# Dependently Typed Programming based on Automated Theorem Proving

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### Abstract

Mella is a minimalistic dependently typed programming language and interactive theorem prover implemented in Haskell. Its main purpose is to investigate the effective integration of automated theorem provers in a pure and simple setting. Such integrations are essential for supporting program development in dependently typed languages. We integrate the equational theorem prover Waldmeister and test it on more than 800 proof goals from the TPTP library. In contrast to previous approaches, the reconstruction of Waldmeister proofs within Mella is quite robust and does not generate a significant overhead to proof search. Mella thus yields a template for integrating more expressive theorem provers in more sophisticated languages.

## 1 Introduction

Dependently typed programming (DTP) languages such as Adga [10] or Epigram [18] are currently receiving considerable attention. By combining the elegance of functional programming with more expressive type systems, they introduce a new mathematically principled style of program development. In contrast to traditional functional programming, types are powerful enough to support detailed specifications of a program's properties. This however requires type-level reasoning that is no longer decidable. DTP languages are at the same time interactive theorem proving (ITP) systems similar to Nuprl [14] or Coq [8]. On the one hand this supports developing programs that are correct by construction. On the other hand it puts an additional burden on programmers.

To support program development at an appropriate level of abstraction, it is essential that programmers can focus on the more high-level creative aspects of proofs, whereas trivial and routine proof tasks are automated. Yet how can this be achieved?

Traditionally, automation is obtained in ITP systems by implementing large libraries of tactics, internally verified solvers and sophisticated simplification techniques, or by using external solvers as oracles. More recently, external automated theorem proving (ATP) systems, satisfiability modulo theories (SMT) solvers and other decision procedures have been integrated in a more trustworthy way into ITP systems by internally reconstructing proofs provided by the

external tools. A prime example is Isabelle's Sledgehammer tool (cf. [9]), which includes a relevance filter for selecting hypotheses, an interface for passing proof tasks to external tools, and a mechanism for internally reconstructing external proofs.

This approach seems particularly promising for DTP languages where it could make program development more lightweight and less time consuming. Unfortunately however, ATP integration for DTP languages is not straightforward. First, state-of-the-art ATP technology is designed for classical reasoning whereas DTP requires constructive logic. Second, the logical kernels of DTP languages tend to be much more complex than those of traditional ITP systems, hence proof reconstruction establishes relatively less trust. Third, proof reconstruction turns out to be highly inefficient in practice due to proof normalisation, whereas in theory it should be linear in the size of input proofs [13].

Due to these issues, ATP integrations for DTP languages certainly deserve to be studied in a radically pure and simple setting. This essentially amounts to building a simple trustworthy DTP language kernel around an ATP system as its most important proof engine. In this paper we focus in particular on the communication between ATP and ITP and the efficiency of proof reconstruction. Our main contributions are as follows.

First, we implement the extended calculus of constructions with universes as a minimalistic DTP language, called Mella, in Haskell. This includes term data types based on de Bruijn indices and a monadic approach to bidirectional type checking and inference.

Second, we design and implement a simple proof scripting language for Mella. It is inspired by Isabelle/Isar and Agda. Apart from commands for executing interactive proofs it allows calling external ATP systems within the Proof General interface [2].

Third, we provide interfaces for executing Mella proofs in the ATP system Waldmeister and for reconstructing Waldmeister proofs within Mella. Proof reconstruction amounts to building a Mella proof term and type checking it; proof normalisation is avoided.

Fourth, we test the performance of the ATP integration on more than 800 proof tasks from the TPTP library [26]. In contrast to previous approaches, proof reconstruction is very effective and does not create a significant overhead to Waldmeister proof search. However, a small number of proof reconstructions currently fail due to dynamic scoping problems.

In many ways, Mella is still a prototype. The DTP language implemented has neither recursion nor data types. It is just expressive enough to support proofs in many-sorted first-order constructive logic with equality. But for the main purpose of this paper—exploring effective ATP integrations for DTP languages—this is certainly no limitation.

Two particular features of Mella proof reconstruction are that proof search and proof normalisation are avoided. Our micro-step reconstruction is in contrast to Isabelle's current macro-step approach based on the internally verified ATP system Metis [16], and it seems more robust and efficient. In contrast to Agda or Coq, we only type check the internal proof terms corresponding to external proofs. These proof terms provide proof certificates that could be further normalised if needed. Whenever type checking succeeds, correctness of Mella's inference system guarantees that normalisation is possible.

# 2 Calculus of Constructions

This section introduces the basics of the calculus of construction, which is Mella's underlying type theory. We assume familiarity with basic type systems [5, 23, 22]. The type inference rules of this calculus are given in Figure 1; its details are explained in the remainder of this section,

$$\begin{aligned} & \text{T-Axiom} \ \overline{+s:_{\uparrow}s'} \ s:s' \in \mathcal{A} \\ & \text{T-Named} \ \frac{\mathbf{x}:T \in \Gamma}{\Gamma \vdash \mathbf{x}:_{\uparrow}T} & \text{T-Unnamed} \ \frac{\Delta ! \, n \equiv_{\beta} T}{\Delta \vdash n:_{\uparrow}T} \\ & \text{T-Weakening} \ \frac{\Gamma ; \Delta \vdash \mathbf{x}:_{\downarrow} T \qquad \Gamma ; \Delta \vdash S:_{\uparrow} s}{\Gamma , \mathbf{y}:S; \Delta \vdash \mathbf{x}:_{\downarrow} T} \ s \in \mathcal{S} \ \text{and y is fresh} \\ & \Delta \text{-Weakening} \ \frac{\Delta \vdash n:_{\uparrow} T}{\Delta' + \Delta \vdash n:_{\uparrow} T} \ \Delta' \ \text{is valid} \\ & \text{T-Abs} \ \frac{\Gamma ; \Delta \vdash S:_{\uparrow} s}{\Gamma ; \Delta \vdash \lambda t:_{\downarrow} \Pi S.T} \ s \in \mathcal{S} \\ & \text{T-App} \ \frac{\Gamma ; \Delta \vdash f:_{\uparrow} \Pi S.T}{\Gamma ; \Delta \vdash f :_{\uparrow} \Pi S.T} \ \Gamma ; \Delta \vdash x:_{\downarrow} S}{\Gamma ; \Delta \vdash f :_{\uparrow} 1!} \\ & \text{T-PI} \ \frac{\Gamma ; \Delta \vdash S:_{\uparrow} s_{1} \qquad \Gamma ; \Delta, S \vdash T:_{\uparrow} s_{2}}{\Gamma ; \Delta \vdash \Pi S.T:_{\uparrow} s_{3}} \ (s_{1}, s_{2}, s_{3}) \in \mathcal{R} \\ & \text{T-Inf} \ \frac{\Gamma ; \Delta \vdash T:_{\uparrow} s}{\Gamma ; \Delta \vdash t:_{\downarrow} T} \ s \in \mathcal{S} \end{aligned}$$

