

# Approximating minimum-power edge-multicovers

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**Abstract.** Given a graph with edge costs, the *power* of a node is the maximum cost of an edge incident to it, and the power of a graph is the sum of the powers of its nodes. Motivated by applications in wireless networks, we consider the following fundamental problem in wireless network design. Given a graph  $G = (V, E)$  with edge costs and degree bounds  $\{r(v) : v \in V\}$ , the **Minimum-Power Edge-Multi-Cover (MPEMC)** problem is to find a minimum-power subgraph  $J$  of  $G$  such that the degree of every node  $v$  in  $J$  is at least  $r(v)$ . We give two approximation algorithms for MPEMC, with ratios  $O(\log k)$  and  $k + 1/2$ , where  $k = \max_{v \in V} r(v)$  is the maximum degree bound. This improves the previous ratios  $O(\log n)$  and  $k + 1$ , and implies ratios  $O(\log k)$  for the **Minimum-Power  $k$ -Outconnected Subgraph** and  $O(\log k \log \frac{n}{n-k})$  for the **Minimum-Power  $k$ -Connected Subgraph** problems; the latter is the currently best known ratio for the min-cost version of the problem.

## 1 Introduction

### 1.1 Motivation and problems considered

Wireless networks are studied extensively due to their wide applications. The power consumption of a station determines its transmission range, and thus also the stations it can send messages to; the power typically increases at least quadratically in the transmission range. Assigning power levels to the stations (nodes) determines the resulting communication network. Conversely, given a communication network, the power required at  $v$  only depends on the farthest node reached directly by  $v$ . This is in contrast with wired networks, in which every pair of stations that communicate directly incurs a cost. An important network property is fault-tolerance, which is often measured by minimum degree or node-connectivity of the network. Node-connectivity is much more central here than edge-connectivity, as it models stations failures. Such power minimization problems were vastly studied; see for example [1, 2, 5, 8, 9] and the references therein for a small sample of papers in this area. The first problem we consider is finding a low power network with specified lower degree bounds. The second problem is the **Min-Power  $k$ -Connected Subgraph** problem. We give approximation algorithms for these problems, improving the previously best known ratios.

**Definition 1.** Let  $(V, J)$  be a graph with edge-costs  $\{c(e) : e \in J\}$ . For a node  $v \in V$  let  $\delta_J(v)$  denote the set of edges incident to  $v$  in  $J$ . The power  $p_J(v)$  of  $v$  is the maximum cost of an edge in  $J$  incident to  $v$ , or 0 if  $v$  is an isolated node of  $J$ ; i.e.,  $p_J(v) = \max_{e \in \delta_J(v)} c(e)$  if  $\delta_J(v) \neq \emptyset$ , and  $p_J(v) = 0$  otherwise. For  $V' \subseteq V$  the power of  $V'$  w.r.t.  $J$  is the sum  $p_J(V') = \sum_{v \in V'} p_J(v)$  of the powers of the nodes in  $V'$ .

Unless stated otherwise, all graphs are assumed to be undirected and simple. Let  $n = |V|$ . Given a graph  $G = (V, E)$  with edge-costs  $\{c(e) : e \in E\}$ , we seek to find a low power subgraph  $(V, J)$  of  $G$  that satisfies some prescribed property. One of the most fundamental problems in Combinatorial Optimization is finding a minimum-cost subgraph that obeys specified degree constraints (sometimes called also “matching problems”) c.f. [10]. Another fundamental property is fault-tolerance (connectivity). In fact, these problems are related, and we use our algorithm for the former as a tool for approximating the latter.

**Definition 2.** Given degree bounds  $r = \{r(v) : v \in V\}$ , we say that an edge-set  $J$  on  $V$  is an  $r$ -edge cover if  $d_J(v) \geq r(v)$  for every  $v \in V$ , where  $d_J(v) = |\delta_J(v)|$  is the degree of  $v$  in the graph  $(V, J)$ .

**Minimum-Power Edge-Multi-Cover (MPEMC):**

*Instance:* A graph  $G = (V, E)$  with edge-costs  $\{c(e) : e \in E\}$ , degree bounds  $r = \{r(v) : v \in V\}$ .

*Objective:* Find a minimum power  $r$ -edge cover  $J \subseteq E$ .

Given an instance of MPEMC, let  $k = \max_{v \in V} r(v)$  denote the maximum requirement.

We now define our connectivity problems. A graph is  $k$ -outconnected from  $s$  if it contains  $k$  internally-disjoint  $sv$ -paths for all  $v \in V \setminus \{s\}$ . A graph is  $k$ -connected if it is  $k$ -outconnected from every node, namely, if it contains  $k$  internally-disjoint  $uv$ -paths for all  $u, v \in V$ .

**Minimum-Power  $k$ -Outconnected Subgraph (MP $k$ OS):**

*Instance:* A graph  $G = (V, E)$  with edge-costs  $\{c(e) : e \in E\}$ , a root  $s \in V$ , and an integer  $k$ .

