Acute Triangulations of the Cuboctahedral Surface

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Abstract

In this paper we prove that the surface of the cuboctahedron can be triangulated into 8 non-obtuse triangles and 12 acute triangles. Furthermore, we show that both bounds are the best possible.

1 Introduction

A triangulation of a two-dimensional space means a collection of (full) triangles covering the space, such that the intersection of any two triangles is either empty or consists of a vertex or of an edge. A triangle is called *geodesic* if all its edges are *segments*, i.e., shortest paths between the corresponding vertices. We are interested only in *geodesic triangulations*, all the members of which are, by definition, geodesic triangles. The number of triangles in a triangulation is called its *size*.

In rather general two-dimensional spaces, like Alexandrov surfaces, two geodesics starting at the same point determine a well defined angle. Our interest will be

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focused on triangulations which are *acute* (resp. *non-obtuse*), which means that the angles of all geodesic triangles are smaller (resp. not greater) than $\frac{\pi}{2}$.

The discussion of acute triangulations has one of its origins in a problem of Stover reported in 1960 by Gardner in his Mathematical Games section of the *Scientific American* (see [4], [5], [6]). There the question was raised whether a triangle with one obtuse angle can be cut into smaller triangles, all of them acute. In the same year, independently, Burago and Zalgaller [1] investigated in considerable depth acute triangulations of polygonal complexes, being led to them by the problem of their isometric embedding into \mathbb{R}^3 . However, their method could not give an estimate on the number of triangles used in the existed acute triangulations. In 1980, Cassidy and Lord [2] considered acute triangulations of the square. Recently, acute triangulations of quadrilaterals [12], trapezoids [18], convex quadrilaterals [3], pentagons [16] and general polygons [11, 17] have also been considered.

On the other hand, compact convex surfaces have also been triangulated. Acute and non-obtuse triangulations of all Platonic surfaces, which are surfaces of the five well-known Platonic solids, have been investigated in [7], [9], and [10]. Recently, Saraf [15] considered the acute triangulations of the polyhedral surfaces again, but there is still no estimate on the size of the existed acute triangulations. Maehara [13] considered the proper acute triangulation of a polyhedral surface and obtained an upper bound of the size of the triangulation, which is determined by the length of the longest edge, the minimum value of the geodesic distance from a vertex to an edge that is not incident to the vertex, and the measure of the smallest face angle in the given polyhedral surface. Furthermore, some other well-known surfaces have also been acutely triangulated, such as flat Möbius strips [19] and flat tori [8].

Combining all the known results for the polyhedral surfaces mentioned above, we are motivated to investigate the non-obtuse and acute triangulations of the surfaces of the Archimedean solids. In this paper we consider the surface of the Archimedean solid cuboctahedron, which is a convex polyhedron with eight triangular faces and six square faces. It has 12 identical vertices, with two triangles and two squares meeting at each, and 24 identical edges, each separating a triangle from a square. For the sake of convenience, let C denote the surface of the cuboctahedron with side length 1. Let \mathscr{T} denote an acute triangulation of C and \mathscr{T}_0 a non-obtuse triangulation of C. Let $|\mathscr{T}|$ and $|\mathscr{T}_0|$ denote the size of \mathscr{T} and \mathscr{T}_0 respectively. We prove that the best possible bounds for $|\mathscr{T}|$ and $|\mathscr{T}_0|$ are 12 and 8 respectively.

2 Non-obtuse triangulations

Theorem 2.1. The surface of the cuboctahedron admits a non-obtuse triangulation with 8 triangles and no non-obtuse triangulation with fewer triangles.

Proof. Fig. 1 describes the unfolded surface C. We fix two vertices a and b, which are the vertices of a diagonal of a square face on C. Let a', b' be the antipodal vertices of a, b respectively. There are six geodesics from a to a' and b to b'. We

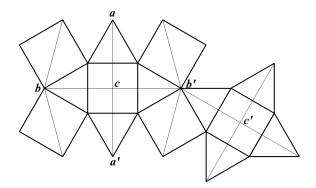


Figure 1: A non-obtuse triangulation of C.

choose those two passing through two triangular faces and one square face. Denote the two intersection points of the geodesics aa' and bb' chosen above by c and c'. Clearly, c and c' are an antipodal pair of vertices on C. Draw the segments from a (resp. a') to b and b'. Thus C is triangulated into 8 non-obtuse triangles: abc, ab'c, abc', ab'c', a'bc, a'b'c, a'bc', a'b'c'.