Figure 1: Typing rules for  $CC\omega$ 

The set of terms of the calculus of constructions (CC) is inductively defined by the following grammar.

$$t ::= x \in V \mid \Pi x : t \cdot t \mid \lambda x : t \cdot t \mid tt$$

Here, V is a set of variables.  $\Pi x:t.t$  is the dependant product type; it essentially amounts to universal quantification.  $\lambda x:t.t$  is lambda abstraction and tt is application. In CC, types themselves are terms. They are distinguished, and their mutual dependencies are expressed, by the type inference rules. Terms that are not types are called *non-type terms*, or briefly *terms* if the context allows. A type of a non-type term is called *proper*, whereas types of types are called *sorts*.

Judgements are expressions  $\Gamma \vdash t : T$ , where  $\Gamma$  is an environment that provides types for variables, t is a term and T a type. They can be proved by the type inference rules.

In CC, every proper type has sort  $\star$ , while  $\star$  is defined to have sort  $\square$ . The dependencies between terms and types for CC can be modelled by the set  $\{(\star, \star), (\star, \square), (\square, \star), (\square, \square)\}$ . The statement  $(\star, \star)$ , for instance, says that terms may depend on terms; the statement  $(\square, \star)$  says that terms may depend on types.

To make our calculus rich enough for DTP we extend it with universes. The calculus of constructions with universes,  $CC\omega$ , extends CC with an infinite set  $\Box_0, \ldots, \Box_n, \ldots$  of sorts [7, 19]. A pure type system (PTS) is given by a triple  $(\mathcal{S}, \mathcal{A}, \mathcal{R})$ , where  $\mathcal{S}$  is a set of sorts and  $\mathcal{A}$  a set of typing relations  $s_1 : s_2$  with  $s_1, s_2 \in \mathcal{S}$ . The set  $\mathcal{R}$  consists of triples  $(s_1, s_2, s_3)$ , where  $s_1, s_2, s_3 \in \mathcal{R}$ . This set, in combination with the typing rule T-PI in Figure 1 controls the dependencies of terms and types. The PTS for  $CC\omega$  [7] is given by:

```
S = \{\star\} \cup \{\Box_i \mid i \in \mathbb{N}\},
A = \{\star : \Box_0\} \cup \{\Box_i : \Box_{i+1} \mid i \in \mathbb{N}\},
R = \{\star \leadsto \star, \star \leadsto \Box_i, \Box_i \leadsto \star \mid i \in \mathbb{N}\} \cup \{(\Box_i, \Box_j, \Box_{\max(i,j)}) \mid i \in \mathbb{N}\},
```

The notation  $s_1 \leadsto s_2$  is shorthand for  $(s_1, s_2, s_2)$ .  $\star \leadsto \star$  means that terms can depend on terms,  $\star \leadsto \Box_i$  means that types can depend on terms (dependent types) and  $\Box_i \leadsto \star$  means that terms can depend on types. The set of all triples  $(\Box_i, \Box_j, \Box_{\max(i,j)})$  defines how types are allowed to depend on types. If a type  $\Box_i$  depends on another type  $\Box_j$ , it must be at the same level in the hierarchy as the highest type  $\Box_{\max(i,j)}$  it depends on.

The syntax of  $CC\omega$  terms is defined by the following grammar, which extends and refines that for CC:

$$t ::= s \in \mathcal{S} \mid n \in \mathbb{N} \mid \mathbf{x} \in V \mid \lambda t \mid \Pi t \cdot t \mid t t \mid t :: t$$
.

We now use de Bruijn indices to represent variables introduced via  $\lambda$  or  $\Pi$  binders, whereas top-level declarations are named [11]. Named variables are written in typewriter font; they are elements of V, the set of valid identifiers. Both named and unnamed variables can have free or bound occurrences. For example, in the term  $\lambda 3$ , the index 3 is free because it is pointing outside the term. A term without free de Bruijn indices is called *locally closed*.

The dependent product type  $\Pi A.B$  corresponds to the logical statement  $\forall a \in A.\ B(a)$ . To prove  $\forall a \in A.\ B(a)$  constructively, one needs to show that for every possible  $a \in A$  an inhabitant of B can be constructed. A function of type  $\Pi A.B$  is therefore a proof of the statement  $\forall a \in A.\ B(a)$ . If B does not depend on A, then the dependent product type is  $A \to B$ . The additional syntax t :: t is type annotation. It allows us to explicitly state that a given term has some type.

