

An Algorithm for Probabilistic Alternating Simulation

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Abstract. In probabilistic game structures, probabilistic alternating simulation (PA-simulation) relations preserve formulas defined in probabilistic alternating-time temporal logic with respect to the behaviour of a subset of players. We propose a partition based algorithm for computing the largest PA-simulation. It is to our knowledge the first such algorithm that works in polynomial time. Our solution extends the generalised coarsest partition problem (GCPP) to a game-based setting with mixed strategies. The algorithm has higher complexities than those in the literature for non-probabilistic simulation and probabilistic simulation without mixed actions, but slightly improves the existing result for computing probabilistic simulation with respect to mixed actions.

1 Introduction

Simulation and bisimulation relations are useful tools in the verification of finite and infinite state systems. State space minimisation modulo these relations is a valuable technique to fight the state explosion problem in model checking, since bisimulation preserves properties formulated in logics like CTL and CTL* [9] while simulation preserves the universal (or safe) fragment of these logics [16].

In some situations, however, it is necessary to model quantitative aspects of a system. It is the case, for instance, in wireless networks, where we often need to assume that there is a chance of connection failure with a given rate. This requires modelling network systems with randomised behaviours (e.g., by pooling a connection after uncertain amount of time to minimise conflict). Another important fact of real-world systems is that environment changes, such as unexpected power-off, are often unpredictable. Therefore, we need to encode appropriate system behaviours to handle such situations, and in order to do so, it is sometimes crucial to employ probabilistic strategies to achieve the best possible outcomes [29]. One simple example is the rock-scissor-paper game where there is no deterministic strategy to win since the other player's move is unknown, but there is a probabilistic strategy, sometimes called *mixed strategy*, to win at least a third of all cases in a row, regardless of what the other player does.³

³ A mixed strategy also ensures an eventual win but deterministic strategies do not.

A probabilistic game structure (PGS) is a model that has probabilistic transitions, and allows the consideration of probabilistic choices of players. The simulation relation in PGSs, called probabilistic alternating simulation (PA-simulation), has been shown to preserve a fragment of probabilistic alternating-time temporal logic (PATL) under *mixed strategies*, which is used in characterising what a group of players can enforce in such systems [30]. In this paper we propose a polynomial-time algorithm for computing the largest PA-simulation, which is, to the best of our knowledge, the first algorithm for computing a simulation relation in probabilistic concurrent games. A PGS combines the modelling of probabilistic transitions from probabilistic automata (PA), and the user interactions from concurrent game structures (GS). In PA, the probabilistic notions of simulation preserve PCTL safety formulas [24]. The *alternating simulation* [3] in GS has been proved to preserve a fragment of ATL^* , under the semantics of *deterministic strategies*. These simulation relations are computable in polynomial time for finite systems [31, 3].

Related work. Efficient algorithms have been proposed for computing the largest simulation (e.g., see [17, 27, 5, 15, 28]) in finite systems, with a variety of time and space complexities. In particular, Gentilini et al. [15] develop an efficient algorithm with an improved time complexity based on the work of Henzinger et al. [17] without losing the optimal space complexity. Van Glabbeek and Ploeger [28] later find a flaw in [15] and propose a non-trivial fix. The best algorithm for simulation in terms of time complexity is [21]. To compute probabilistic simulation, Baier et al. [4] reduce the problem of establishing a weight function for the lifted relation to a maximal flow problem [1]. Cattani and Segala [6] reduce the problem of deciding strong probabilistic bisimulation to LP [22] problems. Zhang et al. [32] develop algorithms with improved time complexity for probabilistic simulations, following [4, 6]. Crafa and Ranzato [10] improve the time complexity of the algorithms of Zhang et al. [32] by applying abstract interpretation. A space efficient probabilistic simulation algorithm is proposed by Zhang [31] using the techniques proposed in [15, 28].

Studies on stochastic games have actually been carried out since as early as the 1950s [25], and a rich literature has developed in recent years (e.g. see [12, 11, 13, 7]). One existing approach called game metrics [14] defines approximation-based simulation relations, with a kernel simulation characterising the logic quantitative μ -calculus ($q\mu$) [11], an extension of modal μ -calculus [20] where each state is assigned a quantitative value in $[0, 1]$ for every formula. However, so far the best solutions in the literature on approximating the simulation as defined in the metrics for concurrent games potentially take exponential time [8]. Although PA-simulation is strictly stronger than the kernel simulation relation of the game metrics in [14], the algorithm presented in the paper has a more tractable complexity result, and we believe that it will benefit the abstraction or refinement based techniques for verifying game-based properties.

Structure of the paper. Sect. 2 defines basic notions that are used in the technical part. In Sect. 3 we propose a solution of calculating largest PA-simulation in

finite PGSSs, based on GCPP. The algorithms on PA-simulation is presented in Sect. 4. We conclude the paper in Sect. 5.

2 Preliminaries

Probabilistic game structures are defined in terms of discrete probabilistic distributions. A *discrete probabilistic distribution* Δ over a finite set S is a function of type $S \rightarrow [0, 1]$, where $\sum_{s \in S} \Delta(s) = 1$. We write $\mathcal{D}(S)$ for the set of all such distributions on a fixed S . For a set $T \subseteq S$, define $\Delta(T) = \sum_{s \in T} \Delta(s)$. Given a finite index set I , a list of distributions $(\Delta_i)_{i \in I}$ and a list of probabilities $(p_i)_{i \in I}$ where, for all $i \in I$, $p_i \in [0, 1]$ and $\sum_{i \in I} p_i = 1$, $\sum_{i \in I} p_i \Delta_i$ is obviously also a distribution. For $s \in S$, \bar{s} is called a *point (or Dirac) distribution* satisfying $\bar{s}(s) = 1$ and $\bar{s}(t) = 0$ for all $t \neq s$. Given $\Delta \in \mathcal{D}(S)$, we define $[\Delta]$ as the set $\{s \in S \mid \Delta(s) > 0\}$, which is the *support* of Δ .

In this paper we assume a set of two players $\{\text{I}, \text{II}\}$ (though our results can be extended to handle a finite set of players as in the standard game structure and ATL semantics [2]), and *Prop* a finite set of propositions.

Definition 1. A probabilistic game structure \mathcal{G} is a tuple $\langle S, s_0, \mathcal{L}, Act, \delta \rangle$, where

- S is a finite set of states, with s_0 the initial state;
- $\mathcal{L} : S \rightarrow 2^{Prop}$ is the labelling function which assigns to each state $s \in S$ a set of propositions that are true in s ;
- $Act = Act_{\text{I}} \times Act_{\text{II}}$ is a finite set of joint actions, where Act_{I} and Act_{II} are, respectively, the sets of actions for players I and II;
- $\delta : S \times Act \rightarrow \mathcal{D}(S)$ is a transition function.