Indeed, noticing that all of those eight triangles are congruent, we only need to show that the triangle abc is non-obutse. By the construction we know that aa' is orthogonal to bb'. So $\angle acb = \frac{\pi}{2}$. Further, it is clear that $\angle abc = \angle bac = \frac{5\pi}{12}$.

We prove now that for any non-obtuse triangulation \mathscr{T}_0 of \mathcal{C} , we always have $|\mathscr{T}_0| \geq 8$. If not, then we have $|\mathscr{T}_0| = 4$ or $|\mathscr{T}_0| = 6$. If $|\mathscr{T}_0| = 4$, then \mathscr{T}_0 has $(4 \times 3)/2 = 6$ edges and, by Euler's formula, 6 - 4 + 2 = 4 vertices. So \mathscr{T}_0 is isomorphic to K_4 ; If $|\mathscr{T}_0| = 6$, then \mathscr{T}_0 is isomorphic to the 1-skeleton of the double pyramid over the triangle. In both cases there are vertices with degree 3. However, at each vertex of \mathcal{C} the total angle is $\frac{5\pi}{3}$, so each vertex in \mathscr{T}_0 has degree at least 4. Clearly, each other vertex of \mathscr{T}_0 also has degree at least 4. Thus we obtain a contradiction.

The proof is complete.

3 Acute triangulations

Theorem 3.1. The surface of the cuboctahedron admits an acute triangulation with 12 triangles.

Proof. Let a', b', c' and d' be four distinct vertices of the cuboctahedron such that $|a'b'| = |b'c'| = |c'd'| = |d'a'| = \sqrt{2}$, where |pq| denotes the intrinsic distance on the surface C between two points p and q. Clearly, the four segments a'b', b'c', c'd' and d'a' determine a cycle which decomposes C into two regions C_1 and C_2 . Take a vertex a (resp. b, c, d) adjacent to both a' (resp. b', c', d') and b' (resp. c', d', a') such that $a, c \in C_1, b, d \in C_2$. Take a point a^* (resp. b^*, c^*, d^*) on a'b' (resp. b'c', c'd', d'a') such that $\angle a'aa^*$ (resp. $\angle b'bb^*, \angle c'cc^*, \angle d'dd^*) = \frac{\pi}{6}$.

We get a triangulation of C with 12 triangles:

 $a^*ab^*, a^*b^*b, a^*bd, a^*dd^*, a^*d^*a, b^*bc^*, b^*c^*c, b^*ca, c^*cd^*, c^*d^*d, c^*db, d^*ca.$

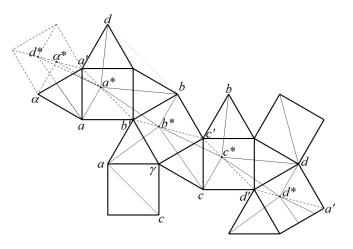


Figure 2: An acute triangulation of C.

There are two shortest paths from a^* to b^* (resp. c^* to d^*); here we choose the path in \mathcal{C}_2 . There are two shortest paths from b^* to c^* (resp. d^* to a^*); here we choose the path in \mathcal{C}_1 , see Fig. 2.

Indeed, the values of the angles around a^*, b^*, c^*, d^* (resp. a, b, c, d) are entirely the same. So we only need to consider the angles around a^* and a respectively.

Firstly, we consider the angles around a^* .

In the triangle $a^*b'b$, $\angle a^*b'b = \frac{\pi}{4} + \frac{\pi}{3} = \frac{7\pi}{12} > \frac{\pi}{2}$, which implies that $\angle b^*a^*b < \angle b'a^*b < \frac{\pi}{2}$.

