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Hamiltonian orthogeodesic alternating paths

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ABSTRACT

Let R be a set of red points and let B be a set of blue points. The point set $P = R \cup B$ is called *equitable* if $||B| - |R|| \leqslant 1$ and it is called *general* if no two points are vertically or horizontally aligned. An *orthogeodesic alternating path* on P is a path such that each edge is an orthogeodesic chain connecting points of different color and such that no two edges cross. We consider the problem of deciding whether a set of red and blue points admits a *Hamiltonian* orthogeodesic alternating path, that is, an orthogeodesic alternating path visiting all points. We prove that every general equitable point set admits a Hamiltonian orthogeodesic alternating path and we present an $\mathcal{O}(n \log^2 n)$-time algorithm for finding such a path, where n is the number of points. On the other hand, we show that the problem is $\mathcal{NP}$-complete if the path must be on the grid (i.e., vertices and bends have integer coordinates). Further, we show that we can approximate the maximum length of an orthogeodesic alternating path on the grid by a factor of 3, whereas we present a family of point sets with n points that do not have a Hamiltonian orthogeodesic alternating path with more than $n/2 + 2$ points. Additionally, we show that it is $\mathcal{NP}$-complete to decide whether a given set of red and blue points on the grid admits an orthogeodesic perfect matching if horizontally aligned points are allowed. This contrasts a recent result by Kano (2009) [9] who showed that this is possible on every general point set.

1. Introduction

Let R be a set of red points and let B be a set of blue points such that $|R| \leqslant |B|$. The point set $P = R \cup B$ is called *equitable* if $|B| - |R| \leqslant 1$ and it is called *balanced* if $|B| = |R|$. The color of a point $p \in P$ is denoted by $c(p)$. An *alternating path* on a set of red and blue points P is a sequence of points $p_1, \dots, p_h$ that is alternatingly colored red and blue, such that p_i is connected to p_{i+1} $(i = 1, \dots, h-1)$ by a Jordan arc. Throughout this paper we only consider the case when the curves corresponding to the edges of the path can only intersect at common endpoints. An *orthogeodesic chain* is a polygonal chain consisting of horizontal and vertical straight-line segments whose total length is equal to the L_1-distance between its endpoints. If the curves representing the edges of a given alternating path are straight-line segments, then the path is called a *straight-line alternating path*; if the curves are orthogeodesic chains, then the path is called an *orthogeodesic alternating path*. Given a set of points P, an alternating path is called *Hamiltonian* if it contains all points in P. Clearly a Hamiltonian alternating path can only exist on an equitable point set. A point set P is called *general* if no two points of P are vertically or horizontally aligned. In this paper, we study combinatorial and algorithmic aspects of Hamiltonian orthogeodesic alternating paths on general point sets.

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1.1. Previous work

The problem of computing a Hamiltonian straight-line alternating path on a given equitable set of points in general position is a classical subject of investigation. Akiyama and Urrutia [3] concentrate on point sets in convex position. They show that it is not always possible to compute a Hamiltonian alternating path on a given equitable point set in convex position and present an $\mathcal{O}(n^2)$-time algorithm that computes a Hamiltonian alternating path if it exists. Abellanas et al. [2] study the case when points are not restricted to be in convex position. They prove that if either the convex hull of P consists of all the red points and no blue points or if the two point sets are linearly separable, that is, if there exists a straight line that separates the red points from the blue points, then a Hamiltonian straight-line alternating path can always be found. Kaneko et al. [7] study the values of n for which every equitable set of n points admits a Hamiltonian alternating path and proved that this happens only for $n \leqslant 12$ and $n = 14$. For any other value of n there exist equitable point sets that do not admit a Hamiltonian alternating path. On the other hand, Cibulka et al. [5] describe arbitrarily large equitable point sets that admit a Hamiltonian straight-line alternating path for every coloring of the points. Non-Hamiltonian alternating paths have also been considered. Abellanas et al. [1] and Kynčl et al. [11] study the values $\ell(n)$ of the length of a longest straight-line alternating path on sets of red and blue points in convex position and provide upper and lower bounds on $\ell(n)$.

Similar problems have been studied for graph families other than paths. Abellanas et al. [2] investigate *alternating spanning trees*, that is, spanning trees on red and blue point sets such that no two edges cross and each edge is a straight-line segment connecting points of different colors. They prove that every point set $P = R \cup B$ admits an alternating spanning tree whose maximum vertex degree is $\mathcal{O}(\frac{|B|}{|R|} + \log |R|)$. Kaneko et al. [8] consider non-planar Hamiltonian alternating cycles allowing edge crossings. They prove that at most $n - 1$ crossings are sufficient to compute a Hamiltonian alternating cycle and that this is worst-case optimal.

Concerning the orthogeodesic version of the problem, Kano [9] shows that every general equitable point set admits a perfect matching such that each edge is an L-shaped orthogonal chain connecting a red point to a blue point. Moreover, the computation of Hamiltonian orthogeodesic paths is also related to the point-set embeddability problem, i.e., the problem of embedding a given graph on a given set of points. An orthogeodesic alternating path can in fact be regarded as an orthogeodesic embedding of a path on the given point set. Orthogeodesic embeddings have been introduced and studied by Katz et al. [10].

1.2. Contribution

We show that every general equitable point set with n points admits a Hamiltonian orthogeodesic alternating path and we present an $\mathcal{O}(n \log^2 n)$-time algorithm that computes such a path. The path has at most two bends per edge which is proved to be worst-case optimal. However, the bends along the edges of the computed path may not have integer coordinates. In fact, we show that deciding whether a set of red and blue grid points P admits a Hamiltonian orthogeodesic alternating path with bends at grid points is $\mathcal{NP}$-complete. Further, we describe an $\mathcal{O}(n \log^2 n)$-time algorithm that computes an orthogeodesic alternating path of length $|P|/3$ with bends at grid points and we show that there are point sets that do not admit an orthogeodesic alternating path with more than $|P|/2 + 2$ points. Finally, we show that if points of P are allowed to be horizontally or vertically aligned, then it is $\mathcal{NP}$-complete to decide whether a balanced point set P has a perfect orthogeodesic alternating matching. This contrasts the result by Kano [9] stating that such a matching always exists if we are not allowed to place more than one point per horizontal or vertical line.

2. Preliminaries

Given a point set P, the *bounding box of* P, denoted as $\mathcal{B}(P)$, is the smallest axis-parallel rectangle enclosing P. The *extremal points* of $\mathcal{B}(P)$ are the points on the top, bottom, left, and right side of $\mathcal{B}(P)$ denoted as p_t, p_b, p_ℓ, and p_b respectively. Let p and q be two points such that $\mathcal{B}(\{p, q\})$ is a non-degenerate rectangle. A *horizontal chain* (*vertical chain*) is a two-bend orthogeodesic chain such that the first and the last segment are horizontal (vertical). Note that a horizontal chain (vertical chain) is uniquely determined by its endpoints and the x-coordinate (y-coordinate) of its vertical (horizontal) segment.

A point set $P = R \cup B$ is called a *butterfly* if it has the following properties, see also Fig. 6(a).

(i) For every two blue points p and q of P, $x(p) < x(q)$ implies $y(p) < y(q)$.
(ii) For every two red points p and q of P, $x(p) < x(q)$ implies $y(p) < y(q)$.
(iii) For every pair consisting of a blue point p and a red point q of P, $x(p) > x(q)$ and $y(p) < y(q)$.

When printed in black and white, the darker dots in our figures represent blue points while the light gray ones represent red points.

3. Hamiltonian orthogeodesic alternating paths

In this section we consider the problem of computing a Hamiltonian orthogeodesic alternating path for a given general equitable set of red and blue points. Of course, for an alternating path to exist it is necessary that the point set is equitable.

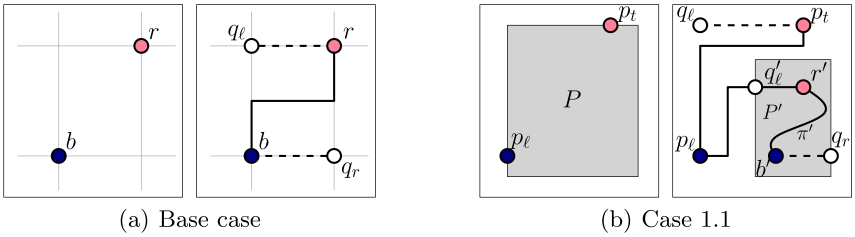


Fig. 1. Illustration for the proof of Theorem 1. Base case and Case 1.1.

On the other hand, we consider general point sets in order to avoid trivial counterexamples. Namely, a set of collinear points that are not alternatingly red and blue does not admit an orthogeodesic alternating path. We present a polynomial time algorithm to construct a Hamiltonian orthogeodesic alternating path with at most two bends per edge, given a general equitable point set. We show that this is also worst-case optimal in terms of bends by showing that there are point sets that do not allow a Hamiltonian orthogeodesic alternating path with at most one bend per edge. We start by proving the following auxiliary lemma for balanced point sets (i.e., point sets with $|R| = |B|$).

Lemma 1. *Let P be a general balanced point set with n red and blue points and let $p_1, \ldots, p_n$ be the sequence of points sorted from left to right. If $c(p_1) = c(p_n)$, then there is an index i with $2 \leqslant i \leqslant n-1$ such that $c(p_i) \neq c(p_1)$ and such that both the point set $P_1 := \{p_1, \ldots, p_i\}$ and the point set $P_2 := \{p_{i+1}, \ldots, p_n\}$ are non-empty and balanced.*

Proof. For every $2 \leqslant j \leqslant n-1$ let $P_j := \{p_1, \ldots, p_j\}$ and assume without loss of generality that $c(p_1)$ and $c(p_n)$ are both red. Let $r(j)$ and $b(j)$ denote the number of red and blue points in P_j, respectively, and let $f(j) := b(j) - r(j)$. Then $f(1) = -1$ and $f(n-1) = 1$. Further, for $2 \leqslant j \leqslant n$, we have $|f(j) - f(j-1)| = 1$ since either $r(j) - r(j-1) = 1$ and $b(j) - b(j-1) = 0$ or $b(j) - b(j-1) = 1$ and $r(j) - r(j-1) = 0$. Let i be the smallest index such that $f(i) = 0$ that is, $r(i) = b(i)$. Hence, P_i is balanced. Notice that since f changes by exactly one unit when going from $j-1$ to j, such an index must exist. Since $P = P_n$ is balanced and P_i is balanced, so is $P \setminus P_i$. Thus we have $P_1 := P_i$ and $P_2 := P \setminus P_i$. Finally, notice that since i is the smallest index where $f(i) = 0$, then $f(i-1) = -1$; this implies that $c(p_i)$ is blue, i.e., $c(p_i) \neq c(p_1)$. $\square$

Note that, for reasons of symmetry, the lemma also yields that there is an index i such that $c(p_i) \neq c(p_n)$ and such that both $P_1 := \{p_1, \ldots, p_{i-1}\}$ and $P_2 := \{p_i, \ldots, p_n\}$ are balanced.

Theorem 1. *Every general equitable point set consisting of n red and blue points admits a Hamiltonian orthogeodesic alternating path. Further, a Hamiltonian orthogeodesic path with at most two bends per edge can be computed in $\mathcal{O}(n \log^2 n)$ time.*

Proof. Let $P = R \cup B$ be a general equitable set of n red and blue points. For convenience, we will describe the constructed path as a path oriented from one endpoint to the other.

Assume first that the point set P is balanced. Every Hamiltonian path π on a balanced point set contains one red endpoint r and one blue endpoint b. Thus, we may refer to the ends of the path as the red end and the blue end. First, we show by induction on the size of P that every balanced point set P admits a Hamiltonian orthogeodesic alternating path π such that the following invariants are maintained.

