

Approximating subset k -connectivity problems

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Abstract

A subset $T \subseteq V$ of terminals is k -connected to a root s in a directed/undirected graph J if J has k internally-disjoint vs -paths for every $v \in T$; T is k -connected in J if T is k -connected to every $s \in T$. We consider the **Subset k -Connectivity Augmentation** problem: given a graph $G = (V, E)$ with edge/node-costs, node subset $T \subseteq V$, and a subgraph $J = (V, E_J)$ of G such that T is k -connected in J , find a minimum-cost augmenting edge-set $F \subseteq E \setminus E_J$ such that T is $(k + 1)$ -connected in $J \cup F$. The problem admits trivial ratio $O(|T|^2)$. We consider the case $|T| > k$ and prove that for directed/undirected graphs and edge/node-costs, a ρ -approximation for **Rooted Subset k -Connectivity Augmentation** implies the following ratios for **Subset k -Connectivity Augmentation**:

- (i) $b(\rho + k) + \left(\frac{3|T|}{|T|-k}\right)^2 H\left(\frac{3|T|}{|T|-k}\right)$;
- (ii) $\rho \cdot O\left(\frac{|T|}{|T|-k} \log k\right)$,

where $b = 1$ for undirected graphs and $b = 2$ for directed graphs, and $H(k)$ is the k th harmonic number. The best known values of ρ on undirected graphs are $\min\{|T|, O(k)\}$ for edge-costs and $\min\{|T|, O(k \log |T|)\}$ for node-costs; for directed graphs $\rho = |T|$ for both versions. Our results imply that unless $k = |T| - o(|T|)$, **Subset k -Connectivity Augmentation** admits the same ratios as the best known ones for the rooted version. This improves the ratios in [19, 14].

1 Introduction

In the **Survivable Network** problem we are given a graph $G = (V, E)$ with edge/node-costs and pairwise connectivity requirements $\{r(u, v) : u, v \in T \subseteq V\}$ on a set T of terminals. The goal is to find a minimum-cost subgraph of G that contains $r(u, v)$ internally-disjoint uv -paths for all $u, v \in T$. In **Rooted Subset k -Connectivity** problem there is $s \in T$ such that $r(s, t) = k$ for all $t \in T \setminus \{s\}$ and $r(u, v) = 0$ otherwise. In **Subset k -Connectivity** problem $r(u, v) = k$ for all $u, v \in T$ and $r(u, v) = 0$ otherwise. In the *augmentation versions*, G contains a subgraph J of cost zero with $r(u, v) - 1$ internally disjoint paths for all $u, v \in T$. A subset $T \subseteq V$ of terminals is k -connected to a root s in a directed/undirected graph J if J has k internally-disjoint vs -paths for every $v \in T$; T is k -connected in J if T is k -connected to every $s \in T$. Formally, the versions of **Survivable Network** we consider are as follows, where we revise our notation to $k \leftarrow k + 1$.

Rooted Subset k -Connectivity Augmentation

Instance: A graph $G = (V, E)$ with edge/node-costs, a set $T \subseteq V$ of terminals, root $s \in T$, and a subgraph $J = (V, E_J)$ of G such that $T \setminus \{s\}$ is k -connected to s in J .

Objective: Find a minimum-cost augmenting edge-set $F \subseteq E \setminus E_J$ such that $T \setminus \{s\}$ is $(k+1)$ -connected to s in $J \cup F$.

Subset k -Connectivity Augmentation

Instance: A graph $G = (V, E)$ with edge/node-costs, subset $T \subseteq V$, and a subgraph $J = (V, E_J)$ of G such that T is k -connected in J .

Objective: Find a minimum-cost augmenting edge-set $F \subseteq E \setminus E_J$ such that T is $(k + 1)$ -connected in $J \cup F$.

The Subset k -Connectivity Augmentation is Label-Cover hard to approximate [9]. It is known and easy to see that for both edge-costs and node-costs, if Subset k -Connectivity Augmentation admits approximation ratio $\rho(k)$ such that $\rho(k)$ is a monotone increasing function, then Subset k -Connectivity admits ratio $k \cdot \rho(k)$. Moreover, for edge costs, if in addition the approximation $\rho(k)$ is w.r.t. a standard setpair/biset LP-relaxation to the problem, then Subset k -Connectivity admits ratio $H(k) \cdot \rho(k)$, where $H(k)$ denotes the k th harmonic number. For edge-costs, a standard LP-relaxation for Survivable Network (due to Frank and Jordán [5]) is:

$$\min \left\{ \sum_{e \in E} c_e x_e : \sum_{e \in E(X, X^*)} x_e \geq r(X, X^*), X, X^* \subseteq V, X \cap X^* = \emptyset, 0 \leq x_e \leq 1 \right\}$$

where $r(X, X^*) = \max\{r(u, v) : u \in X, v \in X^*\}$ and $E(X, X^*)$ is the set of edges in E from X to X^* .