If in state s player I performs action a_1 and player II performs action a_2 then $\delta(s, \langle a_1, a_2 \rangle)$ is the distribution for the next states. During each step the players choose their next moves simultaneously. We define a *mixed action* of player I (II) as a distribution over Act_{I} (Act_{II}), and write Π_{I} (Π_{II}) for the set of mixed actions of player I (II).⁴ In particular, \bar{a} is a *deterministic* mixed action which always chooses a . We lift the transition function δ to handle mixed actions. Given $\pi_1 \in \Pi_{\text{I}}$ and $\pi_2 \in \Pi_{\text{II}}$, for all $s, t \in S$, we have

$$\bar{\delta}(s, \langle \pi_1, \pi_2 \rangle)(t) = \sum_{a_1 \in Act_{\text{I}}, a_2 \in Act_{\text{II}}} \pi_1(a_1) \cdot \pi_2(a_2) \cdot \delta(s, \langle a_1, a_2 \rangle)(t)$$

Example 1. Assume $Prop = \{p\}$. A simple PGS with the initial state s_0 in Fig. 1 can be defined as $\mathcal{G} = \langle S, s_0, \mathcal{L}, Act, \delta \rangle$, where

- $S = \{s_0, s_1, s_2\}$;
- $\mathcal{L}(s_0) = \mathcal{L}(s_1) = \emptyset$ and $\mathcal{L}(s_2) = \{p\}$;

⁴ Note Π_{I} is equivalent to $\mathcal{D}(Act_{\text{I}})$, though we choose a different symbol because the origin of a mixed action is a simplified *mixed strategy* of player I which has type $S^+ \rightarrow \mathcal{D}(Act_{\text{I}})$. A mixed action only considers player I's current step.

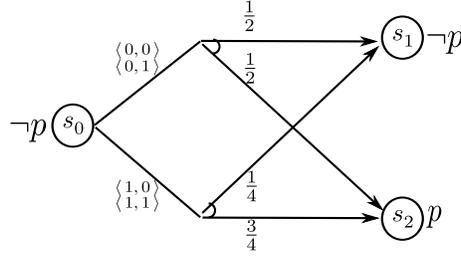


Fig. 1. A probabilistic game structure.

- $Act_I = Act_{II} = \{0, 1\}$;
- $\delta(s_0, \langle 0, 0 \rangle) = \delta(s_0, \langle 0, 1 \rangle) = \Delta$ with $\Delta(s_1) = \Delta(s_2) = \frac{1}{2}$ and $\delta(s_0, \langle 1, 0 \rangle) = \delta(s_0, \langle 1, 1 \rangle) = \Delta'$ with $\Delta'(s_1) = \frac{1}{4}$, $\Delta'(s_2) = \frac{3}{4}$;
- $\delta(s_i, a) = \bar{s}_i$ for $i \in \{1, 2\}$ and $a \in Act$ (s_1 and s_2 are absorbing states).

Definition 2. Given a list of mixed actions $\langle \pi_i \rangle_{i \in I}$ (of player I), $\langle p_i \rangle_{i \in I}$ satisfying $\sum_{i \in I} p_i = 1$, $\sum_{i \in I} p_i \pi_i$ is a mixed action defined by $(\sum_{i \in I} p_i \pi_i)(s)(a) = \sum_{i \in I} p_i \cdot (\pi_i(s)(a))$ for all $s \in S$ and $a \in Act_I$.

Lemma 1. Let $s \in S$, $\pi \in \Pi_I$ and $\sigma = \sum_{i \in I} p_i \sigma_i \in \Pi_{II}$, then $\bar{\delta}(s, \langle \pi, \sigma \rangle) = \sum_{i \in I} p_i \cdot \bar{\delta}(s, \langle \pi, \sigma_i \rangle)$.

Proof. Let $t \in S$, then

$$\begin{aligned}
& \bar{\delta}(s, \langle \pi, \sigma \rangle)(t) \\
&= \sum_{a_1 \in Act_I} \sum_{a_2 \in Act_{II}} \pi(s)(a_1) \cdot \sigma(s)(a_2) \cdot \delta(s, \langle a_1, a_2 \rangle)(t) \\
&= \sum_{a_1 \in Act_I} \sum_{a_2 \in Act_{II}} \pi(s)(a_1) \cdot \sum_{i \in I} p_i \cdot \sigma_i(s)(a_2) \cdot \delta(s, \langle a_1, a_2 \rangle)(t) \\
&= \sum_{i \in I} p_i \cdot \left(\sum_{a_1 \in Act_I} \sum_{a_2 \in Act_{II}} \pi(s)(a_1) \cdot \sigma_i(s)(a_2) \cdot \delta(s, \langle a_1, a_2 \rangle)(t) \right) \\
&= \sum_{i \in I} p_i \cdot \bar{\delta}(s, \langle \pi, \sigma_i \rangle)(t)
\end{aligned}$$

The proof of the following lemma is similar to the above.

Lemma 2. Let $s \in S$, $\pi = \sum_{i \in I} p_i \pi_i \in \Pi_I$ and $\sigma \in \Pi_{II}$, then $\bar{\delta}(s, \langle \pi, \sigma \rangle) = \sum_{i \in I} p_i \cdot \bar{\delta}(s, \langle \pi_i, \sigma \rangle)$.

Simulation relations in probabilistic systems require a definition of *lifting* [18], which extends the relations to the domain of distributions.⁵ Let S, T be two sets and $\mathcal{R} \subseteq S \times T$ be a relation, then $\bar{\mathcal{R}} \subseteq \mathcal{D}(S) \times \mathcal{D}(T)$ is a *lifted relation* defined by $\Delta \bar{\mathcal{R}} \Theta$ if there exists a weight function $w : S \times T \rightarrow [0, 1]$ such that

- $\sum_{t \in T} w(s, t) = \Delta(s)$ for all $s \in S$,
- $\sum_{s \in S} w(s, t) = \Theta(t)$ for all $t \in T$,
- $s \mathcal{R} t$ for all $s \in S$ and $t \in T$ with $w(s, t) > 0$.

⁵ In a probabilistic system without explicit user interactions, state s is simulated by state t if for every $s \xrightarrow{a} \Delta_1$ there exists $t \xrightarrow{a} \Delta_2$ such that Δ_1 is simulated by Δ_2 .

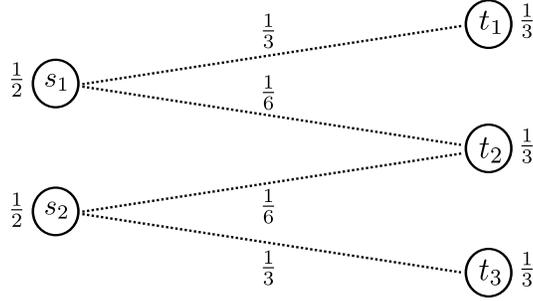


Fig. 2. An example showing how to lift one relation.

The intuition behind the lifting is that each state in the support of one distribution may correspond to a number of states in the support of the other distribution, and vice versa. The example in Fig. 2 is taken from [23] to show how to lift one relation. We have two set of states $S = \{s_1, s_2\}$ and $T = \{t_1, t_2, t_3\}$, and $\mathcal{R} = \{(s_1, t_1), (s_1, t_2), (s_2, t_2), (s_2, t_3)\}$. We have $\Delta \overline{\mathcal{R}} \Theta$, where $\Delta(s_1) = \Delta(s_2) = \frac{1}{2}$ and $\Theta(t_1) = \Theta(t_2) = \Theta(t_3) = \frac{1}{3}$. To check this, we define a weight function w by: $w(s_1, t_1) = \frac{1}{3}$, $w(s_1, t_2) = \frac{1}{6}$, $w(s_2, t_2) = \frac{1}{6}$, and $w(s_2, t_3) = \frac{1}{3}$. The dotted lines indicate the allocation of weights required to relate Δ to Θ via $\overline{\mathcal{R}}$. By lifting in this way, we are able to extend the notion of alternating simulation [3] to a probabilistic setting.

