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To Helena and Thị Mỹ Liên.

The drawings under the parts' title are freely inspired by works of

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- *Jiří Načeradský* and *Jaroslav Nešetřil*.

The image on the first page of the front-matter is an original drawing by *Jaroslav Nešetřil*, which is freely inspired by the ink on paper "Elegant Rocks and Sparse Trees" by *Zhao Mengfu*.

Preface

This text is aimed at doctoral students and researchers, who are interested in Combinatorics and Graph Theory or who would just like to learn about some active topics and trends. But the book may be also interesting to researchers in mathematics, physics, chemistry, computer science, etc. who would seek for an introduction to the tools available for analysis of the properties of discrete structures, and sparse structures particularly. The dichotomy between sparse and dense objects is one of the main paradigm of the whole mathematics which transcends boundaries of particular disciplines. This is also reflected by our book.

The book is organized in three parts, called *Presentation*, *Theory*, and *Applications*.

The first part, *Presentation*, gives a general overview of the covered material and of its relationships with other domains of contemporary mathematics and computer science. In particular, Chap. 2 is devoted to the exposition of some typical examples illustrating the scope of this book.

The second part, *Theory*, is the largest part of the book and it is divided into eleven chapters. Chapter 3 introduces all the relevant notions and results which will be used in the book: basic notions and standard terminology, as well as more involved concepts and constructions (such as homomorphisms, minors, expanders, Ramsey theory, logic, or complexity classes), or more specific considerations on graph parameters, structures, and homomorphism counting. Chapter 4 introduces the specific notions used to study the density properties, *shallow minors*, *shallow topological minors*, or *shallow immersions* of individual graphs, as well as the related fundamental stability results. These results are applied in Chap. 5, and this leads to the *nowhere dense/somewhere dense* classification and to the notion of *classes with bounded expansion* (which are sparser than general nowhere dense classes). This classification is very robust and it can be characterized by virtually all main combinatorial invariants. Several first characterizations are included in Chap. 5, and more characterizations are given in Chaps. 7, 8, 12, and 11. Chapter 5 ends with a discussion about the connection to model theory and the various approaches to handle general relational structures. Although the study of dense graphs frequently relies on the properties of dense homogeneous core structures (like complete graphs or even random graphs), it will be shown that sparse graph properties are intimately related to the properties of trees, and particularly to the ones of bounded height trees. Fundamental results on bounded height trees and, more generally, on graphs with bounded tree-depth are proved in Chap. 6. They open the way to the main decomposition theorem, which is the subject of Chap. 7. The decomposition scheme introduced there, which we call *low tree-depth coloring*, is a deep generalization of the concept of proper coloring. The low tree-depth colorings also lead to an alternative characterization of the nowhere dense/somewhere dense dichotomy. Yet another characterization of this dichotomy is proved in Chap. 8, that relies on the notion of independence through the notion of *quasi-wideness* (which has been introduced in the context of mathematical logic). Chapters 9 and 11 deal with *homomorphism dualities*. Bounded expansion classes are proved to have the richest spectrum of finite dualities and, in the oriented case, they are actually characterized by this property. Meanwhile, Chap. 10 establishes a connection to model theory and deals particularly with relativizations of the homomorphism preservation theorem of first-order logic. A last characterization of the somewhere dense/nowhere dense dichotomy is proved in Chap. 12 by considering the asymptotic logarithmic density of a fixed pattern in the shallow minors of the graph of a class. In a sense, one can view this last result as a characterization of the dichotomy in probabilistic terms. The *Theory* part ends with Chap. 13 where the results of the previous chapters are gathered and put to service in the study of the characteristics of nowhere dense classes, of classes with bounded expansion, and of classes with bounded tree-depth (which are derived from trees with bounded height). It is pleasing to see how these characterizations are nicely related.