In Fig. 2 the planar circle C with diameter bd (the dot line-segment) passes through the midpoint of a'b', say, x'. So, the segment a'b' is tangent to C at x'. Since $a^* \in a'b'$ and $a^* \neq x'$, we have $\angle ba^*d < \frac{\pi}{2}$.

In the quadrilateral $a^*da'd^*$, $\angle da'd^* = \frac{\pi}{3} + \frac{\pi}{2} + \frac{\pi}{3} + \frac{\pi}{4} = \frac{17\pi}{12}$, which implies that $\angle a^*d^*a' + \angle d^*a^*d + \angle a^*da' = \frac{7\pi}{12}$. However, $\angle a^*da' > \angle ada' = \frac{\pi}{12}$. Therefore, $\angle d^*a^*d < \angle d^*a^*d + \angle a^*da' < \frac{7\pi}{12} - \frac{\pi}{12} = \frac{\pi}{2}$.

Denote by α the vertex adjacent to both a and a'. Take a point $\alpha^* \in d'a'$ such that $\angle \alpha^* \alpha a' = \frac{\pi}{6}$. Clearly, the triangle aa^*a' is congruent to the triangle $\alpha \alpha^*a'$,

so we have $|\alpha^* a'| = |a^* a'|$, which implies that $\angle a' a^* \alpha^* = \frac{\pi}{12}$. Then $\angle \alpha^* a^* a = (\pi - \frac{\pi}{6} - \frac{\pi}{4}) - \frac{\pi}{12} = \frac{\pi}{2}$. Noticing that the distance from d^* to a' is further than that from α^* to a', we have $\angle d^* a^* a < \angle \alpha^* a^* a = \frac{\pi}{2}$.

In the triangle aa^*b' , $\angle aa^*b' = \pi - \frac{\pi}{3} - \frac{\pi}{4} = \frac{5\pi}{12}$. In the triangle $a^*b'b^*$, $\angle a^*b'b^* = \frac{\pi}{4} + \frac{\pi}{3} + \frac{\pi}{4} = \frac{5\pi}{6}$. Noticing that $|b'b^*| < \frac{1}{2}|b'c'| = \frac{1}{2}|a'b'| < |a^*b'|$, we have $\angle b'a^*b^* < \frac{2}{2}b'b^*a^*$ and therefore $\angle b'a^*b^* < \frac{\pi}{12}$. So we have $\angle aa^*b^* < \frac{5\pi}{12} + \frac{\pi}{12} = \frac{\pi}{2}$.

Consider now the angles around a. It is clear that $\angle a^*ad^* < \angle a^*a\alpha = \frac{\pi}{2}$ and $\angle a^*ab^* = \angle a^*ab' + \angle b'ab^* < \frac{\pi}{3} + \frac{\pi}{6} = \frac{\pi}{2}$. Let γ denote the vertex adjacent to both c and c'. Then $\angle b^*ac = \angle \gamma ac + \angle b^*a\gamma < \angle \gamma ac + \angle ba\gamma = \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2}$. Finally, $\angle d^*ac = \angle d^*a\alpha + \angle \alpha ac < \frac{\pi}{6} + \frac{\pi}{4} < \frac{\pi}{2}$.

4 No acute triangulation with fewer triangles

Let \mathscr{C} be the 1-skeleton of the cuboctahedron. The graph-theoretic distance $d_{\mathscr{C}}(v, w)$ between the vertices v, w of \mathscr{C} is called the \mathscr{C} -distance between v and w. Let g(u, v)denote a geodesic between two points u and v on the surface \mathscr{C} . We start with the following lemma.

Lemma 4.1. Let u, w_1, w_2 be three vertices of C. Then the angle formed by $g(u, w_1)$ and $g(u, w_2)$ on C is equal to $\frac{\pi}{12}i$, where $i \in \mathbb{Z}$ and $1 \le i \le 20$.