We present a property of lifted relations. that by combining distributions that are lift-related with the same weight on both sides we get the resulting distributions lift-related.

Lemma 3. *Let $\mathcal{R} \subseteq S \times S'$ and $\langle p_i \rangle_{i \in I}$ be a list of values satisfying $\sum_{i \in I} p_i = 1$, $\Delta_i \overline{\mathcal{R}} \Delta'_i$ for $\Delta_i \in \mathcal{D}(S)$ and $\Delta'_i \in \mathcal{D}(S')$ for all i , then $\sum_{i \in I} p_i \Delta_i \overline{\mathcal{R}} \sum_{i \in I} p_i \Delta'_i$.*

Now we present the definition of PA-I-simulation.

Definition 3. *Given a PGS, a probabilistic alternating I-simulation (PA-I-simulation) is a relation $\sqsubseteq \subseteq S \times S$ such that if $s \sqsubseteq t$, then*

- $\mathcal{L}(s) = \mathcal{L}(t)$,
- for all $\pi_1 \in \Pi_I$, there exists $\pi'_1 \in \Pi_I$, such that for all $\pi'_2 \in \Pi_{II}$, there exists $\pi_2 \in \Pi_{II}$, such that $\bar{\delta}(s, \langle \pi_1, \pi_2 \rangle) \sqsubseteq \bar{\delta}(t, \langle \pi'_1, \pi'_2 \rangle)$.

If s PA-I-simulates t and t PA-I-simulates s , we say s and t are PA-I-simulation equivalent.⁶

PA-I-simulation has been shown to preserve a fragment of PATL which covers the ability of player I to enforce certain temporal requirements [30]. For example, if in state s player I can enforce reaching some states satisfying p within 5 transition steps and with probability at least $\frac{1}{2}$, written $s \models \langle\langle I \rangle\rangle^{\geq \frac{1}{2}} \diamond^{\leq 5} p$, then for every state t that simulates s with respect to I, i.e., $s \sqsubseteq t$ by some PA-I-simulation ' \sqsubseteq ', we also have $t \models \langle\langle I \rangle\rangle^{\geq \frac{1}{2}} \diamond^{\leq 5} p$.

⁶ Alternating simulations and equivalences are for player I unless stated otherwise.

General Coarsest Partition Problem

The general coarsest partition problem (GCPP) provides a characterisation of (non-probabilistic) simulation in finite state transition systems [15]. Informally, in this approach, states that are (non-probabilistic) simulation equivalent are grouped into the same block, and all such blocks form a partition over the (finite) state space. Based on the partition, blocks are further related by a partial order \preceq , so that if $P \preceq Q$, then every state in block P is simulated by every state in block Q . The GCPP is to find, for a given PGS, the smallest such set of blocks. In the literature such a methodology yields space efficient algorithms for computing the largest (non-probabilistic) simulation relation in a finite system [15, 28]. Similar methods have been adopted and developed to compute the largest simulation relations in the model of probabilistic automata [31].

We briefly review the basic notions that are required to present the GCPP problem. A *partition* over a set S , is a collection $\Sigma \subseteq \mathcal{P}(S)$ satisfying (1) $\bigcup \Sigma = S$ and (2) $P \cap Q = \emptyset$ for all distinct *blocks* $P, Q \in \Sigma$. Given $s \in S$, write $[s]_\Sigma$ for the block in partition Σ that contains s . A partition Σ_1 is *finer* than Σ_2 , written $\Sigma_1 \triangleleft \Sigma_2$, if for all $P \in \Sigma_1$ there exists $Q \in \Sigma_2$ such that $P \subseteq Q$.

Given a set S , a *partition pair* over S is (Σ, \preceq) where Σ is a partition over S and $\preceq \subseteq \Sigma \times \Sigma$ is a partial order. Write $Part(S)$ for the set of partition pairs on S . If $\mathcal{Y} \triangleleft \Sigma$ and \preceq is a relation on Σ , then $\preceq(\mathcal{Y}) = \{(P, Q) \mid P, Q \in \mathcal{Y}, \exists P', Q' \in \Sigma, P \subseteq P', Q \subseteq Q', P' \preceq Q'\}$ is the relation on \mathcal{Y} *induced* by \preceq . Let (Σ_1, \preceq_1) and (Σ_2, \preceq_2) be partition orders, write $(\Sigma_1, \preceq_1) \leq (\Sigma_2, \preceq_2)$ if $\Sigma_1 \triangleleft \Sigma_2$, and $\preceq_1 \subseteq \preceq_2$ (Σ_1). Define a relation $\sqsubseteq_{(\Sigma, \preceq)} \subseteq S \times S$ as determined by a partition pair (Σ, \preceq) by $s \sqsubseteq_{(\Sigma, \preceq)} t$ iff $[s]_\Sigma \preceq [t]_\Sigma$.

Let $\rightarrow \subseteq S \times S$ be a (transition) relation and $\mathcal{L} : S \rightarrow 2^{Prop}$ a labelling function, then a relation \sqsubseteq is a simulation on S if for all $s, t \in S$ with $s \sqsubseteq t$, we have (1) $\mathcal{L}(s) = \mathcal{L}(t)$ and (2) $s \rightarrow s'$ implies $t \rightarrow t'$ and $s' \sqsubseteq t'$. Let (Σ, \preceq) be a partition pair on S , then it is *stable* with respect to \rightarrow if for all $P, Q \in \Sigma$ with $P \preceq Q$ and $s \in P$ such that $s \rightarrow s'$ with $s' \in P' \in \Sigma$, then there exists $Q' \in \Sigma$ such that for all $t \in Q$, there exists $t' \in Q'$ such that $t \rightarrow t'$. The following result is essential to the GCPP approach, as we derive the largest simulation relation by computing the coarsest partition pair over a finite state space.⁷

Proposition 1. [15, 28] *Let (Σ, \preceq) be a partition pair, then it is stable with respect to \rightarrow iff the induced relation $\sqsubseteq_{(\Sigma, \preceq)}$ is a simulation (with respect to \rightarrow).*

Given a transition relation on a state space there exists a unique largest simulation relation. Thus, solutions to GCPP provide the coarsest stable partition pairs, and they have been proved to characterise the largest simulation relations in non-probabilistic systems [15, 28].

⁷ We choose the word *coarsest* for partition pairs to make it consistent with the standard term GCPP, and it is clear in the context that *coarsest* carries the same meaning as *largest* with respect to the order \leq defined on partition pairs.

3 Solving GCPP in Probabilistic Game Structures

In this section we extend the GCPP framework to characterise PA-simulations in PGSs. Given a PGS $\mathcal{G} = \langle S, s_0, \mathcal{L}, Act, \delta \rangle$, a *partition pair* over \mathcal{G} is (Σ, \preceq) where Σ is a partition over S . Write $Part(\mathcal{G})$ for the set of all partition pairs over S . We show how to compute the coarsest partition pair and prove that it characterises the largest PA-simulation for a given player.

Since in probabilistic systems transitions go from states to distributions over states, we first present a probabilistic version of *stability*, as per [31]. Let $\rightarrow \subseteq S \times \mathcal{D}(S)$ be a probabilistic (transition) relation. For a distribution $\Delta \in \mathcal{D}(S)$ and Σ a partition, write Δ_Σ as a distribution on Σ defined by $\Delta_\Sigma(P) = \Delta(P)$ for all $P \in \Sigma$. Let (Σ, \preceq) be a partition pair, it is *stable* with respect to the relation \rightarrow , if for all $P, Q \in \Sigma$ with $P \preceq Q$ and $s \in P$ such that $s \rightarrow \Delta$, then for all $t \in Q$ there exists $t \rightarrow \Theta$ such that $\Delta_\Sigma \preceq \Theta_\Sigma$.