The third part, *Applications*, concerns both theoretical and algorithmic applications of the concepts and results introduced in the second part. This part opens with Chap. 14 which gives several examples of classes with bounded expansion, such as classical classes defined in the context of geometric graphs and graph drawing, as well as classes admitting bounded non-repetitive colorings. It is also the occasion for a connection with the Erdős-Rényi model of random graphs. Some applications are considered in Chap. 15, such as the existence of linear matching (and more generally unions of long disjoint paths), connection with the Burr-Erdős conjecture, with game coloring, and with spectral graph theory. In Chap. 16, the use of a density driven criterion for the existence of sublinear vertex separators links our study to the sparse model of property testing, via the concept of *hyperfiniteness*.

We provide in Chap. 17 core algorithms related to our study. In particular, we detail a fast iterative algorithm to compute a low tree-depth decomposition, the number of colors being controlled by a polynomial dependence on the densities of the shallow minors of the graph. The fact that this algorithm is nearly linear for sparse classes is one of the main advantages of our approaches. In Chap. 18 we consider algorithmic applications, which mainly derive from the fast low tree-depth coloring algorithm. These cover various well-known algorithmic problems, such as *subgraph isomorphism*, decidability of first-order properties, as well as their counting versions.

The title of the last chapter—*Further Directions*—is self-explanatory.

This book contains some previously unpublished results of the authors, as can be expected in a fast developing field. The extensive literature reflects the multiplicity of connections, applications, and similarities to other parts of mathematics and theoretical computer science.

We included exercises at the end of nearly every chapter. These exercises may complement previous material by a small question but often they suggest further study or extension of the main text. Such exercises may also contain hints for solutions. Some hints are also included at the end of the book.

This book is the result of the collaboration of the authors for over a decade in both Paris and Prague (and elsewhere). This was made possible thanks to the generous support of institutions at both ends: École des Hautes Études en Sciences Sociales, École Normale Supérieure, and Université Paris VI in Paris, as well as the Institute of Theoretical Computer Science (ITI) and the Department of Applied Mathematics (KAM) and most recently by Computer Science Institute (IUUK) of Charles University in Prague. We thank our colleagues for friendly working atmosphere. Particularly, we would like to thank Zdeněk Dvořák, Louis Esperet, Tomáš Gavenčák, Andrew Goodall, Jan van den Heuvel, Ida Kantor, Jíří Matoušek, Reza Naserasr, Melda Nešetřilová (née Hope), and Pascal Ochem for comments to parts of the book.

Paris, Prague, *Jaroslav Nešetřil*
December 2011 *Patrice Ossona de Mendez*

Contents

List of Symbols

We list here most of the symbols throughout this book, together with the page corresponding to the symbol's definition.

Variables

F, G, H	finite loopless undirected graphs, 21
$\vec{G}, \vec{H}$	finite directed graphs, 24
$\mathbb{A}, \mathbb{B}, \mathbb{L}$	Limits of homomorphism equivalence classes, 219
u, v, x, y	vertices, 21
e, f, g	edges, 21
$\mathcal{C}, \mathcal{F}, \mathcal{D}$	classes of graphs, 89
$\mathfrak{C}$	a sequence of infinite graph classes, 93
Σ	a surface, 31
ϱ, ς	graph parameters, 95
$\lambda_1, \lambda_2, \ldots, \lambda_n$	eigenvalues, 37
a, b, c	depth of a shallow (topological) minor, 62
$\mathcal{H}$	a hypergraph, 48
ϕ	formula or sentence, 49
σ	signature, 47
$P(X), Q(X, Y)$	polynomials, 55

Asymptotic Notations

$f = O(g)$	Landau symbol O: asymptotic domination of f by g, 55
$f = \Omega(g)$	asymptotic domination of g by f, 55
$f = \Theta(g)$	asymptotic equivalence of f and g, 55
$f = o(g)$	Landau symbol o: $f/g \to 0$, 55
$f \sim g$	Asymptotic equality, 55
$f \asymp g$	polynomial functional dependence, 55

Special Structures

C_n	cycle of order n (and length n), 21
K_n	complete graph of order n, 21
$K_{n,m}$	complete bipartite graph with parts of size n and m, 21
P_n	path of order n (and length $n-1$), 21
$\vec{P}_n$	directed path of order n (and length $n-1$), 42
$\vec{T}_n$	transitive tournament of order n, 42
$G(n, p(n))$	random graph of order n and edge probability $p(n)$, 314