Proof. For any vertex u of \mathcal{C} , consider all the segments from u to any other vertex v (see Fig. 3). It is easy to see that $d_{\mathscr{C}}(u,v) \leq 3$. If $d_{\mathscr{C}}(u,v) = 1$, then $v \in \{a_1, a_2, a_3, a_4\}$. Clearly g(u, v) is an edge of \mathcal{C} . If $d_{\mathscr{C}}(u, v) = 2$, then $v \in$ $\{b_1, b_2, b_3, b_4, b_5, b_6\}$. Further, if $v = b_2$ or $v = b_5$, then g(u, v) is a diagonal of a square face on the surface; if $v \in \{b_1, b_3, b_4, b_6\}$, then there are two geodesics between u and v. If $d_{\mathscr{C}}(u, v) = 3$, then v = c and there are six geodesics between uand v. Please note that the solid line between u and v in Fig. 3 is not a geodesic. Thus there are 20 geodesics starting from u to any other vertex v on \mathcal{C} , and all of

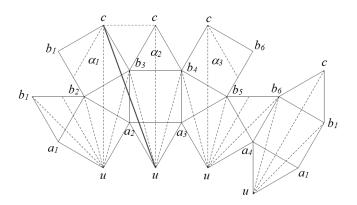


Figure 3: Geodesics starting from a vertex u.

them divide the total angle around u into 20 equal parts. Trivially, each part has angle $\frac{\pi}{12}$. Thus the angle formed by $g(u, w_1)$ and $g(u, w_2)$ on C is equal to $\frac{\pi}{12}i$, where $i \in \mathbb{Z}$ and $1 \leq i \leq 20$.

Lemma 4.2. There is no acute triangulation of C with 8 triangles.

Proof. Suppose there exists an acute triangulation \mathscr{T} of \mathcal{C} containing 8 triangles. By a method similar to that used in the proof of Theorem 2.1, we know that \mathscr{T} is isomorphic to the 1-skeleton of the regular octahedron, where all the vertices have degree 4. Clearly each vertex of \mathscr{T} is a vertex of \mathcal{C} . By Lemma 4.1, it is easily seen that any acute angle in \mathscr{T} is $\frac{\pi}{12}i$, where i = 1, 2, 3, 4, 5. Recall that the total angle at any vertex of \mathcal{C} is $\frac{5\pi}{3}$. Therefore the four angles around each vertex of \mathscr{T} are all isogonal and equal to $\frac{5\pi}{12}$.

Now let v_1, v_2 be two adjacent vertices in \mathscr{T} . Then in both of the triangles having side $g(v_1, v_2)$, all the three angles are equal to $\frac{5\pi}{12}$. In the following we show that in one of them, where the third vertex is denoted by v_3 , there is always a contradiction.

There are three cases to consider.

Case 1.
$$d_{\mathscr{C}}(v_1, v_2) = 1$$

If $\angle v_3 v_1 v_2 = \angle v_3 v_2 v_1 = \frac{5\pi}{12}$, then clearly we have $\angle v_1 v_3 v_2 = \frac{\pi}{6} \neq \frac{5\pi}{12}$, a contradic-



Figure 4: $d_{\mathscr{C}}(v_1, v_2) = 1.$

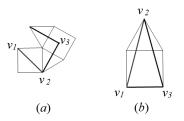


Figure 5: $d_{\mathscr{C}}(v_1, v_2) = 2.$

tion, as shown in Fig. 4.

Case 2. $d_{\mathscr{C}}(v_1, v_2) = 2.$

If $g(v_1, v_2)$ is a diagonal of a square face of C, then $\angle v_3 v_1 v_2 = \angle v_3 v_2 v_1 = \frac{5\pi}{12}$ forces v_3 not to be a vertex of C (see Fig. 5(*a*)), a contradiction. Otherwise, by the proof of Lemma 4.1 we may assume that v_1 is a corner of a square face and $g(v_1, v_2)$ intersects the interior of the square face, as show in Fig. 5(*b*). Let v_3 be another corner of the square face such that $\angle v_3 v_1 v_2 = \frac{5\pi}{12}$. By the proof of *Case*1, we know that $\angle v_1 v_2 v_3 = \frac{\pi}{6}$, a contradiction again.