*Objective:* Find a minimum-power  $k$ -outconnected from  $s$  spanning subgraph  $J$  of  $G$ .

**Minimum-Power  $k$ -Connected Subgraph (MP $k$ CS):**

*Instance:* A graph  $G = (V, E)$  with edge-costs  $\{c(e) : e \in E\}$  and an integer  $k$ .

*Objective:* Find a minimum-power  $k$ -connected spanning subgraph  $J$  of  $G$ .

## 1.2 Our Results

The previous best approximation ratio for MPEMC was  $O(\log n)$  [3]. Our main result improves this ratio to  $O(\log k)$ .

**Theorem 1.** MPEMC admits an  $O(\log k)$ -approximation algorithm.

For small values of  $k$ , the problem admits also the ratios  $k + 1$  for arbitrary  $k$  [2], while for  $k = 1$  the best known ratio is  $k + 1/2 = 3/2$  [4]. Our second result extends the latter ratio to arbitrary  $k$ .

**Theorem 2.** *MPkCS admits a  $(k + 1/2)$ -approximation algorithm.*

For small values of  $k$ , say  $k \leq 6$ , the ratio  $(k + 1/2)$  is better than  $O(\log k)$  because of the constant hidden in the  $O(\cdot)$  term. And overall, our paper gives the currently best known ratios for all values  $k \geq 2$ .

In [5] it is proved that an  $\alpha$ -approximation for MPkCS implies an  $(\alpha + 4)$ -approximation for MPkOS. The previous best ratio for MPkOS was  $O(\log n) + 4 = O(\log n)$  [5] for large values of  $k = \Omega(\log n)$ , and  $k + 1$  for small values of  $k$  [9]. From Theorem 1 we obtain the following.

**Theorem 3.** *MPkOS admits an  $O(\log k)$ -approximation algorithm.*

In [2] it is proved that an  $\alpha$ -approximation for MPkCS and a  $\beta$ -approximation for Min-Cost  $k$ -Connected Subgraph implies a  $(\alpha + 2\beta)$ -approximation for MPkCS. Thus the previous best ratio for MPkCS was  $2\beta + O(\log n)$  [3], where  $\beta$  is the best ratio for MCKCS (for small values of  $k$  better ratios for MPkCS are given in [9]). The currently best known value of  $\beta$  is  $O\left(\log k \log \frac{n}{n-k}\right)$  [7], which is  $O(\log k)$ , unless  $k = n - o(n)$ . From Theorem 1 we obtain the following.

**Theorem 4.** *MPkCS admits an  $O(\beta + \log k)$ -approximation algorithm, where  $\beta$  is the best ratio for MCKCS. In particular, MPkCS admits an  $O\left(\log k \log \frac{n}{n-k}\right)$ -approximation algorithm.*

### 1.3 Overview of the techniques

Let the *trivial solution* for MPkCS be obtained by picking for every node  $v \in V$  the cheapest  $r(v)$  edges incident to  $v$ . It is known and easy to see that this produces an edge set of power at most  $(k + 1) \cdot \text{opt}$ , see [2].

Our  $O(\log k)$ -approximation algorithm uses the following idea. Extending and generalizing an idea from [3], we show how to find an edge set  $I \subseteq E$  of power  $O(\text{opt})$  such that for the residual instance, the trivial solution value is reduced by a constant fraction. We repeatedly find and add such an edge set  $I$  to the constructed solution, while updating the degree bounds accordingly to  $r(v) \leftarrow \max\{r(v) - d_I(v), 0\}$ . After  $O(\log k)$  steps, the trivial solution value is reduced to  $\text{opt}$ , and the total power of the edges we picked is  $O(\log k) \cdot \text{opt}$ . At this point we add to the constructed solution the trivial solution of the residual problem, which at this point has value  $\text{opt}$ , obtaining an  $O(\log k)$ -approximate solution.

Our  $(k + 1/2)$ -approximation algorithm uses a two-stage reduction. The first reduction reduces MPkCS to a constrained version of MPkCS with  $k = 1$ , where we also have lower bounds  $\ell_v$  on the power of each node  $v \in V$ ; these lower bounds are determined by the trivial solution to the problem. We will

show that a  $\rho$ -approximation algorithm to this constrained version implies a  $(k - 1 + \rho)$ -approximation algorithm for MPEMC. The second reduction reduces the constrained version to the Minimum-Cost Edge Cover problem with a loss of  $3/2$  in the approximation ratio. As Minimum-Cost Edge Cover admits a polynomial time algorithm, we get a ratio  $\rho = 3/2$  for the constrained problem, which in turn gives the ratio  $k - 1 + \rho = k + 1/2$  for MPEMC.