The Subset k -Connectivity problem admits trivial ratios $O(|T|^2)$ for both edge-costs and node-costs, by computing for every $u, v \in V$ an optimal edge-set of k internally-disjoint uv -paths (this is essentially a Min-Cost k -Flow problem, that can be solved in polynomial time), and taking the union of the computed edge-sets. We note that for metric edge-costs the problem admits an $O(1)$ ratio [2]. For $|T| \geq k + 1$ the problem can also be decomposed into k instances of Rooted Subset k -Connectivity problems, c.f. [11] for the case $T = V$, where it is also shown that for $T = V$ the number of of Rooted Subset k -Connectivity Augmentation instances can be reduced to $O\left(\frac{|T|}{|T|-k} \log k\right)$, which is $O(\log k)$ unless $k = |T| - o(|T|)$.

Recently, Laekhanukit [14] made an important observation that the method of [11] can be extended for the case of arbitrary $T \subseteq V$. Specifically, he proved that if $|T| \geq 2k$, then $O(\log k)$ instances of Rooted Subset k -Connectivity Augmentation will suffice. Thus for $|T| \geq 2k$, the $O(k)$ -approximation algorithm of [19] for Rooted Subset k -Connectivity Augmentation leads to the ratio $O(k \log k)$ for Rooted Subset k -Connectivity Augmentation. By cleverly exploiting an additional property of the algorithm of [19] (see [14, Lemma 14]), he reduced the ratio to $O(k)$ in the case $|T| \geq k^2$.

However, using a different approach, we will show that all this is not necessary, as for both directed and undirected graphs and edge-costs and node-costs, Subset k -Connectivity Augmentation can be reduced to solving *one* instance (or two instances, in the case of directed graphs) of Rooted Subset k -Connectivity Augmentation and $O\left(\frac{3|T|}{|T|-k}\right)^2 H\left(\frac{3|T|}{|T|-k}\right)$ instances of Min-Cost k -Flow problem. This leads to a much simpler algorithm, improves the result of Laekhanukit [14] for $|T| < k^2$, and applies also for node-costs and directed graphs. In addition, we give a more natural and much simpler extension of the algorithm of [11] for $T = V$, that also enables the same bound $O\left(\frac{|T|}{|T|-k} \log k\right)$ as in [11] for arbitrary T with $|T| \geq k + 1$, and in addition applies also for directed graphs, for node-costs, and for an arbitrary type of edge-costs, e.g., metric costs, or uniform costs, or 0, 1-costs. When we say “0, 1-edge-costs” we mean that the input graph G is complete, and the goal is to add to the subgraph J of G formed by the zero-cost edges a minimum size edge-set F (any edge is allowed) such that $J \cup F$ satisfies the connectivity requirements. Formally, our result is the following.

Theorem 1.1 *For both directed and undirected graphs, and edge-costs and node-costs the following holds. If Rooted Subset k -Connectivity Augmentation admits approximation ratio $\rho = \rho(k, |T|)$, then for $|T| \geq k + 1$ Subset k -Connectivity Augmentation admits the following approximation ratios:*

- (i) $b(\rho + k) + \left(\frac{|T|}{|T|-k}\right)^2 O\left(\log \frac{|T|}{|T|-k}\right)$, where $b = 1$ for undirected graphs and $b = 2$ for directed graphs.
- (ii) $\rho \cdot O\left(\frac{|T|}{|T|-k} \log \min\{k, |T| - k\}\right)$, and this is so also for 0,1-edge-costs.

Furthermore, if for edge-costs the approximation ratio ρ is w.r.t. a standard LP-relaxation for the problem, then so are the ratios in (i) and (ii).

For $|T| > k$, the best known values of ρ on undirected graphs are $O(k)$ for edge-costs and $\min\{O(k \log |T|), |T|\}$ for node-costs [19]; for directed graphs $\rho = |T|$ for both versions. For 0,1-edge-costs $\rho = O(\log k)$ [20] for undirected graphs and $\rho = O(\log |T|)$ [18] for directed graphs. For edge-costs, these ratios are w.r.t. a standard LP-relaxation. Thus Theorem 1.1 implies the following.

Corollary 1.2 *For $|T| \geq k + 1$, Subset k -Connectivity Augmentation admits the following approximation ratios.*

- For undirected graphs, the ratios are $O(k) + \left(\frac{|T|}{|T|-k}\right)^2 O\left(\log \frac{|T|}{|T|-k}\right)$ for edge-costs, $O(k \log |T|) + \left(\frac{|T|}{|T|-k}\right)^2 O\left(\log \frac{|T|}{|T|-k}\right)$ for node-costs, and $\frac{|T|}{|T|-k} \cdot O\left(\log^2 k\right)$ for 0,1-edge-costs.
- For directed graphs, the ratio is $2(|T| + k) + \left(\frac{|T|}{|T|-k}\right)^2 O\left(\log \frac{|T|}{|T|-k}\right)$ for both edge-costs and node-costs, and $\frac{|T|}{|T|-k} \cdot O(\log |T| \log k)$ for 0,1 edge-costs.

For Subset k -Connectivity, the ratios are larger by a factor of $H(k)$ for edge-costs, and by a factor k for node-costs.

Note that except the case of 0,1-edge-costs, Corollary 1.2 is deduced from part (i) of Theorem 1.1. However, part (ii) of Theorem 1.1 might become relevant if Rooted Subset k -Connectivity Augmentation admits ratio better than $O(k)$. In addition, part (ii) applies for *any type* of edge-costs, e.g. metric or 0,1-edge-costs.