Because there are two kinds of variables—named and unnamed ones—judgements take the form  $\Gamma$ ;  $\Delta \vdash t : T$ , where  $\Gamma$  and  $\Delta$  are the typing contexts for named and unnamed variables. The syntax for these contexts is

$$\Gamma ::= \emptyset \mid \Gamma, \mathbf{x} : T, \qquad \Delta ::= \emptyset \mid \Delta, T.$$

Both contexts are lists, but since  $\Gamma$  names must be unique in  $\Gamma$ , it can be treated as a set. We use  $\emptyset$  to represent empty contexts. We often omit empty contexts

and write  $\vdash t: T$  rather than  $\emptyset; \emptyset \vdash t: T$ . We write  $\Gamma, \mathbf{x}: T$  to denote that context  $\Gamma$  is extended with the new binding  $\mathbf{x}: T$ , whereas for  $\Delta$ , only a type is supplied. We write  $\mathbf{x}: T \in \Gamma$  to assert that T is the type of  $\mathbf{x}$  in  $\Gamma$ . We write  $[k \mapsto t']t$  for the substitution of term t' for the index k in the term t.

Unlike variables in  $\Gamma$ , those in  $\Delta$  are nameless and cannot be looked up by name. Instead we define two lookup operators! and!! with and without index shifting to retrieve the types of variables from  $\Delta$ .

$$\Delta, T !! n = \begin{cases} T & \text{if } n = 0, \\ \Delta !! (n-1) & \text{otherwise,} \end{cases} \qquad \Delta ! n = \uparrow^{n+1} (\Delta !! n).$$

 $\uparrow_c^d t$  is the *d*-place shift of a term t above cutoff c [23]. We write  $\uparrow^d t$  if the cutoff is zero. An unnamed context is well-formed only if all the terms within it are either proper types or sorts. Unnamed contexts can be concatenated using the # operator.

To implement this calculus, a bidirectional type checker is used. This means that for any term t of type T, one can either infer the type, written  $t:_{\uparrow}T$ , or check that the term has the type, written  $t:_{\downarrow}T$ . Type inference requires t and returns T, whereas type checking requires both t and T. The two rules T-INF and T-ANN (the rule for type annotations) provide a conversion between type checking and type inference. Bidirectional type checking ensures that the rules are directly implementable without need for further transformation.

# 3 An Extended Calculus

MELLA requires additional features for equational and incremental interactive reasoning: identity types and metavariables. We call this extension  $CC_{\omega}^+$ .

Firstly, we add identity types to  $CC\omega$ . The identity type  $\mathrm{Id}_A(a,b)$  for any type A, where a,b:A, denotes that a and b represent identical proofs of proposition A [20]. This captures propositional equality within Mella and supports equational reasoning. Our identity type corresponds to the implementation of propositional equality as an inductive family in Agda. Several new terms need to be added to the grammar of  $CC\omega$ :

$$t ::= \lambda x \cdot t \mid \dots \mid \text{refl} \mid \text{Id}_t(t,t) \mid \text{elimJ}$$
.

The reflexivity term refl works exactly like refl in Agda [21]. It allows the construction of identity types  $\mathrm{Id}_A(a,b)$  where  $a \equiv_{\beta} b$ . The typing rules for the reflexivity term EQ-Refl and the identity term EQ-ID are as follows [3]:

$$\begin{aligned} & \text{EQ-Refl} \; \frac{\Gamma; \Delta \vdash A :_{\uparrow} s \quad \Gamma; \Delta \vdash a, b :_{\downarrow} A \quad a \equiv_{\beta} b}{\Gamma; \Delta \vdash \text{refl} :_{\downarrow} \operatorname{Id}_{A}(a, b)} \; s \in \mathcal{S} \\ & \text{EQ-ID} \; \frac{\Gamma; \Delta \vdash A :_{\uparrow} s \quad \Gamma; \Delta \vdash a, b :_{\downarrow} A}{\Gamma; \Delta \vdash \operatorname{Id}_{A}(a, b) :_{\uparrow} s} \; s \in \mathcal{S} \end{aligned}$$

The J rule below eliminates identity types [12], which corresponds to the term elimJ. It can be used in combination with refl to define the standard functions of equational logic in Mella, namely, substitutivity, congruence, transitivity and symmetry. Because displaying the J rule with the locally nameless syntax discussed in Section 2 would render it almost unreadable, we present its Mella syntax:

Like Isabelle's proof scripting language Isar, Mella uses two levels of syntax. The inner syntax, surrounded by quotation marks, is used for  $CC_{\omega}^+$  terms. The outer syntax is for proof scripting. For user interaction, the locally nameless term representation is extended to a more readable named representation. The inner syntax is essentially a simplification of Agda's syntax for terms without implicit arguments or mixfix operators.

Secondly, metavariables are used in Mella. Just as in Agda, these represent "holes" within terms that can incrementally be filled in—or refined—during proofs. Metavariables require one final language extension:

$$t ::= \lambda x \cdot t \mid \dots \mid ?$$
.

As an example, consider checking that term  $\lambda$ ? has type  $\Pi \star .\Pi 0.1$ .

$$\frac{\overline{\Gamma; \Delta \vdash \star :_{\uparrow} \Box_{1}} \text{ T-AXIOM }}{\Gamma; \Delta \vdash \lambda ? :_{\downarrow} \Pi \bullet . 1} \text{ T-ABS}$$

When we try to check that ? :  $\downarrow \Pi 0.1$ , the type checker cannot proceed, so it stores a continuation which allows type checking to resume once a term for the metavariable has been supplied. This forms the basis for interactive theorem proving in Mella.

# 4 Automated Theorem Proving Technology

Having outlined the type-theoretic foundations of Mella, we now discuss the ATP technology which serves as its proof engine.

ATP systems have been designed and implemented for many decades, but mainly for classical first-order logic with equations. They provide fully automated proof search based on sophisticated term orderings, rewriting techniques and heuristics. They can often prove mathematical statements of moderate difficulty and deal with large hypothesis sets, which makes them ideally suited for discharging "trivial" first-order proof goals in ITP systems. A prime example of an ATP integration is Isabelle's Sledgehammer tool (cf. [9] for an overview), which calls a number of external ATP systems and SMT solvers. A relevance filter selects hypotheses for the proof, and the external proof output is internally reconstructed to increase trustworthiness. Proof reconstruction is based on the Metis tool [16], an Isabelle-verified automated theorem prover, which replays the external proof search with the hypotheses used by the external provers.

