# Acute Triangulations of the Cuboctahedral Surface 

Xiao Feng Liping Yuan*<br>College of Mathematics and Information Science, Hebei Normal University, 050016 Shijiazhuang, China. lpyuan@mail.hebtu.edu.cn.


#### Abstract

In this paper we prove that the surface of the cuboctahedron can be triangulated into 8 non-obtuse triangles and 12 acute triangles. Furthermore, we show that both bounds are the best possible.


## 1 Introduction

A triangulation of a two-dimensional space means a collection of (full) triangles covering the space, such that the intersection of any two triangles is either empty or consists of a vertex or of an edge. A triangle is called geodesic if all its edges are segments, i.e., shortest paths between the corresponding vertices. We are interested only in geodesic triangulations, all the members of which are, by definition, geodesic triangles. The number of triangles in a triangulation is called its size.

In rather general two-dimensional spaces, like Alexandrov surfaces, two geodesics starting at the same point determine a well defined angle. Our interest will be

[^0]focused on triangulations which are acute (resp. non-obtuse), which means that the angles of all geodesic triangles are smaller (resp. not greater) than $\frac{\pi}{2}$.

The discussion of acute triangulations has one of its origins in a problem of Stover reported in 1960 by Gardner in his Mathematical Games section of the Scientific American (see [4], 5], 6]). There the question was raised whether a triangle with one obtuse angle can be cut into smaller triangles, all of them acute. In the same year, independently, Burago and Zalgaller [1] investigated in considerable depth acute triangulations of polygonal complexes, being led to them by the problem of their isometric embedding into $\mathbb{R}^{3}$. However, their method could not give an estimate on the number of triangles used in the existed acute triangulations. In 1980, Cassidy and Lord [2] considered acute triangulations of the square. Recently, acute triangulations of quadrilaterals [12], trapezoids [18], convex quadrilaterals [3], pentagons [16] and general polygons [11, 17] have also been considered.

On the other hand, compact convex surfaces have also been triangulated. Acute and non-obtuse triangulations of all Platonic surfaces, which are surfaces of the five well-known Platonic solids, have been investigated in [7], [9, and [10]. Recently, Saraf [15] considered the acute triangulations of the polyhedral surfaces again, but there is still no estimate on the size of the existed acute triangulations. Maehara [13] considered the proper acute triangulation of a polyhedral surface and obtained an upper bound of the size of the triangulation, which is determined by the length of the longest edge, the minimum value of the geodesic distance from a vertex to an edge that is not incident to the vertex, and the measure of the smallest face angle in the given polyhedral surface. Furthermore, some other well-known surfaces have also been acutely triangulated, such as flat Möbius strips [19] and flat tori [8].

Combining all the known results for the polyhedral surfaces mentioned above, we are motivated to investigate the non-obtuse and acute triangulations of the surfaces of the Archimedean solids. In this paper we consider the surface of the Archimedean
solid cuboctahedron, which is a convex polyhedron with eight triangular faces and six square faces. It has 12 identical vertices, with two triangles and two squares meeting at each, and 24 identical edges, each separating a triangle from a square. For the sake of convenience, let $\mathcal{C}$ denote the surface of the cuboctahedron with side length 1 . Let $\mathscr{T}$ denote an acute triangulation of $\mathcal{C}$ and $\mathscr{T}_{0}$ a non-obtuse triangulation of $\mathcal{C}$. Let $|\mathscr{T}|$ and $\left|\mathscr{T}_{0}\right|$ denote the size of $\mathscr{T}$ and $\mathscr{T}_{0}$ respectively. We prove that the best possible bounds for $|\mathscr{T}|$ and $\left|\mathscr{T}_{0}\right|$ are 12 and 8 respectively.

## 2 Non-obtuse triangulations

Theorem 2.1. The surface of the cuboctahedron admits a non-obtuse triangulation with 8 triangles and no non-obtuse triangulation with fewer triangles.

Proof. Fig. 1 describes the unfolded surface $\mathcal{C}$. We fix two vertices $a$ and $b$, which are the vertices of a diagonal of a square face on $\mathcal{C}$. Let $a^{\prime}, b^{\prime}$ be the antipodal vertices of $a, b$ respectively. There are six geodesics from $a$ to $a^{\prime}$ and $b$ to $b^{\prime}$. We


Figure 1: A non-obtuse triangulation of $\mathcal{C}$.
choose those two passing through two triangular faces and one square face. Denote the two intersection points of the geodesics $a a^{\prime}$ and $b b^{\prime}$ chosen above by $c$ and $c^{\prime}$. Clearly, $c$ and $c^{\prime}$ are an antipodal pair of vertices on $\mathcal{C}$. Draw the segments
from $a$ (resp. $a^{\prime}$ ) to $b$ and $b^{\prime}$. Thus $\mathcal{C}$ is triangulated into 8 non-obtuse triangles: $a b c, a b^{\prime} c, a b c^{\prime}, a b^{\prime} c^{\prime}, a^{\prime} b c, a^{\prime} b^{\prime} c, a^{\prime} b c^{\prime}, a^{\prime} b^{\prime} c^{\prime}$.

Indeed, noticing that all of those eight triangles are congruent, we only need to show that the triangle $a b c$ is non-obutse. By the construction we know that $a a^{\prime}$ is orthogonal to $b b^{\prime}$. So $\angle a c b=\frac{\pi}{2}$. Further, it is clear that $\angle a b c=\angle b a c=\frac{5 \pi}{12}$.