## 2 An $O(\log k)$ -approximation (proof of Theorem 1)

As in [3], we reduce MPEMC to Bipartite MPEMC, where  $G = (V, E)$  is a bipartite graph with sides  $A, B$ , and  $r(a) = 0$  for every  $a \in A$  (so, only the nodes in  $B$  may have positive degree bound). This is done by taking two copies  $A = \{a_v : v \in V\}$  and  $B = \{b_v : v \in V\}$  of  $V$ , for every edge  $e = uv \in E$  adding the two edges  $a_u b_v$  and  $a_v b_u$  of cost  $c(e)$  each, and for every  $v \in V$  setting  $r(b_v) = r(v)$  and  $r(a_v) = 0$ . It is proved in [3] that this reduction invokes a factor of 2 in the approximation ratio, namely, that a  $\rho$ -approximation for bipartite MPEMC implies a  $2\rho$ -approximation for general MPEMC.

Let  $\text{opt}$  denote the optimal solution value of a problem instance at hand. For  $v \in V$ , let  $w_v$  be the cost of the  $r(v)$ -th least cost edge incident to  $v$  in  $E$  if  $r(v) \geq 1$ , and  $w_v = 0$  otherwise. Given a partial solution  $J$  to Bipartite MPEMC let  $r_J(v) = \max\{r(v) - d_J(v), 0\}$  be the *residual bound* of  $v$  w.r.t.  $J$ . Let

$$R_J = \sum_{b \in B} w_b r_J(b) .$$

The main step in our algorithm is given in the following lemma, which will be proved later.

**Lemma 1.** *There exists a polynomial time algorithm that given an edge set  $J \subseteq E$ , an integer  $\tau$ , and a parameter  $\gamma > 1$ , either correctly establishes that  $\tau < \text{opt}$ , or returns an edge set  $I \subseteq E \setminus J$  such that  $p_I(V) \leq (1 + \gamma)\tau$  and  $R_{J \cup I} \leq \theta R_J$ , where  $\theta = 1 - \left(1 - \frac{1}{\gamma}\right) \left(1 - \frac{1}{e}\right)$ .*

**Lemma 2.** *Let  $J \subseteq E$  and let  $F \subseteq E \setminus J$  be an edge set obtained by picking  $r_J(b)$  least cost edges in  $\delta_{E \setminus J}(b)$  for every  $b \in B$ . Then  $J \cup F$  is an  $r$ -edge-cover and:  $p_F(B) \leq \text{opt}$ ,  $p_F(A) \leq R_J \leq k \cdot \text{opt}$ .*

*Proof.* Since  $F$  is an  $r_J$ -edge-cover,  $J \cup F$  is an  $r$ -edge-cover. By the definition of  $F$ , for any  $r$ -edge-cover  $I$ ,  $p_F(b) \leq w_b \leq p_I(b)$  for all  $b \in B$ . In particular, if  $I$  is an optimal  $r$ -edge-cover, then

$$p_F(B) \leq \sum_{b \in B} w_b \leq \sum_{b \in B} p_I(b) = p_I(B) \leq \text{opt} .$$

Also,

$$R_J = \sum_{b \in B} w_b r_J(b) \leq k \cdot \sum_{b \in B} w_b \leq k \cdot \text{opt} .$$

Finally,  $p_F(A) \leq R_J$  since

$$p_F(A) = \sum_{a \in A} p_F(a) \leq \sum_{a \in A} \sum_{e \in \delta_F(a)} c(e) = \sum_{e \in F} c(e) \leq \sum_{b \in B} w_b r_J(b) = R_J .$$

This concludes the proof of the lemma.  $\square$

Theorem 1 is deduced from Lemmas 1 and 2 as follows. We set  $\gamma$  to be constant strictly greater than 1, say  $\gamma = 2$ . Then  $\theta = 1 - \frac{1}{2} \left(1 - \frac{1}{e}\right)$ . Using binary search, we find the least integer  $\tau$  such that the following procedure computes an edge set  $J$  satisfying  $R_J \leq \tau$ .

*Initialization:*  $J \leftarrow \emptyset$ .

*Loop:* Repeat  $\lceil \log_{1/\theta} k \rceil$  times:

Apply the algorithm from Lemma 2:

- If it establishes that  $\tau < \text{opt}$  then return “ERROR” and STOP.
- Else do  $J \leftarrow J \cup I$ .

After computing  $J$  as above, we compute an edge set  $F \subseteq E \setminus J$  as in Lemma 2. The edge-set  $J \cup F$  is a feasible solution, by Lemma 2. We claim that for any  $\tau \geq \text{opt}$  the above procedure returns an edge set  $J$  satisfying  $R_J \leq \tau$ ; thus binary search indeed applies. To see this, note that  $R_\emptyset \leq k \cdot \text{opt}$  and thus

$$R_J \leq R_\emptyset \cdot \theta^{\lceil \log_{1/\theta} k \rceil} \leq k \cdot \text{opt} \cdot 1/k = \text{opt} \leq \tau .$$