(H1) Assume that π is oriented from its red end r to its blue end b. Let q_ℓ be the point on the left side of the bounding box of P that is horizontally aligned with r and let q_r be the point on the right side of the bounding box of P that is horizontally aligned with b. Then the straight-line segments $\overline{q_\ell r}$ as well as $\overline{bq_r}$ do not intersect π, except in b and r, respectively, as illustrated in Fig. 1(a).
(H2) Assume that π is oriented from its blue end b to its red end r. Let q_ℓ be the point on the left side of the bounding box of P that is horizontally aligned with the blue endpoint b and let q_r be the point on the right side of the bounding box of P that is horizontally aligned with the red end r. Then the straight-line segments $\overline{q_\ell b}$ as well as $\overline{rq_r}$ do not intersect π, except in b and r, respectively.
(H3) Each orthogeodesic chain in π has at most two bends.

In what follows we consider the case when the constructed path π is oriented from its red end to its blue end. As a consequence, we will only need to prove that invariants (H1) and (H3) hold. The case when π is oriented from its blue end to its red end (for which invariants (H2) and (H3) hold) is symmetric.

In the base case of the induction we have $|P| = 2$, that is, P consists of a red point r and a blue point b. We connect r to b by a vertical chain whose horizontal segment is on the line $y = (y(r) + y(b))/2$ as illustrated in Fig. 1(a). Clearly, the invariants are maintained. Now, suppose that the induction hypothesis holds for all balanced point sets with k red and k blue points such that $k > 1$ and $2k < n$. Let p_ℓ, p_r, p_t and p_b denote the leftmost, rightmost, topmost and bottommost

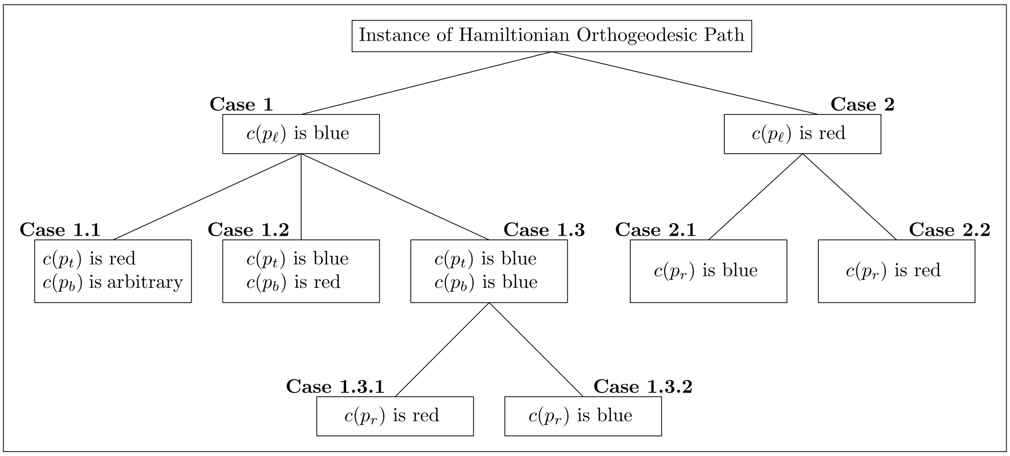
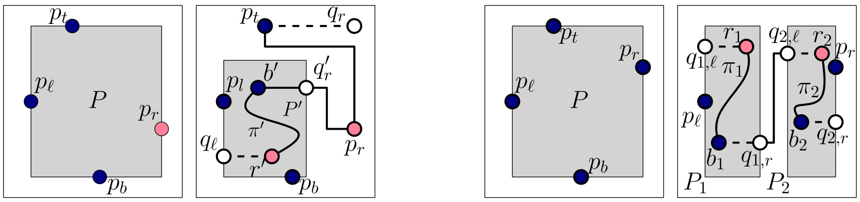


Fig. 2. Case distinction applied in the proof of Theorem 1.



(a) Case 1.3.1 (b) Case 1.3.2

Fig. 3. Illustration for the proof of Theorem 1. Case 1.3.1 and Case 1.3.2.

points on the boundary of the bounding box of P, respectively. Note that some of these points may coincide. We distinguish two cases and several sub-cases as summarized in Fig. 2.

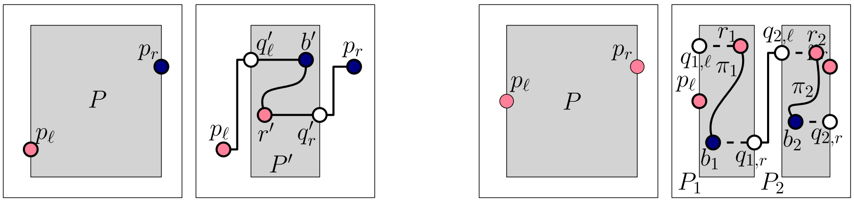
Case 1: *The color of p_ℓ is blue.* We distinguish three sub-cases.

Case 1.1: *The color of p_t is red.* Then $p_t \neq p_\ell$ as illustrated in Fig. 1(b). Let $P' := P \setminus \{p_\ell, p_t\}$. By the induction hypothesis, we can construct a Hamiltonian orthogeodesic alternating path π' on P' starting at a red point $r' \in P'$ and ending at a blue point $b' \in P'$ such that (H1) and (H3) hold. Let q'_ℓ denote the point on the left side of $\mathcal{B}(P')$ that is horizontally aligned with r' and let q'_r denote the point on the right side of $\mathcal{B}(P')$ that is horizontally aligned with b'. Further, let p'_t be the topmost point in P' and let p'_ℓ be the leftmost point in P'. We connect p_t to p_ℓ by a vertical chain whose horizontal segment is located on the line $y = (y(p_t) + y(p'_t))/2$ and we connect p_ℓ to r' by a horizontal chain whose vertical segment is on the vertical line $x = (x(p_\ell) + x(p'_\ell))/2$ as illustrated in Fig. 1(b). The resulting path starts with the red point $r := p_t$ and ends with the blue point $b := b'$. Let q_ℓ be the point on the left side of $\mathcal{B}(P)$ that is horizontally aligned with r. The point q_r that is horizontally aligned with b on the right side of $\mathcal{B}(P)$ coincides with q'_r. Then the segments $\overline{q_\ell r}$ and $\overline{b q_r}$ do not intersect π by construction and by the induction hypothesis, respectively.

Case 1.2: *The color of p_t is blue and the color of p_b is red.* This case is symmetric to Case 1.1 by a reflection at the horizontal axis.

Case 1.3: *The color of p_t is blue and the color of p_b is blue.* We consider two sub-cases depending on the color of p_r.

Case 1.3.1: *The color of p_r is red.* This case is similar to Case 1.1. We have $p_r \neq p_t$ since their colors differ. Let $P' := P \setminus \{p_t, p_r\}$. By the induction hypothesis, we can construct a Hamiltonian orthogeodesic alternating path π' on P' starting at a red point $r' \in P'$ and ending at a blue point $b' \in P'$ such that (H1) and (H3) hold. Let q'_ℓ denote the point on the left side of $\mathcal{B}(P')$ that is horizontally aligned with r' and let q'_r denote the point on the right side of $\mathcal{B}(P')$ that is horizontally aligned with b'. Further, let p'_t be the topmost point in P' and let p'_r be the rightmost point in P'. We connect p_t to p_r by a vertical chain whose horizontal segment is located on the line $y = (y(p_t) + y(p'_t))/2$ and we connect p_r to b' by a horizontal chain whose vertical segment is on the vertical line $x = (x(p_r) + x(p'_r))/2$ as illustrated in Fig. 3(a). The resulting path π starts with the red point $r := r'$ and ends with the blue point $b := p_t$. Let q_r be the point on the right side of $\mathcal{B}(P)$ that is horizontally aligned with b. The point q_ℓ that is horizontally aligned with r on the left side of $\mathcal{B}(P)$ coincides with q'_ℓ. Then the segments $\overline{q_\ell r}$ and $\overline{b q_r}$ do not intersect π by the induction hypothesis and by construction, respectively.



(a) Case 2.1 (b) Case 2.2

Fig. 4. Illustration for the proof of Theorem 1. Case 2.1 and Case 2.2.

Case 1.3.2: *The color of* p_r *is blue.* According to Lemma 1 we can split P into two non-empty balanced point sets P_1 and P_2 that can be separated by a vertical line. By the induction hypothesis, we can compute two Hamiltonian orthogeodesic paths π_1 and π_2 in P_1 and P_2, respectively. Let π_1 be directed from the red point $r_1 \in P_1$ to the blue point $b_1 \in P_1$. Similarly, let π_2 be directed from the red point $p_2 \in P_2$ to the blue point $b_2 \in P_2$. Let $q_{i,\ell}$ and $q_{i,r}$ denote the points on the left and right side of $\mathcal{B}(P_i)$ for $i \in \{1,2\}$, respectively. We connect the paths π_1 and π_2 by a horizontal chain between b_1 and r_2 whose vertical segment is on the line centered between $\mathcal{B}(P_1)$ and $\mathcal{B}(P_2)$ as illustrated in Fig. 3(b). The resulting path π is directed from r_1 to b_2. By the induction hypothesis the segments $\overline{q_{1,\ell}r_1}$ and $\overline{b_2q_{2,r}}$ connecting r_1 and b_2 to the left and right side of $\mathcal{B}(P)$, respectively, do not intersect π.

Case 2: *The color of* p_ℓ *is red.* We distinguish two sub-cases depending on the color of p_r.

Case 2.1: *The color of* p_r *is blue.* Since the size of P is at least four, the point set $P' := P \setminus \{p_\ell, p_r\}$ is non-empty and balanced. By induction hypothesis, we can construct a Hamiltonian orthogeodesic alternating path π' on P' starting at a blue point $b' \in P'$ and ending at a red point $r' \in P'$ such that (H2) and (H3) hold.

Let p'_ℓ and p'_r denote the leftmost and rightmost points in P', respectively. Then we connect p_ℓ to b' by a horizontal chain whose vertical segment is on the line $x = (x(p_\ell) + x(p'_\ell))/2$ and we connect r' to p_r by a horizontal chain whose vertical segment is on the line $x = (x(p_r) + x(p'_r))/2$ as illustrated in Fig. 4(a). Since p_ℓ and p_r are on the left and right side of $\mathcal{B}(P)$ the invariant (H1) is trivially maintained.

Case 2.2: *The color of* p_r *is red.* This case is completely analogous to Case 1.3.2, see Fig. 4(b).

Hence, every general balanced point set P admits a Hamiltonian orthogeodesic alternating path. Next, let P be an unbalanced equitable point set and remember that $|R| < |B|$. First, suppose that there is a blue point p on one of the sides of $\mathcal{B}(P)$, say on the left side. Then the point set $P' := P \setminus \{p\}$ is balanced and we can compute a path π' that starts at a red point r' and ends at a blue point b' satisfying the invariants (H1) and (H3). Further, let q'_ℓ be the point on the left side of $\mathcal{B}(P')$ that is horizontally aligned with r'. Then the segment $\overline{q'_\ell r'}$ does not intersect π'. Hence, we can connect p to r' with a horizontal chain whose vertical segment is on the line $x = (x(p) + x(p'_\ell))/2$, where p'_ℓ denotes the leftmost point in P'.

Second, suppose that all points on the boundary of $\mathcal{B}(P)$ are red. Let $r \notin P$ be an arbitrary point to the left of $\mathcal{B}(P)$, assign to it the red color and let $P' := P \cup \{r\}$. Then P' is balanced and both the leftmost point and the rightmost point of P' are red. Then we can split P' into two non-empty point sets P_1 and P_2 that can be separated by a vertical line such that the rightmost point in P_1 is a blue point according to Lemma 1. Subsequently, we can compute a path π_1 in P_1 starting at r according to Case 2.1 and we can compute a path π_2 satisfying the invariants (H1) and (H3) in P_2. The two paths can be concatenated according to Case 2.2 such that the resulting path π is a Hamiltonian orthogeodesic path on P' starting at r. Since r is an end point of π and since we handled the point set according to Case 2.2, we can safely remove r obtaining a Hamiltonian orthogeodesic path on P.