We prove now that for any non-obtuse triangulation $\mathscr{T}_{0}$ of $\mathcal{C}$, we always have $\left|\mathscr{T}_{0}\right| \geq 8$. If not, then we have $\left|\mathscr{T}_{0}\right|=4$ or $\left|\mathscr{T}_{0}\right|=6$. If $\left|\mathscr{T}_{0}\right|=4$, then $\mathscr{T}_{0}$ has $(4 \times 3) / 2=6$ edges and, by Euler's formula, $6-4+2=4$ vertices. So $\mathscr{T}_{0}$ is isomorphic to $K_{4} ;$ If $\left|\mathscr{T}_{0}\right|=6$, then $\mathscr{T}_{0}$ is isomorphic to the 1 -skeleton of the double pyramid over the triangle. In both cases there are vertices with degree 3. However, at each vertex of $\mathcal{C}$ the total angle is $\frac{5 \pi}{3}$, so each vertex in $\mathscr{T}_{0}$ has degree at least 4. Clearly, each other vertex of $\mathscr{T}_{0}$ also has degree at least 4 . Thus we obtain a contradiction.

The proof is complete.

## 3 Acute triangulations

Theorem 3.1. The surface of the cuboctahedron admits an acute triangulation with 12 triangles.

Proof. Let $a^{\prime}, b^{\prime}, c^{\prime}$ and $d^{\prime}$ be four distinct vertices of the cuboctahedron such that $\left|a^{\prime} b^{\prime}\right|=\left|b^{\prime} c^{\prime}\right|=\left|c^{\prime} d^{\prime}\right|=\left|d^{\prime} a^{\prime}\right|=\sqrt{2}$, where $|p q|$ denotes the intrinsic distance on the surface $\mathcal{C}$ between two points $p$ and $q$. Clearly, the four segments $a^{\prime} b^{\prime}, b^{\prime} c^{\prime}, c^{\prime} d^{\prime}$ and $d^{\prime} a^{\prime}$ determine a cycle which decomposes $\mathcal{C}$ into two regions $\mathcal{C}_{1}$ and $\mathcal{C}_{2}$. Take a vertex $a$ (resp. $b, c, d$ ) adjacent to both $a^{\prime}$ (resp. $b^{\prime}, c^{\prime}, d^{\prime}$ ) and $b^{\prime}$ (resp. $c^{\prime}, d^{\prime}, a^{\prime}$ ) such that $a, c \in \mathcal{C}_{1}, b, d \in \mathcal{C}_{2}$. Take a point $a^{*}\left(\right.$ resp. $\left.b^{*}, c^{*}, d^{*}\right)$ on $a^{\prime} b^{\prime}\left(\right.$ resp. $\left.b^{\prime} c^{\prime}, c^{\prime} d^{\prime}, d^{\prime} a^{\prime}\right)$ such that $\angle a^{\prime} a a^{*}\left(\right.$ resp. $\left.\angle b^{\prime} b b^{*}, \angle c^{\prime} c c^{*}, \angle d^{\prime} d d^{*}\right)=\frac{\pi}{6}$.

We get a triangulation of $\mathcal{C}$ with 12 triangles:
$a^{*} a b^{*}, a^{*} b^{*} b, a^{*} b d, a^{*} d d^{*}, a^{*} d^{*} a, b^{*} b c^{*}, b^{*} c^{*} c, b^{*} c a, c^{*} c d^{*}, c^{*} d^{*} d, c^{*} d b, d^{*} c a$.


Figure 2: An acute triangulation of $\mathcal{C}$.

There are two shortest paths from $a^{*}$ to $b^{*}$ (resp. $c^{*}$ to $d^{*}$ ); here we choose the path in $\mathcal{C}_{2}$. There are two shortest paths from $b^{*}$ to $c^{*}$ (resp. $d^{*}$ to $a^{*}$ ); here we choose the path in $\mathcal{C}_{1}$, see Fig. 2.

Indeed, the values of the angles around $a^{*}, b^{*}, c^{*}, d^{*}$ (resp. $a, b, c, d$ ) are entirely the same. So we only need to consider the angles around $a^{*}$ and $a$ respectively.

Firstly, we consider the angles around $a^{*}$.
In the triangle $a^{*} b^{\prime} b, \angle a^{*} b^{\prime} b=\frac{\pi}{4}+\frac{\pi}{3}=\frac{7 \pi}{12}>\frac{\pi}{2}$, which implies that $\angle b^{*} a^{*} b<$ $\angle b^{\prime} a^{*} b<\frac{\pi}{2}$.

In Fig. 2 the planar circle $C$ with diameter $b d$ (the dot line-segment) passes through the midpoint of $a^{\prime} b^{\prime}$, say, $x^{\prime}$. So, the segment $a^{\prime} b^{\prime}$ is tangent to $C$ at $x^{\prime}$. Since $a^{*} \in a^{\prime} b^{\prime}$ and $a^{*} \neq x^{\prime}$, we have $\angle b a^{*} d<\frac{\pi}{2}$.