Consequently, the least integer  $\tau$  for which the above procedure does not return “ERROR” satisfies  $\tau \leq \text{opt}$ . Thus  $p_J(V) \leq \lceil \log_{1/\theta} k \rceil \cdot (1 + \gamma) \cdot \tau = O(\log k) \cdot \text{opt}$ . Also, by Lemma 2,  $p_F(V) \leq \text{opt} + R_J \leq 2\text{opt}$ . Consequently,

$$p_{J \cup F}(V) \leq p_J(V) + p_F(V) = O(\log k) \cdot \text{opt} + 2\text{opt} = O(\log k) \cdot \text{opt} .$$

In the rest of this section we prove Lemma 1. It is sufficient to prove the statement in the lemma for the residual instance  $((V, E \setminus J), r_J)$  with edge-costs restricted to  $E \setminus J$ ; namely, we may assume that  $J = \emptyset$ . Let  $R = R_\emptyset = \sum_{b \in B} w_b r(b)$ .

**Definition 3.** An edge  $e \in E$  incident to a node  $b \in B$  is  $\tau$ -cheap if  $c(e) \leq \frac{\tau\gamma}{R} \cdot w_b r(b)$ .

**Lemma 3.** Let  $F$  be an  $r$ -edge-cover, let  $\tau \geq p_F(B)$ , and let

$$I = \bigcup_{b \in B} \{e \in \delta_E(b) : c(e) \leq \frac{\tau\gamma}{R} \cdot w_b r(b)\}$$

be the set of  $\tau$ -cheap edges in  $E$ . Then  $R_{I \cap F} \leq R/\gamma$  and  $p_I(B) \leq \gamma\tau$ .

*Proof.* Let  $D = \{b \in B : \delta_{F \setminus I}(b) \neq \emptyset\}$ . Since for every  $b \in D$  there is an edge  $e \in F \setminus I$  incident to  $b$  with  $c(e) > \frac{\tau\gamma}{R} \cdot w_b r(b)$ , we have  $p_{F \setminus I}(b) \geq \frac{\tau\gamma}{R} \cdot w_b r(b)$  for every  $b \in D$ . Thus

$$\tau \geq p_F(B) \geq p_{F \setminus I}(B) = \sum_{b \in D} p_{F \setminus I}(b) \geq \tau \cdot \frac{\gamma}{R} \sum_{b \in D} w_b r(b) .$$

This implies  $\sum_{b \in D} w_b r(b) \leq R/\gamma$ . Note that for every  $b \in B \setminus D$ ,  $\delta_F(b) \subseteq \delta_I(b)$  and hence  $r_{I \cap F}(b) = r_F(b) = 0$ . Thus we obtain:

$$R_{I \cap F} = \sum_{b \in B} w_b r_{I \cap F}(b) = \sum_{b \in D} w_b r_{I \cap F}(b) \leq \sum_{b \in D} w_b r(b) \leq R/\gamma .$$

To see that  $p_I(B) \leq \gamma\tau$  note that

$$p_I(B) = \sum_{b \in B} p_I(b) \leq \frac{\tau\gamma}{R} \sum_{b \in B} w_b r(b) = \frac{\tau\gamma}{R} \cdot R = \tau\gamma .$$

This concludes the proof of the lemma.  $\square$

In [3] it is proved that the following problem, which is a particular case of submodular function minimization subject to matroid and knapsack constraint (see [6]) admits a  $(1 - \frac{1}{e})$ -approximation algorithm.

**Bipartite Power-Budgeted Maximum Edge-Multi-Coverage (BPBMEM):**

*Instance:* A bipartite graph  $G = (A \cup B, E)$  with edge-costs  $\{c(e) : e \in E\}$  and node-weights  $\{w_v : v \in B\}$ , degree bounds  $\{r(v) : v \in B\}$ , and a budget  $\tau$ .

*Objective:* Find  $I \subseteq E$  with  $p_I(A) \leq \tau$  that maximizes

$$\text{val}(I) = \sum_{v \in B} w_v \cdot \min\{d_I(v), r(v)\} .$$

The following algorithm computes an edge set as in Lemma 1.

1. Among the  $\tau$ -cheap edges, compute a  $(1 - \frac{1}{e})$ -approximate solution  $I$  to BPBMEM.
2. If  $R_I \leq \theta R$  then return  $I$ , where  $\theta = 1 - \left(1 - \frac{1}{\gamma}\right) \left(1 - \frac{1}{e}\right)$ ;  
Else declare “ $\tau < \text{opt}$ ”.

Clearly,  $p_I(A) \leq \tau$ . By Lemma 3,  $p_I(B) \leq \gamma\tau$ . Thus  $p_I(V) \leq p_I(A) + p_I(B) \leq (1 + \gamma)\tau$ .

