

# On Formal Specification of Maple Programs<sup>\*</sup>

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**Abstract.** This paper is an example-based demonstration of our initial results on the formal specification of programs written in the computer algebra language *MiniMaple* (a substantial subset of Maple with slight extensions). The main goal of this work is to define a verification framework for *MiniMaple*. Formal specification of *MiniMaple* programs is rather complex task as it supports non-standard types of objects, e.g. symbols and unevaluated expressions, and additional functions and predicates, e.g. runtime type tests etc. We have used the specification language to specify various computer algebra concepts respective objects of the Maple package *DifferenceDifferential* developed at our institute.

## 1 Introduction

We report on a project whose goal is to design and develop a tool to find behavioral errors such as type inconsistencies and violations of method preconditions in programs written in the language of the computer algebra system Maple; for this purpose, these programs need to be annotated with the types of variables and methods contracts [8].

As a starting point, we have defined a substantial subset of the computer algebra language Maple, which we call *MiniMaple*. Since type safety is a prerequisite of program correctness, we have formalized a type system for *MiniMaple* and implemented a corresponding type checker. The type checker has been applied to the Maple package *DifferenceDifferential* [2] developed at our institute for the computation of bivariate difference-differential dimension polynomials. Furthermore, we have defined a language to formally specify the behavior of *MiniMaple* programs. As the next step, we will develop a verification calculus for *MiniMaple*. The other related technical details about the work presented in this paper are discussed in the accompanying paper [7]. For project details and related software, please visit <http://www.risc.jku.at/people/mtkhan/dk10/>.

The rest of the paper is organized as follows: in Section 2, we briefly demonstrate formal type system for *MiniMaple* by an example. In Section 3, we introduce and demonstrate the specification language for *MiniMaple* by an example. Section 4 presents conclusions and future work.

## 2 A Type System for *MiniMaple*

*MiniMaple* procedure parameters, return types and corresponding local (variable) declarations needs to be (manually) type annotated. Type inference would

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be partially possible and is planned as a later goal. The results we derive with type checking Maple can also be applied to Mathematica, as Mathematica has almost the same kinds of runtime objects as Maple.

Listing 1 gives an example of a *MiniMaple* program which we will use in the following section for the discussion of type checking respective formal specification. Also the type information produced by the type system is shown by the mapping  $\pi$  of program variables to types. For other related technical details of the type system, please see [4].

```

1. status:=0;
2. prod := proc(l::list(Or(integer,float)):[integer,float];
3.     #  $\pi=\{l:\text{list}(\text{Or}(\text{integer},\text{float}))\}$ 
4.     global status;
5.     local i, x::Or(integer,float), si::integer:=1, sf::float:=1.0;
6.     #  $\pi=\{\dots, i:\text{symbol}, x:\text{Or}(\text{integer},\text{float}),\dots, \text{status}:\text{anything}\}$ 
7.     for i from 1 by 1 to nops(l) do
8.         x:=l[i]; status:=i;
9.         #  $\pi=\{\dots, i:\text{integer}, \dots, \text{status}:\text{integer}\}$ 
10.        if type(x,integer) then
11.            #  $\pi=\{\dots, i:\text{integer}, x:\text{integer}, si:\text{integer}, \dots, \text{status}:\text{integer}\}$ 
12.            if (x = 0) then
13.                return [si,sf];
14.            end if;
15.            si:=si*x;
16.        elif type(x,float) then
17.            #  $\pi=\{\dots, i:\text{integer}, x:\text{float}, \dots, sf:\text{float}, \text{status}:\text{integer}\}$ 
18.            if (x < 0.5) then
19.                return [si,sf];
20.            end if;
21.            sf:=sf*x;
22.        end if;
23.        #  $\pi=\{\dots, i:\text{integer}, x:\text{Or}(\text{integer},\text{float}),\dots, \text{status}:\text{integer}\}$ 
24.    end do;
25.    #  $\pi=\{\dots, i:\text{symbol}, x:\text{Or}(\text{integer},\text{float}),\dots, \text{status}:\text{anything}\}$ 
26.    status:=-1;
27.    #  $\pi=\{\dots, i:\text{symbol}, x:\text{Or}(\text{integer},\text{float}),\dots, \text{status}:\text{integer}\}$ 
28.    return [si,sf];
29. end proc;
30. result := prod([1, 8.54, 34.4, 6, 8.1, 10, 12, 5.4]);

```

Listing 1: The example *MiniMaple* procedure type-checked

The following problems arise from type checking *MiniMaple* programs:

