# Density classification on infinite lattices and trees 

Ana Bušić ${ }^{*} \quad$ Nazim Fatès ${ }^{\dagger}$ Jean Mairesse ${ }^{\ddagger}$<br>Irène Marcovici ${ }^{\S}$

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#### Abstract

Consider an infinite graph with nodes initially labeled by independent Bernoulli random variables of parameter $p$. We address the density classification problem, that is, we want to design a (probabilistic or deterministic) cellular automaton or a finite-range interacting particle system that evolves on this graph and decides whether $p$ is smaller or larger than $1 / 2$. Precisely, the trajectories should converge to the uniform configuration with only $0^{\prime} s$ if $p<1 / 2$, and only $1^{\prime} s$ if $p>1 / 2$. We present solutions to that problem on $\mathbb{Z}^{d}$, for any $d \geq 2$, and on the regular infinite trees. For $\mathbb{Z}$, we propose some candidates that we back up with numerical simulations.


Keywords. Cellular automata, interacting particle systems, density classification, percolation.

## 1 Introduction

Consider a finite or a countably infinite set of cells, which are spatially arranged according to a group structure $G$. We are interested in the density classification problem, which consists of deciding in a decentralised way, if an initial configuration on $G$ contains more 0 's or more 1's. More precisely, the goal is to design a deterministic or probabilistic dynamical system that evolves in the configuration space $\{0,1\}^{G}$ with a local and homogeneous updating rule and whose trajectories converge to $0^{G}$ or to $1^{G}$ if the initial configuration contains more 0 's or more 1's, respectively. To attack the problem, two natural instantiations

[^0]of dynamical systems are considered, one with synchronous updates of the cells, and one with asynchronous updates. In the first case, time is discrete, all cells are updated at each time step, and the model is known as a Probabilistic Cellular Automaton (PCA) 3. A Cellular Automaton ( $C A$ ) is a PCA in which the updating rule is deterministic. In the second case, time is continuous, cells are updated at random instants, at most one cell is updated at any given time, and the model is known as a (finite range) Interacting Particle System (IPS) 16.

The general spirit of the problem is that of distributed computing: gathering a global information by exchanging only local information. The challenge is two-fold: first, it is impossible to centralise the information (cells are indistinguishable); second, it is impossible to use classical counting techniques (cells contain only a binary information).

The density classification problem was originally introduced for rings of finite size $(G=\mathbb{Z} / n \mathbb{Z})$ and for synchronous models [17. After experimentally observing that finding good rules to perform this task was difficult, it was shown that perfect classification with CA is impossible, that is, there exists no given CA that solves the density classification problem for all values of $n$ [14]. This result however did not stop the quest for the best - although imperfect - models as nothing was known about how well CA could perform. The use of PCA opened a new path [6, 18] and it was shown that there exist PCA that can solve the problem with an arbitrary precision [4]. In the present paper, Prop. 1. we complement the results from [14, 4 by showing that there exists no PCA that solves the density classification problem for all values of $n$.

The challenge is now to extend the research to infinite groups (whose Cayley graphs are lattices or regular trees). First, we need to specify the meaning of "having more 0 's or more 1's" in this context. Consider a random configuration on $\{0,1\}^{G}$ obtained by assigning independently to each cell a value 1 with probability $p$ and a value 0 with probability $1-p$. We say that a model "classifies the density" if the trajectories converge weakly to $1^{G}$ for $p>1 / 2$, and to $0^{G}$ for $p<1 / 2$. A couple of conjectures and negative results exist in the literature. Density classification on $\mathbb{Z}^{d}$ is considered in [2] under the name of "bifurcation". The authors study variants of the famous voter model IPS [16], Ch. V] and they propose two instances that are conjectured to bifurcate. The density classification question has also been addressed for the Glauber dynamic associated to the Ising model at temperature 0, both for lattices and for trees [5, [11, 12. The Glauber dynamic defines an IPS or PCA having $0^{G}$ and $1^{G}$ as invariant measures. Depending on the cases, there is either a proof that the Glauber dynamic does not classify the density, or a conjecture that it does with a proof only for densities sufficiently close to 0 or 1 .

The density classification problem has been approached with different perspectives on finite and infinite groups, as emphasised by the results collected above. For finite groups, the problem is studied per se, as a benchmark for understanding the power and limitations of PCA as a computational model. The community involved is rather on the computer science side. For infinite groups, the goal is to understand the dynamics of specific models that are relevant
in statistical mechanics. The community involved is rather on the theoretical physics and probability theory side.

The aim of the present paper is to investigate how to generalise the finite group approach to the infinite group case. We want to build models of PCA and IPS, as simple as possible, that correct random noise in the initial configuration, even if the density of errors is close to $1 / 2$. We consider the groups $\mathbb{Z}^{d}$, whose Cayley graphs are lattices (Section 3), and the free groups, whose Cayley graphs are infinite regular trees (Section 4). In all cases, except for $\mathbb{Z}$, we obtain both PCA and IPS models that classify the density. To the best of our knowledge, they constitute the first known such examples. The case of $\mathbb{Z}$ is more complicated and could be linked to the so-called positive rates conjecture [8]. We provide some potential candidates for density classification together with simulation experiments (Section 5).

## 2 Defining the density classification problem

Let $(G, \cdot)$ be a finite or countable set of cells equipped with a group structure. Set $\mathcal{A}=\{0,1\}$, the alphabet, and $X=\mathcal{A}^{G}$, the set of configurations. For $x \in X$ and $u \in\{0,1\}$, denote by $|x|_{u}$ the number of occurences of $u$ in $x$.

### 2.1 PCA and IPS

Given a finite set $\mathcal{N} \subset G$, a transition function of neighbourhood $\mathcal{N}$ is a function $f: \mathcal{A}^{\mathcal{N}} \rightarrow \mathcal{A}$. The cellular automaton (CA) $F$ of transition function $f$ is the application $F: X \rightarrow X$ defined by:

$$
\forall x \in X, \forall g \in G, \quad F(x)_{g}=f\left(\left(x_{g \cdot v}\right)_{v \in \mathcal{N}}\right)
$$

When the group $G$ is $\mathbb{Z}^{d}$ or $\mathbb{Z}_{n}=\mathbb{Z} / n \mathbb{Z}$, we denote as usual the law of $G$ by the $\operatorname{sign}+$, so that the definition can be written: $\forall x \in X, \forall k \in \mathbb{Z}^{d}\left(\right.$ resp. $\left.\mathbb{Z}_{n}\right), F(x)_{k}=$ $f\left(\left(x_{k+v}\right)_{v \in \mathcal{N}}\right)$.

Probabilistic cellular automata ( $P C A$ ) are an extension of classical CA: the transition function is now a function $\varphi: \mathcal{A}^{\mathcal{N}} \rightarrow \mathcal{M}(\mathcal{A})$, where $\mathcal{M}(\mathcal{A})$ denotes the set of probability measures on $\mathcal{A}$. At each time step, the cells are updated synchronously and independently, according to a distribution depending on a finite neighbourhood [3]. This defines an application $F: \mathcal{M}(X) \rightarrow \mathcal{M}(X)$. The image of a measure $\mu$ is denoted by $\mu F$.