An integration of ATP systems into DTP languages is, however, much less straightforward, as discussed in the introduction. We therefore start with the simplest case—pure equational logic—for which classical and constructive reasoning coincide. We integrate the Waldmeister system [15], which is highly effective for this fragment and supports sorts<sup>1</sup>.

<sup>&</sup>lt;sup>1</sup>We are using the last publicly available version of Waldmeister, released in 1999.

Waldmeister accepts a set of equations as hypotheses and a single equation as a conclusion. It also requires a term ordering to use rewriting techniques for enhanced proof search. Technically, Waldmeister is based on the unfailing completion procedure [4], a variant of Knuth-Bendix completion [17] that attempts to construct a (ground) canonical term rewrite system from the equational hypotheses. This construction need not be finite, but it is guaranteed that a (rewrite) proof of a valid goal can be found in finite time. Apart from efficient proof search, Waldmeister offers two additional features that benefit an integration into Mella. First, it provides extremely detailed proof output, down to the level of positions and substitutions for rewrites in terms. In contrast to Sledgehammer's macro-step proof reconstruction that replays proof search, we can therefore check individual proof steps efficiently and without search. Second, Waldmeister extracts lemmas from proofs. This memoisation of subproofs further enhances proof reconstruction.

These features can be demonstrated in a simple example from group theory. Let  $(G, \circ, ^{-1}, 1)$  be a group with carrier G, multiplication  $\circ$ , inversion  $^{-1}$  and unit 1. It satisfies axioms of associativity, right identity and right inverse

```
x \circ (y \circ z) = (x \circ y) \circ z, \qquad x \circ 1 = x, \qquad x \circ x^{-1} = 1.
```

Assume that we have implemented groups in Mella and want to prove that every right identity is also a left identity:  $x^{-1} \circ x = x \circ x^{-1}$ . We then need to pass the axioms and the proof goal to Waldmeister and let it search for a proof. Figure 2 shows the Waldmeister input file that corresponds to this proof task.

```
NAME group
MODE PROOF
SORTS
  ANY
SIGNATURE
  e: -> ANY
  i: ANY -> ANY
  f: ANY ANY -> ANY
  a: -> ANY
ORDERING
  i > f > e > a
VARIABLES
  x,y,z: ANY
EQUATIONS
  f(x,e) = x
  f(x,i(x)) = e
  f(f(x,y),z) = f(x,f(y,z))
CONCLUSION
  f(a,i(a)) = f(i(a),a)
```

Figure 2: Waldmeister group input file

The group signature is declared in prefix notation, using sort ANY, and functions f: ANY ANY -> ANY, i:ANY -> ANY and e: -> ANY for multiplication, inverse, and unit. An constant a is also introduced. Waldmeister's term ordering is declared in the ORDERING block: a lexicographic path ordering (lpo) is constructed from a precedence on the group signature and the constant a. The next block declares three variables x, y and z of type ANY. The EQUATIONS block

lists the group axioms in Waldmeister syntax. Finally, the proof goal is declared in Waldmeister syntax for constant a, since universal goals are Skolemised.

After Waldmeister is called, it returns the proof in Figure 3 within milliseconds. Here, the --details flag has been set to obtain precise information for

```
Lemma 1: f(e,i(i(x1))) = x1
  f(e,i(i(x1)))
    by Axiom 2 RL at 1 with {x1 <- x1}
  f(f(x1,i(x1)),i(i(x1)))
    by Axiom 3 LR at e with \{x3 \leftarrow i(i(x1)), x2 \leftarrow i(x1), x1 \leftarrow x1\}
  f(x1,f(i(x1),i(i(x1))))
    by Axiom 2 LR at 2 with \{x1 \leftarrow i(x1)\}
  f(x1.e)
    by Axiom 1 LR at e with \{x1 \leftarrow x1\}
Lemma 2: ...
Lemma 3: ...
Lemma 4: ...
Theorem 1: f(a,i(a)) = f(i(a),a)
  f(a,i(a))
    by Axiom 2 LR at e with \{x1 <-a\}
    by Axiom 2 RL at e with \{x1 \leftarrow i(a)\}
  f(i(a),i(i(a)))
    by Lemma 4 LR at 2 with \{x1 <-a\}
  f(i(a),a)
```

Figure 3: Waldmeister group output file

each proof step. In the third step of the proof of Lemma 1,

```
f(x1,f(i(x1),i(i(x1)))) = f(x1,e)
```

for instance, the right identity axiom f(x1,i(x1)) = e has been used to rewrite from left to right the subterm at position 2 by matching or substituting i(x1) for x1. This level of detail allows efficient micro-step proof reconstruction; the lemmas generated support proof reconstruction by memoisation. Details of the communication between Mella and Waldmeister, in particular proof reconstruction, are covered in the following section.

# 5 Implementing $CC_{\omega}^+$ Terms in Haskell

Users interact with Mella via the Proof General Emacs interface, which is standard for many ITP systems [2]. User level terms with explicit variables are parsed to an internal Haskell representation using de Bruijn indices, as represented by the Haskell data type Index. The complete Haskell implementation can be found online<sup>2</sup>. The data type has a field dbInt for the index and another one, dbName, for the user level variable name. This is useful for pretty-printing.