We conclude this section by mentioning some additional related work. The case $T = V$ of Rooted Subset k -Connectivity problem is the k -Outconnected Subgraph problem; this problem admits a polynomial time algorithm for directed graphs [6], which implies ratio 2 for undirected graphs. For arbitrary T , the problem harder than Directed Steiner Tree [15]. The case $T = V$ of Subset k -Connectivity problem is the k -Connected Subgraph problem. This problem is NP-hard, and the best known ratio for it is $O\left(\log k \log \frac{n}{n-k}\right)$ for both directed and undirected graphs [17]; for the augmentation version of increasing the connectivity by one the ratio in [17] is $O\left(\log \frac{n}{n-k}\right)$. For metric costs the problem admits ratios $2 + \frac{k-1}{n}$ for undirected graphs and $2 + \frac{k}{n}$ for directed graphs [10]. For 0,1-edge-costs the problem is solvable for directed graphs [5], which implies ratio 2 for undirected graphs. The Survivable Network problem is Label-Cover hard [9], and the currently best known non-trivial ratios for it on undirected graphs are: $O(k^3 \log |T|)$ for arbitrary edge-costs by Chuzhoy and Khanna [3], $O(\log k)$ for metric costs due to Cheriyan and Vetta [2], $O(k) \cdot \min\{\log^2 k, \log |T|\}$ for 0,1-edge-costs [20, 13], and $O(k^4 \log^2 |T|)$ for node-costs [19].

2 Proof of Theorem 1.1

We start by proving the following essentially known statement.

Proposition 2.1 *Suppose that Rooted Subset k -Connectivity Augmentation admits an approximation ratio ρ . If for an instance of Subset k -Connectivity Augmentation we are given a set of q edges (when any edge is allowed) and p stars (directed to or from the root) on T whose addition to G makes T $(k + 1)$ -connected, then we can compute a $(\rho p + q)$ -approximate solution F to this instance in polynomial time. Furthermore, for edge-costs, if the ρ -approximation is w.r.t. a standard LP-relaxation, then $c(F) \leq (\rho p + q)\tau^*$, where τ^* is an optimal standard LP-relaxation value for Subset k -Connectivity Augmentation.*

Proof: For every edge uv among the q edges compute a minimum-cost edge-set $F_{uv} \subseteq E \setminus E_J$ such that $J \cup F_{uv}$ contains k internally-disjoint uv -paths. This can be done in polynomial time for both edge and node costs, using a Min-Cost k -Flow algorithm. For edge-costs, it is known that $c(F_{uv}) \leq \tau^*$. Then replace uv by F_{uv} , and note that T remains k -connected. Similarly, for every star S with center s and leaf-set T' , compute an α -approximate augmenting edge-set $F_S \subseteq E \setminus E_J$ such that $J \cup F_S$ contains k internally-disjoint sv -paths (or vs -paths, in the case of directed graphs and S being directed towards the root) for every $v \in T'$. Then replace S by F_S , and note that T remains k -connected. For edge-costs, it is known that if the ρ -approximation for the rooted version is w.r.t. a standard LP-relaxation, then $c(F_S) \leq (\alpha p + q)\tau^*$. The statement follows. \square

Motivated by Proposition 2.1, we consider the following question:

Given a k -connected subset T in a graph J , how many edges and/or stars on T one needs to add to J such that T will become $(k + 1)$ -connected?

We emphasize that we are interested in obtaining *absolute bounds* on the number of edges in the question, expressed in certain parameters of the graph; namely we consider the *extremal graph theory* question and not the *algorithmic problem*. Indeed, the algorithmic problem of adding the minimum number of edges on T such that T will become $(k + 1)$ -connected can be shown to admit a polynomial-time algorithm for directed graphs using the result of Frank and Jordán [5]; this also implies a 2-approximation algorithm for undirected graphs. However, in terms of the parameters $|T|, k$, the result in [5] implies only the trivial bound $O(|T|^2)$ on the the number of edges one needs to add to J such that T will become $(k + 1)$ -connected.

Our bounds will be derived in terms of the family of the “deficient” sets of the graph J . We need some definitions to state our results.

Definition 2.1 *An ordered pair $\hat{X} = (X, X^+)$ of subsets of a groundset V is called a biset if $X \subseteq X^+$; X is the inner part and X^+ is the outer part of \hat{X} , $\Gamma(\hat{X}) = X^+ \setminus X$ is the boundary of \hat{X} , and $X^* = V \setminus X^+$ is the complementary set of \hat{X} .*

Given an instance of Subset k -Connectivity Augmentation we may assume that T is an independent set in J . Otherwise, we obtain an equivalent instance by subdividing every edge $uv \in J$ with $u, v \in T$ by a new node.

Definition 2.2 *Given a k -connected independent set T in a graph $J = (V, E_J)$ let us say that a biset \hat{X} on V is (T, k) -tight in J if $X \cap T, X^* \cap T \neq \emptyset$, X^+ is the union of X and the set of neighbors of X in J , and $|\Gamma(\hat{X})| = k$.*

An edge covers a biset \hat{X} if it goes from X to X^* . By Menger's Theorem, F is a feasible solution to **Subset k -Connectivity Augmentation** if, and only if, F covers the biset-family \mathcal{F} of tight bisets; see [12, 20]. Thus our question can be reformulated as follows:

Given a k -connected independent set T in a graph J , how many edges and/or stars on T are needed to cover the family \mathcal{F} of (T, k) -tight bisets?