The analog of PCA in continuous time are (finite-range) interacting particle systems (IPS) [16. IPS are characterised by a finite neighbourhood $\mathcal{N} \subset G$, and a transition function $f: \mathcal{A}^{\mathcal{N}} \rightarrow \mathcal{A}$ (or $\varphi: \mathcal{A}^{\mathcal{N}} \rightarrow \mathcal{M}(\mathcal{A})$ ). We attach random and independent clocks to the cells of $G$. For a given cell, the instants of $\mathbb{R}_{+}$at which the clock rings form a Poisson process of parameter 1. Let $x^{t}$ be the configuration at time $t \geq 0$ of the process. If the clock at cell $g$ rings at instant $t$, the state of the cell $g$ is updated into $f\left(\left(x_{g \cdot v}^{t}\right)_{v \in \mathcal{N}}\right)$ (or according to the probability measure $\left.\varphi\left(\left(x_{g \cdot v}^{t}\right)_{v \in \mathcal{N}}\right)\right)$. This defines a transition semigroup
$F=\left(F^{t}\right)_{t \in \mathbb{R}_{+}}$, with $F^{t}: \mathcal{M}(X) \rightarrow \mathcal{M}(X)$. If the initial measure is $\mu$, the distribution of the process at time $t$ is given by $\mu F^{t}$.

In a PCA, all cells are updated at each time step, in a "synchronous" way. On the other hand, for an IPS, the updating is "asynchronous". Indeed, the probability of having two clocks ringing at the same instant is 0 .

Observe that PCA are discrete-time Markov chains, while IPS are continuoustime Markov processes. A measure $\mu$ is said to be an invariant mesasure of a process $F$, resp. $\left(F_{t}\right)_{t}$, if $\mu F=\mu$, resp. $\mu F_{t}=\mu$ for all $t \in \mathbb{R}_{+}$.

### 2.2 The density classification problem on $\mathbb{Z}_{n}$

The density classification problem was originally stated as follows: find a finite neighbourhood $\mathcal{N} \subset \mathbb{Z}$ and a transition function $f: \mathcal{A}^{\mathcal{N}} \rightarrow \mathcal{A}$ such that for any integer $n \geq 1$ and any configuration $x \in \mathcal{A}^{\mathbb{Z}_{n}}$, when applying the CA $F$ of transition function $f$ to $x$, the sequence of iterates $\left(F^{k}(x)\right)_{k \geq 0}$ reaches the fixed point $\mathbf{0}=0^{n}$ if $|x|_{0}>|x|_{1}$ and the fixed point $\mathbf{1}=1^{n}$ if $|x|_{1}>|x|_{0}$. Land and Belew [14] have proved that there exists no CA that perfectly performs this density classification task for all values of $n$. We now prove that this negative result can be extended to the PCA. It provides at the same time a new proof for CA as a particular case.

Denote by $\delta_{x}$ the probability measure that corresponds to a Dirac distribution centred on $x$.

Proposition 1. There exists no PCA or IPS that solves perfectly the density classification problem on $\mathbb{Z}_{n}$, that is, for any integer $n \geq 1$, and for any configuration $x \in \mathcal{A}^{\mathbb{Z}_{n}}$, $\left(\delta_{x} F^{t}\right)_{t \geq 0}$ converges to $\delta_{\mathbf{0}}$ if $|x|_{0}>n / 2$ and to $\delta_{\mathbf{1}}$ if $|x|_{1}>n / 2$.
Proof. We carry out the proof for PCA. For IPS, the argument is similar and even simpler. Let us assume that $F$ is a PCA that solves perfectly the density classification problem on $\mathbb{Z}_{n}$. Let $\mathcal{N}$ be the neighbourhood of $F$, and let $\ell$ be such that $\mathcal{N} \subset(-\ell, \ell)$. We will prove that for any $x \in \mathcal{A}^{\mathbb{Z}_{n}}$ (with $n \geq 2 \ell$ ), the number of occurrences of 1 after application of $F$ to $x$ is almost surely equal to $|x|_{1}$. Let us assume that it is not the case. Then, we have:

$$
\begin{equation*}
\exists x, y \in \mathcal{A}^{\mathbb{Z}_{n}},|x|_{1} \neq|y|_{1}, \quad \delta_{x} F(y)>0 \tag{1}
\end{equation*}
$$

Assume for instance that $|y|_{1}>|x|_{1}$ (the case $|y|_{1}<|x|_{1}$ is treated similarly). We first assume that $|x|_{1}=a>n / 2$. For some integers $k \geq 2, m \geq 2 \ell$, let us consider the configuration $z=x^{k} 0^{m} \in \mathcal{A}^{\mathbb{Z}_{k n+m}}$. We have $|z|_{1}=k a$. Let $y_{s}$ be the suffix of length $n-\ell$ of $y$, and let $y_{p}$ be the prefix of length $n-\ell$ of $y$. By applying (1), it follows that:

$$
\exists u, v, u^{\prime}, v^{\prime} \in \mathcal{A}^{\ell}, \quad \delta_{z} F\left(u y_{s} y^{k-2} y_{p} v u^{\prime} 0^{m-2 \ell} v^{\prime}\right)>0 .
$$

Set $w=u y_{s} y^{k-2} y_{p} v u^{\prime} 0^{m-2 \ell} v^{\prime}$. We have $|w|_{1} \geq k|y|_{1}-2 \ell \geq k(a+1)-2 \ell$. For $m$ big enough, if we set $k$ to be the largest integer such that $k(a-n / 2)<m / 2$, we have:

$$
|z|_{1}=k a<\frac{k n+m}{2}, \quad|w|_{1} \geq k(a+1)-2 \ell>\frac{k n+m}{2} .
$$

So, with a positive probability, we can reach a configuration with more ones than zeros starting from a configuration with more zeros than ones. Since $F$ classifies the density with probability 1 , the new configuration can be considered as an initial condition that needs to be classified and will thus almost surely evolve to the fixed point 1, that is, a bad classification will occur, which contradicts our hypothesis.

The case $|x|_{1}=a<n / 2$ is analogous, except that we now consider configurations $z$ of the form $x^{k} 1^{m} \in \mathcal{A}^{\mathbb{Z}_{k n+m}}$ and choose the integers $k, m$ such that $k a+m<(k n+m) / 2$ and $k(a+1)+m-2 \ell>(k n+m) / 2$.

We have proved that for any $x \in \mathcal{A}^{\mathbb{Z}_{n}}$ (with $n \geq \ell$ ), the number of occurrences of ones after application of $F$ to $x$ is almost surely equal to $|x|_{1}$. This is in contradiction with the fact that $F$ classifies the density.

This proposition can be extended to larger dimensions: for any $d \geq 1$, there is no PCA or IPS that classifies perfectly the density on all the groups of the form $\mathbb{Z}_{n_{1}} \times \ldots \times \mathbb{Z}_{n_{d}}$.

### 2.3 The density classification problem on infinite groups

Let us define formally the density classification problem on infinite groups.
We denote by $\mu_{p}$ the Bernoulli measure of parameter $p$, that is, the product measure of density $p$ on $X=\mathcal{A}^{G}$. A realisation of $\mu_{p}$ is obtained by assigning independently to each element of $G$ a label 1 with probability $p$ and a label 0 with probability $1-p$. We denote respectively by $\mathbf{0}$ and $\mathbf{1}$ the two uniform configurations $0^{G}$ and $1^{G}$ and by $\delta_{x}$ the probability measure that corresponds to a Dirac distribution centred on $x$.