Graph Parameters

$\lvert G\rvert$	order of the graph G, 21
$\lVert G\rVert$	size of the graph G, 21
$\alpha(G)$	independence number of G, 58
$\beta^*(G)$	size of a maximum induced matching of G, 344
$\beta(G)$	matching number of G, 14
$\Delta(G)$	maximum degree of G, 21
$\delta(G)$	minimum degree of G, 21
$\chi(G)$	chromatic number of G, 24
$\chi_g(G)$	game chromatic number, 352
$\chi_{rk}(G)$	vertex ranking number of G, 125
$\chi_s(G)$	star chromatic number of G, 147
$\omega(G)$	clique number of G, 39
$b_\epsilon(G)$	ϵ-boundedness of G, 39
$\mathrm{bw}(G)$	band-width of G, 37
$\mathrm{col}(G)$	coloring number of G, 86
$\mathrm{col}_k(G)$	k-coloring number of G, 86
$\mathrm{cr}(G)$	crossing number of G, 319
$\mathrm{cr}(\vec{G})$	cycle rank of the digraph $\vec{G}$, 127
$\bar{d}(G)$	average degree of G, 21
$g(G)$	genus of the graph G, 55
$g_\alpha(G)$	α-vertex expansion of G, 37
$\mathrm{girth}(G)$	minimum length of a cycle of G, 27
$h(G)$	Hadwiger number of G, 33
$\mathrm{Iso}(G)$	edge expansion of G, 37
$\mathrm{mad}(G)$	maximum average degree of G, 24
$\mathrm{pw}(G)$	path-width of G, 34
$\mathrm{qn}(G)$	queue number of G, 321
$r(G)$	Ramsey number of G, 53
$s(G)$	separation number of G, 37
$\mathrm{sn}(G)$	stack number of G, 321
$\mathrm{tw}(G)$	tree-width of G, 34

$\mathrm{wcol}_k(G)$	weak k-coloring number of G, 86
$\langle G \rangle$	profile, Lovász vector, 46

Other Voices, Other Rooms

$\alpha_r(G)$	r-independence number of G, 175
$\chi_p(G)$	p-chromatic number of G, 150
$\Phi_{\mathcal{C}}$	Scattering function of the class $\mathcal{C}$, 176
$\overline{\Phi}_{\mathcal{C}}$	Uniforam scattering function of the class $\mathcal{C}$, 177
$\mathrm{free}(F, G)$	degree of freedom of F in $\mathcal{C}$, 280
$h_i(G)$	maximal order of a clique immersion in G, 33
$h_t(G)$	maximal order of a topological clique minor of G, 33
$\ell\mathrm{dens}(G)$	logarithmic density of G, 97
$s_G(i)$	maximum minimal size of a $\frac{1}{2}$-vertex separator of a subgraph of G of order i, 37
$\mathrm{td}(G)$	tree-depth of G, 115
$\nabla_r(G)$	grad of rank r of G, 66
$\nabla(G)$	maximum edge-density of a minor of G, 66
$\widetilde{\nabla}_r(G)$	top-grad of rank r of G, 67
$\widetilde{\nabla}(G)$	maximum edge-density of a topological minor of G, 67
$\overset{\propto}{\nabla}_{p,q}(G)$	imm-grad of rank (p, q) of G, 84

Functions

$\mathcal{F}(c, t)$	maximum order of a c-colored graph of tree-depth t without non-trivial involutive automorphisms, 132
$\mathcal{F}(t)$	maximum order of a graph of tree-depth t without non-trivial involutive automorphisms, 132
$R(n_1, \dots, n_k)$	Ramsey number, 52
$\mathrm{E}[X]$	expected value of X, 171
$\mathrm{Inj}(A, B)$	set of all injective injective mappings from A to B, 92
$\widehat{f}$	smallest upper continuous concave function greater or equal to f, 370
$f(\mathcal{C})$	$\sup_{G \in \mathcal{C}} f(G)$, 91
$\limsup_{G \in \mathcal{C}} f(G)$	limit superior of f on the class $\mathcal{C}$, 92
$\overline{f}(\mathcal{C})$	limit superior of f on the class $\mathcal{C}$, 92
$\overline{f}(\mathfrak{C})$	limit superior of f on the class sequence $\mathfrak{C}$, 93
$\delta(S)$	cut-set (or cobord) of S, 37
$d_G(v)$	degree of vertex v in the graph G, 21
$d^-(v)$	indegree of v, 24
$\mathrm{dist}_G(x, y)$	shortest path distance of x and y in G, 61
$d^+(v)$	outdegree of v, 24
$\mathrm{height}(x, F)$	height of vertex x in the rooted forest F, 115