Definition 2.3 *The intersection and the union of two bisets \hat{X}, \hat{Y} is defined by $\hat{X} \cap \hat{Y} = (X \cap Y, X^+ \cap Y^+)$ and $\hat{X} \cup \hat{Y} = (X \cup Y, X^+ \cup Y^+)$. Two bisets \hat{X}, \hat{Y} intersect if $X \cap Y \neq \emptyset$; if in addition $X^* \cap Y^* \neq \emptyset$ then \hat{X}, \hat{Y} cross. We say that a biset-family \mathcal{F} is:*

- crossing if $\hat{X} \cap \hat{Y}, \hat{X} \cup \hat{Y} \in \mathcal{F}$ for any $\hat{X}, \hat{Y} \in \mathcal{F}$ that cross.
- k -regular if $|\Gamma(\hat{X})| \leq k$ for every $\hat{X} \in \mathcal{F}$, and if $\hat{X} \cap \hat{Y}, \hat{X} \cup \hat{Y} \in \mathcal{F}$ for any intersecting $\hat{X}, \hat{Y} \in \mathcal{F}$ with $|X \cup Y| \leq |T| - k - 1$.

The following statement is essentially known.

Lemma 2.2 *Let T be a k -connected independent set in a graph $J = (V, E_J)$, and let \hat{X}, \hat{Y} be (T, k) -tight bisets. If $(X \cap T, X^+ \cap T), (Y \cap T, Y^+ \cap T)$ cross or if $|(X \cup Y) \cap T| \leq |T| - k - 1$ then $\hat{X} \cap \hat{Y}, \hat{X} \cup \hat{Y}$ are both (T, k) -tight.*

Proof: The case $(X \cap T, X^+ \cap T), (Y \cap T, Y^+ \cap T)$ was proved in [20] and [14]. The proof of the case $|(X \cup Y) \cap T| \leq |T| - k - 1$ is identical to the proof of [7, Lemma 1.2] where the case $T = V$ is considered. \square

Corollary 2.3 *The biset-family*

$$\mathcal{F} = \{(X \cap T, X^+ \cap T) : (X, X^+) \text{ is a } (T, k)\text{-tight biset in } J\}$$

is crossing and k -regular, and the reverse family $\bar{\mathcal{F}} = \{(T \setminus X^+, T \setminus X) : \hat{X} \in \mathcal{F}\}$ of \mathcal{F} is also crossing and k -regular. Furthermore, if J is undirected then \mathcal{F} is symmetric, namely, $\mathcal{F} = \bar{\mathcal{F}}$.

Given two bisets \hat{X}, \hat{Y} we write $\hat{X} \subseteq \hat{Y}$ and say that \hat{Y} contains \hat{X} if $X \subseteq Y$ or if $X = Y$ and $X^+ \subseteq Y^+$; $\hat{X} \subset \hat{Y}$ and \hat{Y} properly contains \hat{X} if $X \subset Y$ or if $X = Y$ and $X^+ \subset Y^+$.

Definition 2.4 *A biset \hat{C} is a core of a biset-family \mathcal{F} if $\hat{C} \in \mathcal{F}$ and \hat{C} contains no biset in $\mathcal{F} \setminus \{\hat{C}\}$; namely, a core is an inclusion-minimal biset in \mathcal{F} . Let $\mathcal{C}(\mathcal{F})$ be the family of cores of \mathcal{F} and let $\nu(\mathcal{F}) = |\mathcal{C}(\mathcal{F})|$ denote their number.*

Given a biset-family \mathcal{F} and an edge-set I on T , the residual biset-family \mathcal{F}_I of \mathcal{F} consists of the members of \mathcal{F} uncovered by I . We will assume that for any I , the cores of \mathcal{F}_I and of $\bar{\mathcal{F}}_I$ can be computed in polynomial time. For \mathcal{F} being the family of (T, k) -tight bisets this can be implemented in polynomial time using the Ford-Fulkerson Max-Flow Min-Cut algorithm, c.f. [20]. It is known and easy to see that if \mathcal{F} is crossing and/or k -regular, so is \mathcal{F}_I , for any edge-set I .

Definition 2.5 *For a biset-family \mathcal{F} on T let $\nu(\mathcal{F})$ be the maximum number of bisets in \mathcal{F} which inner parts are pairwise-disjoint. For an integer k let $\mathcal{F}^k = \{\hat{X} \in \mathcal{F} : |X| \leq (|T| - k)/2\}$.*

Lemma 2.4 *Let \mathcal{F} be a k -regular biset-family on T and let $\hat{X}, \hat{Y} \in \mathcal{F}^k$ intersect. Then $\hat{X} \cap \hat{Y} \in \mathcal{F}^k$ and $\hat{X} \cup \hat{Y} \in \mathcal{F}$.*

Proof: Since $|X|, |Y| \leq \frac{|T|-k}{2}$, we have $|X \cup Y| = |X| + |Y| - |X \cap Y| \leq |T| - k - 1$. Thus $\hat{X} \cap \hat{Y}, \hat{X} \cup \hat{Y} \in \mathcal{F}$, by the k -regularity of \mathcal{F} . Moreover, $\hat{X} \cap \hat{Y} \in \mathcal{F}^k$, since $|X \cap Y| \leq |X| \leq \frac{|T|-k}{2}$. \square

We will prove the following two theorems that imply Theorem 1.1.