The density classification problem is to find a PCA or an IPS $F$, such that:

$$
\left\{\begin{array}{l}
p<1 / 2 \Longrightarrow \mu_{p} F^{t} \underset{t \rightarrow \infty}{w} \delta_{\mathbf{0}}  \tag{2}\\
p>1 / 2 \Longrightarrow \mu_{p} F^{t} \xrightarrow[t \rightarrow \infty]{w} \delta_{\mathbf{1}}
\end{array} .\right.
$$

The notation $\xrightarrow{w}$ stands for the weak convergence of measures. In our case, the interpretation is that for any finite subset $K \subset G$, the probability that at time $t$, all the cells of $K$ are labelled by 0 (resp. by 1 ) tends to 1 if $p<1 / 2$ (resp. if $p>1 / 2$ ). Or equivalently, that for any single cell, the probability that it is labelled by 0 (resp. by 1) tends to 1 if $p<1 / 2$ (resp. if $p>1 / 2$ ).

### 2.4 From subgroups to groups

Next proposition has the following consequence: given a process that classifies the density on $\mathbb{Z}^{2}$, we can design a new one that classifies on $\mathbb{Z}^{d}$ for $d>2$. The idea is to divide $\mathbb{Z}^{d}$ into $\mathbb{Z}^{2}$-layers and to apply the original process independently on each layer.

Proposition 2. Let $H$ be a subgroup of $G$, and let $F_{H}$ be a process (PCA or IPS) of neighbourhood $\mathcal{N}$ and transition function $f$ that classifies the density
on $\mathcal{A}^{H}$. We denote by $F_{G}$ the process on $\mathcal{A}^{G}$ having the same neighbourhood $\mathcal{N}$ and the same transition function $f$. Then, $F_{G}$ classifies the density on $\mathcal{A}^{G}$.

Proof. Since $H$ is a subgroup, the group $G$ is partitioned into a union of classes $g_{1} H, g_{2} H, \ldots$ We have $\mathcal{N} \subset H$, so that if an element $g \in G$ is in some class $g_{i} H$, then for any $v \in \mathcal{N}$, the element $g \cdot v$ is also in $g_{i} H$. Since $F_{H}$ classifies the density, on each class $g_{i} H$, the process $F_{G}$ satisfies (2). Thus for any cell of $G$, the probability that it is labelled by 0 (resp. by 1 ) tends to 1 if $p<1 / 2$ (resp. if $p>1 / 2)$.

## 3 Classifying the density on $\mathbb{Z}^{2}$ : Toom's rule

To classify the density on $\mathbb{Z}^{2}$, a natural idea is to apply the majority rule on a cell and its four direct neighbours. Unfortunately, this does not work, neither in the CA nor in the IPS version. Indeed, an elementary square of four cells in state 1 on a background of 0 's is a fixed point for the process. For $p \in(0,1)$, monochromatic elementary squares of both colors appear almost surely in the initial configuration which makes the convergence to $\mathbf{0}$ or $\mathbf{1}$ impossible.

Another idea is to apply the majority rule on the four nearest neighbours (excluding the cell itself) and to choose uniformly the new state of the cell in case of equality. In the IPS setting, this process is known as the Glauber dynamics associated to the Ising model. It has been conjectured to classify the density, but the result has been proved only for values of $p$ that are sufficiently close to 0 or 1 [5].

To overcome the difficulty, we consider the majority CA but on the asymmetric neighbourhood $\mathcal{N}=\{(0,0),(0,1),(1,0)\}$. We prove that this CA, known as Toom's rule [3, 7], classifies the density on $\mathbb{Z}^{2}$. Our proof relies on the properties of the percolation clusters on the triangular lattice [10]. We then define an IPS inspired by this local rule and prove with the same techniques that it also classifies the density.

### 3.1 A cellular automaton that classifies the density

Let us denote by maj : $\mathcal{A}^{3} \rightarrow \mathcal{A}$, the majority function, so that maj$(x, y, z)=$ 0 if $x+y+z<2$ and 1 if $x+y+z \geq 2$.
Theorem 1. The cellular automaton $\mathcal{T}: \mathcal{A}^{\mathbb{Z}^{2}} \rightarrow \mathcal{A}^{\mathbb{Z}^{2}}$ defined by:

$$
\mathcal{T}(x)_{i, j}=\operatorname{maj}\left(x_{i, j}, x_{i, j+1}, x_{i+1, j}\right)
$$

for any $x \in \mathcal{A}^{\mathbb{Z}^{2}},(i, j) \in \mathbb{Z}^{2}$, classifies the density.
Proof. By symmetry, it is sufficient to prove that if $p>1 / 2$, then $\left(\mu_{p} \mathcal{T}^{n}\right)_{n \geq 0}$ converges weakly to $\delta_{\mathbf{1}}$.

Let us consider the triangular lattice of sites (vertices) $\mathbb{Z}^{2}$ and bonds (edges) $\left\{\{(i, j),(i, j+1)\},\{(i, j),(i+1, j)\},\{(i+1, j),(i, j+1)\},(i, j) \in \mathbb{Z}^{2}\right\}$. We recall that a 0 -cluster is a subset of connected sites labelled by 0 which is maximal for
inclusion. The site percolation threshold on the triangular lattice is equal to $1 / 2$ so that, for $p>1 / 2$, there exists almost surely no infinite 0 -cluster [10. Thus, if $S_{\mathbf{0}}$ denotes the set of sites labelled by 0 , the set $S_{\mathbf{0}}$ consists almost surely of a countable union $S_{\mathbf{0}}=\cup_{k \in \mathbb{N}} S_{k}$ of finite 0 -clusters. Moreover, the size of the 0 -clusters decays exponentially: there exist some constants $\kappa$ and $\gamma$ such that the probability for a given site to be part of a 0 -cluster of size larger than $n$ is smaller than $\kappa e^{-\gamma n}$, see [10].

Let us describe how the 0-clusters are transformed by the action of the CA. For $S \subset \mathbb{Z}^{2}$, let $1_{S}$ be the configuration defined by $\left(1_{S}\right)_{x}=1$ if $x \in S$ and $\left(1_{S}\right)_{x}=0$ otherwise. Let $\mathcal{T}(S)$ be the subset $S^{\prime}$ of $\mathbb{Z}^{2}$ such that $\mathcal{T}\left(1_{S}\right)=1_{S^{\prime}}$. By a simple symmetry argument, this last equality is equivalent to $\mathcal{T}\left(1_{\mathbb{Z}^{2} \backslash S}\right)=$ $1_{\mathbb{Z}^{2} \backslash S^{\prime}}$. We observe the following.

- The rule does not break up or connect different 0-clusters (proved by Gács [7] Fact 3.1]). More precisely, if $S$ consists of the 0 -clusters $\left(S_{k}\right)_{k}$, then the components of $\mathcal{T}(S)$ are the nonempty sets among $\left(\mathcal{T}\left(S_{k}\right)\right)_{k}$.
- Any finite 0 -cluster disappears in finite time: if $S$ is a finite and connected subset of $\mathbb{Z}^{2}$, then there exists an integer $n \geq 1$ such that $\mathcal{T}^{n}(S)=\emptyset$. This is the eroder property [3].
- Let us consider a 0 -cluster and a rectangle in which it is contained. Then the 0 -cluster always remains within this rectangle. More precisely, if $R$ is a rectangle set, that is, a set of the form $\left\{(x, y) \in \mathbb{Z}^{2} \mid a_{1} \leq x \leq a_{2}, b_{1} \leq\right.$ $\left.y \leq b_{2}\right\}$, and if $S \subset R$, then for all $n \geq 1, \mathcal{T}^{n}(S) \subset R$ (proof by induction).