- Global variables (declarations) can not be type annotated; therefore values of arbitrary types can be assigned to global variables in Maple. Therefore we introduce *global* and *local* contexts to handle the different semantics of the variables inside and outside of the body of a procedure respectively loop.
  - In a *global* context new variables may be introduced by assignments and the types of variables may change arbitrarily by assignments.
  - In a *local* context variables can only be introduced by declarations. The types of variables can only be *specialized* i.e. the new value of a variable should be a sub-type of the declared variable type. The sub-typing relation is observed while specializing the types of variables.
- A predicate **type**( $E, T$ ) (which is true if the value of expression  $E$  has type  $T$ ) may direct the control flow of a program. If this predicate is used in

a conditional, then different branches of the conditional may have different type information for the same variable. We keep track of the type information introduced by the different type tests from different branches to adequately reason about the possible types of a variable. For instance, if a variable  $x$  has type  $\text{Or}(\text{integer}, \text{float})$ , in a conditional statement where the "if" branch is guarded by a test  $\text{type}(x, \text{integer})$ , in the "else" branch  $x$  has automatically type  $\text{float}$ . This automatic type inferencing only applies if an identifier has a union type. A warning is generated, if a test is redundant (always yields true or false).

The type checker has been applied to the Maple package *DifferenceDifferential* [2]. No crucial typing errors have been found but some bad code parts have been identified that can cause problems, e.g., variables that are declared but not used (and therefore cannot be type checked) and variables that have duplicate global and local declarations.

### 3 A Specification Language for *MiniMaple*

Based on the type system presented in the previous section, we have developed a formal specification language for *MiniMaple*. This language is a logical formula language which is based on Maple notations but extended by new concepts. The formula language supports various forms of quantifiers, logical quantifiers (**exists** and **forall**), numerical quantifiers (**add**, **mul**, **min** and **max**) and sequential quantifier (**seq**) representing truth values, numeric values and sequence of values respectively. We have extended the corresponding Maple syntax, e.g., logical quantifiers use typed variables and numerical quantifiers are equipped with logical conditions that filter values from the specified variable range.

Also the language allows to formally specify the behavior of procedures by pre- and post-conditions and other constraints; it also supports loop specifications and assertions. In contrast to specification languages such as Java Modeling Language [3], abstract data types can be introduced to specify abstract concepts and notions from computer algebra.

```
(*@
requires true;
global status;
ensures
  (status = -1 and RESULT[1] = mul(e, e in l, type(e, integer))
  and RESULT[2] = mul(e, e in l, type(e, float))
  and forall(i::integer, 1 <= i and i <= nops(l) and type(l[i], integer) implies l[i] <> 0)
  and forall(i::integer, 1 <= i and i <= nops(l) and type(l[i], float) implies l[i] >= 0.5))
  or
  (1 <= status and status <= nops(l) and RESULT[1] = mul(l[i], i=1..status-1, type(l[i], integer))
  and RESULT[2] = mul(l[i], i=1..status-1, type(l[i], float))
  and ((type(l[status], integer) and l[status]=0) or (type(l[status], float) and l[status] < 0.5))
  and forall(i::integer, 1 <= i and i < status and type(l[i], integer) implies l[i] <> 0)
  and forall(i::integer, 1 <= i and i < status and type(l[i], float) implies l[i] >= 0.5));
@*)
proc(l::list(Or(integer, float))):[integer, float]; ... end proc;
```

Listing 2: The example *MiniMaple* procedure formally specified

Listing 2 gives a formal specification of the example procedure introduced in Section 2. The procedure has no pre-condition as shown in the **requires** clause; the **global** clause says that a global variable *status* can be modified by the body of the procedure. The normal behavior of the procedure is specified in the **ensures** clause. The post condition specifies that, if the complete list is processed then we get the result as the product of all integers and floats in the list but if procedure terminates pre-maturely then we only get the product of integers and floats till the value of variable *status* (index of the input list). For the complete syntax and other details of the formal specification language see [6]. To test the specification language, we have formally specified some parts of the Maple package *DifferenceDifferential* [2] developed at our institute as the main test for the specification language.

## 4 Conclusions

We may use the specification language sketched in this short paper to generate executable assertions that are embedded in *MiniMaple* programs and check at runtime the validity of pre/post conditions. Our main goal, however, is to use the specification language to verify the correctness of *MiniMaple* annotated programs by static analysis, in particular to detect violations of methods preconditions. For this purpose, based on the results of a prior investigation, we intend to use the verification framework Why3 [1] to implement the verification calculus for *MiniMaple*, i.e., to translate *MiniMaple* into the intermediate language of Why3 and to apply its verification condition generator to generate verification conditions and prove their correctness with various back-end provers. Since the verification calculus must be sound, we have defined a formal semantics of *MiniMaple* [5] such that the correctness of the transformation can be shown.