<sup>&</sup>lt;sup>2</sup>http://www.dcs.shef.ac.uk/~alasdair

```
data Index = DB {dbInt :: Int, dbName :: Text} deriving (Show)
instance Eq Index where
    (DB n _) == (DB m _) = n == m
   Next we provide data types for sorts and terms.
data Sort = Star | Box Word deriving (Show, Eq)
data Term = Sort Sort
          | Unnamed Index
          | Named Text
          | Pi Tag Term Term
          | Ann Term Term
          | App Term Term
          | J Term Term Term Term Term Term
          | Id Term Term Term
          | Lam Tag Term
          | Refl
          | Meta Int
          deriving (Eq, Show)
```

Tags are used to attach additional information to terms. Specifically, for Lam and Pi terms, they store the associated user level variables, for instance to provide meaningful error messages. Tags are not relevant for term equality.

Terms can be  $\beta$ -reduced using the nf function. The function for shifting is implemented as follows:

```
shift :: Int -> Int -> Term -> Term
shift d c (Unnamed (DB n name))
    | n < c = Unnamed (DB n name)
    | n >= c = Unnamed (DB (n + d) name)
shift d c (Lam tag f) = Lam tag (shift d (c + 1) f)
shift d c (Pi tag s t) = Pi tag (shift d c s) (shift d (c + 1) t)
shift d c (App f x) = App (shift d c f) (shift d c x)
shift d c (Ann t ty) = Ann (shift d c t) (shift d c ty)
shift d c (Id ty a b) = Id (shift d c ty) (shift d c a) (shift d c b)
shift d c x = x
```

Two additional Haskell functions process metavariables. A first function generates fresh metavariables as they arise in interactive proofs. A second function substitutes user supplied expressions for metavariables. Detailed code can be found at our web site.

The contexts  $\Gamma$  and  $\Delta$  for named and unnamed variables are implemented as follows:

Since the order in which variables are added to named contexts may matter, a custom map data type, OMap, has been implemented to record that information.

Finally, the set  $\mathcal{R}$  which defines the dependencies allowed between types and terms is implemented as follows:

```
setR Star Star = Star
setR (Box n) Star = Star
setR Star (Box n) = Box n
setR (Box n) (Box m) = Box (max n m)
```

# 6 Type Checking and Inference in Haskell

Type checking is performed within a type checking monad. The overall approach is inspired by that of Agda. The type checking monad transformer (TCMT) is a monad transformer stack consisting of the EitherT monad transformer and the StateT transformer. The state monad carries the type checking context (tcmCtx) as well as lists of inference rules used for type checking (tcmTCRules) and type inference (tcmIRules). It also contains a depth value (tcmDepth) for tracing and logging the type checking process (tcmLog). Whenever a metavariable is encountered during type checking, a continuation is added to the state (in tcmMetas). It contains the information required to resume type checking once a user supplies a value for it. As mentioned in Section 5, metavariables must be fresh, so a counter is used for indexing them. The either monad allows handling failures; when type checking fails we use it to return TypeError values.

data MetaContinuation = MC Ctx Int Term

Type checking rules have the form Term -> Term -> TCMT m Bool, where, as mentioned in Section 2, both the term and its tentative type are provided as inputs. Type checking returns True if the terms type check, and False if rule application fails (in which case another rule will be selected). It fails with TypeError if a term does not type check. Type inference rules require only a term t as an input. They return Just T when  $t:_{\uparrow} T$ , and Nothing when the inference rule cannot be applied; TypeError is raised when the rule fails.

Two Haskell functions are used for type checking and type inference with the TCMT monad. The typecheck function takes two terms as arguments and attempts to apply a type checking inference rule. It returns Nothing if no inference rule can be found, and the name of the rule otherwise. The hasType function is similar, but simply fails if no rule can be applied. It is usually called as an infix function, and allows rules to be written in a more declarative fashion.

```
typecheck :: (Functor m, Monad m) => Term -> Term -> TCMT m (Maybe Text)
typecheck t1 t2 = do
    tRules <- tcmTCRules <$> get
    foldM tryRule Nothing tRules
where
    tryRule (Just name) _ = return (Just name)
tryRule Nothing (TCR name rule) = do
    r <- rule t1 t2
    return $ if r then Just name else Nothing</pre>
```

The infer function attempts to infer the type of its argument. It fails if no inference rule can be applied and returns the inferred term otherwise.

As an example of a type checking rule, the code for the T-ABS rule from Figure 1 is shown below. Pattern matching and guards are used to restrict the terms it can be applied to. Each line in the do block then imposes such a condition. validType checks that argType is either a proper type or a sort, while the next line checks that the body of the lambda expression has the correct type. If both these conditions hold, the rule can be applied and True is returned.

```
tAbs :: (Functor m, Monad m) => Term -> Term -> TCMT m Bool
tAbs (Lam tag expr) pi@(Pi _ argType exprType) | inf pi = do
    validType argType
    withUnnamedVar tag argType $ expr 'hasType' exprType
    return True

tAbs _ _ = return False

tAbsRule :: (Functor m, Monad m) => TCRule m
tAbsRule = TCR "T-Abs" tAbs
```

This Haskell infrastructure suffices to implement the  $CC_{\omega}^+$  part of Mella. The ATP integration is described in the next section.

# 7 ATP Integration

Our general approach to ATP integration is depicted in Figure 4. Mella proof tasks are represented as judgements  $\Gamma$ ;  $\Delta \vdash$ ? : T. They encode that from a set

# MELLA TERMS $\Gamma; \Delta \vdash ?: T \\ \hline$ Serialise Reconstruct $\hline$ ATP Input $\hline$ ATP Output $\hline$

Figure 4: Overview of Waldmeister Integration

ATP INTEGRATION

of hypotheses given by the contexts  $\Gamma$  and  $\Delta$  a proof term t—represented by metavariable?—of type T (the proof goal) is to be inferred. This is achieved by serialising  $\Gamma$ ,  $\Delta$  and T and passing them on to Waldmeister. In our group example,  $\Gamma$  and  $\Delta$  contain the group axioms, whereas T contains the proof goal. More generally, the contexts can also contain lemmas that have been proved before. If Waldmeister fails to find a proof within a certain time limit, the user is notified. Otherwise, its proof output is translated into a proof term t in MELLA, which is then type checked. Since Waldmeister produces intermediate lemmas, as we have seen, an additional context  $\Gamma'$  is added to  $\Gamma$ . Constructing a proof term from a Waldmeister proof and type checking it yields proof reconstruction. We now discuss the individual steps in more detail.