Another obstacle in characterising PA-simulation is that the concerned player can only partially determine a transition. That is, after player I performs an action on a state, the exact future distribution on next states depends on an action from player II. Therefore, we need to (again) lift the stability condition for PA-I-simulation from distributions to sets of distributions.

Let $\leq \subseteq S \times S$ be a partial order on a set S , define $\leq_{sm} \subseteq \mathcal{P}(S) \times \mathcal{P}(S)$, by $P \leq_{sm} Q$ if for all $t \in Q$ there exists $s \in P$ such that $s \leq t$. In the literature this definition is known as a ‘Smyth order’ [26]. In a PGS, we ‘curry’ the transition function by defining $\bar{\delta}(s, \pi_1) = \{\bar{\delta}(s, \langle \pi_1, \pi_2 \rangle) \mid \pi_2 \in \Pi_{II}\}$, which is the set of distributions that are possible if player I takes a mixed action $\pi_1 \in \Pi_I$ on $s \in S$.

Definition 4. (*lifted stability*) Let (Σ, \preceq) be a partition pair on S in a PGS, it is *stable with respect to player I’s choice*, if for all $\pi \in \Pi_I$, $P, Q \in \Sigma$ with $P \preceq Q$ and $s \in P$, there exists $\pi' \in \Pi_I$ such that $\bar{\delta}(s, \pi)_\Sigma \preceq_{sm} \bar{\delta}(t, \pi')_\Sigma$ for all $t \in Q$.

Intuitively, the Smyth order captures the way of *behavioral* simulation. That is, if $\bar{\delta}(t, \pi')$ is at least as restrictive as $\bar{\delta}(s, \pi)$, then whatever player I is able to enforce by performing π in s , he can also enforce it by performing π' in t , as player II has *fewer* choices in $\bar{\delta}(t, \pi')$ than in $\bar{\delta}(s, \pi)$. At this point, for the sake of readability, if it is clear from the context, we write W for W_Σ as the distribution W mapped onto partition Σ .

For simulation relations, it is also required that the related states agree on their labelling. Define Σ_0 as the *labelling partition* satisfying for all $s, t \in S$, $\mathcal{L}(s) = \mathcal{L}(t)$ iff $[s]_{\Sigma_0} = [t]_{\Sigma_0}$. Write $Part^0(\mathcal{G}) \subseteq Part(\mathcal{G})$ for the set of partition pairs (Σ, \preceq) satisfying $(\Sigma, \preceq) \leq (\Sigma_0, Id)$, where Id is the identity relation.

Lemma 4. For all $(\Sigma, \preceq) \in Part^0(\mathcal{G})$, if (Σ, \preceq) is a *stable partition pair with respect to player I’s choice* then $\sqsubseteq_{(\Sigma, \preceq)}$ is a PA-I-simulation.

Proof. Straightforward by Definition 4.

Obviously every PA-I-simulation is contained in the relation induced by (Σ_0, Id) , and moreover, the above lemma asserts that every stable partition pair smaller than (Σ_0, Id) is a PA-I-simulation. In the following, we try to compute

the coarsest partition pair by refining (Σ_0, Id) until it stabilises. The resulting stable partition pair can be proved to characterise the largest PA-I-simulation on the state space S as required.

We say t simulates s with respect to player-I's choice on a partition pair (Σ, \preceq) if for all $\pi \in \Pi_I$, there exists $\pi' \in \Pi_I$ such that $\bar{\delta}(s, \pi) \preceq_{Sm} \bar{\delta}(t, \pi')$. For better readability, sometimes we also say t simulates s on (Σ, \preceq) if it is clear from the context, and write $s \sqsubseteq_{(\Sigma, \preceq)}^* t$. Note it is straightforward to show that $\sqsubseteq_{(\Sigma, \preceq)}^*$ is a transitive relation, by definition of \preceq_{Sm} . Let $(\Sigma_1, \preceq_1) \leq (\Sigma_2, \preceq_2)$, we say (Σ_1, \preceq_1) is stable on (Σ_2, \preceq_2) , if for all $P, Q \in \Sigma_1$ with $P \preceq_1 Q$, $s \in P$ and $t \in Q$, t simulates s on (Σ_2, \preceq_2) .

Definition 5. Define an operator $\rho : \text{Part}(\mathcal{G}) \rightarrow \text{Part}(\mathcal{G})$, such that $\rho((\Sigma, \preceq))$ is the largest partition pair $(\Sigma', \preceq') \leq (\Sigma, \preceq)$ that is stable on (Σ, \preceq) .

The operator ρ has the following properties.

Lemma 5. ρ is well defined on $\text{Part}(\mathcal{G})$.

Proof. We show that given a partition pair (Σ, \preceq) on S , $\rho((\Sigma, \preceq))$ is a unique partition pair. Let $P \in \Sigma$. Define $\leq_P \subseteq P \times P$ by $s \leq_P t$ if $s \sqsubseteq_{(\Sigma, \preceq)}^* t$. Then \leq_P is a preorder on P , from which we define a partition pair (Σ_P, \preceq_P) where $\Sigma_P = \{\{t \in P \mid s \leq_P t \wedge t \leq_P s\} \mid s \in P\}$ and $X_1 \preceq_P X_2$ if there exist $s \in X_1$ and $t \in X_2$ such that $s \leq_P t$. Define $\rho((\Sigma, \preceq)) = (\Sigma', \preceq')$ with $\Sigma' = \bigcup_{P \in \Sigma} \Sigma_P$ and $\preceq' = (\preceq \setminus \text{Id})(\Sigma') \cup \bigcup_{P \in \Sigma} \preceq_P$. For the definition of \preceq' , the first part of the union $(\preceq \setminus \text{Id})(\Sigma')$ is the relation on Σ' as induced from the nonreflexive part of \preceq , and in the second part each \preceq_P gives a new relation generated inside block P which is stable on (Σ, \preceq) . Note that each \preceq_P is acyclic, and thus a partial order on Σ_P . This implies that \preceq' is a partial order on Σ' .

We show that (Σ', \preceq') is indeed the largest such partition pair. Suppose there exists (Σ'', \preceq'') such that $(\Sigma'', \preceq'') \leq (\Sigma, \preceq)$ and it is stable on (Σ, \preceq) , we show that $(\Sigma'', \preceq'') \leq (\Sigma', \preceq')$.