$(\#F \subseteq G)$	number of induced copies of F in G, 278
$\sigma_\zeta(F, Y)$	number of ζ-consistent mappings from F to Y, 283
dist_L	left distance in $[\mathrm{Rel}(\sigma)]$ (and $\overline{[\mathrm{Rel}(\sigma)]}$), 208
dist_R	right distance in $[\mathrm{Rel}(\sigma)]$ (and $\overline{[\mathrm{Rel}(\sigma)]}$), 208
dist	full distance in $[\mathrm{Rel}(\sigma)]$ (and $\overline{[\mathrm{Rel}(\sigma)]}$), 208
$\mathrm{dist}_\equiv$	first-order pseudo-metric on $\mathrm{Rel}(\sigma)$, 239
dist_{FO}	first-order distance in $\mathfrak{T}$, 234
dist_{td}	tree-depth distance in $[\mathrm{Rel}(\sigma)]$, 236
d_{FO}	quotient metric of $(\mathfrak{T}_C, \mathrm{dist}_{FO})/\sim_P$, 238
$\mathrm{Hom}(G, H)$	set of all homomorphisms from G to H, 41
$\vartheta(\mathbf{A})$	bijective mapping from $\mathrm{Rel}(\sigma)$ to P, 229
$\mathbf{M}(\phi)$	bijective mapping from P to $\mathrm{Rel}(\sigma)$, 229

Operations

$A(G)$	adjacency matrix of G, 37
$D(G)$	degree matrix of G, 355
$L(G)$	Laplacian of G, 355
$N_d^G(u)$	d-neighborhood of u in G, 61
$Q_k(G_L, y)$	set of vertices that are weakly k-accessible from y, 86
$R_k(G_L, y)$	set of vertices that are k-accessible from y, 86
$G - v$	vertex deletion, 23
G/e	edge contraction, 30
$G \setminus e$	edge deletion, 30
$G[A]$	subgraph of G induced by A, 22
$G_{<k}$	subgraph of G induced by the vertices of degree strictly smaller than k, 22
$G_{\leq k}$	subgraph of G induced by the vertices of degree at most k, 22
$G/\mathcal{P}$	minor of G obtained as the adjacency graph of the parts in $\mathcal{P}$, 62
$\mathrm{clos}(F)$	closure of the rooted forest F, 115
$G \square H$	Cartesian product of G and H, 279
$G \times H$	categorical product of G and H, 40
$G \bullet H$	lexicographic product of G and H, 80
$G + H$	disjoint union (categorical sum) of G and H, 40
$\mathrm{Gaifman}(\mathbf{A})$	Gaifman graph of $\mathbf{A}$, 49
$\mathrm{Inc}(\mathbf{A})$	incidence graph of a relational structure $\mathbf{A}$, 49
$\mathrm{Inc}(\mathcal{H})$	incidence graph of a hypergraph $\mathcal{H}$, 49
$\mathbf{A} \times \mathbf{B}$	categorical product of $\mathbf{A}$ and $\mathbf{B}$, 47
$\mathbf{A} + \mathbf{B}$	disjoint union (categorical sum) of $\mathbf{A}$ and $\mathbf{B}$, 47
$\mathbf{U}(\mathbf{A})$	Feder and Vardi function $\mathbf{U}$, 199