Theorem 2.5 *Let \mathcal{F} be a biset-family on T such that both $\mathcal{F}, \bar{\mathcal{F}}$ are crossing and k -regular. Then there exists a polynomial-time algorithm that computes an edge-cover I of \mathcal{F} of size $|I| = \nu(\mathcal{F}^k) + \nu(\bar{\mathcal{F}}^k) + \left(\frac{3|T|}{|T|-k}\right)^2 H\left(\frac{3|T|}{|T|-k}\right)$. Furthermore, if \mathcal{F} is symmetric then $|I| = \nu(\mathcal{F}^k) + \left(\frac{3|T|}{|T|-k}\right)^2 H\left(\frac{3|T|}{|T|-k}\right)$.*

Theorem 2.6 *Let \mathcal{F} be a biset-family on T such that both \mathcal{F} and $\bar{\mathcal{F}}$ are k -regular. Then there exists a collection of $O\left(\frac{|T|}{|T|-k} \lg \min\{\nu, |T| - k\}\right)$ stars on T which union covers \mathcal{F} , and such a collection can be computed in polynomial time. Furthermore, the total number of edges in the stars is at most $\nu(\mathcal{F}^k) + \nu(\bar{\mathcal{F}}^k) + \left(\frac{|T|}{|T|-k}\right)^2 \cdot O\left(\log \frac{|T|}{|T|-k}\right)$.*

Note that the second statement in Theorem 2.6 implies (up to constants) the bound in Theorem 2.5. However, the proof of Theorem 2.5 is much simpler than the proof of Theorem 2.6, and the proof of Theorem 2.5 is a part of the proof of the second statement in Theorem 2.6.

Let us show that Theorems 2.5 and 2.6 imply Theorem 1.1. For that, all we need is to show that by applying one time the α -approximation algorithm for the Rooted Subset k -Connectivity Augmentation, we obtain an instance with $\nu(\mathcal{F}^k), \nu(\bar{\mathcal{F}}^k) \leq k + 1$. This is achieved by the following procedure due to Khuller and Raghavachari [8] that originally considered the case $T = V$, see also [1, 4, 10]; the same procedure is also used by Laekhanukit in [14].

Choose an arbitrary subset $T' \subseteq T$ of $k + 1$ nodes, add a new node s (the root) and all edges between s and T' of cost zero each, both to G and to J . Then, using the α -approximation algorithm for the Rooted Subset k -Connectivity Augmentation, compute an augmenting edge set F such that $J \cup F$ contains k internally disjoint vs -paths and sv -paths for every $v \in T'$. Now, add F to J and remove s from J . It is a routine to show that $c(F) \leq \text{bopt}$, and that for edge-costs $c(F) \leq b\tau^*$. It is also known that if \hat{X} is a tight biset of the obtained graph J , then $X \cap T', X^* \cap T' \neq \emptyset$, c.f. [1, 14]. Combined with Lemma 2.4 we obtain that $\nu(\mathcal{F}^k), \nu(\bar{\mathcal{F}}^k) \leq |T'| \leq k + 1$ for the obtained instance, as claimed.

3 Proof of Theorem 2.5

Definition 3.1 *Given a biset-family \mathcal{F} on T , let $\Delta(\mathcal{F})$ denote the maximum degree in the hypergraph $\mathcal{F}^{in} = \{X : \hat{X} \in \mathcal{F}\}$ of the inner parts of the bisets in \mathcal{F} . We say that $T' \subseteq T$ is a transversal of \mathcal{F} if $T' \cap X \neq \emptyset$ for every $X \in \mathcal{F}^{in}$; a function $t : T \rightarrow [0, 1]$ is a fractional transversal of \mathcal{F} if $\sum_{v \in X} t(v) \geq 1$ for every $X \in \mathcal{F}^{in}$.*

Lemma 3.1 *Let \mathcal{F} be a crossing biset-family. Then $\Delta(\mathcal{C}(\mathcal{F})) \leq \nu(\bar{\mathcal{F}})$.*

Proof: Since \mathcal{F} is crossing, the members of $\mathcal{C}(\mathcal{F})$ are pairwise non-crossing. Thus if \mathcal{H} is a subfamily of $\mathcal{C}(\mathcal{F})$ such that the intersection of the inner parts of the bisets in \mathcal{H} is non-empty, then $\bar{\mathcal{H}}$ is a subfamily of $\bar{\mathcal{F}}$ such that the inner parts of the bisets in $\bar{\mathcal{H}}$ are pairwise disjoint, so $|\bar{\mathcal{H}}| \leq \nu(\bar{\mathcal{F}})$. The statement follows. \square

Lemma 3.2 *Let T' be a transversal of a biset-family \mathcal{F}' on T and let I' be an edge-set on T obtained by picking for every $s \in T'$ an edge from s to every inclusion member of the set-family $\{X^* : \hat{X} \in \mathcal{F}', s \in X\}$. Then I' covers \mathcal{F}' . Moreover, if \mathcal{F}' is crossing then $|I'| \leq |T'| \cdot \nu(\bar{\mathcal{F}}')$.*