Case 3. $d_{\mathscr{C}}(v_1, v_2) = 3.$

If $g(v_1, v_2)$ passes through two triangular faces and one square face, then we consider the triangle $v_1v_2v_3$ lying above $g(v_1, v_2)$, as shown in Fig. 6(a). Clearly, $\angle v_1v_2v_3 = \angle v_2v_1v_3 = \frac{5\pi}{12}$, but $\angle v_1v_3v_2 = \frac{5\pi}{6}$, a contradiction. If $g(v_1, v_2)$ passes through one triangular face and two square faces, then we consider the triangle $v_1v_2v_3$ lying below $g(v_1, v_2)$, as shown in Fig. 6(b). It is easy to see that $\angle v_1v_2v_3 =$

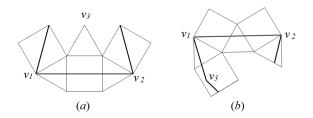


Figure 6: $d_{\mathscr{C}}(v_1, v_2) = 3.$

 $\angle v_2 v_1 v_3 = \frac{5\pi}{12}$ makes v_3 not be a vertex of \mathcal{C} , which contradicts to the fact that each vertex of \mathscr{T} must be a vertex of \mathcal{C} .

The proof is complete.

Lemma 4.3. There is no acute triangulation of C with 10 triangles.

Proof. Suppose that there exists an acute triangulation \mathscr{T} of \mathcal{C} containing 10 triangles. Then \mathscr{T} is isomorphic to the 1-skeleton of the double pyramid over the pentagon. So \mathscr{T} contains a 5-cycle C_5 and all its vertices have degree 4. Clearly, the vertices of C_5 must be the vertices of \mathcal{C} . For the sake of convenience, let $V(C_5)$ denote the set of all vertices of C_5 and $E(C_5)$ denote the set of all edges of C_5 . Furthermore, we have the following fact.

Fact. The angles formed by any two adjacent edges of C_5 are between $\frac{2\pi}{3}$ and π .

If u, v are two adjacent vertices of C, we call u, v an *adjacent pair* of C. In order to prove Lemma 4.3, we prove the following properties about the cycle C_5 mentioned above at first.

Proposition 4.4. $V(C_5)$ contains at least two adjacent pairs of C.

Proof. We first show that $V(C_5)$ contains at least one adjacent pair of \mathcal{C} . Suppose that $u \in V(C_5)$, as shown in Fig. 3. If $\{a_1, a_2, a_3, a_4\} \cap V(C_5) \neq \emptyset$, then clearly $V(C_5)$ contains an adjacent pair. If not, then the other four vertices of C_5 come

from $b_1, b_2, b_3, b_4, b_5, b_6$ and c. It is easy to see that among those four vertices there must be at least one adjacent pair of C.

Now let $v_1, v_2 \in V(C_5)$ be an adjacent pair of \mathcal{C} , as shown in Fig. 7. If $\{u_1, u_2, u_3, u_4, u_5\} \cap V(C_5) \neq \emptyset$, then clearly $V(C_5)$ contains another adjacent pair of \mathcal{C} and the proposition is proved. Otherwise, the other three vertices of C_5 come from the remained five vertices $\overline{u}_1, \overline{u}_2, \overline{u}_3, \overline{u}_4$ and \overline{u}_5 of \mathcal{C} . It's not hard to see that among any three vertices from $\overline{u}_1, \overline{u}_2, \overline{u}_3, \overline{u}_4$ and \overline{u}_5 there must be one adjacent pair of \mathcal{C} .

Proposition 4.5. Let $v_i, v_j \in V(C_5)$.

(a) If
$$d_{\mathscr{C}}(v_i, v_j) = 1$$
, then $g(v_i, v_j) \in E(C_5)$;

(b) If $d_{\mathscr{C}}(v_i, v_j) = 3$, then $g(v_i, v_j) \notin E(C_5)$;

(c) If $d_{\mathscr{C}}(v_i, v_j) = 2$ and $g(v_i, v_j)$ is a diagonal of a square face of \mathcal{C} , then $g(v_i, v_j) \notin E(C_5)$.