Relations

Classes and Sets

$\mathrm{Forb_h}(\mathcal{F})$	Class of graphs with no homomorphic image of a graph in $\mathcal{F}$, 90
$(\mathbf{A} \to)$	structures with a homomorphism from $\mathbf{A}$, 205
$(\to \mathbf{A})$	structures with a homomorphism to $\mathbf{A}$, 205
$\mathrm{Inc}(\mathcal{C})$	Class of the incidence graphs of relational structures in $\mathcal{C}$, 111
$\mathcal{T}_t$	class of all graphs with tree-depth at most t, 134
$\mathcal{G}raph$	class of all (isomorphism types of) finite graphs, 40
$\mathrm{Rel}(\sigma)$	Category of all finite σ-structures, 47
$\mathfrak{Rel}(\sigma)$	class of all (finite or infinite) σ-structures, 233
$\mathrm{Tree}(\sigma)$	Class of all finite σ-trees, 49
$[\mathcal{G}raph]$	poset of all hom-equivalence classes of graphs, 43
$\overline{[\mathrm{Rel}(\sigma)]}_L$	left completion of $[\mathrm{Rel}(\sigma)]$, 212
$\overline{[\mathrm{Rel}(\sigma)]}_R$	right completion of $[\mathrm{Rel}(\sigma)]$, 218
$\overline{[\mathrm{Rel}(\sigma)]}$	full completion of $[\mathrm{Rel}(\sigma)]$, 219
$\mathcal{C}^{\nabla}$	resolution of $\mathcal{C}$, 94
$\mathcal{C}^{\widetilde{\nabla}}$	topological resolution of $\mathcal{C}$, 94
$\mathcal{C}^{\overset{\propto}{\nabla}}$	immersion resolution of $\mathcal{C}$, 94
$\mathfrak{P}$	class of all closed PP-theories, 231
$\mathfrak{T}$	class of all theories, 233
$\mathfrak{T}_C$	class of all complete theories, 233
$\mathfrak{T}_F$	class of all complete theories with a finite model, 233
$\mathfrak{T}_{FMP}$	class of all complete theories with finite model property, 233

Posets

$x \vee y$	join of x and y, 203
$x \wedge y$	meet of x and y, 203
$x \leq_F y$	partial order induced by a rooted forest F, 115
A^{ℓ}	set of lower bounds of A, 204
$\mathcal{F}^{\star}$	Ideal dual to the filter $\mathcal{F}$, 205
$\mathcal{I}^{\star}$	Filter dual to the ideal $\mathcal{I}$, 205
A^{u}	set of upper bounds of A, 204
${}_{\downarrow}[\mathbf{A}]$	lower set of $[\mathrm{Rel}(\sigma)]$ defined as the elements $\leq_h \mathbf{A}$, 203
$[\mathbf{A}]^{\uparrow}$	upper set of $[\mathrm{Rel}(\sigma)]$ defined as the elements $\geq_h \mathbf{A}$, 203

Logic

$\phi \vdash \psi$	entailment relation, 231
$G \equiv H$	elementary equivalence, 50
$G \equiv^n H$	n-back-and-forth equivalence, 50
$\mathbf{A} \models \phi$	$\mathbf{A}$ satisfies ϕ, 49

qcount(ϕ)	quantifier count of ϕ, 49
qrank(ϕ)	quantifier rank of ϕ, 49
Mod(T)	class of the models of the theory T, 233
Th($\mathbf{A}$)	theory of $\mathbf{A}$, 233
FO	Class of all σ-sentences, 229
FO^n	Class of all σ-sentences with quantifier rank at most n, 229
P	Class of all primitive positive sentences, 229
P^n	Class of all primitive positive sentences with quantifier rank at most n, 229

Complexity Classes

AC^i	unbounded fanin $O(\log^i(n))$-depth circuits, 57
AW[$*$]	alternating W[$*$], 399
FO	first-order logic, 57
FPT	fixed-parameter tractable, 399
L	deterministic logarithmic space, 56
NC^i	Nick's Classes: $O(\log^i(n))$ time on a a polynomial number of processors, 57
NL	non-deterministic logarithmic space, 56
NP	non-deterministic polynomial time, 56
P	deterministic polynomial time, 56
PSPACE	polynomial space, 56
W[1]	weighted analogue of NP, 399
W[t]	non-deterministic fixed-parameter hierarchy, 399
W[$*$]	union of the W[t]'s, 399