Finally, we show that we can compute a Hamiltonian orthogeodesic path according to the preceding case distinction in $\mathcal{O}(n\log^2 n)$ time. We sort the points with respect to their x- and y-coordinates, respectively, in $\mathcal{O}(n\log n)$ time and we maintain two arrays X and Y containing the points in sorted order. Then each point p can be addressed by two integers $h(p)$ and $v(p)$ denoting the index of p in the horizontal array X and the vertical array Y, respectively. That is we have $X_{h(p)} = p$ and $Y_{v(p)} = p$. Further, we maintain two spatial data structures D_R and D_B with $\mathcal{O}(n\log n)$ initialization time supporting orthogonal range queries in $\mathcal{O}(\log n)$ query time [4] for the blue and red points, respectively.

We assume that each recursive call of our algorithm receives as input the point set P and its bounding box $\mathcal{B}(P)$ represented by two intervals $[i, j]$ and $[k, l]$ where $i = h(p_l)$, $j = h(p_r)$, $k = v(p_b)$, and $l = v(p_t)$ and p_ℓ, p_r, p_t and p_b are the extremal points of $\mathcal{B}(P)$. Notice that some of the extremal points may coincide.

First we consider all cases, except for Case 1.3.2 and Case 2.2. In these cases we recurse on a point set $P' := P \setminus \{p_1, p_2\}$, where p_1 and p_2 are two extremal points of $\mathcal{B}(P)$. We need to compute the bounding box of P', i.e., the two intervals $[i', j']$ and $[k', l']$ corresponding to the extremal points of $\mathcal{B}(P')$. To this aim we consider the axis-aligned rectangle R obtained by removing from $[i, j]$ and $[k, l]$ the indices in $\{i, j, k, l\}$ corresponding to p_1 and p_2. For example, assume that the extremal points are all distinct and we are in Case 1.1; then $p_1 = p_\ell$ and $p_2 = p_t$ and the intervals defining R are $[i + 1, j]$ and

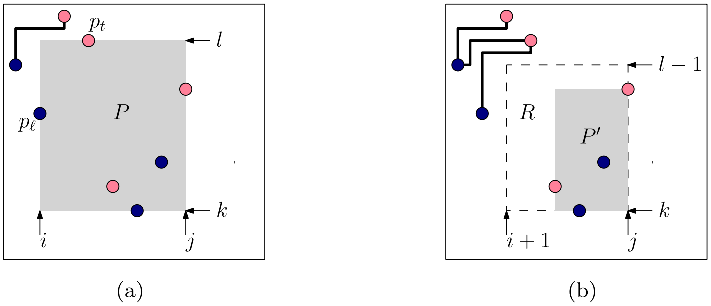


Fig. 5. Computation of $\mathcal{B}(P')$. (a) The bounding box $\mathcal{B}(P)$ of P is defined by the two intervals $[i, j]$ and $[k, l]$. (b) The rectangle R defined by the two intervals $[i+1, j]$ and $[k, l-1]$ (shown dashed in the picture) contains $\mathcal{B}(P')$ but does not coincide with it.

$[k, l-1]$. Notice that, R contains $\mathcal{B}(P')$ but it may not coincide with $\mathcal{B}(P')$ (see Fig. 5); thus we repeatedly reduce the size of R until it coincides with $\mathcal{B}(P')$.

Let $m = |P'| = |P| - 2$ and let $[a, b]$ and $[c, d]$ be the horizontal and vertical interval, respectively, defining the rectangle R. Suppose that the topmost side of R does not contain a point of P'; we perform a binary search on the vertical array $Y[c \dots d]$ in order to locate the topmost point of P'. Let x be the index of $Y[c \dots d]$ considered at the generic iteration of the binary search and let p be the corresponding point; we query the spatial data structures with the rectangle R' corresponding to the intervals $[a, b]$ and $[c, x]$. If the number of points in R' is equal to m and p is a point of P', then we have found the topmost point of P'. In order to test whether p is in P' we can test whether p is contained in R because we chose R as a rectangle containing exactly the points in P'. Since $v(p) \in [c, d]$ by definition, it is sufficient to test whether $h(p) \in [a, b]$. An analogous binary search is performed for each side of R that does not contain a point of P'. The number of steps for each binary search is $\mathcal{O}(\log n)$ and each step can be performed in $\mathcal{O}(\log n)$; thus we can find the extremal points of P' in $\mathcal{O}(\log^2 n)$ time.

In the remaining Cases 1.3.2 and 2.2, we need to split the point set according to Lemma 1. Note that we can compute the number of red and blue points in a given rectangle in $\mathcal{O}(\log n)$ time using the spatial data structures D_R and D_B. That is, given an index i into the horizontal array of the points, we can compute the function $f(i)$ defined according to Lemma 1 in $\mathcal{O}(\log n)$ time. Then we can find an index i splitting the point set with the desired properties by binary search on the indices in $\mathcal{O}(\log^2 n)$ time.

Since each operation of the algorithm can be implemented to run in $\mathcal{O}(\log^2 n)$ time and since each type of operation is executed at most n times, the running time of the algorithm is in $\mathcal{O}(n \log^2 n)$. $\square$

If P is an equitable point set such that all points have even integer coordinates (or odd integer coordinates), then the algorithm computes a Hamiltonian orthogeodesic alternating path such that every bend has integer coordinates, that is, it computes a Hamiltonian path on the grid. Moreover, whenever the horizontal and vertical distance of each pair of points is at least two, we can modify the algorithm to compute a path on the grid as well. Instead of mapping the horizontal and vertical segments of the vertical and horizontal chains onto the bisector of two points as described in the proof of Theorem 1, we instead map it to any grid line between the points. Note that, if the horizontal and vertical distance of any pair of points in P is at least two, then there must be at least one unoccupied grid line between the points. Hence, we obtain the following result.

Corollary 1. *Let P be an equitable set of n red and blue grid points such that $\min\{|x(p) - x(q)|, |y(p) - y(q)|\} \geqslant 2$ for all $p, q \in P$ with $p \neq q$. Then P admits a Hamiltonian orthogeodesic path π with at most two bends per edge such that each bend is located at a grid point. Further, there is an $\mathcal{O}(n \log^2 n)$-time algorithm that constructs such a path.*

The path computed by the algorithm in the proof of Theorem 1 has two bends per edge. Next, we show that this is worst-case optimal by showing that there are point sets that do not admit a Hamiltonian orthogeodesic alternating path with one bend per edge.

Theorem 2. *For every $n \geqslant 5$ there exists a general equitable point set of size n that does not admit a Hamiltonian orthogeodesic alternating path with at most one bend per edge.*

Proof. Consider a butterfly point set P with at least five points and let π be a Hamiltonian orthogeodesic alternating path on P. Since $|R| \leqslant |B|$, every Hamiltonian path on P contains at least one blue point that is not an endpoint of π. Let b be an internal blue point of π. Then b is connected to two red points r_1 and r_2 by two orthogeodesic chains χ_1 and χ_2, respectively, as illustrated in Fig. 6(b). One of the two chains, say χ_1, must have a horizontal segment incident to b while the other chain, that is χ_2, must have a vertical segment incident to b. If b is connected to r_1 and r_2 by orthogeodesic chains with at most one bend, then χ_1 must be attached to r_1 by a vertical segment, while χ_2 must be attached to r_2 by

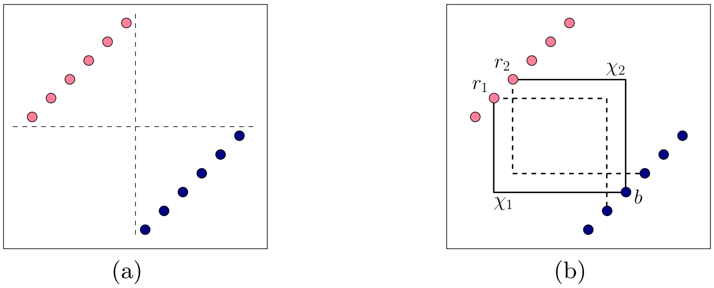


Fig. 6. (a) A butterfly point set. (b) Illustration for the proof of Theorem 2. If point p is connected to r_1 and r_2 with two 1-bend orthogeodesic chains, then r_1 and r_2 cannot be connected to any other blue point by an orthogeodesic chain without introducing a crossing. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

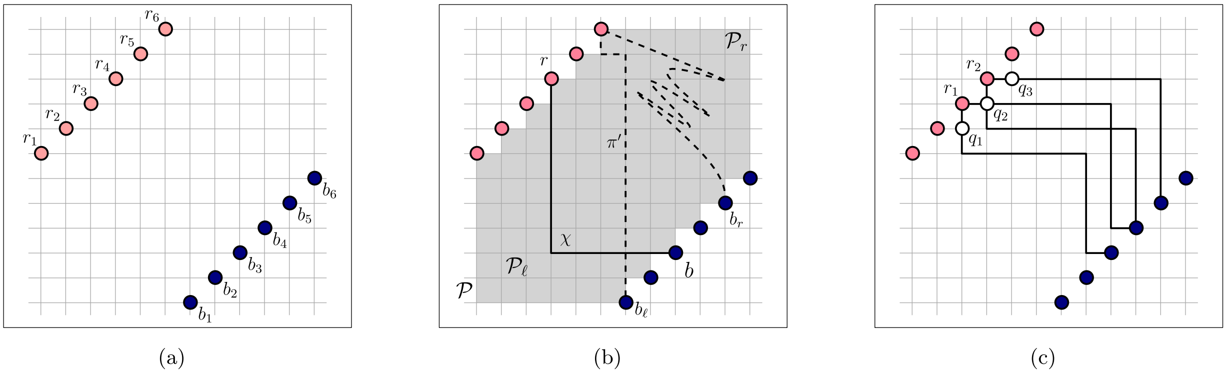


Fig. 7. Illustration for the proof of Theorem 3. (a) Butterfly one the grid. (b) If a path starts neither in blue point that is neither the leftmost nor the rightmost blue point, then there must be a crossing. (c) Two internal red points of the path cannot be next to each other on the grid, since the four incident orthogeodesic chains can use only three distinct points incident to the red points. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

a horizontal segment. Since all red points are above and to the left of all the blue points and since the two chains cannot cross, we have $x(r_1) < x(r_2)$. At least one of the points r_1 and r_2 must be connected to a blue point b' distinct from b since π can have at most one red point as an endpoint due to $|R| \leqslant |B|$. If $x(b') < x(b)$ then the orthogeodesic chain connecting r_1 or r_2 to b' would cross chain χ_1. If $x(b') > x(b)$ then the orthogeodesic chain connecting r_1 or r_2 to b' would cross chain χ_2 as illustrated in Fig. 6(b). Thus, two bends are worst-case optimal. □

4. Hamiltonian orthogeodesic alternating paths on the grid

While we have seen that we can always construct a Hamiltonian orthogeodesic alternating path on the grid if the horizontal and vertical distance between any pair of points is at least 2, a Hamiltonian orthogeodesic path does not always exist if we drop this requirement and consider point sets whose points are neither horizontally nor vertically aligned.

Theorem 3. *For every $n \geqslant 5$, there exists a general equitable set P of n red and blue grid points that does not admit a Hamiltonian orthogeodesic alternating path on the grid.*

Proof. Let P be a butterfly point set with at least five points with the additional property that the points have integer coordinates. Let $R := \{r_1, \ldots, r_{|R|}\}$ and $B := \{b_1, \ldots, b_{|B|}\}$ denote the red and blue points and let P be such that $x(r_i) = i$, $y(r_i) = i$ for all $1 \leqslant i \leqslant |R|$ as well as $x(b_i) = |R| + i$ and $y(b_i) = i - |B|$ for $1 \leqslant i \leqslant |B|$, respectively, as illustrated in Fig. 7(a).