Proof: The statement that I' covers \mathcal{F}' is obvious. If \mathcal{F}' is crossing, then for every $s \in T$ the inclusion-minimal members of $\{X^* : \hat{X} \in \mathcal{F}', s \in X\}$ are pairwise-disjoint, hence their number is at most $\nu(\bar{\mathcal{F}}')$. The statement follows. \square

Lemma 3.3 *Let \mathcal{F} be a k -regular biset-family on T . Then the following holds.*

- (i) $\nu(\mathcal{F}) \leq \nu(\mathcal{F}^k) + \frac{2|T|}{|T|-k}$.
- (ii) *If $\nu(\mathcal{F}_{\{e\}}^k) = \nu(\mathcal{F}^k)$ holds for every edge e on T then $\nu(\mathcal{F}^k) \leq \frac{|T|}{|T|-k}$.*
- (iii) *There exists a polynomial time algorithm that finds a transversal T' of $\mathcal{C}(\mathcal{F})$ of size at most $|T'| \leq \left(\nu(\mathcal{F}^k) + \frac{2|T|}{|T|-k}\right) \cdot H(\Delta(\mathcal{C}(\mathcal{F})))$.*

Proof: Part (i) is immediate.

We prove (ii). Let $\hat{C} \in \mathcal{C}(\mathcal{F}^k)$ and let \hat{U}_C be the union of the bisets in \mathcal{F}^k that contain \hat{C} and contain no other member of $\mathcal{C}(\mathcal{F}^k)$. If $|U_C| \leq |T| - k - 1$ then $\hat{U}_C \in \mathcal{F}$, by the k -regularity of \mathcal{F} . In this case $\nu(\mathcal{F}_{\{e\}}^k) \leq \nu(\mathcal{F}^k) - 1$ for any edge from C to U_C^* . Hence $|U_C| \geq |T| - k$ must hold for every $\hat{C} \in \mathcal{C}(\mathcal{F})$. By Lemma 2.4, the sets in the set family $\{U_C : \hat{C} \in \mathcal{C}(\mathcal{F})\}$ are pairwise disjoint. The statement follows.

We prove (iii). Let T^k be an inclusion-minimal transversal of \mathcal{F}^k . By Lemma 2.4, $|T^k| = \nu(\mathcal{F}^k)$. Setting $t(v) = 1$ if $v \in T^k$ and $t(v) = \frac{2}{|T|-k}$ otherwise, we obtain a fractional transversal of $\mathcal{C}(\mathcal{F})$ of value at most $\nu(\mathcal{F}^k) + \frac{2|T|}{|T|-k}$. Consequently, the greedy algorithm of Lovász [16] finds a transversal T' as claimed. \square

The algorithm for computing I as in Theorem 2.5 starts with $I = \emptyset$ and then continues as follows.

Phase 1

While there exists an edge e on T such that $\nu(\mathcal{F}_{I \cup \{e\}}^k) \leq \nu(\mathcal{F}_I^k) - 1$, or such that $\nu(\bar{\mathcal{F}}_{I \cup \{e\}}^k) \leq \nu(\bar{\mathcal{F}}_I^k) - 1$, add e to I .

Phase 2

Find a transversal T' of $\mathcal{C}(\mathcal{F}')$ as in Lemma 3.3(iii), where $\mathcal{F}' = \mathcal{F}_I$. Then find an edge-cover I' of \mathcal{F}' as in Lemma 3.2 and add I' to I .

The edge-set I computed covers \mathcal{F} by Lemma 3.2. Clearly, the number of edges in I at the end of Phase 1 is at most $\nu(\mathcal{F}^k) + \nu(\bar{\mathcal{F}}^k)$, and is at most $\nu(\mathcal{F}^k)$ if \mathcal{F} is symmetric. Now we bound the size of I' . Note that at the end of Phase 1 we have $\nu(\mathcal{F}_I^k), \nu(\bar{\mathcal{F}}_I^k) \leq \frac{|T|}{|T|-k}$ (by Lemma 3.3(ii)) and thus $\nu(\bar{\mathcal{F}}_I) \leq \frac{3|T|}{|T|-k}$ (by Lemma 3.3(i)) and $\Delta(\mathcal{C}(\mathcal{F}_I)) \leq \nu(\bar{\mathcal{F}}_I) \leq \nu(\bar{\mathcal{F}}_I^k) + \frac{2|T|}{|T|-k} \leq \frac{3|T|}{|T|-k}$ (by Lemma 3.1). Consequently, $|T'| \leq \left(\nu(\mathcal{F}_I^k) + \frac{2|T|}{|T|-k}\right) \cdot H(\Delta(\mathcal{C}(\mathcal{F}_I))) \leq \frac{3|T|}{|T|-k} \cdot H\left(\frac{3|T|}{|T|-k}\right)$. From this we get $|I'| \leq |T'| \cdot \nu(\bar{\mathcal{F}}_I) \leq \left(\frac{3|T|}{|T|-k}\right)^2 \cdot H\left(\frac{3|T|}{|T|-k}\right)$.

The proof of Theorem 2.5 is now complete.

4 Proof of Theorem 2.6

We start by analyzing the performance of a natural Greedy Algorithm for covering $\nu(\mathcal{F}^k)$, that starts with $I = \emptyset$ and while $\nu(\mathcal{F}_I^k) \geq 1$ adds to I a star S for which $\nu(\mathcal{F}_{I \cup S}^k)$ is minimal. It is easy to see that the algorithm terminates since any star with center s in the inner part of some core of \mathcal{F}_I^k and edge set $\{vs : v \in T \setminus \{s\}\}$ reduces the number of cores by one. The proof of the following statement is similar to the proof of the main result of [11].