- Let $P \in \Sigma''$ and $s \in P$, then there exists $P' \in \Sigma'$ such that $s \in P'$. First we have $P \subseteq [s]_\Sigma$ by $\Sigma'' \triangleleft \Sigma$. For all $t \in P$, we have $s \sqsubseteq_{(\Sigma, \preceq)}^* t$ and $t \sqsubseteq_{(\Sigma, \preceq)}^* s$, by P stable on (Σ, \preceq) . By definition we have $s \leq_{[s]_\Sigma} t$ and $t \leq_{[s]_\Sigma} s$, and thus $t \in P'$. Therefore, $P \subseteq P'$. This proves $\Sigma'' \triangleleft \Sigma'$.
- Let $P, Q \in \Sigma''$ and $P \preceq'' Q$. Since $\Sigma'' \triangleleft \Sigma'$, there exist $P', Q' \in \Sigma'$ such that $P \subseteq P'$ and $Q \subseteq Q'$. We need to show that $P' \preceq' Q'$. Taking $s_1 \in P'$ and $s_2 \in Q'$, we show that $s_1 \sqsubseteq_{(\Sigma, \preceq)}^* s_2$. Let $t_1 \in P$ and $t_2 \in Q$, we have $t_1 \sqsubseteq_{(\Sigma, \preceq)}^* t_2$. Also within $[s_1]_\Sigma$ we have $s_1 \preceq_{[s_1]_\Sigma} t_1$, and within $[s_2]_\Sigma$ we have $t_2 \preceq_{[s_2]_\Sigma} s_2$. As both $\preceq_{[s_1]_\Sigma}$ and $\preceq_{[s_2]_\Sigma}$ are contained in $\sqsubseteq_{(\Sigma, \preceq)}^*$, We apply transitivity to get $s_1 \sqsubseteq_{(\Sigma, \preceq)}^* s_2$. Therefore, $P' \preceq' Q'$. This shows that $(P, Q) \in \preceq'(\Sigma')$, and thus $\preceq'' \subseteq \preceq'(\Sigma'')$. □

The following lemma is used in the proof of Lemma 7.

Lemma 6. *If $(\Sigma_1, \preceq_1) \leq (\Sigma_2, \preceq_2)$ and there are distributions Δ, Δ' satisfying $\Delta_{\Sigma_1} \overline{\preceq_1} \Delta'_{\Sigma_1}$, then $\Delta_{\Sigma_2} \overline{\preceq_2} \Delta'_{\Sigma_2}$.*

Proof. (sketch) By reusing the same weight function for \preceq_1 on the partition Σ_1 for \preceq_2 on the coarser partition Σ_2 . \square

Lemma 7. *ρ is monotonic on $(Part^0(\mathcal{G}), \leq)$.*

Proof. Let $(\Sigma_1, \preceq_1) \leq (\Sigma_2, \preceq_2)$, $(\Sigma'_1, \preceq'_1) = \rho((\Sigma_1, \preceq_1))$, $(\Sigma'_2, \preceq'_2) = \rho((\Sigma_2, \preceq_2))$. We show that $(\Sigma'_1, \preceq'_1) \leq (\Sigma'_2, \preceq'_2)$.

We first prove that (Σ'_1, \preceq'_1) is stable on (Σ_2, \preceq_2) . Let $P, Q \in \Sigma'_1$ such that $P \preceq'_1 Q$, then for all $s \in P$, $t \in Q$ and $\pi \in \Pi_1$, there exists $\pi' \in \Pi_1$ such that $\bar{\delta}(s, \pi) \overline{\preceq_1} \bar{\delta}(t, \pi')$. Then by Lemma 6, we also have $\bar{\delta}(s, \pi) \overline{\preceq_2} \bar{\delta}(t, \pi')$. By definition of ρ , we have that the partition pair (Σ'_2, \preceq'_2) is the unique largest partition pair that is stable on (Σ_2, \preceq_2) . As (Σ'_1, \preceq'_1) is stable on (Σ_2, \preceq_2) , it must be the case that $(\Sigma'_1, \preceq'_1) \leq (\Sigma'_2, \preceq'_2)$. \square

Lemma 4 ensures that for all $(\Sigma, \preceq) \in Part^0(\mathcal{G})$, $\sqsubseteq_{(\Sigma, \preceq)}$ is a PA-I-simulation if $\rho((\Sigma, \preceq)) = (\Sigma, \preceq)$, i.e., (Σ, \preceq) is a fixpoint of ρ . However, we still need to find the largest PA-I-simulation. The following result indicates that if S is finite, the coarsest stable partition pair achieved by repetitively applying ρ on (Σ_0, Id) indeed yields the largest PA-I-simulation.⁸ Define $\rho^0(X) = X$ and $\rho^{n+1}(X) = \rho(\rho^n(X))$ for partition pairs X .

Theorem 1. *Let $(\Sigma, \preceq) = \bigcap_{i \in \mathbb{N}} \rho^i((\Sigma_0, \text{Id}))$, then $\sqsubseteq_{(\Sigma, \preceq)}$ is the largest PA-I-simulation on \mathcal{G} .*

Proof. (sketch) Let \sqsubseteq^+ be the largest PA-I-simulation on \mathcal{G} . Define a set $\Sigma^+ = \{\{t \in S \mid s \sqsubseteq^+ t \wedge t \sqsubseteq^+ s\} \mid s \in S\}$. Since \sqsubseteq^+ is the largest PA-I-simulation, it can be shown that \sqsubseteq^+ is reflexive, symmetric and transitive within each block $P \in \Sigma^+$. Moreover, we define a relation \preceq^+ by $P \preceq^+ Q$ if there exists $s \in P$ and $t \in Q$ such that $s \sqsubseteq^+ t$, and it can be shown that \preceq^+ is a partial order on Σ^+ . Then (Σ^+, \preceq^+) forms a partition pair on \mathcal{G} , and furthermore, it is stable, and we also have $(\Sigma^+, \preceq^+) \leq (\Sigma_0, \text{Id})$.

We apply ρ on both sides. By Lemma 7 (monotonicity), and (Σ^+, \preceq^+) being stable, we have $(\Sigma^+, \preceq^+) = \rho^i((\Sigma^+, \preceq^+)) \leq \rho^i((\Sigma_0, \text{Id}))$ for all $i \in \mathbb{N}$. As $Part(\mathcal{G})$ is finite, there exists $j \in \mathbb{N}$, such that $\rho^j((\Sigma_0, \text{Id})) = \rho^{j+1}((\Sigma_0, \text{Id}))$. Therefore, $\rho^j((\Sigma_0, \text{Id}))$ is a stable partition pair, and $\sqsubseteq_{\rho^j((\Sigma_0, \text{Id}))}$ is a PA-I-simulation by Lemma 4. Straightforwardly we have $\sqsubseteq^+ \subseteq \sqsubseteq_{\rho^j((\Sigma_0, \text{Id}))}$. Since \sqsubseteq^+ is the largest PA-I-simulation by assumption, we have $\sqsubseteq^+ = \sqsubseteq_{\rho^j((\Sigma_0, \text{Id}))}$, and the result directly follows. \square

⁸ The following proof resembles the classical paradigm of finding the least fixpoint in an ω -chain of a complete partial order by treating (Σ_0, Id) as \perp . However, here we also need that fixpoint to represent the largest PA-I-simulation.

4 A Decision Procedure for PA-I-Simulation

Efficient algorithms for simulation in the non-probabilistic setting sometimes apply predecessor based methods [17, 15] for splitting blocks and refining partitions. This method can no longer be applied for simulations in the probabilistic setting, as the transition functions now map a state to a state distribution rather than a single state, and simulation relation needs to be *lifted* to handle distributions. The algorithms in [32, 31] follow the approaches in [4] by reducing the problem of deciding a weight function on lifted relations to checking the value of a maximal flow problem. This method, however, does *not* apply to combined transitions, where a more general solution is required. Algorithms for deciding probabilistic bisimulations [6] reduce the problem on checking weight functions with combined choices to solutions in linear programming (LP), which are known to be decidable in polynomial time [19].⁹

In our approach, simulation relations are characterised by partition pairs in the solutions to the GCPP. Starting from the initial partition pair (Σ_0, Id) , we gradually refine the partition by checking whether each pair of states in the same block can simulate each other with respect to player I's choice on a chosen pivot block. When deciding whether s is able to simulate t , we need to examine potentially infinitely many mixed actions in Π_I . This problem can be moderated by the following observations. First we show that for s to be simulated by t , it is only required to check all deterministic choices of player I on s .