In the quadrilateral $a^{*} d a^{\prime} d^{*}, \angle d a^{\prime} d^{*}=\frac{\pi}{3}+\frac{\pi}{2}+\frac{\pi}{3}+\frac{\pi}{4}=\frac{17 \pi}{12}$, which implies that $\angle a^{*} d^{*} a^{\prime}+\angle d^{*} a^{*} d+\angle a^{*} d a^{\prime}=\frac{7 \pi}{12}$. However, $\angle a^{*} d a^{\prime}>\angle a d a^{\prime}=\frac{\pi}{12}$. Therefore, $\angle d^{*} a^{*} d<\angle d^{*} a^{*} d+\angle a^{*} d a^{\prime}<\frac{7 \pi}{12}-\frac{\pi}{12}=\frac{\pi}{2}$.

Denote by $\alpha$ the vertex adjacent to both $a$ and $a^{\prime}$. Take a point $\alpha^{*} \in d^{\prime} a^{\prime}$ such that $\angle \alpha^{*} \alpha a^{\prime}=\frac{\pi}{6}$. Clearly, the triangle $a a^{*} a^{\prime}$ is congruent to the triangle $\alpha \alpha^{*} a^{\prime}$,
so we have $\left|\alpha^{*} a^{\prime}\right|=\left|a^{*} a^{\prime}\right|$, which implies that $\angle a^{\prime} a^{*} \alpha^{*}=\frac{\pi}{12}$. Then $\angle \alpha^{*} a^{*} a=$ $\left(\pi-\frac{\pi}{6}-\frac{\pi}{4}\right)-\frac{\pi}{12}=\frac{\pi}{2}$. Noticing that the distance from $d^{*}$ to $a^{\prime}$ is further than that from $\alpha^{*}$ to $a^{\prime}$, we have $\angle d^{*} a^{*} a<\angle \alpha^{*} a^{*} a=\frac{\pi}{2}$.

In the triangle $a a^{*} b^{\prime}, \angle a a^{*} b^{\prime}=\pi-\frac{\pi}{3}-\frac{\pi}{4}=\frac{5 \pi}{12}$. In the triangle $a^{*} b^{\prime} b^{*}, \angle a^{*} b^{\prime} b^{*}=$ $\frac{\pi}{4}+\frac{\pi}{3}+\frac{\pi}{4}=\frac{5 \pi}{6}$. Noticing that $\left|b^{\prime} b^{*}\right|<\frac{1}{2}\left|b^{\prime} c^{\prime}\right|=\frac{1}{2}\left|a^{\prime} b^{\prime}\right|<\left|a^{*} b^{\prime}\right|$, we have $\angle b^{\prime} a^{*} b^{*}<$ $\angle b^{\prime} b^{*} a^{*}$ and therefore $\angle b^{\prime} a^{*} b^{*}<\frac{\pi}{12}$. So we have $\angle a a^{*} b^{*}<\frac{5 \pi}{12}+\frac{\pi}{12}=\frac{\pi}{2}$.

Consider now the angles around $a$. It is clear that $\angle a^{*} a d^{*}<\angle a^{*} a \alpha=\frac{\pi}{2}$ and $\angle a^{*} a b^{*}=\angle a^{*} a b^{\prime}+\angle b^{\prime} a b^{*}<\frac{\pi}{3}+\frac{\pi}{6}=\frac{\pi}{2}$. Let $\gamma$ denote the vertex adjacent to both $c$ and $c^{\prime}$. Then $\angle b^{*} a c=\angle \gamma a c+\angle b^{*} a \gamma<\angle \gamma a c+\angle b a \gamma=\frac{\pi}{4}+\frac{\pi}{4}=\frac{\pi}{2}$. Finally, $\angle d^{*} a c=\angle d^{*} a \alpha+\angle \alpha a c<\frac{\pi}{6}+\frac{\pi}{4}<\frac{\pi}{2}$.

## 4 No acute triangulation with fewer triangles

Let $\mathscr{C}$ be the 1 -skeleton of the cuboctahedron. The graph-theoretic distance $d_{\mathscr{C}}(v, w)$ between the vertices $v, w$ of $\mathscr{C}$ is called the $\mathscr{C}$-distance between $v$ and $w$. Let $g(u, v)$ denote a geodesic between two points $u$ and $v$ on the surface $\mathcal{C}$. We start with the following lemma.

Lemma 4.1. Let $u, w_{1}, w_{2}$ be three vertices of $\mathcal{C}$. Then the angle formed by $g\left(u, w_{1}\right)$ and $g\left(u, w_{2}\right)$ on $\mathcal{C}$ is equal to $\frac{\pi}{12} i$, where $i \in \mathbb{Z}$ and $1 \leq i \leq 20$.