First, note that, since each orthogeodesic chain is contained inside the bounding box of its endpoints and since the endpoints have different color, π must be contained in the polygon $\mathcal{P}$ defined by the union of the rectangles spanned by all pairs of red and blue points, respectively, as illustrated in Fig. 7(b).

Next, we show that the endpoints of π must be leftmost or rightmost red or blue points, respectively. Suppose, for instance, that π starts in a blue point b such that there is a blue point b_ℓ to the left and a blue point b_r to the right of b, respectively. Let π' denote the sub-path of π with endpoints b_ℓ and b_r. Further, let r be the unique red point adjacent to b and let χ be the orthogeodesic chain connecting b and r. Clearly χ bisects $\mathcal{P}$ into two sub-polygons $\mathcal{P}_\ell$ and $\mathcal{P}_r$ such that b_ℓ is contained in $\mathcal{P}_\ell$ and b_r is contained in $\mathcal{P}_r$. That is, the path π' must cross χ as illustrated in Fig. 7(b). Hence, the endpoints of π must be leftmost or rightmost red or blue points, respectively, as claimed.

Finally, we claim that π contains at least one pair of internal red points r_1 and r_2 such that $x(r_2) = x(r_1) + 1$. If n is odd, π has two blue points as its endpoints since we assumed $|R| \leqslant |B|$. Hence we can find two red points with the desired properties. Otherwise, if n is even we have $n \geqslant 6$ and π has both a red and a blue point as its endpoints. Since we argued that the endpoints must be leftmost or rightmost red and blue points, respectively, we can find a pair of red points with the desired properties since the red endpoint of π must be the leftmost or rightmost point of the at least three red points.

Let r_1 and r_2 be two internal red vertices of π such that $x(r_2) = x(r_1) + 1$, that is, $y(r_2) = y(r_1) + 1$. Since r_1 and r_2 are internal points of π, both r_1 and r_2 are adjacent to two orthogeodesic chains each. Since all blue points are to the right of and below r_1 and r_2, respectively, these chains must occupy the horizontal and vertical grid lines starting at r_1 and r_2 in the downward and rightward direction, respectively. That is, these chains must occupy the grid points one unit to the right and one unit below r_1 and r_2, respectively. However, this implies that the four distinct orthogeodesic chains incident to p_1 and p_2 must occupy three distinct points $q_1 := (x(r_1), y(r_1) - 1)$, $q_2 := (x(r_1) + 1, y(r_1)) = (x(r_2), y(r_2) - 1)$ and $q_3 := (x(r_2) + 1, y(r_2))$. Thus, at least two of the chains must intersect as illustrated in Fig. 7(c).

We conclude by observing that any butterfly grid point set with at most four vertices does admit a Hamiltonian orthogeodesic alternating path on the grid since such a path never contains two internal red or blue points, respectively. □

Motivated by Theorem 3, we study the Hamiltonian Orthogeodesic Alternating Path on the Grid problem, that is, the problem of deciding whether a given general equitable set of grid points admits a Hamiltonian orthogeodesic alternating path on the grid. Surprisingly it turns out that this problem is $\mathcal{NP}$-complete. If we are allowed to place more than one point on a horizontal or vertical line, we can show that it is even $\mathcal{NP}$-complete to decide whether there exists an orthogeodesic alternating perfect matching. This contrasts a result by Kano [9] stating that such a matching always exists if we are not allowed to place more than one point per horizontal or vertical line. First, we show that Hamiltonian Orthogeodesic Alternating Path on the Grid is contained in $\mathcal{NP}$.

Lemma 2. Hamiltonian Orthogeodesic Alternating Path on the Grid *is contained in* $\mathcal{NP}$.

Proof. In order to show that Hamiltonian Orthogeodesic Alternating Path on the Grid is contained in $\mathcal{NP}$, we introduce the notion of a *bottommost* orthogeodesic path and show that the problem can be reduced to deciding whether there is a bottommost Hamiltonian orthogeodesic alternating path on P. A candidate solution of this problem can be encoded in polynomial space and verified in polynomial time.

We use the following terminology. We say that a point $p \in \mathbb{R}^2$ *k-dominates* the points in the k-th quadrant of the orthogonal coordinate system with origin at p ($k \in \{1, 2, 3, 4\}$). The unbounded range corresponding to the k-th quadrant is called *k-cone* and p is called the *apex* of the cone. Given a set Q of points, we refer to the union of the k-cones of the points in Q as the *orthogeodesic k-hull of* Q as illustrated in Fig. 8(a). Further, by $Q^{\nwarrow}$ we denote the set of points resulting from translating the points in Q one unit to the left and one unit to the top and by $Q^{\nearrow}$ we denote the set of points resulting from translating the points in Q one unit to the right and one unit to the top.

Assume we are given an orthogeodesic path $\pi = (P, E)$ on a point set P. Let e be an edge of π; we denote the leftmost point of e by e^- and its rightmost point by e^+. Edge e is called *upward* if $y(e^+) \geqslant y(e^-)$, otherwise, it is called *downward*. We define the partial order $\prec$ on E such that for $e_1, e_2 \in E$ we have $e_1 \prec e_2$ if and only if there is a vertical line intersecting e_1 below e_2. The path π is called *bottommost orthogeodesic path* if and only if each edge e is embedded as the bottommost orthogeodesic chain with respect to $\prec$. By this we mean, that each upward edge e connecting p and q is embedded on the orthogeodesic 4-hull of the point set $\{p, q\} \cup P_e^{\nwarrow} \cup B^{\nwarrow}$, and each downward edge is embedded on the orthogeodesic 2-hull of the point set $\{p, q\} \cup P_e^{\nearrow} \cup B^{\nearrow}$, where P_e is the union of the endpoints of all edges that are smaller than e with respect to $\prec$ and B is the set of bends induced by the bottommost chains of these edges. That is, each smallest edge with respect to $\prec$ is embedded as an L-shaped chain consisting of one horizontal and one vertical straight-line segment.

Suppose that π is a bottommost orthogeodesic path. Then it is easy to see that the number of bends of each edge is bounded by a linear function in the number of points as follows. First, note that each of the bends placed at an apex of a 2-cone or at an apex of a 4-cone, respectively, is obtained by translating a point of the original point set. Further, the total number of bends along an edge is at most twice this number of bends plus one. Hence, all bends of π are points of the set

$$\widehat{P} := \left\{(x+i, y+i), (x+i, y-i) \,\middle|\, (x, y) \in P \wedge 1 \leqslant i \leqslant n-1\right\}$$

since π has $n - 1$ edges. It follows that the number of bends of π is bounded by a polynomial function in the number of points of P.

Further, it is clear that every path can be transformed into a bottommost path without changing the partial order $\prec$ as follows. Suppose that $\pi = (P, E)$ is a Hamiltonian orthogeodesic alternating path and let $\prec$ be the partial order on the edges of E defined as above, that is $e' \prec e$ if and only there is a vertical line that intersects e' below e for $e', e \in E$. Assume that $e \in E$ is an upward edge that is not embedded as the bottommost orthogeodesic chain with respect to $\prec$ but all edges $e' \in E$ with $e' \prec e$ are embedded as the bottommost orthogeodesic chain. The case, when e is a downward edge can be handled similarly.

Let χ be the bottommost orthogeodesic chain between e^- and e^+ with respect to $\prec$. We claim that every vertical line intersecting e intersects χ below e or it intersects both chains in the same point. Suppose that there is a vertical line ℓ

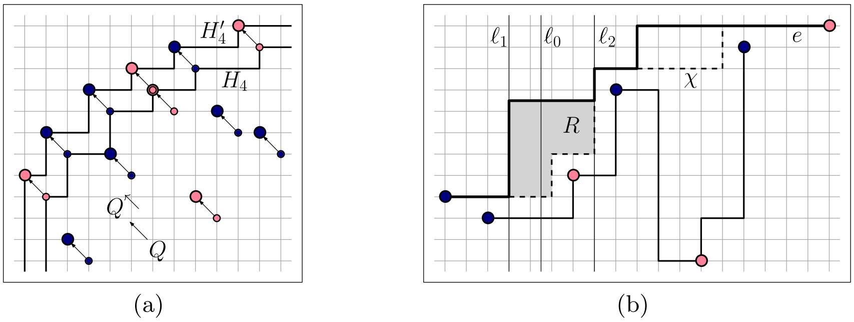


Fig. 8. Illustrations for the proof of Lemma 2. (a) Point set Q (small points) and 4-hull H_4 of Q as well as $Q^{\nwarrow}$ (large points) and 4-hull H'_4 of $Q^{\nwarrow}$. (b) An edge e of a Hamiltonian orthogeodesic alternating path that is not embedded as the bottommost orthogeodesic chain χ.

such that ℓ intersects χ above e. Since e is embedded according to $\prec$, it must be embedded above or on the orthogeodesic 4-hull of the points $\{e^-, e^+\} \cup P_e^{\nwarrow}$ where P_e denotes the union of the endpoints of all edges that are smaller than e with respect to $\prec$. Otherwise, we would find an edge $e' \prec e$ and a vertical line that intersects e' above e or that intersects both e' and e in the same point. On the other hand, if at least one bend b of e is embedded in the region that is 4-dominated by the point set B of all bends corresponding to edges that are smaller than e with respect to $\prec$, then we again find a vertical line intersecting e below some edge e' with $e' \prec e$. Hence e must be embedded above or on χ defined as the orthogeodesic 4-hull of $\{p, q\} \cup P_e^{\nwarrow} \cup B^{\nwarrow}$. Hence every vertical line intersecting e either intersects χ below e or it intersects both chains in the same point.

Since $e \neq \chi$ we can find a vertical line ℓ_0 such that ℓ_0 intersects e above χ as illustrated in Fig. 8(b). Let ℓ_2 be the rightmost line to the right of ℓ_0 such that ℓ_2 intersects e and χ in the same point. Further, let ℓ_1 be the leftmost vertical line to the left of ℓ_0 such that ℓ_1 intersects both e and χ in the same point. Note that ℓ_1 and ℓ_2 are well-defined since the vertical lines through e^- and e^+ have the desired property, respectively. Further, let R be the region enclosed between ℓ_1, ℓ_2, χ and e as illustrated in Fig. 8(b). We claim that R is empty. Note that each edge intersecting the vertical strip between ℓ_1 and ℓ_2 is comparable to e with respect to $\prec$. Then R does not contain any edge e' with $e \prec e'$ by the definition of $\prec$. On the other hand, all edges e' with $e' \prec e$ are below χ which is below e. Hence, R is empty and we can substitute e by a new orthogeodesic chain consisting of the first part of e between e^- and ℓ_1, the last part of e between ℓ_2 and e^+ as well as an intermediate part consisting of the sub-chain of χ between ℓ_1 and ℓ_2 and the vertical segment of ℓ_2 between χ and e. By iteratively applying this argument to any smallest edge with respect to $\prec$ that is not embedded as the bottommost orthogeodesic chain, we can iteratively contract the space between e and χ such that e will eventually be embedded as the bottommost orthogeodesic chain.

Thus, the problem of deciding the Hamiltonian Orthogeodesic Alternating Path on the Grid problem is equivalent to deciding whether there is a bottommost Hamiltonian orthogeodesic alternating path on P. Such a path can be uniquely encoded by the sequence of points along the path and the partial order $\prec$. Given this, we can check if the uniquely determined bottommost path is a Hamiltonian orthogeodesic path in polynomial time by computing the path and checking planarity. Hence, the problem is in $\mathcal{NP}$. $\square$

We are now ready to prove that Hamiltonian Orthogeodesic Alternating Path on the Grid is $\mathcal{NP}$-complete.