Let us now consider all the 0-clusters for which the minimal enveloping rectangle contains the origin $(0,0)$. By the exponential decay of the size of the clusters, one can prove that the number of such 0 -clusters is almost surely finite. Indeed, the probability that the point of coordinates $(m, n)$ is a part of such a cluster is smaller than the probability for this point to belong to a 0 -cluster of size larger than $\max (|m|,|n|)$. And since

$$
\sum_{(m, n) \in \mathbb{Z}^{2}} \kappa e^{-\gamma \max (|m|,|n|)}<4 \kappa \sum_{m \in \mathbb{N}}\left(m e^{-\gamma m}+\sum_{n \geq m} e^{-\gamma n}\right)<\infty
$$

we can apply the Borel-Cantelli lemma to obtain the result. Let $T_{0}$ be the maximum of the time needed to erase these 0 -clusters. The random variable $T_{0}$ is almost surely finite, and after $T_{0}$ time steps, the site $(0,0)$ will always be labelled by a 1 . As the argument can be generalised to any site, it ends the proof.

We point out that Toom's CA classifies the density despite having many different invariant measures. For example:

- Any configuration $x$ that can be decomposed into monochromatic NorthEast paths (that is, $x_{i, j}=x_{i, j+1}$ or $x_{i, j}=x_{i+1, j}$ for any $i, j$ ) is a fixed point and $\delta_{x}$ is an invariant measure.


Figure 1: Illustration of the definition of the IPS.

- Let $y$ be the checkerboard configuration defined by $y_{i, j}=0$ if $i+j$ is even and $y_{i, j}=1$ otherwise, and let $z$ be defined by $z_{i, j}=1-y_{i, j}$. Since we have $\mathcal{T}(y)=z$ and $\mathcal{T}(z)=y$, the two configurations $y$ and $z$ form a periodic orbit and $\left(\delta_{y}+\delta_{z}\right) / 2$ is an invariant measure.


### 3.2 An interacting particle system that classifies the density

We now define an IPS for which we use the same steps as above to prove that it classifies the density.

Note that the exact IPS analog of Toom's rule might classify the density but the above proof does not carry over since, in some cases, different 0-clusters may merge. To overcome the difficulty, we introduce a different IPS with a new neighbourhood of size 7: the cell itself and the 6 cells that are connected to it in the triangular lattice defined in the previous section.

For $\alpha \in \mathcal{A}$, set $\bar{\alpha}=1-\alpha$.
Theorem 2. Let us consider the following IPS: for a configuration $x \in \mathcal{A}^{\mathbb{Z}^{2}}$, we update the state of the cell $(i, j)$ by applying the majority rule on the North-East-Centre neighbourhood, except in the following cases (for which we keep the state unchanged):

1. $x_{i, j}=x_{i-1, j+1}=x_{i+1, j-1}=\bar{x}_{i, j+1}=\bar{x}_{i+1, j}$ and ( $x_{i, j-1}=\bar{x}_{i, j}$ or $\left.x_{i-1, j}=\bar{x}_{i, j}\right)$,
2. $x_{i, j}=x_{i-1, j+1}=x_{i, j-1}=\bar{x}_{i, j+1}=\bar{x}_{i+1, j}=\bar{x}_{i+1, j-1}$ and $x_{i-1, j}=\bar{x}_{i, j}$,
3. $x_{i, j}=x_{i-1, j}=x_{i+1, j-1}=\bar{x}_{i, j+1}=\bar{x}_{i+1, j}=\bar{x}_{i-1, j+1}$ and $x_{i, j-1}=\bar{x}_{i, j}$.

This IPS classifies the density.
The three cases for which we always keep the state unchanged are illustrated below for the case where $x_{i, j}=1$ (central cell). In the first case, we allow to flip the central cell if and only if the two cells marked by a dashed circle are also labelled by 1. Otherwise, the updating could connect two different 0-clusters and break up the 1 -cluster to which the cell $(i, j)$ belongs to. The second and third cases are analogous.

The proof is similar to the one of Theorem 1 but involves some additional technical points.

Proof. We assume as before that $p>1 / 2$. Like the CA of the previous section, the new process that we have defined has the property not to break up or connect different clusters. Furthermore, if we consider a 0 -cluster and the smallest rectangle in which it is contained, we can check again that the 0 -cluster will never go beyond this rectangle. As before, we only need to prove that any finite 0-cluster disappears almost surely in finite time to conclude the proof. We consider a realisation of the trajectory of the IPS with initial density $\mu_{p}$. We associate to any finite 0 -cluster $C \subset \mathbb{Z}^{2}$ the point $v(C)=\max \{(i, j) \in C\}$, where the order is the lexicographic order on the coordinates (we set $v(\emptyset)=(-\infty,-\infty)$ ). The point $v(C)$ is thus the upmost point of $C$ among its rightmost points. Let us consider at time 0 some finite 0-cluster $C_{0}$. We denote by $C_{t}$ the state of this cluster at time $t$.

Claim. The value $v\left(C_{t}\right)$ is nonincreasing. Moreover, if $t \geq 0$ is such that $C_{t} \neq \emptyset$, then there exists almost surely a time $t^{\prime}>t$ such that $v\left(C_{t^{\prime}}\right)<v\left(C_{t}\right)$.

Let us prove the claim. Let us denote by $x \in \mathcal{A}^{\mathbb{Z}^{2}}$ a configuration attained at some time $t$, and let $(i, j)=v\left(C_{t}\right)$. By definition of $v\left(C_{t}\right)$, if a cell of coordinate $\left(i+1, j^{\prime}\right)$ is connected to a cell of $C_{t}$, then $x_{i+1, j^{\prime}}=1$. Either we have also $x_{i+1, j^{\prime}+1}=1$ and the cell $\left(i+1, j^{\prime}\right)$ will not flip. Or $x_{i+1, j^{\prime}+1}=0$, but in this case, since $\left(i+1, j^{\prime}+1\right)$ does not belong to $C_{t}, x_{i, j^{\prime}+1}=1$ and the cell of $C_{t}$ to which is connected $\left(i+1, j^{\prime}\right)$ is necessarily $\left(i, j^{\prime}\right)$. So, $x_{i, j^{\prime}}=0$ and $x_{i+1, j^{\prime}-1}=1$, once again by definition of $v\left(C_{t}\right)$. Depending on the value of $x_{i+2, j^{\prime}-1}$, either rule 1 or rule 2 forbids the cell $\left(i+1, j^{\prime}\right)$ to flip. In the same way, we can prove that if a cell of coordinate $\left(i, j^{\prime}\right), j^{\prime}>j$ is connected to $C_{t}$, then it is not allowed to flip. This proves that $v\left(C_{t}\right)$ is nonincreasing. In order to prove the second part of the claim, we need to show that the cell $(i, j)$ will almost surely be flipped in finite time. By definition of $(i, j)=v\left(C_{t}\right)$, we know that $x_{i, j+1}=x_{i+1, j}=x_{i+1, j-1}=1$. The cell $(i, j)$ will thus be allowed to flip, except if $x_{i-1, j+1}=x_{i, j-1}=0$ and $x_{i-1, j}=1$. But in that case, the cell ( $i-1, j$ ) will end up flipping, except if $x_{i-1, j-1}=x_{i-2, j+1}=1, x_{i-2, j}=0$, and so on. Let $W_{n}=\{(i-n, j),(i-1-n, j+1),(i-n, j-1)\}$. If for each $n$, the cells of $W_{n}$ are in the state $(n \bmod 2)$, then none of the cell $(i-n, j)$ is allowed to flip (see Figure 2a). But recall now that the initial measure is $\mu_{p}$. There exists almost surely an integer $n \geq 0$ such that the initial state of the cell $(i-n, j)$ is not $(n \bmod 2)$. Let $m(t)$ be the smallest integer $n$ whose value at time $t$ is not $n \bmod 2$. Then, one can easily check that $m(t)$ is non-increasing, and that it reaches 0 in finite time. Thus, the cell $(i, j)$ ends up flipping and we have proved the claim.