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## Preface

This textbook is intended for use by students of physics, physical chemistry, and theoretical chemistry. The reader is presumed to have a basic knowledge of atomic and quantum physics at the level provided, for example, by the first few chapters in our book *The Physics of Atoms and Quanta*. The student of physics will find here material which should be included in the basic education of every physicist. This book should furthermore allow students to acquire an appreciation of the breadth and variety within the field of molecular physics and its future as a fascinating area of research.

For the student of chemistry, the concepts introduced in this book will provide a theoretical framework for that entire field of study. With the help of these concepts, it is at least in principle possible to reduce the enormous body of empirical chemical knowledge to a few basic principles: those of quantum mechanics. In addition, modern physical methods whose fundamentals are introduced here are becoming increasingly important in chemistry and now represent indispensable tools for the chemist. As examples, we might mention the structural analysis of complex organic compounds, spectroscopic investigation of very rapid reaction processes or, as a practical application, the remote detection of pollutants in the air.

April 1995

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# Hamiltonian Mechanics unter besonderer Berücksichtigung der höheren Lehranstalten

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**Abstract.** The abstract should summarize the contents of the paper using at least 70 and at most 150 words. It will be set in 9-point font size and be inset 1.0 cm from the right and left margins. There will be two blank lines before and after the Abstract. . .

**Keywords:** computational geometry, graph theory, Hamilton cycles

## 1 Fixed-Period Problems: The Sublinear Case

With this chapter, the preliminaries are over, and we begin the search for periodic solutions to Hamiltonian systems. All this will be done in the convex case; that is, we shall study the boundary-value problem

$$\begin{aligned} \dot{x} &= JH'(t, x) \\ x(0) &= x(T) \end{aligned}$$

with  $H(t, \cdot)$  a convex function of  $x$ , going to  $+\infty$  when  $\|x\| \rightarrow \infty$ .

### 1.1 Autonomous Systems

In this section, we will consider the case when the Hamiltonian  $H(x)$  is autonomous. For the sake of simplicity, we shall also assume that it is  $C^1$ .

We shall first consider the question of nontriviality, within the general framework of  $(A_\infty, B_\infty)$ -subquadratic Hamiltonians. In the second subsection, we shall look into the special case when  $H$  is  $(0, b_\infty)$ -subquadratic, and we shall try to derive additional information.

**The General Case: Nontriviality.** We assume that  $H$  is  $(A_\infty, B_\infty)$ -subquadratic at infinity, for some constant symmetric matrices  $A_\infty$  and  $B_\infty$ , with  $B_\infty - A_\infty$  positive definite. Set:

$$\gamma := \text{smallest eigenvalue of } B_\infty - A_\infty \tag{1}$$

$$\lambda := \text{largest negative eigenvalue of } J \frac{d}{dt} + A_\infty . \tag{2}$$

Theorem 1 tells us that if  $\lambda + \gamma < 0$ , the boundary-value problem:

$$\begin{aligned} \dot{x} &= JH'(x) \\ x(0) &= x(T) \end{aligned} \quad (3)$$

has at least one solution  $\bar{x}$ , which is found by minimizing the dual action functional:

$$\psi(u) = \int_0^T \left[ \frac{1}{2} (A_o^{-1}u, u) + N^*(-u) \right] dt \quad (4)$$

on the range of  $\Lambda$ , which is a subspace  $R(\Lambda)_L^2$  with finite codimension. Here

$$N(x) := H(x) - \frac{1}{2} (A_\infty x, x) \quad (5)$$

is a convex function, and

$$N(x) \leq \frac{1}{2} ((B_\infty - A_\infty)x, x) + c \quad \forall x. \quad (6)$$

**Proposition 1.** *Assume  $H'(0) = 0$  and  $H(0) = 0$ . Set:*

$$\delta := \liminf_{x \rightarrow 0} 2N(x) \|x\|^{-2}. \quad (7)$$

*If  $\gamma < -\lambda < \delta$ , the solution  $\bar{u}$  is non-zero:*

$$\bar{x}(t) \neq 0 \quad \forall t. \quad (8)$$

*Proof.* Condition (7) means that, for every  $\delta' > \delta$ , there is some  $\varepsilon > 0$  such that

$$\|x\| \leq \varepsilon \Rightarrow N(x) \leq \frac{\delta'}{2} \|x\|^2. \quad (9)$$

It is an exercise in convex analysis, into which we shall not go, to show that this implies that there is an  $\eta > 0$  such that

$$f \|x\| \leq \eta \Rightarrow N^*(y) \leq \frac{1}{2\delta'} \|y\|^2. \quad (10)$$

**Fig. 1.** This is the caption of the figure displaying a white eagle and a white horse on a snow field

Since  $u_1$  is a smooth function, we will have  $\|hu_1\|_\infty \leq \eta$  for  $h$  small enough, and inequality (10) will hold, yielding thereby:

$$\psi(hu_1) \leq \frac{h^2}{2} \frac{1}{\lambda} \|u_1\|_2^2 + \frac{h^2}{2} \frac{1}{\delta'} \|u_1\|^2 . \quad (11)$$

