of S.

Of these definitions it can be deduced that all points $p \in E$ unmistakeably belong to one of the three classes previously described. At the same time it can be said that the **exterior of *S*** coincides with the **interior of the compliment of *S***.

With these three definitions presented, we find ourselves able to move on to the important concept of **adherence**. A point is $p \in E$ named adherence point of S if in each range of p there are points of S. The set of all the adherence points of S is designated **closure** or **close of S**. The **close** of S comes given by the union of the opening of S with the border of S.

In the same way, we will say that a set S is named **open** when it fulfils one of the following conditions, equivalent to each other:

a) All points of S are interior.
b) The opening of S coincides with S. It is enough that the opening contains S.
c) The border of S is contained within S.

A set S is named **closed** when it fulfils one of the following conditions, also equivalent to each other:

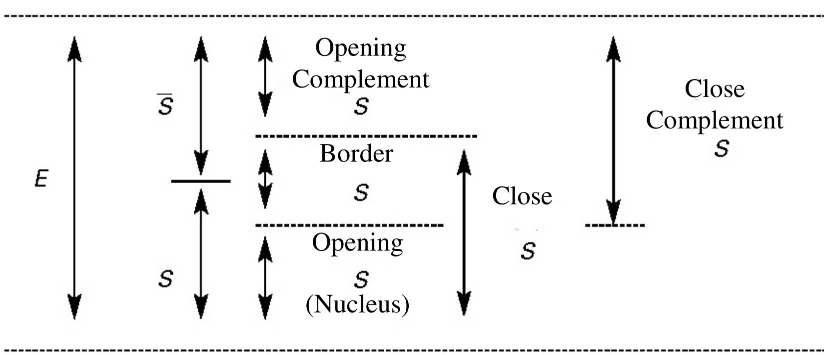
a) The set S contains all its adherence points.
b) The set S coincides with the close of S. It is enough to demand that the close of S is contained within S.
c) The border of S is found contained in S.

It is possible that a set S is open and closed at the same time.

A simple figure can help us to visualise the equivalence of the three conditions, as much in relation with the **open** as with the **closed**.

Let us look at this through the following formulae:

$E = S$ + complement S.
= open of S + close of the complement of S.
= close of S + open of the complement of S.



We can conclude by making the following theorems:

1 A set S is open if, and only if, the complement of S is closed. A set S is closed if, and only if, the complement of S is open.
2 A set S is open if, and only if, it is a range of all its points.

Everything which we have just stated brings us to consider a "**topological space** as a set in which we have selected a series of subsets, which we call open, and which fulfil the same properties as the open subsets of any **metric space**"[3].

Brief Reference to Metric Spaces

Remember that a **metric space** is a pair *(E, d)* formed by the set *E* and an application
$d(E \cdot E \rightarrow R)$ which fulfils the following axioms:

a) $d(p, q) \geq 0$, $d(p, q) = 0$ if and only if $p = q$.
The distance between the two points is never negative and is only zero if and only if the two points are the same.

b) $d(p, q) = d(q, p)$, for any $p, q \in E$.
The distance is symmetric.

c) $d(p, q) + d(q, r) \geq d(p, r)$, for any $p, q, r \in E$.
The distance fulfils triangular inequality.

The elements of the set *E* are named **points of space** and *d* **metric** on *E*.

For indicative purposes only we will remind ourselves of two of the spaces which appear with the greatest frequency in the works consulted or studied. These are **euclidean space** and **Hilbert's H space**.