Proof. (a) We fix two vertices v_1 , v_2 such that $v_1, v_2 \in V(C_5)$ and $d_{\mathscr{C}}(v_1, v_2) = 1$. Suppose the contrary that $g(v_1, v_2) = v_1v_2 \notin E(C_5)$. Then there is a vertex of \mathcal{C} , say u, such that it is adjacent to both v_1 and v_2 in C_5 . Let the five neighbors of v_1 , v_2 in \mathcal{C} be u_i , i = 1, 2, 3, 4, 5, as shown in Fig. 7. By the Fact it is easy to see that $u \neq u_i$ (i = 1, 2, 3, 4, 5). Now denote the five remained vertices of \mathcal{C} by \overline{u}_i (i = 1, 2, 3, 4, 5). For the sake of convenience, let η_i denote the value of the smaller angle formed by v_1 , \overline{u}_i and v_2 on \mathcal{C} . Since $\eta_1 \leq \frac{\pi}{4} + \frac{\pi}{12} = \frac{\pi}{3}$, $\eta_3 \leq \frac{\pi}{12} + \frac{\pi}{3} + \frac{\pi}{12} = \frac{\pi}{2}$, $\eta_5 \leq \frac{\pi}{4} + \frac{\pi}{12} = \frac{\pi}{3}$, by the Fact we have $u \notin \{\overline{u}_1, \overline{u}_3, \overline{u}_5\}$. Noticing that $\eta_2 \leq \frac{\pi}{3} + \frac{\pi}{12} + \frac{\pi}{3} = \frac{\pi}{6} + \frac{\pi}{2} + \frac{\pi}{12} = \frac{3\pi}{4}$, we may assume that $u = \overline{u}_2$. Then by the Fact we have $\frac{2\pi}{3} < \eta_2 < \pi$, and therefore $\eta_2 = \frac{3\pi}{4}$ (by Lemma 4.1). Now let v'_1 be the other adjacent vertex of v_1 in C_5 . In order to ensure $\frac{2\pi}{3} < \angle \overline{u}_2 v_1 v'_1 < \pi$, that is, $\angle \overline{u}_2 v_1 v'_1 \in \{\frac{3\pi}{4}, \frac{5\pi}{6}, \frac{11\pi}{12}\}$, it is easy to check that $g(v_1, v'_1)$ always intersects $g(v_2, \overline{u}_2)$, a contradiction. Similarly, if $u = \overline{u}_4$ we also obtain a contradiction.

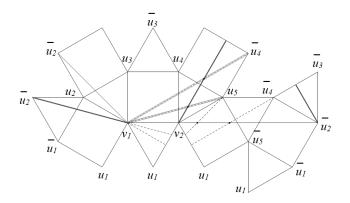


Figure 7: An adjacent pair of C.

(b) Without loss of generality, we assume that $v_i = u$, $v_j = c$, as shown in Fig. 3. Now suppose the contrary that $g(v_i, v_j) = g(u, c) \in E(C_5)$. For the sake of convenience, denote by u' the other adjacent vertex of u in C_5 . Since $\frac{2\pi}{3} < \angle u'uc < \pi$, by Lemma 4.1, we have $\angle u'uc \in \{\frac{3\pi}{4}, \frac{5\pi}{6}, \frac{11\pi}{12}\}$. There are two cases to consider.

Case 1. g(u, c) passes through two triangular faces and one square face.

We consider the rightmost geodesic g(u, c) in Fig. 3. If $\angle u'uc = \frac{3\pi}{4}$ or $\angle u'uc = \frac{11\pi}{12}$, then $u' = b_3$ or $u' = b_4$. Clearly, $d_{\mathscr{C}}(b_3, c) = d_{\mathscr{C}}(b_4, c) = 1$. By (a), we have $g(b_3, c) \in E(C_5)$ or $g(b_3, c) \in E(C_5)$. Thus we obtain a 3-cycle ucb_3u or ucb_4u , a contradiction. If $\angle u'uc = \frac{5\pi}{6}$, then u' = c. We obtain a 2-cycle ucu, a contradiction again.