Theorem 4. Hamiltonian Orthogeodesic Alternating Path on the Grid *is $\mathcal{NP}$-complete.*

Proof. The problem is in $\mathcal{NP}$ as stated in Lemma 2. We show $\mathcal{NP}$-hardness by reduction from 3-Partition using similar techniques as in [10]. An instance of 3-Partition consists of a multi-set $A = \{a_1, \ldots, a_{3m}\}$ of $3m$ positive integers, each in the range $(B/4, B/2)$, where $B = (\sum_{i=1}^{3m} a_i)/m$, and the question is whether there exists a partition of A into m subsets $A_1, \ldots, A_m$ of A, each of cardinality 3, such that the sum of the numbers in each subset is B. Since 3-Partition is *strongly* $\mathcal{NP}$-hard [6], we may assume that B is bounded by a polynomial in m.

Given an instance A of 3-Partition, we construct a corresponding instance $P = R \cup B$ of the Hamiltonian Orthogeodesic Alternating Path on the Grid problem such that P allows for a Hamiltonian orthogeodesic alternating path if and only if there exists a partition of A with the desired properties as follows.

A sequence $p_1, \ldots, p_k$ of diagonally aligned grid points is called *k-spaced* if the Euclidean distance between subsequent points p_i and p_{i+1} is exactly $k\sqrt{2}$ for all $1 \leqslant i \leqslant k-1$. The point set P of the constructed instance consists of four different types of points, called *hinge points*, *element points*, *mask points* and *partition points*, and is aligned on a regular sawtooth-pattern with $3m+2$ teeth, numbered $T_0, \ldots, T_{3m+1}$ from left to right. The point set, as well as the sawtooth-pattern and the teeth are illustrated in Fig. 9.

Let L be some integer to be specified later. Each tooth T_i consists of a diagonal segment with slope 1 of length $L\sqrt{2}$, denoted by S_i, and a diagonal segment with slope -1 of length $(2L+1)\sqrt{2}$. Hence, the tips of the teeth are aligned along

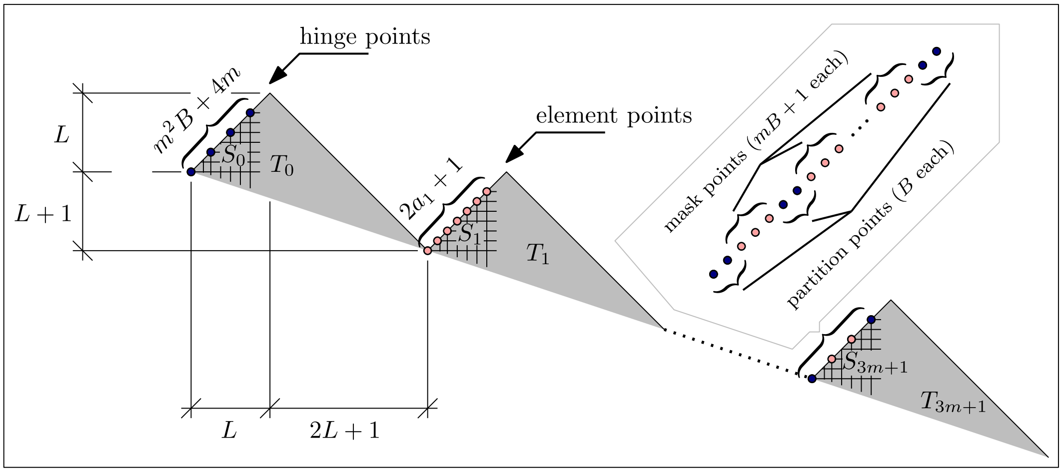


Fig. 9. Point set used in the reduction. Each shaded triangle constitutes a tooth T_i. All points are arranged on the ascending slope S_i of T_i.

a line with negative slope such that the tip of T_i is below the lowest point of S_{i-1} for $1 \leqslant i \leqslant 3m+1$. We construct our point set as follows.

Along S_0, we align $m^2B + 4m$ 2-spaced blue hinge points starting at the leftmost point of S_0. For each element a_i we align $2a_i + 1$ 1-spaced red element points along S_i. Further, we align m sets of B 2-spaced blue partition points along S_{3m+1}, each acting as a partition. These partitions are separated by $m-1$ sequences of $mB + 1$ 2-spaced red mask points which will act as a sort of "dot mask" separating partitions that are consecutive along S_{3m+1}. The maximal sequences of blue points along S_{3m+1} are called *partitions* and the maximal sequences of red points along S_{3m+1} are called *masks*. By construction P contains $m^2B + mB + 4m - 1$ red and $m^2B + mB + 4m$ blue points and, thus, is equitable with one more blue point. Hence, any alternating path must start and end with a blue point and all red points must be interior points of the path. This implies that every red point must be connected to exactly two blue points.

We now show that there is a partition of A with the desired properties if and only if the point set contains a Hamiltonian orthogeodesic alternating path. A high-level illustration of the reduction is given in Fig. 10. Assume that we are given a Hamiltonian orthogeodesic alternating path π on P. First, consider the red mask points. In each mask there must be one mask point that is connected to a blue hinge point on S_0. To see this, note that there are $mB + 1$ red mask points in each of the $m-1$ masks, each of which is adjacent to two blue points on π. Further, each blue point can have at most two adjacent red points on π. Hence, the red mask points of a fixed mask are adjacent to a total of at least $mB + 1$ blue points. Since there are only mB blue points in total on S_{3m+1}, the red mask points of each of the masks must be adjacent to at least one blue hinge point each. Each edge between a red mask point and a blue hinge point is called a *partitioner*. Consider the partial order $\prec$ on the edges of π such that $e' \prec e$ if there is a vertical line intersecting e' below e. Since there is a vertical line ℓ_0 that is intersected by all partitioners, the partitioners are totally ordered by $\prec$. Let the sequence of partitioners sorted with respect to $\prec$ be given by $\mathcal{P}_1, \ldots, \mathcal{P}_k$. Note that $k \geqslant m-1$. For convenience we extend the partitioners by imaginary horizontal lines towards the left and right. Then the k partitioners partition the plane into $k+1$ regions $R_1, \ldots, R_k + 1$ such that R_1 is the region below $\mathcal{P}_1$, R_i is the region bounded by $\mathcal{P}_{i-1}$ and $\mathcal{P}_i$ and R_{k+1} is the region above $\mathcal{P}_{k+1}$.

Next, consider the element points. Since the element points corresponding to a single element are 1-spaced, no partitioner can pass between them on a grid line. Hence, the partitioners will partition the element points according to the element sizes, such that all element points corresponding to a single element are contained in the same partition. Each element a_i can then be associated with a unique index $f(i)$ such that all element points corresponding to a_i are contained in $R_{f(i)}$. However, we still need to show that each partition contains the correct number of element points.

Let D_i be the diagonal line through S_i and let H_i^+ and H_i^- denote the upper and lower half-planes defined by D_i, respectively. We claim that each group of $2a_i + 1$ element points corresponding to element a_i can have at most $2a_i + 2$ blue incidences in H_i^+. Each of these incidences is an orthogeodesic chain starting either with a horizontal segment to the left or with a vertical segment towards the top. These segments can be covered by grid points adjacent to the element points. As there are only $2a_i + 2$ such grid points, the claim holds.

Recall that the element points must be interior points of the path since P contains more blue points than red points, that is, each red element point must be adjacent to two blue points in π. Since the group of element points corresponding to a_i must therefore have $4a_i + 2$ blue incidences in total, and since it can have at most $2a_i + 2$ blue incidence in H^+, it must have at least $2a_i$ blue incidences in H_i^-. Thus, the union of all red element points must have a total of $2mB$ blue incidences on S_{3m+1}. On the other hand, there are only mB blue points on S_{3m+1}, each of which must have two red incidences. This implies that element a_i has exactly $2a_i$ incidences in H_i^- and that the blue partition points are connected only to the element points.

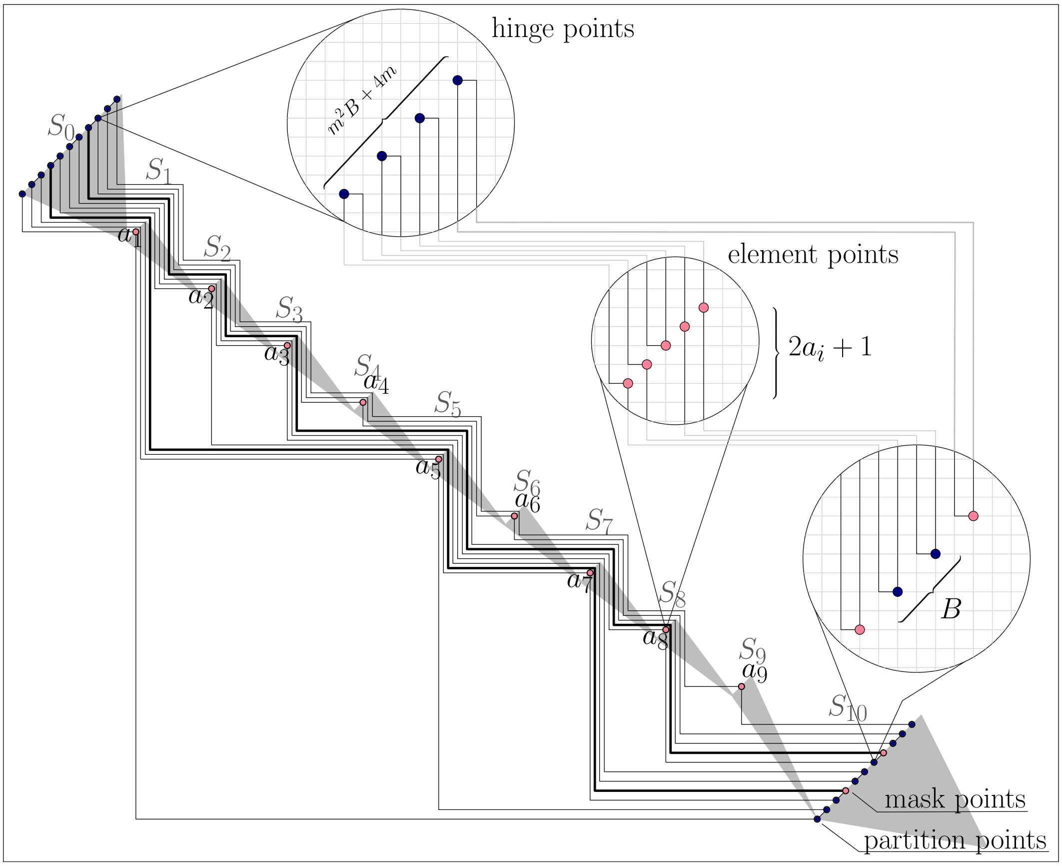


Fig. 10. A high-level illustration of an exemplary reduction from 3-Partition to Hamiltonian Orthogeodesic Alternating Path on the Grid using the instance $A_1 = \{a_1, a_5, a_7\}$, $A_2 = \{a_2, a_3, a_8\}$, $A_3 = \{a_4, a_6, a_9\}$ (not to scale). Details are depicted in the circles.

Finally, consider the partitions. Clearly the B partition points corresponding to a fixed partition on S_{3m+1} must all be contained in the same region R_j. Otherwise, some partitioner, say $\mathcal{P}_s$ must pass between two partition points. However, such a partitioner would have to connect to a red mask point above or to the left of it, both resulting in an orthogonal chain that would not be orthogeodesic. Since there are B blue points in each of the partitions, the number of element points must add up to $2B$, that is, the corresponding elements add up to B and thus yield a valid 3-partition of A.

Conversely, suppose that we are given a valid partition $A_1, \ldots, A_m$ of A according to 3-Partition. Then we can find a Hamiltonian orthogeodesic alternating path as follows. We iteratively embed the orthogeodesic chains such that each chain is drawn as the bottommost orthogeodesic chain (as defined in the proof of Lemma 2) that runs one grid unit above all orthogeodesic chains embedded so far, as illustrated in Fig. 10.