Proof. For any vertex $u$ of $\mathcal{C}$, consider all the segments from $u$ to any other vertex $v$ (see Fig. (3). It is easy to see that $d_{\mathscr{C}}(u, v) \leq 3$. If $d_{\mathscr{C}}(u, v)=1$, then $v \in\left\{a_{1}, a_{2}, a_{3}, a_{4}\right\}$. Clearly $g(u, v)$ is an edge of $\mathcal{C}$. If $d_{\mathscr{C}}(u, v)=2$, then $v \in$ $\left\{b_{1}, b_{2}, b_{3}, b_{4}, b_{5}, b_{6}\right\}$. Further, if $v=b_{2}$ or $v=b_{5}$, then $g(u, v)$ is a diagonal of a square face on the surface; if $v \in\left\{b_{1}, b_{3}, b_{4}, b_{6}\right\}$, then there are two geodesics between $u$ and $v$. If $d_{\mathscr{C}}(u, v)=3$, then $v=c$ and there are six geodesics between $u$ and $v$. Please note that the solid line between $u$ and $v$ in Fig. 3 is not a geodesic. Thus there are 20 geodesics starting from $u$ to any other vertex $v$ on $\mathcal{C}$, and all of


Figure 3: Geodesics starting from a vertex $u$.
them divide the total angle around $u$ into 20 equal parts. Trivially, each part has angle $\frac{\pi}{12}$. Thus the angle formed by $g\left(u, w_{1}\right)$ and $g\left(u, w_{2}\right)$ on $\mathcal{C}$ is equal to $\frac{\pi}{12} i$, where $i \in \mathbb{Z}$ and $1 \leq i \leq 20$.

Lemma 4.2. There is no acute triangulation of $\mathcal{C}$ with 8 triangles.

Proof. Suppose there exists an acute triangulation $\mathscr{T}$ of $\mathcal{C}$ containing 8 triangles. By a method similar to that used in the proof of Theorem [2.1, we know that $\mathscr{T}$ is isomorphic to the 1-skeleton of the regular octahedron, where all the vertices have degree 4 . Clearly each vertex of $\mathscr{T}$ is a vertex of $\mathcal{C}$. By Lemma 4.1, it is easily seen that any acute angle in $\mathscr{T}$ is $\frac{\pi}{12} i$, where $i=1,2,3,4,5$. Recall that the total angle at any vertex of $\mathcal{C}$ is $\frac{5 \pi}{3}$. Therefore the four angles around each vertex of $\mathscr{T}$ are all isogonal and equal to $\frac{5 \pi}{12}$.

Now let $v_{1}, v_{2}$ be two adjacent vertices in $\mathscr{T}$. Then in both of the triangles having side $g\left(v_{1}, v_{2}\right)$, all the three angles are equal to $\frac{5 \pi}{12}$. In the following we show that in one of them, where the third vertex is denoted by $v_{3}$, there is always a contradiction.

There are three cases to consider.
Case 1. $d_{\mathscr{C}}\left(v_{1}, v_{2}\right)=1$.
If $\angle v_{3} v_{1} v_{2}=\angle v_{3} v_{2} v_{1}=\frac{5 \pi}{12}$, then clearly we have $\angle v_{1} v_{3} v_{2}=\frac{\pi}{6} \neq \frac{5 \pi}{12}$, a contradic-


Figure 4: $\quad d_{\mathscr{C}}\left(v_{1}, v_{2}\right)=1$.


Figure 5: $\quad d_{\mathscr{C}}\left(v_{1}, v_{2}\right)=2$.
tion, as shown in Fig. (4)
Case 2. $d_{\mathscr{G}}\left(v_{1}, v_{2}\right)=2$.
If $g\left(v_{1}, v_{2}\right)$ is a diagonal of a square face of $\mathcal{C}$, then $\angle v_{3} v_{1} v_{2}=\angle v_{3} v_{2} v_{1}=\frac{5 \pi}{12}$ forces $v_{3}$ not to be a vertex of $\mathcal{C}$ (see Fig. $5(a)$ ), a contradiction. Otherwise, by the proof of Lemma 4.1 we may assume that $v_{1}$ is a corner of a square face and $g\left(v_{1}, v_{2}\right)$ intersects the interior of the square face, as show in Fig. 5(b). Let $v_{3}$ be another corner of the square face such that $\angle v_{3} v_{1} v_{2}=\frac{5 \pi}{12}$. By the proof of Case1, we know that $\angle v_{1} v_{2} v_{3}=\frac{\pi}{6}$, a contradiction again.

Case 3. $d_{\mathscr{C}}\left(v_{1}, v_{2}\right)=3$.
If $g\left(v_{1}, v_{2}\right)$ passes through two triangular faces and one square face, then we consider the triangle $v_{1} v_{2} v_{3}$ lying above $g\left(v_{1}, v_{2}\right)$, as shown in Fig. 6(a). Clearly, $\angle v_{1} v_{2} v_{3}=\angle v_{2} v_{1} v_{3}=\frac{5 \pi}{12}$, but $\angle v_{1} v_{3} v_{2}=\frac{5 \pi}{6}$, a contradiction. If $g\left(v_{1}, v_{2}\right)$ passes through one triangular face and two square faces, then we consider the triangle $v_{1} v_{2} v_{3}$ lying below $g\left(v_{1}, v_{2}\right)$, as shown in Fig. $6(b)$. It is easy to see that $\angle v_{1} v_{2} v_{3}=$


Figure 6: $\quad d_{\mathscr{C}}\left(v_{1}, v_{2}\right)=3$.
$\angle v_{2} v_{1} v_{3}=\frac{5 \pi}{12}$ makes $v_{3}$ not be a vertex of $\mathcal{C}$, which contradicts to the fact that each vertex of $\mathscr{T}$ must be a vertex of $\mathcal{C}$.

The proof is complete.

Lemma 4.3. There is no acute triangulation of $\mathcal{C}$ with 10 triangles.