A Mella file consists of a list of commands delimited by periods, each of which can be processed and undone individually by Proof General. There are about 20 commands available to the user, which can be displayed using the commands command. The help command provides a documentation for every command in the system. The command fun introduces a new top-level function or value. The following commands, for instance, introduce an identity function and a constant function in Mella.

To declare a theorem and start a proof, the **theorem** command is used. It takes the name of the theorem and its type T. To prove the theorem, the user must construct a proof term t such that  $t:_{\downarrow} T$ . Proofs are built up incrementally from commands and terms that may themselves contain metavariables.

As an example, assume we want to prove that

$$f(x, g(y, g(x, z))) = x$$
 and  $g(x, f(y, f(x, z))) = x$ 

imply

$$f(x, g(y, x)) = x.$$

A "manual" Mella proof without using Waldmeister is as follows:

normalize proof. describe proof.

MELLA commands can be terms, which are surrounded by quotation marks, theorem definitions, function definitions, or command expressions. The command intro args generates a term of the form  $\lambda \ args \rightarrow ?$ . The command

```
= "f x (g y (g x (f t1 (f x t2))))" by "ax2 x t1 t2" at '2,2RL'.
```

says that the left-hand term in the proof goal is equal to the term provided by applying the second axiom at position 2,2 to variable x from right to left, in a notation similar to Waldmeister. The second proof step is similar. The remaining step is reflexivity of equality. The commands normalize proof and describe proof normalise the proof and print out the proof term (which we do not show). We can also use the agda command to compile MELLA files into Agda files. This is very useful for testing the correctness of our implementation.

Command expressions form a large part of Mella's syntax. Examples are intro, = and qed as displayed above. Commands consist of a command name, followed by zero or more arguments and a list of keywords. Each keyword can again be associated with a list of arguments:

```
\verb|command| arg_1...arg_n : \verb|keyword1| karg_1...karg_n : \verb|keyword2| ...
```

Alternatively to the above manual proof we can use Waldmeister to prove our goal.

```
theorem example : "...".
  intro A f g ax1 ax2 x y.
  waldmeister :signature f g x y :axioms ax1 ax2 :kbo :timeout 2.
  qed.
```

The waldmeister command is now used to instantiate the metavariable opened by the intro command. Waldmeister is given the functions and values it may use in the proof via the :signature keyword, which maps to the SIGNATURE section of the Waldmeister input file. The axioms to be used when constructing the proof are listed after the :axioms keyword, and are used in the EQUATIONS

section of the Waldmeister input file. The :kbo option tells Waldmeister to use a Knuth-Bendix ordering as the syntactic ordering for terms (based on the precedence given by the order of expressions declared after :signature). Finally, the :timeout keyword lets one specify the amount of time Waldmeister will be given for proof search.

We now describe proof reconstruction. As already mentioned, Waldmeister splits proofs into lemmas. While this process is primarily intended to increase readability, it also enhances proof reconstruction by memoising subproofs.

The Waldmeister output for the example proof above is shown below. Waldmeister renames variables in its output, so during reconstruction, the renamed variables must be matched with the correct variables within Mella. The proof shows that the term \$5(\$1,\$4(\$0,\$1)) is equal to \$1\$. It consists of two steps: First, Waldmeister applies Axiom 2 from right to left at position 2.2 in \$5(\$1,\$4(\$0,\$1)), which results in the term shown on the next line. Secondly, Waldmeister uses Axiom 1 to reduce the term down to \$1\$, proving the goal.

```
Theorem 1: s5(s1,s4(s0,s1)) = s1

s5(s1,s4(s0,s1))

by Axiom 2 RL at 2.2 with {x3 <- y, x2 <- z, x1 <- s1} s5(s1,s4(s0,s4(s1,s5(z,s5(s1,y)))))

by Axiom 1 LR at e with {x3 <- s5(z,s5(s1,y)), x2 <- s0, x1 <- s1} s1
```

To prove a goal x=y, each step of a proof applies a lemma or axiom to a subterm of x. In Mella this requires us to use the inference rules for congruence (to select the subterm) and symmetry (to choose the direction). If neither congruence nor symmetry is required for a step, they are omitted from the proof output, as is the case for the second step above. The above Waldmeister proof has two steps, hence we need to use transitivity to join both steps together, resulting in the final reconstructed Mella proof term below. This proof term is somewhat unreadable; it has been indented to make the structure of the proof clearer.

# 8 Proof Experiments

We tested the Waldmeister integration on 850 proof goals from the TPTP library [26], among them 115 on Boolean algebras (BOO), 156 on lattices (LAT), 415 on groups (GRP), 106 on relation algebras (REL) and 58 on rings (RNG). The letters in brackets indicate the name given to these problem sets in TPTP. The library contains non-theorems and non-equational theorems that are beyond Waldmeister's scope. In fact, in our experiments, Waldmeister has not been able to find proofs for all goals for principal reasons, but may also have failed to find proofs of equational theorems due to timeout. Here, however, we are only interested in relative success rates for proof reconstruction, that is, the

number or percentage of successful Waldmeister proofs that Mella was able to reconstruct, and in the running times of proof reconstruction relative to proof search. The outcome of these experiments are shown in Table 1.