We start with an arbitrary partition, say A_1 containing elements a_i, a_j, a_k such that $i < j < k$. First, we consider the element a_i. We draw an alternating path starting at the leftmost hinge point using the first a_i partition points, the $2a_i + 1$ element-points corresponding to a_i as well as the leftmost $a_i + 2$ hinge points on S_0. The path alternates between the hinge points and the partition points, visiting the element points in between and ends at a blue hinge point. We proceed accordingly for elements a_j and a_k, respectively, in this order. Next, we embed a sequence of edges corresponding to partitioners. We start with the blue hinge point that we ended after visiting the last element point of a_k and we alternatingly visit consecutive blue hinge points and red mask points, ending again, at a blue hinge point. The remaining elements are handled in an analogous manner. Since each edge is embedded as the bottommost orthogeodesic chain, it is below all points to be inserted in later iterations. Further, let the parameter L used earlier on in the construction be defined as $L := 2m^2B + 2mB + 8m - 1$, that is, L is equal to the number of points of P. This implies that there are at least $L - 1$ unoccupied grid lines between any pair of element points corresponding to different elements. Since the constructed path has $L - 1$ edges, we therefore did not introduce any crossings. Hence, we have constructed a Hamiltonian orthogeodesic alternating path on P. $\square$

Note that we may add another red point above and to the left of all points in P to make the point set balanced. Using arguments analogous to the arguments used in the proof of Theorem 4 we can show the following corollary.

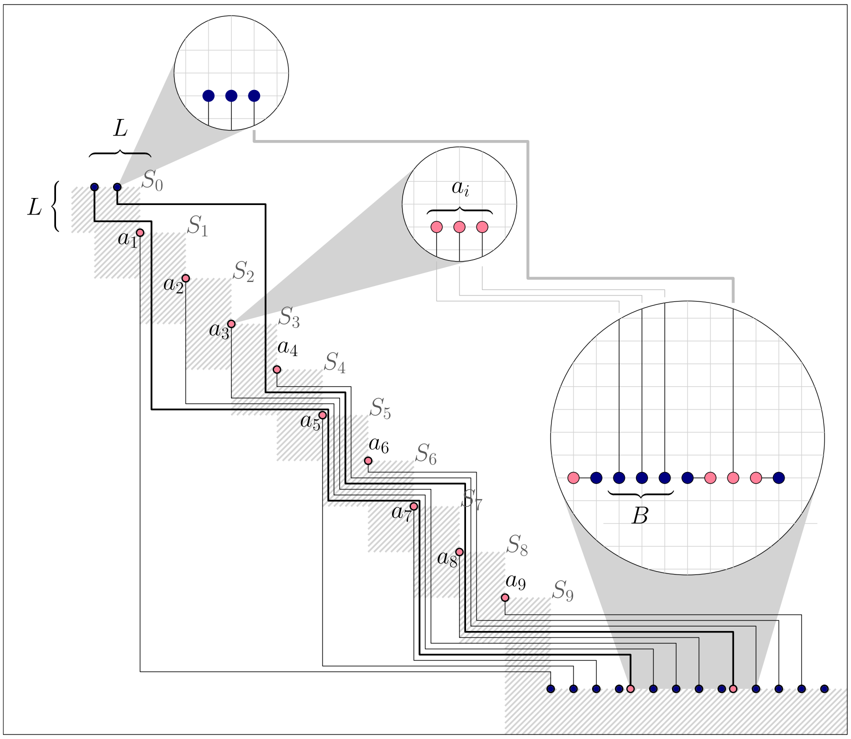


Fig. 11. Example of the reduction from 3-Partition to Perfect Orthogeodesic Alternating Matching on Grid using $A_1 = \{a_1, a_5, a_7\}$, $A_2 = \{a_2, a_3, a_8\}$, and $A_3 = \{a_4, a_6, a_9\}$ (not to scale).

Corollary 2. *It is $\mathcal{NP}$-complete to decide whether a given balanced set of red and blue grid points such that no two points are on a common horizontal or vertical line allows for a Hamiltonian orthogeodesic alternating cycle if bends are only allowed at grid points.*

Kano [9] showed that every balanced set of red and blue points such that no two points are on a common horizontal or vertical line admits a perfect orthogeodesic alternating matching consisting of L-shaped orthogonal chains. Hence such a matching is completely on the grid whenever the points are grid points. Surprisingly, the problem becomes $\mathcal{NP}$-complete if the points are allowed to be horizontally and vertically aligned. The proof for the following theorem is similar to the proof of Theorem 4.

Theorem 5. *Given an arbitrary balanced set of red and blue grid points, it is $\mathcal{NP}$-complete to decide whether there is a perfect orthogeodesic alternating matching on the grid.*

Proof. Showing containment in $\mathcal{NP}$ is analogous to the proof of Lemma 2. We show that the problem is $\mathcal{NP}$-hard by reduction from 3-Partition similar to the proof of Theorem 4. Given an instance A of 3-Partition, we construct a corresponding instance $P = R \cup B$ of the Perfect Orthogeodesic Alternating Matching on Grid problem as illustrated in Fig. 11 such that P allows for a perfect orthogeodesic alternating matching if and only if there exists a partition of A with the desired properties.

A set of horizontally aligned grid points is called *k-spaced* if the Euclidean distance between two adjacent points is exactly k. As in the proof of Theorem 4, the point set P consists of four different types of points, called *hinge points*, *element points*, *mask points* and *partition points*, yet now the point set is aligned on a regular staircase on the grid with $3m + 1$ stairs such that each stair has width and height $L := \lceil B/2 \rceil + 3m$. We number the stairs $S_0, \ldots, S_{3m}$ starting at the top. Then we construct our point set as follows.

On the horizontal line of the topmost stair S_0, we align $m - 1$ 1-spaced blue hinge points starting at the leftmost point of S_0. For each element a_i we align a_i 1-spaced red element points along the horizontal stair S_i. On the bottom line of the staircase, we align m sets of $B + 2$ 1-spaced blue partition points, each acting as a partition. These partitions are each separated by three 1-spaced red mask points that are placed at distance 1 from the partitions and which will act as a sort of "dot mask" separating the partitions.

Clearly, the instance is balanced. Since we consider matchings, each red point must be connected to exactly one blue point. We show that there is a perfect alternating matching if and only if there is a partition of A with the desired properties. Assume we are given a perfect orthogeodesic alternating matching.

First, consider the red mask points in the middle of each mask. Each of those red mask points can only be connected to one of the hinge points since it is flanked by red points on both sides. Due to the horizontal alignment of both the mask points and the hinge points, these incidences are uniquely determined. We call any orthogeodesic chain connecting a mask point and a hinge point a *partitioner*. All the remaining red mask points must connect to the unique adjacent blue point on the bottommost stair since all other blue points are already matched to the red mask points in the middle.

Since the element points corresponding to a single element are 1-spaced, no partitioner may pass between them and, hence, the partitioners partition the elements such that the element points corresponding to a single element are all in the same partition.

Now consider the red element points. Each of these points must be connected to a blue partition point, since these points are the only remaining blue points. Since the elements have been partitioned by the partitioners and since there are exactly B blue points in each partition, it is clear that the existence of the matching implies the existence of a partition of A, which is obtained from the matching in a straightforward manner.

Conversely, given a partition, we can easily construct a valid perfect orthogeodesic alternating matching. As in the proof of Theorem 4, each orthogeodesic chain is drawn as the bottommost geodesic that runs above all geodesics drawn so far. We start with the element a_i with the smallest index in A_1. We connect the element points of a_i to the leftmost a_i blue partition points that have not yet been used. Then we proceed in the same manner with the second and third element from the first partition. After that we draw the partitioner connecting the leftmost hinge point with the leftmost middle mask point. We proceed accordingly with the remaining partitions. Due to the space we reserved between the elements along the staircase, it is clear, that we can draw all orthogeodesic chains in the desired way. $\square$

5. Long orthogeodesic alternating paths on the grid

Motivated by the hardness of deciding whether a given equitable set of red and blue points admits a Hamiltonian orthogeodesic alternating path on the grid, we consider the following optimization problem. Given a general equitable set of red and blue grid points P, we wish to find a subset $P' \subseteq P$ of maximum size such that P' admits a Hamiltonian orthogeodesic alternating path on the grid. First we show that there are point sets of size n for which P' contains at most $\lfloor n/2 \rfloor + 2$ points.

Theorem 6. *For every $n \geqslant 6$ there exists a general equitable point set P consisting of n red and blue points such that the largest point set $P' \subseteq P$ admitting a Hamiltonian orthogeodesic path on the grid has $\lfloor n/2 \rfloor + 2$ points.*

Proof. Consider a butterfly point set P consisting of n red and blue points on the grid. Let $R := \{r_1, \ldots, r_{|R|}\}$ and $B := \{b_1, \ldots, b_{|B|}\}$ denote the red and blue points of P and let P be such that $|R| \leqslant |B|$ and such that $x(r_i) = i$, $y(r_i) = i$ for all $1 \leqslant i \leqslant |R|$ as well as $x(b_i) = |R| + i$ and $y(b_i) = i - |B|$ for $1 \leqslant i \leqslant |B|$, respectively, as illustrated in Fig. 7(a). Further, let π be an orthogeodesic path with maximum length on a point set $P' \subseteq P$. Since the points are not horizontally and vertically aligned, each edge consists of at least two straight-line segments. Thus, each straight-line segment of π is incident to at most one point in P'. Let S denote the set of straight-line segments of π that are incident to a point in P' and let each segment in S be colored according to the unique point to which it is incident. That is, S contains $|P'| - 1$ red and $|P'| - 1$ blue segments. Each of the segments in S covers a grid point adjacent to the unique point in P' to which it is incident. Since all blue points are to the right and below all red points, each orthogeodesic chain incident to a red point p covers a grid point one unit to the right or one unit below r. Similarly, each orthogeodesic chain incident to a blue point b covers a grid point one unit to the left or one unit above b. Thus, the red straight-line segments of π cover a total of $|R| + 1$ distinct grid points adjacent to the red grid points and the blue straight-line segments of π cover a total of $|B| + 1$ distinct points adjacent to the blue grid points. It follows that there are at most $|R| + 1$ red segments, that is, π contains at most $|R| + 1$ edges and at most $|R| + 2$ points. Since $|R| + |B| = n$ and $|R| \leqslant |B| \lfloor n/2 \rfloor + 2 \leqslant |R| + 2$. Hence, there cannot be an orthogeodesic alternating path containing more than $\lfloor n/2 \rfloor + 2$ points.

Next we show that this is tight by proving that we can construct an orthogeodesic path of this length on P. Let $f(i)$ be defined as follows

$$f(i) := \begin{cases} \frac{i}{2} - 1 & \text{if } i \text{ is even,} \\ \frac{i-1}{2} & \text{if } i \text{ is odd.} \end{cases}$$

We start by connecting b_1 to r_1 by an L-shaped orthogeodesic chain consisting of a horizontal segment incident to b_1 and a vertical segment incident to r_1. For $1 \leqslant i \leqslant f(|R|)$ we connect r_{2i-1} to b_{2_i} by a horizontal chain whose vertical segment is one unit to the right of r_{2i-1} and we connect b_{2i} to r_{2i+1} by a vertical chain whose horizontal segment is one unit above b_{2i} as illustrated in Fig. 12(a).

Case 1: *$|R|$ is odd.* Then the constructed path has $2f(|R|) + 1 = |R|$ edges and ends in $r_{2f(|R|)+1} = r_{|R|}$. We connect $r_{|R|}$ to $b_{|R|}$ by an L-shaped edge composed of a horizontal segment incident to $r_{|R|}$ and a vertical segment incident to $b_{|R|}$, which yields an orthogeodesic path on $|R| + 2 \geqslant \lfloor n/2 \rfloor + 2$ points in total.