Proof. Suppose that there exists an acute triangulation $\mathscr{T}$ of $\mathcal{C}$ containing 10 triangles. Then $\mathscr{T}$ is isomorphic to the 1 -skeleton of the double pyramid over the pentagon. So $\mathscr{T}$ contains a 5 -cycle $C_{5}$ and all its vertices have degree 4. Clearly, the vertices of $C_{5}$ must be the vertices of $\mathcal{C}$. For the sake of convenience, let $V\left(C_{5}\right)$ denote the set of all vertices of $C_{5}$ and $E\left(C_{5}\right)$ denote the set of all edges of $C_{5}$. Furthermore, we have the following fact.

Fact. The angles formed by any two adjacent edges of $C_{5}$ are between $\frac{2 \pi}{3}$ and $\pi$.

If $u, v$ are two adjacent vertices of $\mathcal{C}$, we call $u, v$ an adjacent pair of $\mathcal{C}$. In order to prove Lemma 4.3, we prove the following properties about the cycle $C_{5}$ mentioned above at first.

Proposition 4.4. $V\left(C_{5}\right)$ contains at least two adjacent pairs of $\mathcal{C}$.

Proof. We first show that $V\left(C_{5}\right)$ contains at least one adjacent pair of $\mathcal{C}$. Suppose that $u \in V\left(C_{5}\right)$, as shown in Fig. 3. If $\left\{a_{1}, a_{2}, a_{3}, a_{4}\right\} \cap V\left(C_{5}\right) \neq \emptyset$, then clearly $V\left(C_{5}\right)$ contains an adjacent pair. If not, then the other four vertices of $C_{5}$ come
from $b_{1}, b_{2}, b_{3}, b_{4}, b_{5}, b_{6}$ and $c$. It is easy to see that among those four vertices there must be at least one adjacent pair of $\mathcal{C}$.

Now let $v_{1}, v_{2} \in V\left(C_{5}\right)$ be an adjacent pair of $\mathcal{C}$, as shown in Fig. 7. If $\left\{u_{1}, u_{2}, u_{3}, u_{4}, u_{5}\right\} \cap V\left(C_{5}\right) \neq \emptyset$, then clearly $V\left(C_{5}\right)$ contains another adjacent pair of $\mathcal{C}$ and the proposition is proved. Otherwise, the other three vertices of $C_{5}$ come from the remained five vertices $\bar{u}_{1}, \bar{u}_{2}, \bar{u}_{3}, \bar{u}_{4}$ and $\bar{u}_{5}$ of $\mathcal{C}$. It's not hard to see that among any three vertices from $\bar{u}_{1}, \bar{u}_{2}, \bar{u}_{3}, \bar{u}_{4}$ and $\bar{u}_{5}$ there must be one adjacent pair of $\mathcal{C}$.

Proposition 4.5. Let $v_{i}, v_{j} \in V\left(C_{5}\right)$.
(a) If $d_{\mathscr{C}}\left(v_{i}, v_{j}\right)=1$, then $g\left(v_{i}, v_{j}\right) \in E\left(C_{5}\right)$;
(b) If $d_{\mathscr{C}}\left(v_{i}, v_{j}\right)=3$, then $g\left(v_{i}, v_{j}\right) \notin E\left(C_{5}\right)$;
(c) If $d_{\mathscr{C}}\left(v_{i}, v_{j}\right)=2$ and $g\left(v_{i}, v_{j}\right)$ is a diagonal of a square face of $\mathcal{C}$, then $g\left(v_{i}, v_{j}\right) \notin E\left(C_{5}\right)$.

Proof. (a) We fix two vertices $v_{1}, v_{2}$ such that $v_{1}, v_{2} \in V\left(C_{5}\right)$ and $d_{\mathscr{C}}\left(v_{1}, v_{2}\right)=1$. Suppose the contrary that $g\left(v_{1}, v_{2}\right)=v_{1} v_{2} \notin E\left(C_{5}\right)$. Then there is a vertex of $\mathcal{C}$, say $u$, such that it is adjacent to both $v_{1}$ and $v_{2}$ in $C_{5}$. Let the five neighbors of $v_{1}, v_{2}$ in $\mathcal{C}$ be $u_{i}, i=1,2,3,4,5$, as shown in Fig. 7. By the Fact it is easy to see that $u \neq u_{i}(i=1,2,3,4,5)$. Now denote the five remained vertices of $\mathcal{C}$ by $\bar{u}_{i}$ $(i=1,2,3,4,5)$. For the sake of convenience, let $\eta_{i}$ denote the value of the smaller angle formed by $v_{1}, \bar{u}_{i}$ and $v_{2}$ on $\mathcal{C}$. Since $\eta_{1} \leq \frac{\pi}{4}+\frac{\pi}{12}=\frac{\pi}{3}, \eta_{3} \leq \frac{\pi}{12}+\frac{\pi}{3}+\frac{\pi}{12}=\frac{\pi}{2}$, $\eta_{5} \leq \frac{\pi}{4}+\frac{\pi}{12}=\frac{\pi}{3}$, by the Fact we have $u \notin\left\{\bar{u}_{1}, \bar{u}_{3}, \bar{u}_{5}\right\}$. Noticing that $\eta_{2} \leq \frac{\pi}{3}+\frac{\pi}{12}+\frac{\pi}{3}=$ $\frac{\pi}{6}+\frac{\pi}{2}+\frac{\pi}{12}=\frac{3 \pi}{4}$, we may assume that $u=\bar{u}_{2}$. Then by the Fact we have $\frac{2 \pi}{3}<\eta_{2}<\pi$, and therefore $\eta_{2}=\frac{3 \pi}{4}$ (by Lemma 4.1). Now let $v_{1}^{\prime}$ be the other adjacent vertex of $v_{1}$ in $C_{5}$. In order to ensure $\frac{2 \pi}{3}<\angle \bar{u}_{2} v_{1} v_{1}^{\prime}<\pi$, that is, $\angle \bar{u}_{2} v_{1} v_{1}^{\prime} \in\left\{\frac{3 \pi}{4}, \frac{5 \pi}{6}, \frac{11 \pi}{12}\right\}$, it is easy to check that $g\left(v_{1}, v_{1}^{\prime}\right)$ always intersects $g\left(v_{2}, \bar{u}_{2}\right)$, a contradiction. Similarly, if $u=\bar{u}_{4}$ we also obtain a contradiction.