ĺ	Waldmeister				Proof Reconstruction		
	CPU Time	Timeout	Error	Unprovable	Fail	Success	%
BOO	300	9	51	1	10	44	81.5
	30	41	21	1	10	42	80.8
	10	62	0	1	10	42	80.8
	5	64	0	1	10	40	80
	1	66	0	1	10	38	79.2
LAT	300	91	14	0	11	40	78.4
	30	107	0	0	11	38	77.6
	10	110	0	0	11	35	76.1
	5	110	0	0	11	35	76.1
	1	113	0	0	9	34	79.1
GRP	300	39	4	0	213	159	42.7
	30	53	1	0	202	159	44.0
	10	57	0	0	199	159	44.4
	5	66	0	0	192	157	45
	1	91	0	0	185	139	42.9
REL	300	20	2	0	2	82	97.6
	30	34	0	0	2	70	97.2
	10	58	0	0	2	46	95.8
	5	61	0	0	0	45	100
	1	66	0	0	0	40	100
RNG	300	35	5	0	0	18	100
	30	41	0	0	0	17	100
	10	44	0	0	0	14	100
	5	44	0	0	0	14	100
	1	45	0	0	0	13	100

Table 1: Proof Reconstruction Experiments

The first column in the table shows the TPTP problem sets. The first four columns are related to Waldmeister. The first of them shows the Waldmeister CPU time limits for proof search—1s, 5s, 10s, 30s and 300s. The second one gives the number of proofs searches that exceeded the time limit. The third one gives the number of proofs that aborted, for instance, due to out of memory errors. In the case of Boolean algebras, the fourth row shows that Waldmeister refuted one proof goal. The final three columns contain data on proof reconstruction. The first of them shows the number of proofs for which reconstruction failed; the second one the number of successfully reconstructed proofs. The rowwise sums of these columns give the numbers of successful Waldmeister proofs. The third row gives the percentage of successful proof reconstructions.

First, it turns out that the CPU time limit for Waldmeister has little impact on success rates. The number of successful Waldmeister proofs increases only slightly with proof search time; the success rates for reconstruction remain almost unaffected. This suggests that there is little correlation between proof search time and the difficulty of reconstructing the resulting proof. Waldmeister could spend a long time traversing a search space only to find a very short and simple proof which is trivial to reconstruct.

Second, success rates are surprisingly different for different problem sets. For groups, proof reconstruction was particularly poor, succeeding only 45% of the time for proofs returned after a 5 second timeout. For rings and relation

algebras, reconstruction succeeded almost always, with a 100% reconstruction success rate at 5 seconds. For lattices and Boolean algebras reconstruction was also overall successful; it is 80% for Boolean algebras and 76.1% for lattices (again with a 5 second timeout). Some explanations for this are given below.

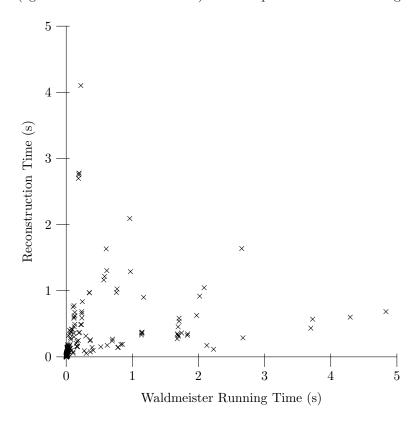


Figure 5: Waldmeister running times versus proof reconstruction times

Next we have investigated the correlation between proof search and proof reconstruction times. A graph is plotted in Figure 5. Unfortunately, these times were very short for most of our proofs, which makes it very difficult to draw convincing conclusions. For some proofs, proof search took rather long whereas reconstruction was fast. In other cases, proof search was fast, but the proof could not be reconstructed or type checked efficiently. We have inspected the proof for each goal that took longer than 2s to reconstruct. In each of these cases, either proof terms are extremely long, with more than 100 lemmas, or there are extremely large substitutions.

As an example, consider the following line from Figure 3:

### = by Axiom 2 LR at 2 with $\{x1 \leftarrow i(x1)\}$

In the substitution  $\mathtt{x1} <-\mathtt{i}(\mathtt{x1})$ , for instance, the term  $\mathtt{i}(\mathtt{x1})$  can be enormous. In fact, our experiments contain substitutions of terms thousands of characters long, resulting in extremely large and unwieldy lemmas. This underscores the benefit of Waldmeister's lemma generation, which allows us to type check each one individually. As soon as proof terms become large, type checking slows

down. These observations confirm what one would expect: proof reconstruction times depend on proof sizes rather than proof search times, whereas proof search time and proof size are often only weakly correlated. Proof length, however, is not a key factor for using ATP systems in DTP program development. Ultimately, our experiments suggest that the Waldmeister integration into Mella is feasible, and proof reconstruction yields little overhead to proof search.

There are several reasons why proof reconstruction may fail. Firstly, Waldmeister sometimes introduces fresh Skolem constants in proofs. These currently cannot be handled by the proof reconstruction code and cause it to fail. More precisely, such constants, which are dynamically generated by Waldmeister can currently not be associated with an environment during proof reconstruction. Secondly, rules such as the right inverse axiom  $x \circ x^{-1} = 1$  for groups, when applied from right to left to a (sub)term 1, can lead to "inventing" fresh variables x in a Waldmeister proof. Mella would then have to introduce this value to the type signature of the lemma and supply it as a parameter. This currently assigns lemmas the wrong types in proofs and causes proof reconstruction to fail. For certain problem sets such as groups, "creative" proof steps of this kind seem particularly frequent, whereas in others (such as Boolean algebras, relation algebras or rings), they are present, but seem less significant.