The example of Figure 2 b illustrates how the proof works. Here, no cell of the cluster $C_{t}$ is allowed to flip, but since the cells on the right and on the top of $v\left(C_{t}\right)$ cannot flip either, $v\left(C_{t}\right)$ does not increase. The cell at the left of $v\left(C_{t}\right)$ will end up flipping, and $v\left(C_{t}\right)$ will then be allowed to flip.

Since we know that a 0 -cluster cannot go beyond its enveloping rectangle, a

(a)

(b)

Figure 2: Illustration of the proof of Theorem 2
direct consequence of the claim is that any 0 -cluster disappears in finite time. This allows us to conclude the proof in the same way as for the majority cellular automaton.

## 4 Classifying the density on regular trees

Consider the finitely presented group $T_{n}=\left\langle a_{1}, \ldots, a_{n} \mid a_{i}^{2}=1\right\rangle$. The Cayley graph of $T_{n}$ is the infinite $n$-regular tree. For $n=2 k$, we also consider the free group with $k$ generators, that is, $T_{2 k}^{\prime}=\left\langle a_{1}, \ldots, a_{k} \mid \cdot\right\rangle$. The groups $T_{2 k}$ and $T_{2 k}^{\prime}$ are not isomorphic, but they have the same Cayley graph.

### 4.1 Shortcomings of the nearest neighbour majority rules

For odd values of $n$, a natural candidate for classifying the density is to apply the majority rule on the $n$ neighbours of a cell. But it is proved that neither the CA (see [12] for $n=3,5$, and 7) nor the IPS (see [11] for $n=3$ ) classify the density.

For $n=4$, a natural candidate would be to apply the majority on the four neighbours and the cell itself. We now prove that it does not work either.

Proposition 3. Consider the group $T_{4}^{\prime}=\langle a, b \mid \cdot\rangle$. Consider the majority CA or IPS with neighbourhood $\mathcal{N}=\left\{1, a, b, a^{-1}, b^{-1}\right\}$. For $p \in(1 / 3,2 / 3)$, the trajectories do not converge weakly to a uniform configuration.

Proof. If $p \in(1 / 3,2 / 3)$, then we claim that at time 0 , there are almost surely infinite chains of zeros and infinite chains of ones that are fixed. Let us choose some cell labelled by 1. Consider the (finite or infinite) subtree of 1's originating from this cell viewed as the root. If we forget the root, the random tree is
exactly a Galton-Watson process. Indeed, the expected number of children of a node is $3 p$ and since $3 p>1$, this Galton-Watson process survives with positive probability. Consequently, there exists almost surely an infinite chain of ones at time 0 somewhere in the tree. In the same way, since $3(1-p)>0$, there exists almost surely an infinite chain of zeros.

As for $\mathbb{Z}^{2}$, we get round the difficulty by keeping the majority rule but choosing a non-symmetrical neighbourhood.

### 4.2 A rule that classifies the density on $T_{4}^{\prime}$

In this section, we consider the free group $T_{4}^{\prime}=\langle a, b \mid \cdot\rangle$, see Fig. 3 (a).
Theorem 3. The cellular automaton $F: \mathcal{A}^{T_{4}^{\prime}} \rightarrow \mathcal{A}^{T_{4}^{\prime}}$ defined by:

$$
F(x)_{g}=\operatorname{maj}\left(x_{g a}, x_{g a b}, x_{g a b^{-1}}\right)
$$

for any $x \in \mathcal{A}^{T_{4}^{\prime}}, g \in T_{4}^{\prime}$, classifies the density.
Proof. We consider a realisation of the trajectory of the CA with initial distribution $\mu_{p}$. Let us denote by $X_{g}^{n}$ the random variable describing the state of the cell $g$ at time $n$. Since the process is homogeneous, it is sufficient to prove that $X_{1}^{n}$ converges almost surely to 0 if $p<1 / 2$ and to 1 if $p>1 / 2$. Let us denote by $h:[0,1] \rightarrow[0,1]$ the function that maps a given $p \in[0,1]$ to the probability $h(p)$ that $\operatorname{maj}(X, Y, Z)=1$ when $X, Y, Z$ are three independent Bernoulli random variables of parameter $p$. An easy computation provides $h(p)=3 p^{2}-2 p^{3}$, and one can check that the sequence $\left(h^{n}(p)\right)_{n \geq 0}$ converges to 0 if $p<1 / 2$ and to 1 if $p>1 / 2$.

We prove by induction on $n \in \mathbb{N}$ that for any $k \in \mathbb{N}$, the family $\mathcal{E}_{k}(n)=$ $\left\{X_{u_{1} u_{2} \ldots u_{k}}^{n} \mid u_{1}, u_{2}, \ldots, u_{k} \in\left\{a, a b, a b^{-1}\right\}\right\}$ consists of independent Bernoulli random variables of parameter $h^{n}(p)$. By definition of $\mu_{p}$, the property is true at time $n=0$. Let us assume that it is true at some time $n \geq 0$, and let us fix some $k \geq 0$. Two different elements of $\mathcal{E}_{k}(n+1)$ can be written as the majority on two disjoint triples of $\mathcal{E}_{k+1}(n)$. The fact that the triples are disjoint is a consequence of the fact that $\left\{a, a b, a b^{-1}\right\}$ is a code: a given word $g \in G$ written with the elementary patterns $a, a b, a b^{-1}$ can be decomposed in only one way as a product of such patterns. By hypothesis, the family $\mathcal{E}_{k+1}(n)$ is made of i.i.d. Bernoulli variables of parameter $h^{n}(p)$, so the variables of $\mathcal{E}_{k}(n+1)$ are independent Bernoulli random variables of parameter $h^{n+1}(p)$. Consequently, the process $F$ classifies the density on $T_{4}^{\prime}$.

Let us mention that from time $n \geq 1$, the field $\left(X_{g}^{n}\right)_{g \in G}$ is not i.i.d. For example, $X_{1}^{1}$ and $X_{a b^{-1} a^{-1}}^{1}$ are not independent since both of them depend on $X_{a}^{0}$.

On $T_{2 k}^{\prime}=\left\langle a_{1}, \ldots, a_{k} \mid \cdot\right\rangle$, one can either apply Prop. 2 to obtain a cellular automaton that classifies the density, or define a new CA by the following formula: $F(x)_{g}=\operatorname{maj}\left(x_{g a_{1}}, x_{g a_{1} a_{2}}, x_{g a_{1} a_{2}^{-1}}, \ldots, x_{g a_{1} a_{k}}, x_{g a_{1} a_{k}^{-1}}\right)$ and check that it is also classifies the density.


Figure 3: The cellular automata described by Theorem 3 and Theorem 4

It is also possible to adapt the above proof to show that the IPS with the same local rule also classifies the density.