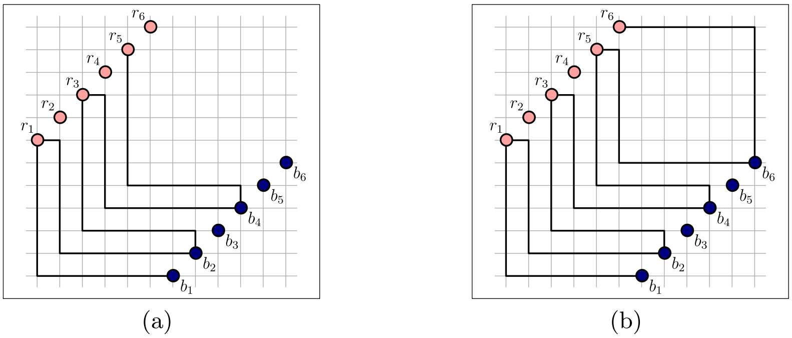


Fig. 12. Illustration for the proof of Theorem 6 depicting a butterfly point set on the grid and a longest orthogeodesic alternating path.

Case 2: $|R|$ *is even.* Then the constructed path has $2f(|R|)+1 = |R|-1$ edges and ends in $r_{2f(|R|)+1} = r_{|R|-1}$ as illustrated in Fig. 12(a). We connect $r_{|R|-1}$ to $b_{|R|}$ by a horizontal chain whose vertical segment is one unit to the right of $r_{|R|-1}$. Further, we connect $b_{|R|}$ to $r_{|R|}$ by an L-shaped edge composed of a horizontal segment incident to $r_{|R|}$ and a vertical segment incident to $b_{|R|}$, as illustrated in Fig. 12(b). The constructed path has $|R|+1$ edges and $|R|+2 \geqslant \lfloor n/2 \rfloor + 2$ points.

This concludes the proof. □

In Corollary 1 we noted that we can always find an orthogeodesic alternating path on the grid for any point set P such that each pair of points has a horizontal or vertical distance of at least two. Given an equitable point set P that does not satisfy this property, we can always find an equitable point set $P' \subseteq P$ such that P' has at least $|P|/9$ points and such that every pair of points in P' has a horizontal and a vertical distance of at least two, respectively. We can achieve this as follows. We scan the points from left to right. Let p be the current point during the scanning. We remove the point following p in the left-to-right order and in order to keep the point set equitable we arbitrarily pick another point with color distinct from $c(p)$ and remove this point as well. The resulting point set remains equitable and contains at least $|P|/3$ points. We then repeat this process vertically such that the remaining point set has at least $|P|/9$ points.

In what follows we describe an algorithm to compute a path of length $|P|/3$. To this aim, we slightly modify the algorithm described in Section 3 (in the variant suggested before Corollary 1). The algorithm will be modified so that it uses only grid lines and removes points during the execution if they would obstruct the drawing of an edge. To keep the point set balanced, we always remove pairs of points with different colors. We show that each point on the path computed by the algorithm can be charged with at most two removed points. That is, the length of the computed path is at least $|P|/3$. We prove first the following auxiliary lemma that will represent the base case of the inductive proof of the next theorem.

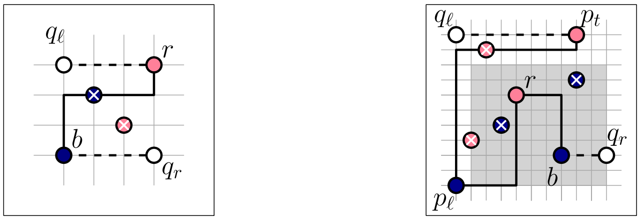
Lemma 3. *Let P be a balanced point set consisting of at least four and at most twelve points. Then P admits an orthogeodesic alternating path π satisfying the invariants* (H1) *and* (H2) *from the proof of Theorem 1 such that each point of π is charged with at most two points from P not on π.*

Proof. First, suppose that $4 \leqslant |P| \leqslant 6$. Let p_t be the topmost point and let q_b be the bottommost point with color distinct from p_t. Then $y(p_t) - y(q_b) \geqslant 2$ and we can connect p_t to q_b by a vertical chain whose horizontal segment is one unit below p_t as illustrated in Fig. 13(a). Clearly, the invariants (H1) and (H2) are maintained. Further, since P has at most six points and the constructed path π has two points, each of the two points can be charged with at most two points not on π as claimed.

Second, suppose that $8 \leqslant |P| \leqslant 12$. We show that we can find an alternating path with the desired properties consisting of at least four points. If $|P| \leqslant 12$, then such a path contains all but at most eight points. Thus, each point of the path can be charged with at most two points not on the path. Without loss of generality we assume that $|P| = 8$. Otherwise we can always find a balanced subset of P consisting of eight points. We make a case distinction similar to the case distinction in the proof of Theorem 1. We let p_ℓ, p_r, p_t and p_b denote the leftmost, rightmost, topmost and bottommost points in P. Note, that some of these points may coincide.

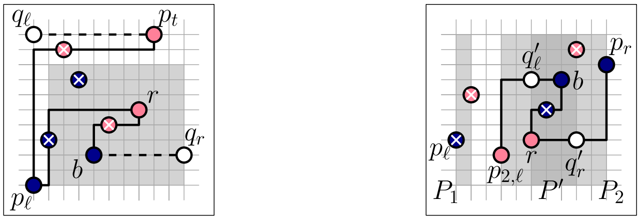
Case 1: *The color of p_ℓ is blue.* We distinguish three cases based on the colors of p_t and p_b, respectively.

Case 1.1: *The color of p_t is red.* Let p denote the point one unit below p_t distinct from p_ℓ if it exists. Let $P' := P \setminus \{p_\ell, p_t, p\}$. That is, P' is equitable and contains at least five points. Let the points in P' be denoted by the sequence $p_1, \dots, p_k$, sorted from left to right and let σ denote the sequence of colors of $p_1, \dots, p_k$ from left to right. We connect



(a) $|P| = 4$ (b) Case 1.1.1

Fig. 13. Illustrations for the proof of Lemma 3 according to the case distinction.



(a) Case 1.1.2 (b) Case 1.3.2a

Fig. 14. Illustrations for the proof of Lemma 3 according to the case distinction.

p_t to p_ℓ as illustrated in Fig. 13(b) by a vertical chain whose horizontal segment is one unit below p_t. We distinguish two cases based on the sequence σ of colors.

Case 1.1.1: *The sequence σ contains a red point r followed by a blue point b.* Then we connect r to b by an L-shaped orthogeodesic chain that is horizontally attached to r and vertically attached to b and we connect p_ℓ to r by an L-shaped orthogeodesic chain that is horizontally attached to p_ℓ and vertically attached to r as illustrated in Fig. 13(b).

Case 1.1.2: *All blue points are at the beginning of the sequence and all red points are at the end of the sequence.* First, suppose that the set P' contains three blue points and two red points. Then the point set $P' \setminus \{p_1\}$ contains a balanced set P'' of four points in which we can find a orthogeodesic alternating path consisting of two points r and b that satisfies the invariants (H1) and (H2). Thus, we can connect p_ℓ to r by a horizontal chain whose vertical segment is on the vertical line through p_1 as illustrated in Fig. 14(a).

Second, suppose that P' contains three red points and two blue points. Let b be the bottommost blue point in $P' \setminus \{p_1\}$. Then there are either two red points above or two red points below b, respectively. Suppose that there are two red points above b. The other case can be handled similarly. Let r be the topmost red point. Then $y(r) - y(b) \geqslant 2$. Hence, we can connect r to b by a vertical chain whose horizontal segment is one unit below r (analogous to what shown in Fig. 14(a)). Finally, we connect p_ℓ to r by a horizontal chain whose vertical segment is on the vertical line through p_1.

Case 1.2: *The color of p_b is red.* This case is symmetric to Case 1.1.

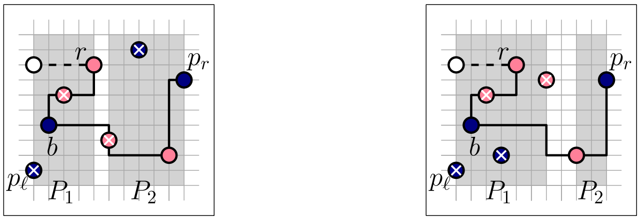
Case 1.3: *The color of p_t is red and the color of p_b is red.* We distinguish two cases based on the color of p_r.

Case 1.3.1: *The color of p_r is red.* This case is analogous to Case 1.1, except that we attach the newly created edges at the blue end of the path instead of at the red end of the path.

Case 1.3.2: *The color of p_r is blue.* By symmetry we can find a horizontal partition of P into two balanced point sets P_1 and P_2, respectively such that the leftmost point $p_{2,\ell}$ of P_2 is red. We distinguish three cases based on the size of $|P_2|$.

Case 1.3.2a: $|P_2| = 6$. Let $P' := P_2 \setminus \{p_{2,\ell}, p_r\}$ as illustrated in Fig. 14(b). Then P' is balanced and contains at least four vertices and we can find an orthogeodesic alternating path consisting of two vertices r and b such that both the horizontal segment $\overline{q'_\ell b}$ and the horizontal segment $\overline{r q'_r}$ do not intersect the edge between r and b, where q'_ℓ denotes the point on the left side of $\mathcal{B}(P')$ that is horizontally aligned with b and q'_r denotes the point on the right side of $\mathcal{B}(P')$ that is horizontally aligned with r. That is, we can connect $p_{2,\ell}$ to b by an L-shaped orthogeodesic chain that is vertically attached to $p_{2,\ell}$ and horizontally attached to b and we can connect r to p_r by an L-shaped orthogeodesic chain that is horizontally attached to r and vertically attached to p_r as illustrated in Fig. 14(b). Then we have constructed a path consisting of four vertices.

Case 1.3.2b: $|P_2| = 4$. Then $|P_1| = 4$ and we can find a path consisting of a red point r and a blue point b with the desired properties in P_1. Let p be the rightmost red point in P_2. Then we can connect b to p by a horizontal chain



(a) Case 1.3.2b (b) Case 1.3.2c

Fig. 15. Illustrations for the proof of Lemma 3 according to the case distinction.

whose vertical segment is on the vertical line of the leftmost red point $p_{2,\ell}$ in P_2 as illustrated in Fig. 15(a). Note that $p_{2,\ell} \neq p$ since P_2 contains 2 red points of which p is the rightmost.

Case 1.3.2c: $|P_2| = 2$. Then $|P_1| = 6$ and the leftmost five points in P_1 must contain a balanced point set P_1' of four vertices. Then we can find an orthogeodesic chain in P_1' connecting a red point r and a blue point b in P_1'. Then we connect $p_{2,\ell}$ to p_r by an L-shaped chain that is horizontally attached to $p_{2,\ell}$ and vertically attached to p_r and we connect b to $p_{2,\ell}$ by a horizontal chain whose vertical segment is on the vertical line through the rightmost point of P_1 as illustrated in Fig. 15(b).

Case 2: *The color of p_ℓ is red.* We distinguish two cases based on the color of p_r.

Case 2.1: *The color of p_r is blue.* Note that the point set P has the same properties as the point set P' in Case 1.3.2a. Therefore, we can handle this case in exactly the same way illustrated in Fig. 14(b), except that need to charge two points less to the computed path.

Case 2.2: *The color of p_r is red.* This case is analogous to Case 1.3.2, except that we exchange the roles of P_1 and P_2.

In all cases we constructed a path with four vertices satisfying the invariants (H1) and (H2). Since P contains at most twelve vertices, at most eight vertices are not contained in the path. Hence, each point of the path can be charged with at most two of these points. □

Now we turn to the proof of Theorem 7.