As an example, consider the proof discussed in Section 7:

```
theorem proof : "(A : *) (f g : A -> A -> A)

-> (axiom1 : (x y z : A) -> Id A (f x (g y (g x z))) x)

-> (axiom2 : (x y z : A) -> Id A (g x (f y (f x z))) x)

-> (x y : A) -> Id A (f x (g y x)) x".

intro A f g ax1 ax2 x y.

waldmeister :signature f g x y :axioms ax1 ax2 :kbo :timeout 2.

qed.
```

Waldmeister uses the following lemma in its proof:

```
Lemma 1: s10(x1,s14(x2,x1)) = x1

s10(x1,s14(x2,x1))

= by Axiom 7 RL

s10(x1,s14(x2,s14(x1,s10(z,s10(x1,y)))))

= by Axiom 8 LR

x1
```

The second line of this proof introduces the new variables y and z. They are not mentioned in that lemma's type, hence the lemma cannot be easily reconstructed. We have implemented heuristics that guess instances of correct type for z and y (in this case x1 and x2) which are present in the context. In this particular lemma, these heuristics make proof reconstruction succeed. In many other case, we still obtain confusing error messages.

## 9 Related Work

The general question of proof automation for ITPs is covered in a wide variety of literature. Barendregt and Barendsen [6] identify three approaches, namely accepting, skeptical, and autarkic. The accepting approach uses ATPs and SMT

solvers as oracles, requiring no proof output. The skeptical approach requires that external tools provide evidence or certificates which allow ITP systems to internally reconstruct external proofs to increase trust. The autarkic approach solely relies on internal implementations of solvers and provers or alternatively by verifying external tools.

The accepting approach has, for many years, been pursued in the PVS ITP system, for instance by integrating the Yices SMT solver [25]. However, this is often insufficient for constructive logic as proofs have computational content and may require execution.

The autarkic approach is the ideal, as an internally verified solver is guaranteed to produce correct output. The omega, tauto and ring tactics in Coq, and Isabelle's blast and metis tactics for instance, are autarkic. The disadvantage of this approach however is clear: there is a need to efficiently re-implement provers in the proof system.

The approach taken in this paper approach is skeptical. We believe this yields an adequate balance between efficiency and trust. Our approach is heavily inspired by Isabelle's Sledgehammer tool, which however is predominantly based on macro-step proof reconstruction. Additionally, ATP integration in Mizar—so far without proof reconstruction—is currently under development [24]. The skeptical approach has also been used in the context of dependent types, in a Waldmeister integration into Agda [13]. The relative inefficiency of this approach due to Agda proof normalisation is another main inspiration for Mella. Work on proof irrelevance in the most recent version of Agda, may however lead to a solution to this problem within Agda. More recently, using the skeptical approach, an SMT solver has been integrated into Coq [1].

## 10 Conclusion and Future Work

We have integrated the equational theorem prover Waldmeister into the prototypical dependently typed programming language Mella which is based on the extended calculus of constructions with universes. In contrast to previous approaches, where theorem provers were added a posteriori to existing ITP systems complement existing internal tactics and proof strategies, we take the ATP system as a core proof engine for the programming language and build the language around it. As a user front end we have implemented a proof scripting language in the Proof General environment. This provides an interface between Mella and Waldmeister. Since Waldmeister provides highly detailed proof output we can perform micro-step proof reconstruction, translating the proof output into a Mella proof term and type checking that term.

Proof terms in Mella are not normalised. On the one hand, this makes proof reconstruction much more efficient. On the other hand this yields a proof certificate rather than a proper normalised proof. The strong normalisation property of the underlying type system, however, guarantees that all proofs that have been successfully checked can also be normalised. In the case of an equational proof this amounts to a refl term.

In sum, our findings suggest that integrating ATP systems into DTP languages can be very beneficial for program development in this setting, and that the approach taken with Mella may serve as a template for future approaches to integrate more expressive ATP systems in more sophisticated DTP languages.

There are various interesting directions for future work.

First, already the minimalist formalism of Mella without recursion or data types requires proofs in full multi-sorted first-order constructive logic with equations. However, state-of-the-art ATP systems are essentially all based on classical first-order logic and often do not support sorts. Our current Waldmeister integration deals only with multi-sorted equational logic, a fragment where classical and constructive reasoning coincide. While using classical ATP systems for more expressive fragments of first-order logic, such as Harrop formulae, is still possible, specific ATP systems for constructive or intuitionistic logic should be designed for applications in DTP.

Second, many state-of-the-art ATP systems adhere to a common input standard (TPTP), but many of them do not provide any detailed proof output or use a proprietary format. Detailed proof output is often perceived as detrimental to proof search efficiency. In the context of DTP, however, its absence is detrimental to proof reconstruction. As Sledgehammer shows, macro-step proof reconstruction, that is, replaying proof search with an internally verified theorem prover, has the disadvantage that many proofs provided by the external ATPs will not be accepted by the ITP system. Our proof experiments show that micro-step reconstruction of individual proofs steps is superior to this approach, but it requires detailed ATP output. Proof standardisation as in the TSTP project [27] is a valuable step in this direction. While sheer proof power was the main emphasis of ATP development in the past, applications in the context of ITP systems requires this to be balanced with detailed proof output and support for types.

Third, in its current version, MELLA still suffers from the fact that Wald-meister proofs which introduce new constants or variables cannot always be reconstructed. We could work around this by reconstructing proofs as they are, with additional constants and variables included, and proving that such reconstructed proofs are equivalent to the desired proofs. The simple heuristics currently used should further be refined to cover more proofs. Alternatively, when heuristics fail, the presence of lemmas in Waldmeister proof outputs allows local manual proof reconstruction. Often, reconstruction failures are caused by a very small number of lemmas. These could be replaced by metavariables so that the proof can be delegated to users. Thus, even when and ATP system cannot completely finish a proof, it might still produce a number of simpler proof goals for the user and at least simplify the global proof goal.

Fourth, MELLA needs to be extended with features found in more sophisticated DTP and ITP tools. First we could extend  $CC_{\omega}^+$  with data types, induction or  $\Sigma$ -types. Alternatively we could extend the proof scripting language by adding more automation, or by providing a more structured method of proof construction, similar to Isar. Some features like induction might only require proof management such as induction tactics, and would not affect the ATP integration, while others, such as the addition of  $\Sigma$ -types seem to require modifications to how ATP systems are integrated.

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