$\mathbf{R^n}$ **Euclidean space** is perhaps the most widespread metric space. Its points are given under the form e = $(p_1, p_2, \ldots.p_n)$, in which $p_1, p_2, \ldots.p_n$ are arbitrary real numbers. The distance in these spaces comes from the well-known formula:

$$d(e, n) = \sqrt{\sum_{i=1}^{n}(q_i - p_i)^2}$$

It is unquestionable that the first two axioms are fulfilled. Let us see if the same occurs with the third, which is to say:

$$\sqrt{\sum_{i=1}^{n}(r_i - p_i)^2} \leq \sqrt{\sum_{i=1}^{n}(q_i - p_i)^2} + \sqrt{\sum_{i=1}^{n}(r_i - p_i)^2}$$

Let us make $(q_i - p_i) = a_i$, $(r_i - q_i) = b_i$. The previous expression can be deduced from:

[3] Mascaró, F.; Monterde, J.; Nuño, J. J., and Sivera, R.: *Introducció a la topología*. Ed. Universitat de Valencia, 1997, page. 51.

$$\sqrt{\sum_{i=1}^{n}(a_i + b_i)^2} \leq \left(\sqrt{\sum_{i=1}^{n} a_i^2} + \sqrt{\sum_{i=1}^{n} b_i^2}\right)^2 = \sum_{i=1}^{n} a_i^2 + \sum_{i=1}^{n} b_i^2 + 2\sqrt{\sum_{i=1}^{n} a_i^2 \cdot \sum_{i=1}^{n} b_i^2}$$

$$2\sum_{i=1}^{n} a_i \cdot b_i \leq 2\sqrt{\sum_{i=1}^{n} a_i^2 \cdot \sum_{i=1}^{n} b_i^2}$$

$$\sum_{i=1}^{n}(a_i \cdot b_i)^2 \leq \sum_{i=1}^{n} a_i^2 \cdot \sum_{i=1}^{n} b_i^2$$

This final inequality is that known as the Cauchy-Schwarz inequality.

We will now, finally, refer to **Hilbert's H space**. This space is formed by all the successions of real numbers $h = (p_1, p_2....)$, so that the sum of its squares $\sum p_i^2 (i = 1,2,...)$ is convergent. In this case the distance is the same as the euclidean:

$$d(h, \eta) = \sqrt{\sum (q_i - p_i)^2}$$

Here it is warranted to state that the convergence of the infinite series has moved inside the root.

In this way one has:

$$\sum_{i=1}^{N}(q_i - p_i)^2 = \sum_{i=1}^{N} q_i^2 + \sum_{i=1}^{N} p_i^2 - 2\sum_{i=1}^{N} p_i \cdot q_i$$

Upon creating N, following the established hypothesis, the first two addends remain enclosed, whereas the third is in this way as a consequence of the Cauchy-Schwarz inequality:

$$\left(\sum_{i=1}^{N} p_i \cdot q_i\right)^2 \leq \sum_{i=1}^{N} p_i^2 \cdot \sum_{i=1}^{N} q_i^2$$

Therefore, in this way, the series placed within the root is convergent and the distance is found to be adequately defined, as the axioms 1) and 2) fulfil this need with simple observation and 3) is easily tested by the steps to the limit in the euclidean space formula.

From this brief reminder we should indicate that **metric spaces** are not general enough to describe whatever "form" of space, as others exist in which it is not possible to make each pair of elements of the real numbers such as distance between each coincide. To this respect it is enough to observe the axiomatic of the metric spaces to confirm the intervention of real numbers in them. In a crude way we can say that **all metric space** is a **topological space**. In other words, a defined metric space on a set in R induces a topological space. On the other hand the reciprocal proposal that every topological space proceeds from an adequate metric space is not true.

In this brief summary we have tried, with a selection of some historical and elemental aspects as close as possible to intuition, to drive **topological reasoning** to that point in which a strict formal rigor is imposed, capable of describing one of the outlines with a greater level of abstraction than modern mathematics provides: **topological spaces** apt for the treatment of economics and management systems.

To achieve this it has been considered as opportune to begin the basic content of this work with the description of pretopological axiomatics, first from a perspective which, abusing language, could be called **orthodox**, to later move on to its transformation attempting to generalise to deal with situations immerse in a context of **uncertainty**: both aspects will constitute the content of the first part of our work.