### 4.3 A rule that classifies the density on $T_{3}$

We now consider the group $T_{3}=\left\langle a, b, c \mid a^{2}=b^{2}=c^{2}=1\right\rangle$.
Theorem 4. The cellular automaton $F: \mathcal{A}^{T_{3}} \rightarrow \mathcal{A}^{T_{3}}$ defined by:

$$
F(x)_{g}=\operatorname{maj}\left(x_{g a b}, x_{g a c}, x_{g a c b c}\right)
$$

for any $x \in \mathcal{A}^{T_{3}}, g \in T_{3}$, classifies the density.
Proof. The proof is analogous to the previous case. We prove by induction on $n \in \mathbb{N}$ that for any $k \in \mathbb{N}$, that the family $\mathcal{E}_{k}(n)=\left\{X_{u_{1} u_{2} \ldots u_{k}}^{n} \mid u_{1}, u_{2}, \ldots, u_{k} \in\right.$ $\{a b, a c, a c b c\}\}$ consists of independent Bernoulli random variables of parameter $h^{n}(p)$, the key point being that $\{a b, a c, a c b c\}$ is a code.

Once again, as explained in Prop. 2, since we have a solution on $T_{3}$, we obtain a CA that classifies the density for any $T_{n}, n \geq 3$, by applying exactly the same rule. The corresponding IPS on $T_{n}$ also classifies the density.

## 5 Classifying the density on $\mathbb{Z}$

The one-dimensional case appears as much more difficult than the other cases and we are not aware of any solution to the density classification problem on $\mathbb{Z}$.

However, if we slightly change the formulation of the problem, simple solutions do exist. We first give one such modification and then go back to the original problem and describe three models, two CA and one PCA, that are conjectured to classify the density. We also provide some preliminary analytical results as well as experimental confirmations of these results by using numerical simulations.

In the examples below, the traffic cellular automaton, rule 184 according to Wolfram's notation, plays a central role. It is the CA with neighborhood $\mathcal{N}=\{-1,0,1\}$ and local function traf defined by:

| $x, y, z$ | 111 | 110 | 101 | 100 | 011 | 010 | 001 | 000 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\operatorname{traf}(x, y, z)$ | 1 | 0 | 1 | 1 | 1 | 0 | 0 | 0 |

This CA can be seen as a simple model of traffic flow on a single lane: the cars are represented by 1's moving one step to the right if and only if there are no cars directly in front of them. It is a density-preserving rule.

### 5.1 An exact solution with weakened conditions

On finite rings, several models have been proposed that solve relaxed variants of the density classification problem. We concentrate on one of these models introduced in [15]. The original setting is modified since the model operates on an extended alphabet, and the criterium for convergence is also weakened. Modulo this relaxation, it solves the problem on finite rings $\mathbb{Z}_{n}$. We show the same result on $\mathbb{Z}$.

Proposition 4. Consider the cellular automaton $F$ on the alphabet $\mathcal{B}=\mathcal{A}^{2}$, with neighbourhood $\mathcal{N}=\{-1,0,1\}$, and local function $f=\left(f_{1}, f_{2}\right)$ defined by:

$$
f_{1}(x, y, z)=\operatorname{traf}\left(x_{1}, y_{1}, z_{1}\right) \quad ; \quad f_{2}(x, y, z)= \begin{cases}0 & \text { if } x_{1}=y_{1}=0  \tag{3}\\ 1 & \text { if } x_{1}=y_{1}=1 \\ y_{2} & \text { otherwise }\end{cases}
$$

The projections $\mu_{p} F^{n}\left(\mathcal{A}^{\mathbb{Z}} \times \cdot\right)$ converge to $\delta_{\mathbf{0}}$ if $p<1 / 2$ and to $\delta_{\mathbf{1}}$ if $p>1 / 2$.
Intuitively, the CA operates on two tapes: on the first tape, it simply performs the traffic rule; on the second tape, what is recorded is the last occurrence of two consecutive zeros or ones in the first tape. If $p<1 / 2$, then, on the first tape, there is convergence to configurations which alternate between patterns of types $0^{k}$ and $(10)^{\ell}$. Consequently, on the second tape, there is convergence to the configuration $\delta_{\mathbf{0}}$. We formalise the argument below.

Proof. Let $T: \mathcal{A}^{\mathbb{Z}} \rightarrow \mathcal{A}^{\mathbb{Z}}$ be the traffic CA, see above. Following an idea of Belitsky and Ferrari [1], we define the recoding $\psi: \mathcal{A}^{\mathbb{Z}} \rightarrow\{-1,0,1\}^{\mathbb{Z}}$ by $\psi(x)_{i}=1-x_{i}-x_{i-1}$. Consider $\left(\psi \circ T^{n}(x)\right)_{n \geq 0}$, the recodings of the trajectory of the CA originating from $x \in\{0,1\}^{\mathbb{Z}}$. There is a convenient alternative way to describe $\left(\psi \circ T^{n}(x)\right)_{n \geq 0}$. It corresponds to the trajectories in the so-called

Ballistic Annihilation model: 1 and -1 are interpreted as particles that we call respectively positive and negative particles. Negative particles move one cell to the left at each time step while positive particles move one cell to the right; and when two particles of different types meet, they annihilate.

Consider the Ballistic Annihilation model with initial condition $\mu_{p} \psi$ for $p>$ $1 / 2$. The density of negative particles is $p^{2}$, while the density of positive particles is $(1-p)^{2}$. During the evolution, the density of positive particles decreases to 0 , while the density of negative particles decreases to $2 p-1$. In particular, the negative particles that will never disappear have density $2 p-1$ (see [1] for details). We can track back the position at time 0 of the "eternal" negative particles. Let $X$ be the (random) position of the first eternal particle on the right of cell 0 . After time $X$, the column 0 in the space-time diagram contains only 0 or -1 values. This key point is illustrated in the figure below.


We now go back to the traffic CA with initial condition distributed according to $\mu_{p}$ for $p>1 / 2$ and concentrate on two consecutive columns of the space-time diagram. The property tells us that after some almost surely finite time, the columns contain only the patterns 11,01 , or 10 .

For the CA defined by Eq. 3 with an initial condition distributed according to a measure $\mu$ satisfying $\mu\left(\cdot \times \mathcal{A}^{\mathbb{Z}}\right)=\mu_{p}$ for $p>1 / 2$, the above key point gets translated as follows: in any given column of the space-time diagram, after some a.s. finite time, the column contains only the letters $(0,1)$ or $(1,1)$. In particular, $\mu_{p} F^{t}\left(\mathcal{A}^{\mathbb{Z}} \times \cdot\right)$ converges weakly to $\delta_{\mathbf{1}}$ if $p>1 / 2$.

### 5.2 Density classifier candidates on $\mathbb{Z}$

The GKL cellular automaton. The Gács-Kurdyumov-Levin (GKL) cellular automaton is the CA with neighbourhood $\mathcal{N}=\{-3,-1,0,1,3\}$ defined by

$$
\operatorname{gkl}(x)_{k}= \begin{cases}\operatorname{maj}\left(x_{k}, x_{k+1}, x_{k+3}\right) & \text { if } x_{k}=1 \\ \operatorname{maj}\left(x_{k}, x_{k-1}, x_{k-3}\right) & \text { if } x_{k}=0\end{cases}
$$

for any $x \in \mathcal{A}^{\mathbb{Z}}, k \in \mathbb{Z}$.
The GKL CA is known to be one of the best performing CA for the density classification on finite rings (see Fig. 4). It has also been proven to have the eroder property: if the initial configuration contains only a finite number of ones (resp. zeros), then it reaches $\mathbf{0}$ (resp. 1) in finite time, see 9 .


Figure 4: Two space-time diagrams of GKL (top) and Kari's PCA (bottom) for $n=149$. Initial condition with density $70 / 149$ (left) and $77 / 149$ (right).

Kari traffic cellular automaton. This CA is defined by the composition of the two following rules applied sequentially at each time step: (a) apply the traffic rule, (b) change the 1 into a 0 in every pattern 0010 and the 0 into a 1 in every pattern 1011 (see Fig. 44).