Theorem 7. *Let P be an equitable set of grid points. There is an $\mathcal{O}(n \log^2 n)$-time algorithm that computes an equitable set $P' \subseteq P$ with $|P'| \geqslant |P|/3$ that admits a Hamiltonian orthogeodesic alternating path on the grid.*

Proof. Our proof is based on a case distinction similar to the proof of Theorem 1. Note that we can find an orthogeodesic alternating path on the grid containing two vertices for every equitable point set with at most three points by simply connecting an arbitrary red point to an arbitrary blue point by an L-shaped edge. Thus, we only consider point sets with at least four points.

First, we consider only the balanced case. That is, for every general balanced point set consisting of $n = 2k$ points with $k \geqslant 1$, we prove that we can compute an orthogeodesic alternating path on the grid containing at least $|P|/3$ points such that the invariants (H1) and (H2) given in the proof of Theorem 1 are maintained and such that each point on the path is charged at most two points that are not on the path. The proof is by induction on n. We let the base cases be all balanced point sets with at least four and at most twelve points. By Lemma 3 we can always find an orthogeodesic alternating path with the desired properties on such a point set.

Next, suppose that the induction hypothesis holds for all $2k < n$ such that $k \geqslant 2$ and $n \geqslant 14$. We make a case distinction according to the proof of Theorem 1 and as illustrated in Fig. 4. Additionally, we use the definitions and terminology according to the proof of Theorem 1. That is, by p_ℓ, p_r, p_b and p_t we denote the leftmost, rightmost, bottommost and topmost points in P, respectively.

Case 1: *The color of p_ℓ is blue.* We distinguish three sub-cases.

Case 1.1: *The color of p_t is red.* Let $Q \subseteq P$ be the points distinct from p_ℓ and p_t on the horizontal line one unit below p_t and on the vertical line one unit to the right of p_ℓ, respectively, that is $|Q| \leqslant 2$ and $p_\ell, p_t \notin Q$.

Let Q' be an arbitrary set of two points from $P \setminus (Q \cup \{p_\ell, p_t\})$ such that $P' := P \setminus (Q \cup Q')$ is balanced. Then P' contains at least eight points and we can apply the induction hypothesis to P'. The at most four points in $Q \cup Q'$ will be charged to the two new vertices on the path.

We connect p_ℓ to the path computed in P' as in Case 1.1 in the proof of Theorem 1 and as illustrated in Fig. 1(b).

Case 1.2: *The color of p_t is blue and the color of p_b is red.* This case is obtained from Case 1.1 by a reflection.

Case 1.3: *The color of p_t is blue and the color of p_b is blue.* We consider two sub-cases depending on the color of p_r.

Case 1.3.1: *The color of p_r is red.* This case is similar to Case 1.1. Instead of attaching two new points to the left side of the path, we attach two more edges at the right side of the path. Clearly, we must remove at most four points for two newly created edges.

Case 1.3.2: *The color of p_r is blue.* Let P_1 and P_2 be a partition of P according to Lemma 1. By symmetry we can choose P_2 such that the leftmost point $p_{2,\ell}$ in P_2 is red. Suppose that the vertical line one unit to the left of $p_{2,\ell}$ is occupied by a point p. Clearly, $p \in P_1$.

First, suppose that P_1 contains at most four vertices. Then P_2 contains at least ten vertices and we can handle P_2 according to Case 2.1 of Theorem 1 without charging the first and the last point of the resulting path with any removed points since we can use L-shaped edges incident to these points, which do not cross any other points. Note that the point set $P'_2 := P_2 \setminus \{p_{2,\ell}, p_r\}$ still contains at least eight points and thus we can apply the induction hypothesis on P'_2 after handling P_2 according to Case 2.2 of Theorem 1. That is, we can remove all points in P_1 and charge the removal to p_ℓ.

Otherwise P_1 contains at least six points, that is, we can remove p and an arbitrary point p' whose color is different from $c(p)$ and apply the induction hypothesis to the set $P'_1 := P_1 \setminus \{p, p'\}$ since $|P'_1| \geqslant 4$. If P_2 has only two vertices, we directly connect $p_{2,\ell}$ to p_r by an L-shaped edge and charge the removal of p and p' to $p_{2,\ell}$. If P_2 has four vertices, then we can also connect $p_{2,\ell}$ to p_r by an L-shaped edge and charge the removal of p, p' and the remaining two points in $P_2 \setminus \{p_{2,\ell}, p_r\}$ to $p_{2,\ell}$ and p_r, respectively. Finally, if P_2 contains at least six vertices, we can handle P_2 according to Case 2.1 of Theorem 1 without charging $p_{2,\ell}$ and p_r since we can apply the induction hypothesis to the point set $P' := P_2 \setminus \{p_{2,\ell}, p_r\}$ containing at least four vertices.

The sub-paths constructed for P_1 and P_2, if any, are connected according to Case 1.3.2 of the proof of Theorem 1 and as illustrated in Fig. 3(b).

Case 2: *The color of p_ℓ is red.* We consider two sub-cases.

Case 2.1: *The color p_r is blue.* Note that P has at least 14 points, that is we can apply the induction hypothesis to $P \setminus \{p_\ell, p_r\}$. Then we connect p_ℓ to b' and r' to p_r using L-shaped edges. Clearly, this satisfies invariant (H1) since p_ℓ and p_r at on the left and right side of the bounding box of P, respectively.

Case 2.2: *The color of p_r is red.* This case is similar to Case 1.3.2, except that we exchange the roles of P_1 and P_2.

The unbalanced case can be handled similar to the proof of Theorem 1. Suppose that P is an unbalanced equitable point set consisting of at least five red and blue points. We may assume without loss of generality that $|B| = |R| + 1$. First, consider the case that one of the points p on the boundary of $\mathcal{B}(P)$ is blue. Assume without loss of generality that p is on the left side. Then we can compute a path as described earlier for the balanced set of points $P' := P \setminus \{p\}$ and connect p to this path by an L-shaped orthogeodesic chain that is vertically attached to p and horizontally attached to the red end of the path computed for P'.

Second, consider the case that all points on the boundary are red. Then we add a new red point r to the left of P and consider the point set $P' := P \cup \{r\}$. Then P' has at least six points such that the leftmost two points are red. We split P' into two point sets P_1 and P_2 according to Lemma 1 such that the rightmost point $p_{1,r}$ in P_1 is blue. Clearly, P_1 contains at least four points, since the leftmost two points are red. First, suppose that P_1 contains exactly four points $p_1, \ldots, p_4$ sorted from left to right. Then the color of $r = p_1$ and p_2 is red and the color of p_3 and p_4 is blue, respectively. If P_2 contains only two vertices, then P contains only six points. Then we can pick any balanced subset of points from P and compute an orthogeodesic alternating path with two points and we are done. If P_2 contains at least four points, we can compute a path in P_2 the by induction hypothesis starting with a red point r_2 and ending in a blue point b_2. Then we connect p_2 to p_3 by an L-shaped chain that is vertically attached to p_2 and horizontally attached to p_3 and we connect p_3 to r_1 by an L-shaped chain that is vertically attached to p_3 and horizontally attached to r_2. The removed points p_1 and p_4 are charged to p_3. Second, suppose that P_1 contains at least six points. Then we can handle P_1 according to Case 2.1 of Theorem 1 without charging r with the removal of any point. Finally, since r is only used as and endpoint of the path, if it is used at all, and since it is not charged with the removal of any points, we can safely remove it again.

Note that we can decide which points to remove before recursing on the point sets from which we removed the points. Therefore, the rest of the algorithm can be implemented and analyzed as in the proof of Theorem 1. Thus, the algorithm can be implemented to run in $\mathcal{O}(n \log^2 n)$ time. □

6. Concluding remarks

In this paper, we studied the existence of Hamiltonian orthogeodesic alternating paths. We proved that such a path can always be computed in $\mathcal{O}(n \log^2 n)$ time on every general point set. However, if we require both points and bends to be placed on the grid, then we proved that this problem is $\mathcal{NP}$-complete. Analogous $\mathcal{NP}$-completeness results holds for Hamiltonian orthogeodesic alternating cycles and matchings. Finally, we studied the problem of finding an orthogeodesic alternating path on the grid with maximum length and we presented a factor-3 approximation algorithm whose running time is $\mathcal{O}(n \log^2 n)$.

We conclude by observing that the results in this paper can be interpreted in terms of orthogeodesic point-set embeddability. From this point of view, the problem can be defined as follows. Let G be a planar graph whose vertices are properly 2-colored (adjacent vertices have distinct colors) and let P be a set of red and blue points. An orthogeodesic point-set embedding of G on P is a planar drawing such that each edge is drawn as an orthogeodesic chain and each vertex is represented by a point of P with the same color. Clearly, a trivial necessary condition for the existence of such a point-set embedding is that the number of red and blue points is equal to the number of red and blue vertices of G, respectively. If P satisfies this necessary condition, we say that P is *compatible* with G. Also, G must be a bipartite graph otherwise it cannot be properly 2-colored.

Theorem 1 implies that if G is a path, then G admits an orthogeodesic point-set embedding on every general point set P. It is natural to ask what is the largest family of planar bipartite graphs that admit an orthogeodesic point-set embedding on every compatible general point set. We can prove that this family is the family of paths.

Theorem 8. *Let G be a properly 2-colored planar graph. If G admits an orthogeodesic point-set embedding on every general point set compatible with G, then G is a forest of paths.*

Proof. First, note that if G has less than four vertices the theorem is trivially true. Assume then the G has at least four vertices. If G is not a forest of paths, then either G contains a degree-3 vertex, or it contains at least one cycle.

Let P be a butterfly. Any orthogeodesic chain connecting a blue point b to a red point must have either a horizontal segment incident to b from the right or a vertical segment incident to b from above. Thus, at most two orthogeodesic chains from a blue point b to the red points can be incident to b in an orthogeodesic fashion. Analogously, at most two orthogeodesic chains from a red point r to blue points can be incident to r without crossing. It follows that G cannot have a degree-3 vertex. Thus G must contain at least one cycle C. Since G is bipartite, each cycle of G has even length and at least four vertices. Suppose that C admits an orthogeodesic point-set embedding on (a subset of) P and assume that C has more than four vertices. Let b be a blue point that is neither the leftmost nor the rightmost blue point. Let r_1 be one of the two red points adjacent to b in C. Let χ_1 be the orthogeodesic chains connecting b to r_1. Since b is neither the leftmost blue point nor the rightmost blue point, there should be an orthogeodesic chain connecting a point to the right of χ_1 to a point to the left of χ_1. However, no orthogeodesic chain can connect a point to the left of χ_1 to a point to the right of χ_1 without crossing χ_1. Assume now that C has exactly four vertices. Let $r_1, r_2 \in R$ and $b_1, b_2 \in B$ be the four points on which C is embedded, then both r_1 and r_2 must be connected with orthogeodesic chains to both b_1 and b_2. This is not possible without creating a crossing. □

7. Open problems

The algorithm that we presented for computing a Hamiltonian orthogeodesic alternating path off the grid needs two bends on some of the edges. While it is not always possible to construct a path with at most one bend per edge, it would be interesting to characterize the point sets admitting such a path and to devise an efficient algorithm for computing such a path, if it exists. Further, it is an interesting open problem to study the gap between the factor-3 approximation algorithm of the longest orthogeodesic alternating path and the presented upper bound for the worst-case ratio of an approximation algorithm.

Motivated by Theorem 8 one can characterize the point sets that support 2-colored orthogeodesic point-set embeddings for classes of graphs wider than paths, such as cycles, trees and planar or outer-planar graphs. Also, it would be interesting to study the complexity of deciding whether a given 2-colored planar graph admits an orthogeodesic point-set embedding on a given compatible point set.

Finally, we can consider similar problems for more than two colors. *Straight-line* alternating paths on multi-colored point sets have been studied by Merino et al. [12] but orthogeodesic alternating paths do not seem to have been considered on multi-colored point sets as of yet.

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