Now we show that if  $\tau \geq \text{opt}$  then  $R_I \leq \theta R$ . Let  $F$  be the set of cheap edges in some optimal solution. Then  $p_F(A) \leq \text{opt} \leq \tau$ . By Lemma 3  $R_F \leq R/\gamma$ , namely,  $F$  reduces  $R$  by at least  $R \left(1 - \frac{1}{\gamma}\right)$ . Hence our  $(1 - \frac{1}{e})$ -approximate solution  $I$  to BPBMEM reduces  $R$  by at least  $R \left(1 - \frac{1}{e}\right) \left(1 - \frac{1}{\gamma}\right)$ . Consequently, we have  $R_I \leq R - R \left(1 - \frac{1}{e}\right) \left(1 - \frac{1}{\gamma}\right) = \theta R$ , as claimed.

The proof of Theorem 1 is complete.

### 3 A $\left(k + \frac{1}{2}\right)$ -approximation (proof of Theorem 2)

We say that an edge set  $F \subseteq E$  covers a node set  $U \subseteq V$ , or that  $F$  is a  $U$ -cover, if  $\delta_F(v) \neq \emptyset$  for every  $v \in U$ . Consider the following auxiliary problem:

**Restricted Minimum-Power Edge-Cover**

*Instance:* A graph  $G = (V, E)$  with edge-costs  $\{c(e) : e \in E\}$ ,  $U \subseteq V$ , and degree bounds  $\{\ell_v : v \in U\}$ .

*Objective:* Find a power assignment  $\{\pi(v) : v \in V\}$  that minimizes  $\sum_{v \in V} \pi(v)$ , such that  $\pi(v) \geq \ell_v$  for all  $v \in U$ , and such that the edge set  $F = \{e = uv \in E : \pi(u), \pi(v) \geq c(e)\}$  covers  $U$ .

Later, we will prove the following lemma.

**Lemma 4.** *Restricted Minimum-Power Edge-Cover admits a  $3/2$ -approximation algorithm.*

Theorem 2 is deduced from Lemma 4 and the following statement.

**Lemma 5.** *If Restricted Minimum-Power Edge-Cover admits a  $\rho$ -approximation algorithm, then Minimum-Power Edge-Multi-Cover admits a  $(k-1+\rho)$ -approximation algorithm.*

*Proof.* Consider the following algorithm.

1. Let  $\pi(v)$  be the power assignment computed by the  $\rho$ -approximation algorithm for Restricted Minimum-Power Edge-Cover with  $U = \{v \in V : r(v) \geq 1\}$  and bounds  $\ell_v = w_v$  for all  $v \in U$ . Let  $F = \{e = uv \in E : \pi(u), \pi(v) \geq c(e)\}$ .
2. For every  $v \in V$  let  $I_v$  be the edge-set obtained by picking the least cost  $r_F(v)$  edges in  $\delta_{E \setminus F}(v)$  and let  $I = \cup_{v \in V} I_v$ .

Clearly,  $F \cup I$  is a feasible solution to Minimum-Power Edge-Multi-Cover. Let  $\text{opt}$  denote the optimal solution value for Minimum-Power Edge-Multi-Cover. In what follows note that  $\pi(V) \leq \rho \cdot \text{opt}$  and that  $\sum_{v \in V} w_v \leq \text{opt}$ .

We claim that

$$p_{I \cup F}(V) \leq \pi(V) + (k-1) \cdot \text{opt} .$$

As  $\pi(V) \leq \rho \cdot \text{opt}$ , this implies  $p_{I \cup F}(V) \leq (\rho + k - 1) \cdot \text{opt}$ .

For  $v \in V$  let  $\Gamma_v$  be the set of neighbors of  $v$  in the graph  $(V, I_v)$ . The contribution of each edge set  $I_v$  to the total power is at most  $p_{I_v}(\Gamma_v) + p_{I_v}(v)$ . Note that  $\pi(v) \geq p_{I_v}(v)$  and  $\pi(v) \geq p_F(v)$  for every  $v \in V$ , hence  $p_{F \cup I_v}(v) \leq \pi(v)$ . This implies

$$p_{F \cup I}(V) \leq \sum_{v \in V} (\pi(v) + p_{I_v}(\Gamma_v)) = \pi(V) + \sum_{v \in V} p_{I_v}(\Gamma_v) .$$

Now observe that  $|\Gamma_v| = |I_v| = r_F(v) \leq k-1$  and that  $p_{I_v}(u) \leq w_v$  for every  $u \in \Gamma_v$ . Thus

$$p_{I_v}(\Gamma_v) \leq (k-1) \cdot w_v \quad \forall v \in V .$$

Finally, using the fact that  $\sum_{v \in V} w_v \leq \text{opt}$ , we obtain

$$p_{F \cup I}(V) \leq \pi(V) + \sum_{v \in V} p_{I_v}(\Gamma_v) \leq \pi(V) + (k-1) \sum_{v \in V} w_v \leq \pi(V) + (k-1) \cdot \text{opt} .$$

This finishes the proof of the lemma. □

In the rest of this section we prove Lemma 4.

We reduce **Restricted Minimum-Power Edge-Cover** to the following problem that admits an exact polynomial time algorithm, c.f. [10].