Like GKL, Kari traffic CA has a neighbourhood of radius 3. Both CA also share the combined symmetry consisting in swapping 0 and 1 and right and left. Kari traffic has also the eroder property and it appears to have comparable qualities to GKL concerning the density classification task, see [15. Kari traffic CA is closely related to Kurka's modified version of GKL [13].

The majority-traffic probabilistic cellular automaton. The majoritytraffic PCA of parameter $\alpha \in(0,1)$ is the PCA of neighbourhood $\mathcal{N}=\{-1,0,1\}$ and local function:

$$
\varphi(x, y, z)=\alpha \delta_{\operatorname{maj}(x, y, z)}+(1-\alpha) \delta_{\operatorname{traf}(x, y, z)}
$$

In words, at each time step, we choose, independently for each cell, to apply the majority rule with probability $\alpha$ and the traffic rule with probability $1-\alpha$ (see Fig. 5 .

The majority-traffic PCA has been introduced by Fatès [4] who has proved that it "classifies" the density on a finite ring with an arbitrary precision: for


Figure 5: Two space-time diagrams of the majority-traffic PCA for $\alpha=0.1$ and $n=149$. The same initial condition with density $70 / 149$ is used. The case seen on the right is a rare event (evolution towards a bad classification).
any $n \in \mathbb{N}$ and any $\varepsilon>0$, there exists a value $\alpha_{n, \varepsilon}$ of the parameter such that on $\mathbb{Z}_{n}$, the PCA converges to the right uniform configuration with probability greater than $1-\varepsilon$.

Conjecture 1. The GKL CA, the Kari traffic $C A$, and the majority-traffic $P C A$ with $0<\alpha<\alpha_{c}$ (for some $0<\alpha_{c} \leq 1 / 2$ ) classify the density.

### 5.3 Invariant Measures

Following ideas developed by Kurka [13], we can give a precise description of the invariant measures of these PCA.

Proposition 5. For the majority-traffic PCA and for Kari traffic CA, the extremal invariant measures are $\delta_{\mathbf{0}}, \delta_{\mathbf{1}}$, and $\left(\delta_{(01)^{\mathbb{Z}}}+\delta_{(10)^{Z}}\right) / 2$. For $G K L$, on top of these three measures, there exist extremal invariant measures of density $p$ for any $p \in[1 / 3,2 / 3]$.

Proof. Majority-traffic PCA. Let us consider the majority-traffic PCA $P$ of parameter $\alpha \in(0,1)$. We denote by $\left[x_{0}, \ldots, x_{n}\right]_{k}$ the cylinder set of all configurations $y \in \mathcal{A}^{\mathbb{Z}}$ satisfying $y_{k+i}=x_{i}$ for $0 \leq i \leq n$. Let $\mu$ be any shiftinvariant measure. An exhaustive search shows that if at time 1 , we observe the cylinder $[100]_{0}$ then there are only eight possible cylinders of size 5 at time 0 , that are:

$$
\begin{aligned}
& {[01100]_{-1},[10000]_{-1},[10001]_{-1},[10010]_{-1}} \\
& {[10100]_{-1},[11000]_{-1},[11001]_{-1},[11100]_{-1}}
\end{aligned}
$$

If we weight each cylinder by the probability to reach $[100]_{0}$ from them, we obtain the following expression:

$$
\begin{aligned}
\mu P[100]= & \alpha(1-\alpha) \mu[01100]+(1-\alpha) \mu[10000]+(1-\alpha) \mu[10001]+(1-\alpha) \mu[10010] \\
& +\alpha \mu[10100]+\alpha^{2} \mu[11000]+\alpha^{2} \mu[11001]+\alpha(1-\alpha) \mu[11100] .
\end{aligned}
$$

Since the measure $\mu$ is supposed to be shift-invariant, we do not need to specify the position of the cylinders: we denote by $\mu\left[x_{0}, \ldots, x_{n}\right]$ the value $\mu\left(\left[x_{0}, \ldots, x_{n}\right]_{k}\right)$ which does not depend on $k \in \mathbb{Z}$. Gathering the terms with the same coefficient, we have:

$$
\begin{aligned}
\mu P[100] & =(1-\alpha)(\mu[100]-\mu[10011])+\alpha \mu[10100]+\alpha(1-\alpha) \mu[1100]+\alpha^{2} \mu[1100] \\
& =(1-\alpha)(\mu[100]-\mu[10011])+\alpha \mu[10100]+\alpha \mu[1100] .
\end{aligned}
$$

Some more rearrangements provide:

$$
\begin{aligned}
\mu P[100] & =(1-\alpha)(\mu[100]-\mu[10011])+\alpha(\mu[100]-\mu[00100]) \\
& =\mu[100]-(1-\alpha) \mu[10011]-\alpha \mu[00100] .
\end{aligned}
$$

This proves that the sequence $\left(\mu P^{n}[100]\right)_{n \geq 0}$ is non-increasing. Let us assume that $\mu P=\mu$. Then, $\mu[10011]=\mu[00100]=0$.

Let us consider the cylinder [ $10^{n} 0011$ ] for some $n \geq 2$. If we apply the majority rule on each cell except on the second cell from the left, then after $n$ iterations, we reach the cylinder [10011]. Since this occurs with a positive probability, we obtain that for any $n \geq 0, \mu\left[10^{n} 0011\right]=0$. This provides: $\mu[0011]=\mu[00011]=\mu[000011]=\ldots=\mu\left[0^{n} 11\right]$ for any $n \geq 2$. Consequently, $\mu[0011]=0$. From a cylinder of the form $\left[00(10)^{n} 11\right]$, if we choose to apply the majority rule on each cell, then we reach the cylinder [0011] in $n$ steps. Thus, $\mu\left[00(10)^{n} 11\right]=0$ for any $n \geq 0$. It follows that $\mu$ can be written as the sum $\mu=\mu_{0}+\mu_{1}$ of two invariant measures, where $\mu_{0}$ charges only the subshift $\Sigma_{0}=\left\{x \in \mathcal{A}^{\mathbb{Z}} \mid \forall k \in \mathbb{Z}, x_{k} x_{k+1} \neq 00\right\}$ and $\mu_{1}$ the subshift $\Sigma_{1}=\left\{x \in \mathcal{A}^{\mathbb{Z}} \mid \forall k \in\right.$ $\left.\mathbb{Z}, x_{k} x_{k+1} \neq 11\right\}$. Let us assume that $\mu[00]=0$ (which is the case for $\mu_{0}$ ). In the same way that we have computed $\mu P[110]$, we can compute $\mu P[11]$, and we obtain:

$$
\begin{aligned}
\mu P[11] & =\alpha \mu[0110]+\alpha \mu[1110]+\alpha \mu[1101]+\mu[1011]+\mu[0111]+\mu[1111] \\
& =\alpha \mu[110]+\alpha \mu[1101]+\mu[11]-\mu[0011] \\
& =\mu[11]+\alpha \mu[110]+\alpha \mu[1101] .
\end{aligned}
$$

By hypothesis, $\mu P=\mu$, so that the last equality implies that $\mu[110]=0$.
In all cases, if $\mu$ is a shift-invariant measure such that $\mu P=\mu$, then $\mu[00]=$ $\mu(\mathbf{0}), \mu[11]=\mu(\mathbf{1})$ and $\mu[01]=\mu[10]=\mu\left((01)^{\mathbb{Z}}\right)=\mu\left((10)^{\mathbb{Z}}\right)$.