Lemma 8. *Let (Σ, \preceq) be a partition pair, then t simulates s on (Σ, \preceq) if for all $a \in \text{Act}_I$, there exists $\pi \in \Pi_I$ such that $\bar{\delta}(s, \bar{a}) \preceq_{Sm} \bar{\delta}(t, \pi)$.*

Proof. (sketch) By definition, t simulates s on (Σ, \preceq) if for all $\pi_1 \in \Pi_I$ there exists $\pi_2 \in \Pi_I$ such that $\bar{\delta}(s, \pi_1) \preceq_{Sm} \bar{\delta}(t, \pi_2)$. Since $\pi_1(s) \in \mathcal{D}(\text{Act}_I)$, for each $a_1 \in [\pi_1(s)]$, we have some $\pi_3 \in \Pi_I$ such that $\bar{\delta}(s, a_1) \preceq_{Sm} \bar{\delta}(t, \pi_3)$, and we get π_2 by combining all such mixed actions π_3 , by applying Lemma 3 and Lemma 1.

The next lemma says when checking a Smyth order $\bar{\delta}(s, \pi) \preceq_{Sm} \bar{\delta}(t, \pi')$, it suffices to focus on player II's deterministic choices in $\bar{\delta}(t, \pi')$ as all probabilistic choices of player II can be represented as interpolations from deterministic choices.

Lemma 9. *$\bar{\delta}(s, \pi) \preceq_{Sm} \bar{\delta}(t, \pi')$ if for all $a \in \text{Act}_II$, there exists $\pi'' \in \Pi_{II}$ such that $\bar{\delta}(s, \langle \pi, \pi'' \rangle) \preceq \bar{\delta}(t, \langle \pi', \bar{a} \rangle)$.*

Proof. (sketch) Similar to the proof of the above lemma by combining all the mixed actions in Π_{II} .

Combining the above two lemmas, we have the following.

Lemma 10. *Let (Σ, \preceq) be a partition pair, then t simulates s with respect to player-I's choice on (Σ, \preceq) if for all $a_1 \in \text{Act}_I$, there exists $\pi_1 \in \Pi_I$ such that for all $a_2 \in \text{Act}_{II}$, there exists $\pi_2 \in \Pi_{II}$ such that $\bar{\delta}(s, \langle \bar{a}_1, \pi_2 \rangle) \preceq \bar{\delta}(t, \langle \pi_1, \bar{a}_2 \rangle)$.*

⁹ The maximal flow problem is a special instance of an LP problem, which can be solved more efficiently [1].

The following lemma states how to check if the action a can be followed by a mixed action from Π_I . Given a finite set S and \preceq a partial order on S , we define $[s]_{\preceq} = \{t \in S \mid s \preceq t\}$, called the *up-closure* of s . Finding a weight function for two distributions on a partition pair can be encoded in LP, with linearity of the constraints guaranteed by Lemma 10.

Lemma 11. *Given a partition pair (Σ, \preceq) , two states $s, t \in S$ and $a \in Act_I$, there exists $\pi \in \Pi_I$ such that $\bar{\delta}(s, \bar{a}) \bar{\preceq}_{S^m} \bar{\delta}(t, \pi)$, iff the following LP has a solution:*

Let $Act_I = \{a_1, a_2, \dots, a_\ell\}$, $Act_{II} = \{b_1, b_2, \dots, b_m\}$ and $\Sigma = \{B_1, B_2, \dots, B_n\}$

$$\sum_{i=1}^{\ell} \alpha_i = 1 \quad (1)$$

$$\forall i = 1, 2, \dots, \ell : 0 \leq \alpha_i \leq 1 \quad (2)$$

$$\forall j = 1, 2, \dots, m : \sum_{k=1}^m \beta_{j,k} = 1 \quad (3)$$

$$\forall j, k = 1, 2, \dots, m : 0 \leq \beta_{j,k} \leq 1 \quad (4)$$

Moreover, $\forall x, y = 1, 2, \dots, n : j = 1, 2, \dots, m :$

$$0 \leq w_{x,y,j} \leq 1 \quad (5)$$

$$\forall B_z \in \Sigma : \sum_{k=1}^m \beta_{j,k} \cdot \delta(s, \langle a, b_k \rangle)(B_z) = \sum_{z'=1}^n w_{z,z',j} = \sum_{B_{z'} \in [B_z]_{\preceq}} w_{z,z',j} \quad (6)$$

$$\forall B_z \in \Sigma : \sum_{i=1}^{\ell} \alpha_i \cdot \delta(t, \langle a_i, b_j \rangle)(B_z) = \sum_{z'=1}^n w_{z',z,j} \quad (7)$$

Informally, $\alpha_1, \alpha_2, \dots, \alpha_\ell$ are used to ‘guess’ a mixed action from player I, as constrained in Eq. 1 and Eq. 2. To establish the Smyth order $\bar{\preceq}_{S^m}$, for every player II action b_j with $j = 1, 2, \dots, m$, we ‘guess’ a mixed action from Act_{II} represented by $\beta_{j,1}, \beta_{j,2}, \dots, \beta_{j,m}$, as constrained in Eq. 3 and Eq. 4. Then for every player II action b_j , we use $w_{x,y,j}$ to represent the weight function that is required to establish the lifted relation $\bar{\preceq}$ for distributions $\sum_{k=1}^m \beta_{j,k} \cdot \delta(s, \langle a, b_k \rangle)$ and $\sum_{i=1}^{\ell} \alpha_i \cdot \delta(t, \langle a_i, b_j \rangle)$, by Eq. 5, Eq. 6 and Eq. 7. In particular, the additional condition in Eq. 6 is to ensure that every non-zero $w_{z,z',j}$ must imply $B_z \preceq B_{z'}$ as required by the weight function.

Proof. (of Lemma 11)

(\Leftarrow) Suppose the above LP has a solution, by Lemma 10, we show that there exists a player I mixed action $\pi \in \Pi_I$, such that for all $b_j \in Act_{II}$, there exists a player II mixed action $\sigma_j \in \Pi_{II}$ such that $\bar{\delta}(s, \langle \bar{a}, \sigma_j \rangle) \bar{\preceq} \bar{\delta}(t, \langle \pi, b_j \rangle)$.