**Minimum-Cost Edge-Cover:**

*Instance:* A multi-graph (possibly with loops)  $G = (U, E)$  with edge-costs  $\{c(e) : e \in E\}$ .

*Objective:* Find a minimum cost edge-set  $F \subseteq E$  that covers  $U$ .

Our reduction is not approximation ratio preserving, but incurs a loss of  $3/2$  in the approximation ratio. That is, given an instance  $(G, c, U, \ell)$  of **Restricted Minimum-Power Edge-Cover**, we construct in polynomial time an instance  $(G', c')$  of **Minimum-Cost Edge-Cover** such that:

- (i) For any  $U$ -cover  $I'$  in  $G'$  corresponds a feasible solution  $\pi$  to  $(G, c, U, \ell)$  with  $\pi(V) \leq c'(I')$ .
- (ii)  $\text{opt}' \leq 3\text{opt}/2$ , where  $\text{opt}$  is an optimal solution value to **Restricted Minimum-Power Edge-Cover** and  $\text{opt}'$  is the minimum cost of a  $U$ -cover in  $G'$ .

Hence if  $I'$  is an optimal (min-cost) solution to  $(G', c')$ , then  $\pi(V) \leq c'(I') \leq 3\text{opt}/2$ .

Clearly, we may set  $\ell_v = 0$  for all  $v \in V \setminus U$ . For  $I \subseteq E$  let

$$D(I) = \sum_{v \in V} \max\{p_I(v) - \ell_v, 0\}.$$

Here is the construction of the instance  $(G', c')$ , where  $G' = (U, E')$  and  $E'$  consists of the following three types of edges, where for every edge  $e' \in E'$  corresponds a set  $I(e') \subseteq E$  of one edge or of two edges.

1. For every  $v \in U$ ,  $E'$  has a loop-edge  $e' = vv$  with  $c'(vv) = \ell_v + D(\{vu\})$  where  $vu$  is an arbitrary chosen minimum cost edge in  $\delta_E(v)$ . Here  $I(e') = \{vu\}$ .
2. For every  $uv \in E$  such that  $u, v \in U$ ,  $E'$  has an edge  $e' = uv$  with  $c'(uv) = \ell_u + \ell_v + D(\{uv\})$ . Here  $I(e') = \{uv\}$ .
3. For every pair of edges  $ux, xv \in E$  such that  $c(ux) \geq c(xv)$ ,  $E'$  has an edge  $e' = uv$  with  $c'(uv) = \ell_v + \ell_u + D(\{ux, xv\})$ . Here  $I(e') = \{ux, xv\}$ .

**Lemma 6.** *Let  $I' \subseteq E'$  be a  $U$ -cover in  $G'$ , let  $I = \cup_{e \in I'} I(e)$ , and let  $\pi$  be a power assignment defined on  $V$  by  $\pi(v) = \max\{p_I(v), \ell_v\}$ . Then  $\pi(V) \leq c'(I')$ ,  $I$  is a  $U$ -cover in  $G$ , and  $\pi$  is a feasible solution to  $(G, c, U, \ell)$ .*

*Proof.* We have that  $I$  is a  $U$ -cover in  $G$ , by the definition of  $I$  and since  $I(e')$  covers both endnodes of every  $e' \in E'$ . By the definition of  $\pi$ , we have that  $I \subseteq \{e = uv \in E : \pi(u), \pi(v) \geq c(e)\}$ . Hence  $\pi$  is a feasible solution to  $(G, c, U, \ell)$ , as claimed.

We prove that  $\pi(V) \leq c'(I')$ . For  $e' = uv \in E'$  let  $\ell(e') = \ell_v$  if  $e'$  is a type 1 edge, and  $\ell(e') = \ell_u + \ell_v$  otherwise. Note that  $\pi(v) = \max\{p_I(v), \ell(v)\} = \ell_v + \max\{p_I(v) - \ell(v), 0\}$ , hence

$$\pi(V) = \sum_{v \in U} \ell_v + \sum_{v \in V} \max\{p_I(v) - \ell(v), 0\} = \sum_{v \in U} \ell_v + D(I) .$$

By the definition of  $\ell(e')$  and since  $I'$  is a  $U$ -cover  $\sum_{v \in U} \ell_v \leq \sum_{e' \in I'} \ell(e')$ . Also,  $D(I) = D(\cup_{e' \in I'} I(e'))$ , by the definition of  $I$ . Thus we have

$$\sum_{v \in U} \ell_v + D(I) \leq \sum_{e' \in I'} \ell(e') + D(\cup_{e' \in I'} I(e')) .$$