Case 2. g(u, c) passes through one triangular face and two square faces.

We consider the second leftmost geodesic g(u, c) (the vertical one) in Fig. 3. If $\angle u'uc = \frac{5\pi}{6}$, then u' = c and we obtain a 2-cycle ucu, a contradiction. If $\angle u'uc \in \{\frac{3\pi}{4}, \frac{11\pi}{12}\}$, then $u' = b_5$ or $u' = b_6$. If $u' = b_6$, by (a), we obtain a contradiction; If $u' = b_5$, let u'' be the other adjacent vertex of u' in C_5 . Thus $\angle uu'u'' \in \{\frac{3\pi}{4}, \frac{5\pi}{6}, \frac{11\pi}{12}\}$. If $\angle uu'u'' = \frac{3\pi}{4}$, then $u'' = b_2$ and the geodesic $g(b_5, b_2)$ in $E(C_5)$ must pass through the faces $\alpha_1, \alpha_2, \alpha_3$, which intersects g(u, c) in its interior. This is impossible in C_5 . If $\angle uu'u'' = \frac{5\pi}{6}$, then u'' = c and we obtain a 3-cycle ub_5cu , a contradiction again.

If $\angle uu'u'' = \frac{11\pi}{12}$, then $u'' = b_2$. Let u''' be the other adjacent vertex of u'' in C_5 . We know that $\angle u'u''u''' \in \{\frac{3\pi}{4}, \frac{5\pi}{6}, \frac{11\pi}{12}\}$ which implies that $u''' \in \{a_3, b_5, u\}$. Clearly, $u''' \neq b_5$ and $u''' \neq u$. If $u''' = a_3$, then $g(b_2, a_3)$ in $E(C_5)$ intersects g(u, c) in its interior, which is a contradiction.

(c) Without loss of generality, let $v_i = u_1$, $v_j = u_2$ and $g(u_1, u_2)$ is a diagonal of a square face of \mathcal{C} , as shown in Fig. 7. Now suppose the contrary that $g(u_1, u_2) \in E(C_5)$. Let u'_1 be the other adjacent vertex of u_1 in C_5 . By the Fact and Lemma 4.1, we have $\angle u'_1 u_1 u_2 \in \{\frac{3\pi}{4}, \frac{5\pi}{6}, \frac{11\pi}{12}\}$. If $\angle u'_1 u_1 u_2 \in \{\frac{3\pi}{4}, \frac{11\pi}{12}\}$, then $u'_1 = \overline{u}_3$. Clearly, $d_{\mathscr{C}}(u_1, \overline{u}_3) = 3$, which contradicts to (b). If $\angle u'_1 u_1 u_2 = \frac{5\pi}{6}$, then $u'_1 = u_5$. Let u'' be the other adjacent vertex of u'_1 in C_5 . Noticing that $d_{\mathscr{C}}(u_1, u_5) = 2$ and $g(u_1, u_5)$ is a diagonal of a square face of \mathcal{C} , by the above discussion we know that if $\angle u_1 u_5 u''_1 \in \{\frac{3\pi}{4}, \frac{11\pi}{12}\}$, then there is a contradiction; if $\angle u_1 u_5 u''_1 = \frac{5\pi}{6}$, then $u''_1 = \overline{u}_3$. Repeating the above process again and we obtain a 4-cycle $u_1 u_5 \overline{u}_3 u_2 u_1$, which is a contradiction.

Proposition 4.6. C_5 has only one possible configuration as shown in Fig. 8.

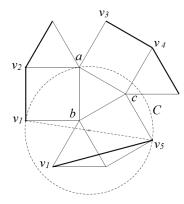


Figure 8: The 5-cycle C_5 .

Proof. Denote the vertices of C_5 by v_i (i = 1, 2, 3, 4, 5), and v_i , v_{i+1} are adjacent in C_5 (i + 1 takes modulo 5). By Proposition 4.4 and 4.5, there are two edges of C_5 ,

say e_1 and e_2 , which are edges of C. There are two cases to consider.