Lemma 4.1 *Let \mathcal{F} be a k -regular biset-family and let \mathcal{S} be the collection of stars computed by the Greedy Algorithm. Then*

$$|\mathcal{S}| = O\left(\frac{|T|}{|T| - k} \ln \min\{\nu(\mathcal{F}^k), |T| - k\}\right).$$

Recall that given $\hat{C} \in \mathcal{C}(\mathcal{F}_I^k)$ we denote by \hat{U}_C the union of the bisets in \mathcal{F}_I^k that contain \hat{C} and contain no other member of $\mathcal{C}(\mathcal{F}_I^k)$, and that by Lemma 2.4, the sets in the set-family $\{U_C : \hat{C} \in \mathcal{C}(\mathcal{F})\}$ are pairwise disjoint.

Definition 4.1 ([11]) *Let us say that $s \in V$ out-covers $\hat{C} \in \mathcal{C}(\mathcal{F}^k)$ if $s \in U_C^*$.*

Lemma 4.2 *Let \mathcal{F} be k -regular biset-family and let $\nu = \nu(\mathcal{F}^k)$.*

- (i) *There is $s \in T$ that out-covers at least $\nu\left(1 - \frac{k}{|T|}\right) - 1$ members of $\mathcal{C}(\mathcal{F}^k)$.*
- (ii) *Let s out-cover the members of $\mathcal{C} \subseteq \mathcal{C}(\mathcal{F}^k)$ and let S be a star with one edge from s to the inner part of each member of \mathcal{C} . Then $\nu(\mathcal{F}_S^k) \leq \nu(\mathcal{F}_S^k) - |\mathcal{C}|/2$.*

Consequently, there exists a star S on T such that

$$\nu(\mathcal{F}_S^k) \leq \frac{1}{2} \left(1 + \frac{k}{|T|}\right) \cdot \nu + \frac{1}{2} = \alpha \cdot \nu + \beta. \quad (1)$$

Proof: We prove (i). Consider the hypergraph $\mathcal{H} = \{T \setminus \Gamma(\hat{U}_C) : \hat{C} \in \mathcal{C}(\mathcal{F}^k)\}$. Note that the number of members of $\mathcal{C}(\mathcal{F}^k)$ out-covered by any $v \in T$ is at least the degree of s in \mathcal{H} minus 1. Thus all we need to prove is that there is a node $s \in T$ whose degree in \mathcal{H} is at least $\nu\left(1 - \frac{k}{|T|}\right)$. For every $C \in \mathcal{C}(\mathcal{F})$ we have $|T \setminus \Gamma(\hat{U}_C)| \geq |T| - k$, by the k -regularity of \mathcal{F} . Hence the bipartite incidence graph of \mathcal{H} has at least $\nu(|T| - k)$ edges, and thus has a node $s \in T$ of degree at least $\nu\left(1 - \frac{k}{|T|}\right)$, which equals the degree of s in \mathcal{H} . Part (i) follows.

We prove (ii). It is sufficient to show that every $\hat{C} \in \mathcal{C}(\mathcal{F}_S^k)$ contains some $\hat{C}' \in \mathcal{C}(\mathcal{F}^k) \setminus \mathcal{C}$ or contains at least two members in \mathcal{C} . Clearly, \hat{C} contains some $\hat{C}' \in \mathcal{C}(\mathcal{F}^k)$. We claim that if $\hat{C}' \in \mathcal{C}$ then \hat{C} must contain some $\hat{C}'' \in \mathcal{C}(\mathcal{F}^k)$ distinct from \hat{C}' . Otherwise, $\hat{C} \in \mathcal{F}^k(C)$. But as S covers all members of $\mathcal{F}^k(C)$, $\hat{C} \notin \mathcal{F}_S^k$. This is a contradiction. \square

Let us use parameters $\alpha, \beta, \gamma, \delta$ and j set to

$$\alpha = \frac{1}{2} \left(1 + \frac{k}{|T|} \right) \quad \beta = \frac{1}{2} \quad \gamma = 1 - \frac{k}{|T|} = 2(1 - \alpha) \quad \delta = 1,$$

and j is the minimum integer such that $\alpha^j \left(\nu - \frac{\beta}{1-\alpha} \right) \leq \frac{2}{1-\alpha}$ (note that $\alpha < 1$), namely,

$$j = \left\lceil \frac{\ln \frac{1}{2} (\nu(1-\alpha) - \beta)}{\ln(1/\alpha)} \right\rceil \leq \left\lceil \frac{\ln \frac{1}{2} \nu(1-\alpha)}{\ln(1/\alpha)} \right\rceil. \quad (2)$$

We assume that $\nu \geq \frac{2+\beta}{1-\alpha}$ to have $j \geq 0$ (otherwise Lemma 4.1 follows). Note that $\frac{\beta}{1-\alpha} = \frac{|T|}{|T|-k}$.