Figure 7: An adjacent pair of $\mathcal{C}$.
(b) Without loss of generality, we assume that $v_{i}=u, v_{j}=c$, as shown in Fig. 3. Now suppose the contrary that $g\left(v_{i}, v_{j}\right)=g(u, c) \in E\left(C_{5}\right)$. For the sake of convenience, denote by $u^{\prime}$ the other adjacent vertex of $u$ in $C_{5}$. Since $\frac{2 \pi}{3}<\angle u^{\prime} u c<\pi$, by Lemma 4.1, we have $\angle u^{\prime} u c \in\left\{\frac{3 \pi}{4}, \frac{5 \pi}{6}, \frac{11 \pi}{12}\right\}$. There are two cases to consider.

Case 1. $g(u, c)$ passes through two triangular faces and one square face.
We consider the rightmost geodesic $g(u, c)$ in Fig. 3. If $\angle u^{\prime} u c=\frac{3 \pi}{4}$ or $\angle u^{\prime} u c=$ $\frac{11 \pi}{12}$, then $u^{\prime}=b_{3}$ or $u^{\prime}=b_{4}$. Clearly, $d_{\mathscr{C}}\left(b_{3}, c\right)=d_{\mathscr{C}}\left(b_{4}, c\right)=1$. By $(a)$, we have $g\left(b_{3}, c\right) \in E\left(C_{5}\right)$ or $g\left(b_{3}, c\right) \in E\left(C_{5}\right)$. Thus we obtain a 3 -cycle $u c b_{3} u$ or $u c b_{4} u$, a contradiction. If $\angle u^{\prime} u c=\frac{5 \pi}{6}$, then $u^{\prime}=c$. We obtain a 2 -cycle $u c u$, a contradiction again.

Case 2. $g(u, c)$ passes through one triangular face and two square faces.
We consider the second leftmost geodesic $g(u, c)$ (the vertical one) in Fig. 3. If $\angle u^{\prime} u c=\frac{5 \pi}{6}$, then $u^{\prime}=c$ and we obtain a 2 -cycle $u c u$, a contradiction. If $\angle u^{\prime} u c \in$ $\left\{\frac{3 \pi}{4}, \frac{11 \pi}{12}\right\}$, then $u^{\prime}=b_{5}$ or $u^{\prime}=b_{6}$. If $u^{\prime}=b_{6}$, by $(a)$, we obtain a contradiction; If $u^{\prime}=b_{5}$, let $u^{\prime \prime}$ be the other adjacent vertex of $u^{\prime}$ in $C_{5}$. Thus $\angle u u^{\prime} u^{\prime \prime} \in\left\{\frac{3 \pi}{4}, \frac{5 \pi}{6}, \frac{11 \pi}{12}\right\}$. If $\angle u u^{\prime} u^{\prime \prime}=\frac{3 \pi}{4}$, then $u^{\prime \prime}=b_{2}$ and the geodesic $g\left(b_{5}, b_{2}\right)$ in $E\left(C_{5}\right)$ must pass through the faces $\alpha_{1}, \alpha_{2}, \alpha_{3}$, which intersects $g(u, c)$ in its interior. This is impossible in $C_{5}$. If $\angle u u^{\prime} u^{\prime \prime}=\frac{5 \pi}{6}$, then $u^{\prime \prime}=c$ and we obtain a 3 -cycle $u b_{5} c u$, a contradiction again.