Case 1. e_1 and e_2 are adjacent in C.

Suppose that $e_1 = v_1v_2$ and $e_2 = v_2v_3$. Without loss of generality, we may assume that $v_1 = a_1$, $v_2 = u$, as shown in Fig. 3. Please keep in mind that any angle formed by two adjacent edges of C_5 is between $\frac{2\pi}{3}$ and π . Thus we have $v_3 = a_3$. By Proposition 4.5, we know that $d_{\mathscr{C}}(v_3, v_4) = 1$, or $d_{\mathscr{C}}(v_3, v_4) = 2$ and $g(v_3, v_4)$ passes through one triangular face and one square face, which implies that $v_4 = b_4$ or c.

If $v_4 = b_4$ and $d_{\mathscr{C}}(v_4, v_5) = 1$, then $v_5 = c$ and we obtain a 5-cycle $a_1ua_3b_4ca_1$, as the configuration described in Fig. 8. If $v_4 = b_4$ and $d_{\mathscr{C}}(v_4, v_5) = 2$, then $v_5 = b_1$. Since $d_{\mathscr{C}}(v_5, v_1) = 1$, by Proposition 4.5(a), $g(v_5, v_1) = v_5v_1 \in E(C_5)$. Thus we obtain a 5-cycle $a_1ua_3b_4b_1a_1$. If $v_4 = c$ and $d_{\mathscr{C}}(v_4, v_5) = 1$, then $v_5 = b_1$, we obtain a 5-cycle $a_1ua_3cb_1a_1$. If $v_4 = c$ and $d_{\mathscr{C}}(v_4, v_5) = 2$, then $v_5 = a_1$, which is a contradiction.

Case 2. e_1 and e_2 are not adjacent in C.

Suppose that $e_1 = v_1v_2$ and $e_2 = v_3v_4$. Without loss of generality, we may assume that $v_1 = a_1$, $v_2 = u$, as shown in Fig. 3. By Proposition 4.5, we know that $d_{\mathscr{C}}(v_2, v_3) = 1$, or $d_{\mathscr{C}}(v_2, v_3) = 2$ and $g(v_2, v_3)$ passes through one triangular face and one square face. That is, $v_3 = a_3$ or b_4 . If $v_3 = a_3$, then the discussion is same to that in *Case* 1. If $v_3 = b_4$, then clearly $v_4 = c$. This situation has been discussed in Case 1.

Now we are back to the proof of Lemma 4.3. Clearly, a 5-cycle C_5 described above decomposes \mathcal{C} into two regions, and one of them is shown in Fig. 8. Without loss of generality, let v_6 be the vertex of the acute triangulation \mathscr{T} lying in this region. Since $\angle v_6 v_2 v_1 < \frac{\pi}{2}$, v_6 can not lie in the triangular face av_2v_3 and the square face av_3v_4c except for the edge ac. Further, since $\angle v_6v_1v_2 < \frac{\pi}{2}$, $\angle v_6v_3v_4 < \frac{\pi}{2}$ and $\angle v_6v_4v_3 < \frac{\pi}{2}$, v_6 must lie in the triangular face abc. Clearly, $v_6 \notin \{a, b, c\}$ and the edge $g(v_1, v_6)$ of \mathscr{T} must intersect the square face v_1bav_2 . In Fig. 8, let C be the planar circle with diameter v_1v_5 (here v_1v_5 is the dash segment instead of the geodesic). It is easy to see that v_6 lies in the interior of the upper semi-disc bounded by C and the dash segment v_1v_5 . As a result, we have $\angle v_1v_6v_5 > \frac{\pi}{2}$, which contradicts to the fact that \mathscr{T} is an acute triangulation. Therefore, there is no acute triangulation of \mathcal{C} with ten triangles.

Combining Theorem 2.1, 3.1, Lemma 4.2, 4.3, we obtain the following main theorem immediately.

Theorem 4.7. The surface of the cuboctahedron admits an acute triangulation with 12 triangles, and there is no acute triangulation with fewer triangles.

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