It is easy to see that

$$D(\cup_{e' \in I'} I(e')) \leq \sum_{e' \in I'} D(I(e')) .$$

Finally, note that  $\ell(e') + D(I(e')) = c'(e')$  for every  $e' \in I'$  (if  $e'$  is a type 1 edge, this follows from our assumption that  $\ell_v \geq \min\{c(e) : e \in \delta_E(v)\}$ ). Combining we get

$$\begin{aligned} \pi(V) &= \sum_{v \in U} \ell_v + D(I) \leq \\ &\leq \sum_{e' \in I'} \ell(e') + D(\cup_{e' \in I'} I(e')) \leq \\ &\leq \sum_{e' \in I'} \ell(e') + \sum_{e' \in I'} D(I(e')) = \\ &= \sum_{e' \in I'} (\ell(e') + D(I(e'))) = \\ &= \sum_{e' \in I'} c'(e') = c'(I') . \end{aligned}$$

□

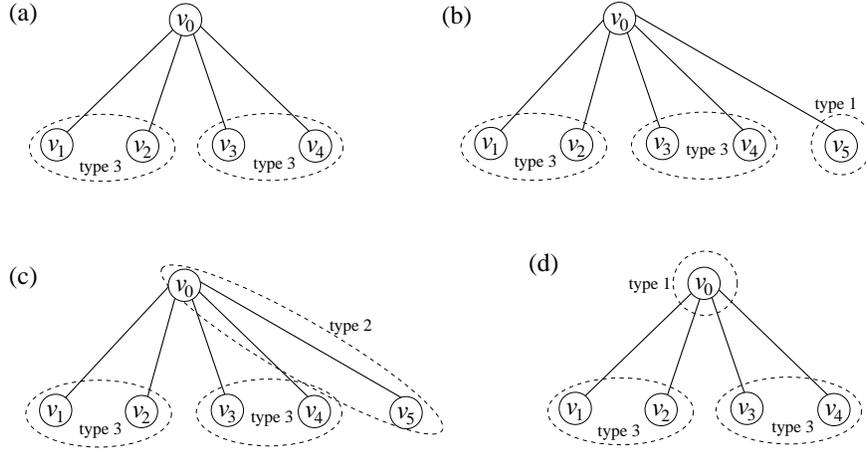
**Lemma 7.** *Let  $\{\pi(v) : v \in V\}$  be a feasible solution to an instance  $(G, c, U, \ell)$  of Restricted Minimum-Power Edge-Cover. Then there exists a  $U$ -cover  $I'$  in  $G'$  such that  $c'(I') \leq 3\pi(V)/2$ .*

*Proof.* Let  $I \subseteq \{e = uv \in E : \pi(u), \pi(v) \geq c(e)\}$  be an inclusion minimal  $U$ -cover. We may assume that  $\pi(v) = \max\{p_I(v), \ell_v\}$  for every  $v \in V$ . Since any inclusion minimal  $U$ -cover is a collection of node disjoint stars, it is sufficient to prove the statement for the case when  $I$  is a star. Then  $I$  has at most one node not in  $U$ , and if there is such a node, then it is the center of the star, if  $|I| \geq 2$ ; in the case  $I$  consists of a single edge  $e$ , then we define the center of  $I$  to be the endnode of  $e$  in  $V \setminus U$  if such exists, or an arbitrary endnode of  $e$  otherwise.

We define a  $U$ -cover  $I'$  in  $G'$ , and show that

$$c'(I') \leq \frac{3}{2} \sum_{v \in V} \max\{p_I(v), \ell_v\} = \frac{3}{2} \pi(V). \quad (1)$$

Let  $v_0$  be the center of  $I$  and let  $\{v_i : 1 \leq i \leq d\}$  be the leaves of  $I$  ordered by descending order of costs  $c(v_0 v_i) \geq c(v_0 v_{i+1})$ . The  $U$ -cover  $I' \subseteq E'$  is defined as follows. We cover each pair  $v_{2i-1}, v_{2i}$ ,  $i = 1, \dots, \lfloor d/2 \rfloor$ , by a type 3 edge. This covers all the nodes except  $v_0$ , and maybe  $v_d$  if  $d$  is odd. We add an additional edge  $f$  of type 1 or 2, if there are nodes in  $U$  ( $v_0$  and/or  $v_d$ ) that remain uncovered by the picked type 3 edges. Formally, we have the following 4 cases, see Figure 1.



**Fig. 1.** Illustration to the definition of the  $U$ -cover  $I'$ .

1.  $d$  is even and  $v_0 \notin U$ , see Figure 1(a). Then  $U$  is covered by type 3 edges.
2.  $d$  is odd, and  $v_0 \notin U$ , see Figure 1(b). Then we add a type 1 edge  $f$  to cover  $v_d$ .
3.  $d$  is odd and  $v_0 \in U$ , see Figure 1(c). Then we add a type 2 edge  $f$  to cover  $v_0, v_d$ .
4.  $d$  is even and  $v_0 \in U$ , see Figure 1(d). Then we add a type 1 edge  $f$  to cover  $v_0$ .