From the solution of LP, a player I mixed action π can be defined by $\pi(a_i) = \alpha_i$ for all $a_i \in Act_I$, satisfying $\sum_{i=1}^{\ell} \alpha_i = 1$, by Eq. 1 and Eq. 2. For each player II action $b_j \in Act_{II}$, the mixed action σ_j can be defined as $\sigma_j(b_k) = \beta_{j,k}$, satisfying $\sum_{k=1}^m \beta_{j,k} = 1$, by Eq. 3 and Eq. 4. Next we show for all $1 \leq j \leq m$, $\bar{\delta}(s, \langle \bar{a}, \sigma_j \rangle) \preceq \bar{\delta}(t, \langle \pi, \bar{b}_j \rangle)$, which is equivalent to $\sum_{k=1}^m \sigma_j(b_k) \cdot \delta(s, \langle a, b_k \rangle) \preceq \sum_{i=1}^{\ell} \pi(a_i) \cdot \delta(t, \langle a_i, b_j \rangle)$ by Lemma 1 and Lemma 2. Given the partition $\Sigma = \{B_1, B_2, \dots, B_n\}$, a weight function $w : \Sigma \times \Sigma \rightarrow [0, 1]$ can be defined by $w(B_x, B_y) = w_{x,y,j}$ for all $1 \leq x, y \leq n$. The conditions on weighted sums are given in Eq. 6 and Eq. 7. We show that $w(B_x, B_y) = w_{x,y,j} > 0$ implies $B_x \preceq B_y$. Suppose $w(B_x, B_y) > 0$ and $B_x \not\preceq B_y$, then $B_y \notin \lfloor B_x \rfloor_{\preceq}$, which would imply $\sum_{x'=1}^n w_{x,x',j} > \sum_{B_{x'} \in \lfloor B_x \rfloor_{\preceq}} w_{x,x',j}$, contradicting Eq. 6.

(\Rightarrow) Suppose there exists a player I mixed action $\pi \in \Pi_I$, such that for all $b_j \in Act_{II}$, there exists a player II mixed action $\sigma_j \in \Pi_{II}$ such that $\bar{\delta}(s, \langle \bar{a}, \sigma_j \rangle) \preceq \bar{\delta}(t, \langle \pi, \bar{b}_j \rangle)$. By Lemma 1 and Lemma 2, equivalently, we have $\sum_{k=1}^m \sigma_j(b_k) \cdot \delta(s, \langle a, b_k \rangle) \preceq \sum_{i=1}^{\ell} \pi(a_i) \cdot \delta(t, \langle a_i, b_j \rangle)$. We show the above LP constraints have a solution.

First we let $\alpha_i = \pi(a_i)$ for each $a_i \in Act_I$, which satisfies Eq. 1 and Eq. 2. Similarly, for each $1 \leq j \leq m$, let $\beta_{j,k} = \sigma_j(b_k)$ for all $1 \leq k \leq m$, which satisfies Eq. 3 and Eq. 4. For each $1 \leq j \leq m$, given the partition $\Sigma = \{B_1, B_2, \dots, B_n\}$, the existing weight function $w : \Sigma \times \Sigma \rightarrow [0, 1]$ satisfies

- (a) for all $B_x \in \Sigma$, $\sum_{y=1}^n w(B_x, B_y) = \sum_{k=1}^m \sigma_j(b_k) \cdot \delta(s, \langle a, b_k \rangle)(B_x)$,
- (b) for all $B_y \in \Sigma$, $\sum_{x=1}^n w(B_x, B_y) = \sum_{i=1}^{\ell} \pi(a_i) \cdot \delta(t, \langle a_i, b_j \rangle)(B_y)$,
- (c) for all $B_x, B_y \in \Sigma$, $w(B_x, B_y) > 0$ implies $B_x \preceq B_y$.

For each $1 \leq j \leq m$ we let $w_{x,y,j} = w(B_x, B_y)$. It is then clear that after replacing each $\sigma_j(b_k)$ by $\beta_{j,k}$ in (a), we get the left equality of Eq. 6. Similarly, after replacing each $\pi(a_i)$ by α_i in (b) we get Eq. 7. Next we show that $\sum_{y=1}^n w_{x,y,j} = \sum_{B_y \in \lfloor B_x \rfloor_{\preceq}} w_{x,y,j}$. First we have $\sum_{B_y \in \lfloor B_x \rfloor_{\preceq}} w_{x,y,j} \leq \sum_{y=1}^n w_{x,y,j}$ by $\lfloor B_x \rfloor_{\preceq} \subseteq \Sigma$. If $\sum_{B_y \in \lfloor B_x \rfloor_{\preceq}} w_{x,y,j} < \sum_{y=1}^n w_{x,y,j}$, there would be some $B_y \notin \lfloor B_x \rfloor_{\preceq}$ and $w_{x,y,j} > 0$, which implies $w(B_x, B_y) > 0$ and $B_x \not\preceq B_y$, contradicting (c).

We define a predicate CanFollow such that $\text{CanFollow}((\Sigma, \preceq), s, t, a)$ decides whether there exists a mixed action of player I from t which simulates action $a \in Act_I$ from s on the partition pair (Σ, \preceq) . CanFollow establishes an LP problem from its parameters (see Lemma 11). We further define a predicate CanSim which decides whether a state simulates another with respect to player I's choice on (Σ, \preceq) for all actions in Act_I , i.e., $\text{CanSim}((\Sigma, \preceq), s, t)$ returns *true* if $\text{CanFollow}((\Sigma, \preceq), s, t, a)$ returns *true* for all $a \in Act_I$.

Algorithm 1 defines a function Split which refines a block $B \in \Sigma$ into a partition pair corresponding to the maximal simulation that is stable on (Σ, \preceq) . It starts with the finest partition and the identity relation (as the final relation is reflexive). For each pair of blocks in the partition, we check if they can simulate each other by picking up a state from each block. (The choice of a state is arbitrary, because all states within the same block are simulation equivalent on

Algorithm 1 Refining a block to make it stable on a partition pair

INPUT: a partition pair (Σ, \preceq) , a block $B \in \Sigma$ OUTPUT: a partition pair (Σ_B, \preceq_B) on B **function** Split $((\Sigma, \preceq), B)$ **begin** $\Sigma_B := \{\{s\} \mid s \in B\}; \preceq_B := \{(s, s) \mid s \in B\}; \Sigma' := \emptyset; \preceq' := \emptyset$ **while** $\Sigma_B \neq \Sigma' \vee \preceq_B \neq \preceq'$ **do** $\Sigma' := \Sigma_B; \preceq' := \preceq_B$ **for each** *distinct* $B_1, B_2 \in \Sigma_B$ **do***pick any* $s_1 \in B_1$ *and* $s_2 \in B_2$ **if** $(\text{CanSim}((\Sigma, \preceq), s_1, s_2) \wedge \text{CanSim}((\Sigma, \preceq), s_2, s_1))$ **then** $\Sigma_B := \Sigma_B \setminus \{B_1, B_2\} \cup \{B_1 \cup B_2\}$ $\preceq_B := \preceq_B \cup \{(X, B_1 \cup B_2) \mid X \in \Sigma : (X, B_1) \in \preceq_B \vee (X, B_2) \in \preceq_B\}$ $\cup \{(B_1 \cup B_2, X) \mid X \in \Sigma : (B_1, X) \in \preceq_B \vee (B_2, X) \in \preceq_B\}$ $\cup \{(B_i, X), (X, B_i) \mid X \in \Sigma : (B_i, X), (X, B_i) \in \preceq_B \wedge i \in \{1, 2\}\}$ **else if** $(\text{CanSim}((\Sigma, \preceq), s_1, s_2))$ **then** $\preceq_B := \preceq_B \cup \{(B_2, B_1)\}$ **else if** $(\text{CanSim}((\Sigma, \preceq), s_2, s_1))$ **then** $\preceq_B := \preceq_B \cup \{(B_1, B_2)\}$ **endfor****endwhile****return** (Σ_B, \preceq_B) **end**

(Σ, \preceq) . If the two state are simulation equivalent on (Σ, \preceq) then we merge the two blocks as well as all incoming and outgoing relation in the current partial order. If only one simulates the other we add an appropriate pair into the current ordering. This process continues until the partition pair stabilises, when no more merging of partitions can happen or any more pair can be added to \preceq_B , which means the resulting partition pair (Σ_B, \preceq_B) is maximal.