Lemma 4.3 *Let $0 \leq \alpha < 1$, $\beta \geq 0$, $\nu_0 = \nu$, and for $i \geq 1$ let*

$$\nu_{i+1} \leq \alpha \nu_i + \beta \quad s_i = \gamma \nu_{i-1} - \delta.$$

Then $\nu_i \leq \alpha^i \left(\nu - \frac{\beta}{1-\alpha} \right) + \frac{\beta}{1-\alpha}$ and $\sum_{i=1}^j s_i \leq \frac{1-\alpha^j}{1-\alpha} \cdot \gamma \left(\nu - \frac{\beta}{1-\alpha} \right) + j \left(\frac{\gamma\beta}{1-\alpha} - \delta \right)$. Moreover, if j is given by (2) then $\nu_j \leq \frac{2+\beta}{1-\alpha} = \frac{5|T|}{|T|-k}$ and $\sum_{i=1}^j s_i \leq 2 \left(\nu - \frac{|T|}{|T|-k} \right)$.

Proof: Unraveling the recursive inequality $\nu_{i+1} \leq \alpha \nu_i + \beta$ in the lemma we get:

$$\nu_i \leq \alpha^i \nu + \beta \left(1 + \alpha + \dots + \alpha^{i-1} \right) = \alpha^i \nu + \beta \frac{1 - \alpha^i}{1 - \alpha} = \alpha^i \left(\nu - \frac{\beta}{1 - \alpha} \right) + \frac{\beta}{1 - \alpha}.$$

This implies $s_i \leq \gamma \left(\nu - \frac{\beta}{1-\alpha} \right) \alpha^{i-1} + \frac{\gamma\beta}{1-\alpha} - \delta$, and thus

$$\begin{aligned} \sum_{i=1}^j s_i &\leq \gamma \left(\nu - \frac{\beta}{1-\alpha} \right) \sum_{i=1}^j \alpha^{i-1} + j \left(\frac{\gamma\beta}{1-\alpha} - \delta \right) \\ &= \gamma \left(\nu - \frac{\beta}{1-\alpha} \right) \cdot \frac{1 - \alpha^j}{1 - \alpha} + j \left(\frac{\gamma\beta}{1-\alpha} - \delta \right) \end{aligned}$$

If j is given by (2) then $\nu_j \leq \alpha^j \left(\nu - \frac{\beta}{1-\alpha} \right) + \frac{\beta}{1-\alpha} \leq \frac{2}{1-\alpha} + \frac{\beta}{1-\alpha} = \frac{2+\beta}{1-\alpha}$, and

$$\begin{aligned} \sum_{i=1}^j s_i &\leq \frac{1 - \alpha^j}{1 - \alpha} \cdot \gamma \left(\nu - \frac{\beta}{1-\alpha} \right) + j \left(\frac{\gamma\beta}{1-\alpha} - \delta \right) \\ &\leq 2 \left(\nu - \frac{\beta}{1-\alpha} \right) = 2 \left(\nu - \frac{|T|}{|T|-k} \right). \end{aligned}$$

□

We now finish the proof of Lemma 4.1. At each one of the first j iterations we out-cover at least $\nu \left(\mathcal{F}_I^k \right) \left(1 - \frac{k}{|T|} \right) - 1$ members of $\mathcal{C} \left(\mathcal{F}_I^k \right)$, by Lemmas 4.2. In each one of the consequent iterations, we can reduce $\nu \left(\mathcal{F}_I^k \right)$ by at least one, if we choose the center of the star in C for some $\hat{C} \in \mathcal{C} \left(\mathcal{F}_I^k \right)$. Thus using Lemma 4.3, performing the necessary computations, and substituting the values of the parameters, we obtain that the number of stars in \mathcal{S} is bounded by

$$j + \nu_j \leq \left\lceil \frac{\ln \frac{1}{2} \nu(1-\alpha)}{\ln(1/\alpha)} \right\rceil + \frac{5|T|}{|T|-k} = O \left(\frac{|T|}{|T|-k} \ln \min\{\nu, |T|-k\} \right).$$

Now we discuss a variation of this algorithm that produces \mathcal{S} with a small number of leaves. Here at each one of the first j iterations we out-cover *exactly* $\nu \left(1 - \frac{k}{|T|}\right) - 1$ min-cores. For that, we need be able to compute the bisets \hat{U}_C , and such a procedure can be found in [14]. The number of edges in the stars at the end of this phase is at most $2 \left(\nu - \frac{|T|}{|T|-k}\right)$ and $\nu_j \leq \frac{5|T|}{|T|-k}$. In the case of non-symmetric \mathcal{F} and/or directed edges, we apply the same algorithm on $\bar{\mathcal{F}}^k$. At this point, we apply Phase 2 of the algorithm from the previous section. Since the number of cores of each one of $\mathcal{F}_I^k, \bar{\mathcal{F}}_I^k$ is now $O\left(\frac{|T|}{|T|-k}\right)$, the size of the transversal T' computed is bounded by $|T'| = O\left(\frac{|T|}{|T|-k} \cdot \log \frac{|T|}{|T|-k}\right)$. The number of stars is at most the size $|T'|$, while the number of edges in the stars is at most $|T'| \cdot \nu(\bar{\mathcal{F}}_I) = \left(\frac{|T|}{|T|-k}\right)^2 \cdot O\left(\log \frac{|T|}{|T|-k}\right)$.

This concludes the proof of Theorem 2.6.

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