If $\angle u u^{\prime} u^{\prime \prime}=\frac{11 \pi}{12}$, then $u^{\prime \prime}=b_{2}$. Let $u^{\prime \prime \prime}$ be the other adjacent vertex of $u^{\prime \prime}$ in $C_{5}$. We know that $\angle u^{\prime} u^{\prime \prime} u^{\prime \prime \prime} \in\left\{\frac{3 \pi}{4}, \frac{5 \pi}{6}, \frac{11 \pi}{12}\right\}$ which implies that $u^{\prime \prime \prime} \in\left\{a_{3}, b_{5}, u\right\}$. Clearly, $u^{\prime \prime \prime} \neq b_{5}$ and $u^{\prime \prime \prime} \neq u$. If $u^{\prime \prime \prime}=a_{3}$, then $g\left(b_{2}, a_{3}\right)$ in $E\left(C_{5}\right)$ intersects $g(u, c)$ in its interior, which is a contradiction.
(c) Without loss of generality, let $v_{i}=u_{1}, v_{j}=u_{2}$ and $g\left(u_{1}, u_{2}\right)$ is a diagonal of a square face of $\mathcal{C}$, as shown in Fig. 7. Now suppose the contrary that $g\left(u_{1}, u_{2}\right) \in$ $E\left(C_{5}\right)$. Let $u_{1}^{\prime}$ be the other adjacent vertex of $u_{1}$ in $C_{5}$. By the Fact and Lemma 4.1, we have $\angle u_{1}^{\prime} u_{1} u_{2} \in\left\{\frac{3 \pi}{4}, \frac{5 \pi}{6}, \frac{11 \pi}{12}\right\}$. If $\angle u_{1}^{\prime} u_{1} u_{2} \in\left\{\frac{3 \pi}{4}, \frac{11 \pi}{12}\right\}$, then $u_{1}^{\prime}=\bar{u}_{3}$. Clearly, $d_{\mathscr{C}}\left(u_{1}, \bar{u}_{3}\right)=3$, which contradicts to $(b)$. If $\angle u_{1}^{\prime} u_{1} u_{2}=\frac{5 \pi}{6}$, then $u_{1}^{\prime}=u_{5}$. Let $u^{\prime \prime}$ be the other adjacent vertex of $u_{1}^{\prime}$ in $C_{5}$. Noticing that $d_{\mathscr{C}}\left(u_{1}, u_{5}\right)=2$ and $g\left(u_{1}, u_{5}\right)$ is a diagonal of a square face of $\mathcal{C}$, by the above discussion we know that if $\angle u_{1} u_{5} u_{1}^{\prime \prime} \in\left\{\frac{3 \pi}{4}, \frac{11 \pi}{12}\right\}$, then there is a contradiction; if $\angle u_{1} u_{5} u_{1}^{\prime \prime}=\frac{5 \pi}{6}$, then $u_{1}^{\prime \prime}=\bar{u}_{3}$. Repeating the above process again and we obtain a 4 -cycle $u_{1} u_{5} \bar{u}_{3} u_{2} u_{1}$, which is a contradiction.

Proposition 4.6. $C_{5}$ has only one possible configuration as shown in Fig. 8.


Figure 8: The 5-cycle $C_{5}$.

Proof. Denote the vertices of $C_{5}$ by $v_{i}(i=1,2,3,4,5)$, and $v_{i}, v_{i+1}$ are adjacent in $C_{5}$ (i+1 takes modulo 5). By Proposition 4.4 and 4.5, there are two edges of $C_{5}$,
say $e_{1}$ and $e_{2}$, which are edges of $\mathcal{C}$. There are two cases to consider.
Case 1. $e_{1}$ and $e_{2}$ are adjacent in $\mathcal{C}$.
Suppose that $e_{1}=v_{1} v_{2}$ and $e_{2}=v_{2} v_{3}$. Without loss of generality, we may assume that $v_{1}=a_{1}, v_{2}=u$, as shown in Fig. 3. Please keep in mind that any angle formed by two adjacent edges of $C_{5}$ is between $\frac{2 \pi}{3}$ and $\pi$. Thus we have $v_{3}=a_{3}$. By Proposition 4.5, we know that $d_{\mathscr{C}}\left(v_{3}, v_{4}\right)=1$, or $d_{\mathscr{C}}\left(v_{3}, v_{4}\right)=2$ and $g\left(v_{3}, v_{4}\right)$ passes through one triangular face and one square face, which implies that $v_{4}=b_{4}$ or $c$.

If $v_{4}=b_{4}$ and $d_{\mathscr{C}}\left(v_{4}, v_{5}\right)=1$, then $v_{5}=c$ and we obtain a 5 -cycle $a_{1} u a_{3} b_{4} c a_{1}$, as the configuration described in Fig. 8, If $v_{4}=b_{4}$ and $d_{\mathscr{C}}\left(v_{4}, v_{5}\right)=2$, then $v_{5}=b_{1}$. Since $d_{\mathscr{C}}\left(v_{5}, v_{1}\right)=1$, by Proposition 4.5 (a), $g\left(v_{5}, v_{1}\right)=v_{5} v_{1} \in E\left(C_{5}\right)$. Thus we obtain a 5 -cycle $a_{1} u a_{3} b_{4} b_{1} a_{1}$. If $v_{4}=c$ and $d_{\mathscr{C}}\left(v_{4}, v_{5}\right)=1$, then $v_{5}=b_{1}$, we obtain a 5 -cycle $a_{1} u a_{3} c b_{1} a_{1}$. If $v_{4}=c$ and $d_{\mathscr{C}}\left(v_{4}, v_{5}\right)=2$, then $v_{5}=a_{1}$, which is a contradiction.

Case 2. $e_{1}$ and $e_{2}$ are not adjacent in $\mathcal{C}$.
Suppose that $e_{1}=v_{1} v_{2}$ and $e_{2}=v_{3} v_{4}$. Without loss of generality, we may assume that $v_{1}=a_{1}, v_{2}=u$, as shown in Fig. 3. By Proposition 4.5, we know that $d_{\mathscr{C}}\left(v_{2}, v_{3}\right)=1$, or $d_{\mathscr{C}}\left(v_{2}, v_{3}\right)=2$ and $g\left(v_{2}, v_{3}\right)$ passes through one triangular face and one square face. That is, $v_{3}=a_{3}$ or $b_{4}$. If $v_{3}=a_{3}$, then the discussion is same to that in Case 1. If $v_{3}=b_{4}$, then clearly $v_{4}=c$. This situation has been discussed in Case 1.