Algorithm 2 is based on the functionality of **Split** in Algorithm 1. Starting from the partition (Σ_0, Id) , which is identified as $(\{\{t \mid \mathcal{L}(t) = \mathcal{L}(s)\} \mid s \in S\}, \{(B, B) \mid B \in \Sigma_0\})$, the algorithm computes a sequence of partition pairs $(\Sigma_1, \preceq_1), (\Sigma_2, \preceq_2) \dots$ until it stabilises, which is detected by checking the condition $\Sigma \neq \Sigma' \vee \preceq \neq \preceq'$. At each iteration we have $(\Sigma_{i+1}, \preceq_{i+1}) \leq (\Sigma_i, \preceq_i)$. Moreover, $(\Sigma_{i+1}, \preceq_{i+1})$ is the maximal partition pair that is stable on (Σ_i, \preceq_i) , because by Algorithm 1, the splitting of each block B in Σ_i creates a maximal partition pair (Σ_B, \preceq_B) of B that is stable on (Σ_i, \preceq_i) , and the new partition pair $(\Sigma_{i+1}, \preceq_{i+1})$ is formed by merging all such maximal pairs as well as by taking into account the previous relation represented by (Σ_i, \preceq_i) . Intuitively, we have $(\Sigma_{i+1}, \preceq_{i+1}) = \rho((\Sigma_i, \preceq_i))$, where ρ is the operator as per Definition 5. The correctness of the algorithm is then justified by Theorem 1, which states that it converges to the coarsest partition pair that is contained in (Σ_0, Id) and returns a representation of the largest PA-I-simulation.

Algorithm 2 Computing the Generalised Coarsest Partition Pair

INPUT: a probabilistic game structure $\mathcal{G} = \langle S, s_0, \mathcal{L}, Act, \delta \rangle$ OUTPUT: a partition pair (Σ, \preceq) on S **function** GCPP (\mathcal{G})**begin** $\Sigma := \{\{t \mid \mathcal{L}(t) = \mathcal{L}(s)\} \mid s \in S\}; \preceq := \{(B, B) \mid B \in \Sigma\}$ $\Sigma' := \emptyset; \preceq' := \emptyset$ **while** $\Sigma \neq \Sigma' \vee \preceq \neq \preceq'$ **do** $\Sigma' := \Sigma; \preceq' := \preceq$ **for each** $B \in \Sigma$ **do** $(\Sigma_B, \preceq_B) := \text{Split}((\Sigma', \preceq'), B)$ $\Sigma := \Sigma \setminus \{B\} \cup \Sigma_B$ $\preceq := \preceq \cup \preceq_B$ $\cup \{(B', X) \mid X \in \Sigma : B' \in \Sigma_B : (B, X) \in \preceq\}$ $\cup \{(X, B') \mid X \in \Sigma : B' \in \Sigma_B : (X, B) \in \preceq\}$ $\setminus \{(B, X), (X, B) \mid X \in \Sigma : (X, B), (B, X) \in \preceq\}$ **endfor****endwhile****return** (Σ, \preceq) **end**

Space complexity. For a PGS $\langle S, s_0, \mathcal{L}, Act, \delta \rangle$, it requires $\mathcal{O}(|S|)$ to store the state space and $\mathcal{O}(|S|^2 \cdot |Act|)$ for the transition relation, since for each $s \in S$ and $\langle a_1, a_2 \rangle \in Act$ it requires an array of size $\mathcal{O}(|S|)$ to store a distribution. Recording a partition pair takes $\mathcal{O}(|S|^2 + |S|^2)$ as the first part is needed to record for each state which equivalence class in the partition it belongs, and the second part is needed for the partial order relation \preceq . The computation from (Σ_i, \preceq_i) to $(\Sigma_{i+1}, \preceq_{i+1})$ can be done in-place which only requires additional constant space to track if the partition pair has been modified during each iteration. Another extra space-consuming part is for solving LP constrains, which we assume has space usage $\mathcal{O}(\gamma(N))$ where $N = 1 + |Act_I| + |Act_{II}| + |Act_{II}|^2 + |S|^2 \cdot |Act_{II}| + 3 \cdot |S| \cdot |Act_{II}|$ is the number of linear constraints at most, and $\gamma(N)$ some polynomial. The space complexity roughly sums up to $\mathcal{O}(|S|^2 \cdot |Act| + \gamma(|Act|^2 + |S|^2 \cdot |Act|))$. The first part $\mathcal{O}(|S|^2 \cdot |Act|)$ for the PGS itself can be considered optimal, while the second part depends on the efficiency of the LP algorithm being used.

Time complexity. The number of variables in the LP problem in Lemma 11 is $|Act_I| + |Act_{II}|^2 + |S|^2 \cdot |Act_{II}|$, and the number of constraints is bounded by $1 + |Act_I| + |Act_{II}| + |Act_{II}|^2 + |S|^2 \cdot |Act_{II}| + 3 \cdot |S| \cdot |Act_{II}|$. The predicate **CanSim** costs $|Act_I|$ times LP solving. Each **Split** invokes at most $|B|^2$ testing of **CanSim** where B is a block in Σ . Each iteration of GCPP splits all current blocks, and the total number of comparisons within each iteration of GCPP is bounded by $|S|^2$. (However it seems heuristics on the existing partition can achieve a speed close to linear in practice by caching previous **CanSim** checks [32].) The number of iterations is bounded by $|S|$. This gives us time complexity which is in the

worst case to solve $\mathcal{O}(|S|^3 \cdot |Act_I|)$ many such LP problems, each of which has $\mathcal{O}(|Act|^2 + |S|^2 \cdot |Act|)$ constraints.

Remark. By removing the interaction between players (i.e., the alternating part), our algorithm downgrades to a partition-based algorithm computing the largest *strong* probabilistic simulation relation in probabilistic automata, where *combined transitions* are needed. This simplified setting is equivalent to removing choices from player Π from PGS. (Informally, we let $|Act_\Pi| = 1$.) Now the time complexity is to solve $\mathcal{O}(|S|^3 \cdot |Act|)$ many such LP problems, each of which has $\mathcal{O}(|Act| + |S|^2)$ constraints. The algorithm of [32] for computing strong probabilistic simulation has time complexity of solving $\mathcal{O}(|S|^2 \cdot m)$ LP problems, where m is the size of the transition relation comparable to $\mathcal{O}(|S|^2 \cdot |Act|)$. They have $\mathcal{O}(|S|^2)$ constraints for each LP instance. Note that the space-efficient algorithm [31] for probabilistic simulation (*without* combined transitions) has the same space complexity but better time complexity than ours, which is due to the reduction to the maximal flow problem.

5 Conclusion

We have presented a partition-based algorithm to compute the largest probabilistic alternating simulation relation in finite probabilistic game structures. To the best of our knowledge, our work presents the first polynomial-time algorithm for computing a relation in probabilistic systems considering (concurrently) mixed choices from players. As aforementioned, PA-simulation is known as stronger than the simulation relation characterising quantitative μ -calculus [14], though it is still a conservative approximation which has a reasonable complexity to be useful in verification of game-based properties.

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