Consider a type 3 edge  $v_{2i-1} v_{2i} \in I'$ . Let  $q_i = \max\{c(v_{2i-1} v_0) - \ell_{v_0}, 0\}$ . Note that  $c'(v_{2i-1} v_{2i}) \leq \pi(v_{2i-1}) + \pi(v_{2i}) + q_i$ . The key point is that

$$q_i \leq \frac{1}{2} (\pi(v_{2i-3}) + \pi(v_{2i-2})) \quad i = 2, \dots, \lfloor d/2 \rfloor.$$

This is since  $q_i \leq c(v_0 v_{2i-1}) \leq \frac{1}{2}(c(v_0 v_{2i-3}) + c(v_0 v_{2i-2}))$  while  $c(v_0 v_j) \leq \pi(v_j)$ . Therefore,

$$\sum_{i=1}^{d/2} c'(v_{2i-1} v_{2i}) \leq \sum_{i=1}^{d/2} [\pi(v_{2i-1}) + \pi(v_{2i}) + q_i] \leq \sum_{i=1}^{2\lfloor d/2 \rfloor} \pi(v_i) + q_1 + \frac{1}{2} \sum_{i=1}^{d-2} \pi(v_i)$$

Now, we prove that (1) hold in each one of our four cases.

1.  $v_0 \notin U$  and  $d$  is even. Note that  $q_1 \leq c(v_0 v_1) \leq \pi(v_0)$ . Then:

$$c'(I') = \sum_{i=1}^{d/2} c'(e_i) \leq \frac{3}{2} \sum_{i=1}^d \pi(v_i) + q_1 \leq \frac{3}{2} \sum_{i=1}^d \pi(v_i) + \pi(v_0) \leq \frac{3}{2} \sum_{i=0}^d \pi(v_i)$$

2.  $v_0 \notin U$  and  $d$  is odd. In this case  $f = v_d v_d$  is a loop type 1 edge, so  $c'(f) \leq \pi(v_d) + \max(c(v_0 v_d) - \ell_{v_0}, 0)$ . This implies

$$\begin{aligned} q_1 + c'(f) &\leq c(v_0 v_1) + c(v_0 v_d) + \pi(v_d) \leq \pi(v_0) + \frac{1}{2}[\pi(v_0) + \pi(v_d)] + \pi(v_d) \\ &= \frac{3}{2}(\pi(v_0) + \pi(v_d)) . \end{aligned}$$

Thus

$$c'(I') = \sum_{i=1}^{d/2} c'(e_i) + c'(f) \leq \frac{3}{2} \sum_{i=1}^{d-1} \pi(v_i) + c'(f) + q_1 \leq \frac{3}{2} \sum_{i=0}^d \pi(v_i)$$

3.  $v_0 \in U$  and  $d$  is odd. In this case  $f = v_0 v_d$ , so  $c'(f) \leq \max(\ell_{v_0}, c(v_0 v_d)) + \pi(v_d)$ . This implies  $q_1 + c'(f) \leq c(v_0 v_1) + c(v_0 v_d) + \pi(v_d) \leq \frac{3}{2}(\pi(v_0) + \pi(v_d))$ . Thus

$$c'(I') = \sum_{i=1}^{d/2} c'(e_i) + c'(f) \leq \frac{3}{2} \sum_{i=1}^{d-1} \pi(v_i) + c'(f) + q_1 \leq \frac{3}{2} \sum_{i=0}^d \pi(v_i) .$$

4.  $v_0 \in U$  and  $d$  is even. In this case  $f = v_0 v_0$  is a loop type 1 edge, so  $c'(f) \leq \ell_{v_0} + c(v_0 v_d) \leq \ell_{v_0} + \frac{1}{2}(\pi(v_{d-1}) + \pi(v_d))$ . This implies  $q_1 + c'(f) \leq c(v_0 v_1) + \frac{1}{2}(\pi(v_{d-1}) + \pi(v_d))$ . Thus

$$\begin{aligned} c'(I') &= \sum_{i=1}^{d/2} c'(e_i) + c'(f) \leq \sum_{i=1}^d \pi(v_i) + \frac{1}{2} \sum_{i=1}^{d-2} \pi(v_i) + q_1 + c'(f) \\ &\leq \frac{3}{2} \sum_{i=1}^d \pi(v_i) + \pi(v_0) \leq \sum_{i=0}^d \pi(v_i) . \end{aligned}$$

This concludes the proof of the lemma.  $\square$

As was mentioned, Lemmas 6 and 7 imply Lemma 4. Lemmas 4 and 5 imply Theorem 2, hence the proof of Theorem 2 is now complete.

## 4 Conclusions and open problems

The main result of this paper is a new approximation algorithm for MPEMC with ratio  $O(\log k)$ . This improves the ratio  $O(\log(nk)) = O(\log n)$  of [3]. We also gave a  $(k+1/2)$ -approximation algorithm, which is better than our  $O(\log k)$ -approximation algorithm for small values of  $k$  (roughly  $k \leq 6$ ).

The main open problem is whether the ratio  $O(\log k)$  shown in this paper is tight, or the problem admits a constant ratio approximation algorithm.

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