Now we are back to the proof of Lemma 4.3. Clearly, a 5 -cycle $C_{5}$ described above decomposes $\mathcal{C}$ into two regions, and one of them is shown in Fig. 8. Without loss of generality, let $v_{6}$ be the vertex of the acute triangulation $\mathscr{T}$ lying in this region. Since $\angle v_{6} v_{2} v_{1}<\frac{\pi}{2}$, $v_{6}$ can not lie in the triangular face $a v_{2} v_{3}$ and the square face $a v_{3} v_{4} c$ except for the edge $a c$. Further, since $\angle v_{6} v_{1} v_{2}<\frac{\pi}{2}, \angle v_{6} v_{3} v_{4}<\frac{\pi}{2}$ and $\angle v_{6} v_{4} v_{3}<\frac{\pi}{2}, v_{6}$ must lie in the triangular face $a b c$. Clearly, $v_{6} \notin\{a, b, c\}$ and the edge $g\left(v_{1}, v_{6}\right)$ of $\mathscr{T}$ must intersect the square face $v_{1} b a v_{2}$. In Fig. 8, let $C$
be the planar circle with diameter $v_{1} v_{5}$ (here $v_{1} v_{5}$ is the dash segment instead of the geodesic). It is easy to see that $v_{6}$ lies in the interior of the upper semi-disc bounded by $C$ and the dash segment $v_{1} v_{5}$. As a result, we have $\angle v_{1} v_{6} v_{5}>\frac{\pi}{2}$, which contradicts to the fact that $\mathscr{T}$ is an acute triangulation. Therefore, there is no acute triangulation of $\mathcal{C}$ with ten triangles.

Combining Theorem 2.1, 3.1, Lemma 4.2, 4.3, we obtain the following main theorem immediately.

Theorem 4.7. The surface of the cuboctahedron admits an acute triangulation with 12 triangles, and there is no acute triangulation with fewer triangles.

Acknowledgements. The second author gratefully acknowledges financial supports by NSF of China (10701033, 10426013); program for New Century Excellent Talents in University, Ministry of Education of China; the Plan of Prominent Personnel Selection and Training for the Higher Education Disciplines in Hebei Province; and WUS Germany (Nr. 2161). She is also indebt to Beijing University for the financial support during her academic visit there.

## References

[1] Y. D. Burago, V. A. Zalgaller, Polyhedral embedding of a net (Russian), Vestnik Leningrad. Univ. 15 (1960) 66-80.
[2] C. Cassidy and G. Lord, A square acutely triangulated, J. Recr. Math. 13 (1980/81) 263-268.
[3] M. Cavicchioli, Acute triangulations of convex quadrilaterals, Discrete Appl. Math., 160 (2012) 1253-1256.
[4] M. Gardner, Mathematical games, A fifth collection of "brain-teasers", Sci. Amer. 202 (2) (1960) 150-154.
[5] M. Gardner, Mathematical games, The games and puzzles of Lewis Carroll, and the answers to February's problems, Sci. Amer. 202 (3) (1960) 172-182.
[6] M. Gardner, New Mathematical Diversions, Mathematical Association of America, Washington D.C., 1995.
[7] T. Hangan, J. Itoh and T. Zamfirescu, Acute triangulations, Bull. Math. Soc. Sci. Math. Roumanie 43 (91) No. 3-4 (2000) 279-285.
[8] J. Itoh, L. Yuan, Acute triangulations of flat tori, Europ. J. Comb. 30 (2009) 1-4.
[9] J. Itoh, T. Zamfirescu, Acute triangulations of the regular dodecahedral surface Europ. J. Comb. 28 (2007) 1072-1086.
[10] J. Itoh, T. Zamfirescu, Acute triangulations of the regular icosahedral surface, Discrete Comput. Geom. 31 (2004) 197-206.
[11] H. Maehara, Acute triangulations of polygons, Europ. J. Comb. 23 (2002) 4555.
[12] H. Maehara, On acute triangulations of quadrilaterals, Proc. JCDCG 2000; Lecture Notes Comp. Sci. 2098 (2001) 237-354.
[13] H. Maehara, On a proper acute triangulation of a polyhedral surface, Discrete Math., 311(17) (2011), 1903-1909.
[14] W. Manheimer, Solution to Problem E1406: Dissecting an obtuse triangle into acute triangles, Amer. Math. Monthly 67 (1960) 923.
[15] S. Saraf, Acute and nonobtuse triangulations of polyhedral surfaces, Europ. J. Comb. 30 (2009) 833-840.
[16] L. Yuan, Acute triangulations of pentagons, Bull. Math. Soc. Sci. Math. Roumanie, 53(101) (2010) 393-410.
[17] L. Yuan, Acute triangulations of polygons, Discrete Comput. Geom. 34 (2005) 697-706.
[18] L. Yuan, Acute triangulations of trapezoids, Discrete Appl. Math., 158 (2010) 1121-1125.
[19] L. Yuan, T. Zamfirescu, Acute triangulations of Flat Möbius strips, Discrete Comput. Geom. 37 (2007) 671-676.


[^0]:    *Corresponding author