Kari traffic CA. If at time 1, we observe the pattern 100 at position 0 , then, at time 0 , that is to say before the application of Kari's CA, this same pattern was present at position -1 . Indeed, one can check that none of the cell of the pattern 100 can have been obtained by the transformation (b) (see the definition of Kari traffic CA), so that one has just to consider the possible history of 100 by the traffic CA. In the same way, one can prove that if at time 1 , we observe the pattern 110 at position 0 , then, at time 0 , this same pattern was present at position 1 . Let $\mu$ be a shift-invariant measure such that $\mu K=\mu$, where $K$ denotes Kari traffic CA. A consequence of the result on the patterns 100 and 110 that we have just described is that $\mu K^{n+1}[110 x 100]=0$ for any
$n \geq 0$ and any $x \in \mathcal{A}^{n}$. But since $\mu K^{n+1}=\mu$, we obtain $\mu[110 x 100]=0$ for any word $x$ on the alphabet $\mathcal{A}$. Once again, we can write $\mu=\mu_{0}+\mu_{1}$ where $\mu_{0}$ and $\mu_{1}$ are two invariant measures defined on $\Sigma_{0}$ and $\Sigma_{1}$.

Let us consider a configuration of $\Sigma_{0}$, that is, without the pattern 00. By the traffic rule, each 0 of the configuration will move one cell to the left. Then by rule 1 , if a 0 is at distance greater than 2 from the next 0 on its right, it is erased by rule (b). The result follows.

GKL. Any word $x \in \mathcal{A}^{\mathbb{Z}}$ that is a concatenation of the patterns $u=001$ and $v=011$ is a fixed point of the GKL cellular automaton: if $x_{n}=0$, then either $x_{n-1}=0$ or $x_{n-3}=0$ so that $F(x)_{n}=0$ and if $x_{n}=1$, then either $x_{n+1}=1$ or $x_{n+3}=1$ so that $F(x)_{n}=1$. As a consequence, GKL has extremal invariant measures of density $p$ for any $p \in[1 / 3,2 / 3]$.

To summarize, the majority-traffic and Kari traffic CA have a simpler set of invariant measures. It does not rule out GKL as a candidate for solving the density classification task, but rather indicates that it could be easier to prove the result for majority-traffic or Kari traffic CA.

### 5.4 Experimental results

Conjecture 1 was first motivated by the observation of the space-time diagrams, see Fig. 4 and 5. We provide some numerical results that support this conjecture. For a given ring size $n$, we generate an initial configuration $x$ by assigning to each cell the state 1 with a probability $p$ and the state 0 with probability $1-p$. Let us denote by $d(x)$ the actual density of 1 in the configuration $x$. We let the system evolve until it reaches a fixed point $\mathbf{0}$ or $\mathbf{1}$ and see if the fixed point is $\mathbf{0}$ for $d(x)<1 / 2$ and $\mathbf{1}$ for $d(x)>1 / 2$. The quality $Q(n)$ corresponds to the proportion of good classifications on a given ring of size $n$.

Figure $\sqrt{6}$ shows the evolution of $Q(n)$, each value of $Q(n)$ being evaluated over 100000 samples. For the three rules, the plots are in agreement with the hypothesis that the asymptotic value of $Q(n)$ is 1 . From a qualitative point of view, we observe that for all values of $d$ the quality decreases before increasing, but this is only a border phenomenon for very small ring sizes. We also observe that when the initial density $d$ increases from 0.45 to 0.48 , the value of $n$ needed to attain a given quality $Q(n)$ increases dramatically. For $d=0.49$, the change of derivative of the curve $Q(n)$ becomes hardly visible. However, our belief is that $Q(n)$ will approach one as the lattice size grows, no matter how close $p$ is to the critical density $1 / 2$. To see why this holds, consider the error rate $\operatorname{err}(d)$ obtained as the probability to make a bad classification when the initial configuration is equal to $d$. We experimentally observed that, as $n$ grows, the function $\operatorname{err}(d)$, which is defined for values $k / n$ with $k \in\{0, \ldots, n\}$, approaches a Bell curve whose mean is centred on $1 / 2$ and whose tail progressively approaches the 0 -axis (see Fig. 7). At the same time, for a fixed $p$, the probability $p(k / n)$ that the initial configuration has a density of ones equal to $k / n$ follows a binomial distribution of parameter $p$. We can thus calculate the global error rate $E(n)=1-Q(n)$ with $E(n)=\sum_{k=0}^{n} \operatorname{err}(k / n) \cdot p(k / n)$. Intuitively, it can be seen


Figure 6: Experimental determination of the quality of classification $Q(n)$ as a function of ring size $n$. Cells are initialised with a probability $p$ to be in state 1. Each point represents an average computed out 100000 trajectories.


Figure 7: Majority-traffic rule with $\alpha=0.25$ : Evolution of the error rate as a function of the initial density when doubling the ring size.
that as $n$ grows to infinity, the two distributions $\operatorname{err}(k / n)$ and $p(k / n)$ progressively separate as their mean value is different and their variance approaches 0 . As a consequence, for larger values of $n$, the value $E(n)$ progressively vanishes and the quality approaches 1 .

By contrast, there are other PCA, such as the rule which was originally studied by Fukś 6, and which consists in doing a copy of the right or left neighbour with a fixed probability $p<1 / 2$, and the identity otherwise. This rules preserves the density in average at each time step for finite rings (see details in [4]). As a consequence, we have $Q(n)=\max (p, 1-p)$ which implies that the increase of $n$ does not improve the average performance of the system. In the infinite case, the preservation of the density is exact and can not allow the system to classify the density. We experimentally observed the same qualitative behaviour for the rule used by Schüle [18] or for the Majority-Traffic rule for $\alpha>$ $1 / 2$. We believe that there exists a strong relationship between the asymptotic behaviour of the rule on finite rings and the ability to classify the density on $\mathbb{Z}$.

### 5.5 Link with the positive rates "conjecture"

The difficulty of classifying the density on $\mathbb{Z}$ is related to the difficulty of the ergodicity problem on $\mathbb{Z}$. By definition, a PCA or an IPS has positive rates if all its local probability transitions are different from 0 and 1 . In $\mathbb{Z}^{2}$, there exist
positive rates PCA and IPS that are non-ergodic (for instance, a "positive rates version" of Toom's rule [3]). It had been a long standing conjecture that all positive rates PCA and IPS on $\mathbb{Z}$ are ergodic. Gács disproved the conjecture by exhibiting a complex counter-example with several invariant measures, but with an alphabet of cardinality $2^{18}$ instead of 2 [8]. If we knew a process that classifies the density on $\mathbb{Z}$, it could pave the way to exhibit simple examples of positive rates processes that are non-ergodic.

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[^0]:    *INRIA/ENS, 23, avenue d'Italie, CS 81321, 75214 Paris Cedex 13, France. E-mail: Ana.Busic@inria.fr.
    ${ }^{\dagger}$ INRIA Nancy - Grand-Est, LORIA, Nancy Université, BP 239, 54506, Vandœuvre-lèsNancy, France. E-mail: Nazim.Fates@loria.fr.
    ${ }^{\ddagger}$ LIAFA, CNRS and Université Paris Diderot - Paris 7, Case 7014, 75205 Paris Cedex 13, France. E-mail: Jean.Mairesse@liafa.jussieu.fr.
    §LIAFA, CNRS and Université Paris Diderot - Paris 7, Case 7014, 75205 Paris Cedex 13, France. E-mail: Irene.Marcovici@liafa.jussieu.fr (corresponding author).