If we choose  $\delta'$  close enough to  $\delta$ , the quantity  $(\frac{1}{\lambda} + \frac{1}{\delta'})$  will be negative, and we end up with

$$\psi(hu_1) < 0 \quad \text{for } h \neq 0 \text{ small} . \quad (12)$$

On the other hand, we check directly that  $\psi(0) = 0$ . This shows that 0 cannot be a minimizer of  $\psi$ , not even a local one. So  $\bar{u} \neq 0$  and  $\bar{u} \neq \Lambda_o^{-1}(0) = 0$ .  $\square$

**Corollary 1.** *Assume  $H$  is  $C^2$  and  $(a_\infty, b_\infty)$ -subquadratic at infinity. Let  $\xi_1, \dots, \xi_N$  be the equilibria, that is, the solutions of  $H'(\xi) = 0$ . Denote by  $\omega_k$  the smallest eigenvalue of  $H''(\xi_k)$ , and set:*

$$\omega := \text{Min} \{ \omega_1, \dots, \omega_k \} . \quad (13)$$

If:

$$\frac{T}{2\pi} b_\infty < -E \left[ -\frac{T}{2\pi} a_\infty \right] < \frac{T}{2\pi} \omega \quad (14)$$

then minimization of  $\psi$  yields a non-constant  $T$ -periodic solution  $\bar{x}$ .

We recall once more that by the integer part  $E[\alpha]$  of  $\alpha \in \mathbb{R}$ , we mean the  $a \in \mathbb{Z}$  such that  $a < \alpha \leq a + 1$ . For instance, if we take  $a_\infty = 0$ , Corollary 2 tells us that  $\bar{x}$  exists and is non-constant provided that:

$$\frac{T}{2\pi} b_\infty < 1 < \frac{T}{2\pi} \quad (15)$$

or

$$T \in \left( \frac{2\pi}{\omega}, \frac{2\pi}{b_\infty} \right) . \quad (16)$$

*Proof.* The spectrum of  $\Lambda$  is  $\frac{2\pi}{T}\mathbb{Z} + a_\infty$ . The largest negative eigenvalue  $\lambda$  is given by  $\frac{2\pi}{T}k_o + a_\infty$ , where

$$\frac{2\pi}{T}k_o + a_\infty < 0 \leq \frac{2\pi}{T}(k_o + 1) + a_\infty . \quad (17)$$

Hence:

$$k_o = E \left[ -\frac{T}{2\pi} a_\infty \right] . \quad (18)$$

The condition  $\gamma < -\lambda < \delta$  now becomes:

$$b_\infty - a_\infty < -\frac{2\pi}{T}k_o - a_\infty < \omega - a_\infty \quad (19)$$

which is precisely condition (14).  $\square$

**Lemma 1.** *Assume that  $H$  is  $C^2$  on  $\mathbb{R}^{2n} \setminus \{0\}$  and that  $H''(x)$  is non-degenerate for any  $x \neq 0$ . Then any local minimizer  $\tilde{x}$  of  $\psi$  has minimal period  $T$ .*

*Proof.* We know that  $\tilde{x}$ , or  $\tilde{x} + \xi$  for some constant  $\xi \in \mathbb{R}^{2n}$ , is a  $T$ -periodic solution of the Hamiltonian system:

$$\dot{x} = JH'(x) . \quad (20)$$

There is no loss of generality in taking  $\xi = 0$ . So  $\psi(x) \geq \psi(\tilde{x})$  for all  $\tilde{x}$  in some neighbourhood of  $x$  in  $W^{1,2}(\mathbb{R}/T\mathbb{Z}; \mathbb{R}^{2n})$ .

But this index is precisely the index  $i_T(\tilde{x})$  of the  $T$ -periodic solution  $\tilde{x}$  over the interval  $(0, T)$ , as defined in Sect. 2.6. So

$$i_T(\tilde{x}) = 0 . \quad (21)$$

Now if  $\tilde{x}$  has a lower period,  $T/k$  say, we would have, by Corollary 31:

$$i_T(\tilde{x}) = i_{kT/k}(\tilde{x}) \geq ki_{T/k}(\tilde{x}) + k - 1 \geq k - 1 \geq 1 . \quad (22)$$

This would contradict (21), and thus cannot happen.  $\square$

*Notes and Comments.* The results in this section are a refined version of [1]; the minimality result of Proposition 14 was the first of its kind.

To understand the nontriviality conditions, such as the one in formula (16), one may think of a one-parameter family  $x_T$ ,  $T \in (2\pi\omega^{-1}, 2\pi b_\infty^{-1})$  of periodic solutions,  $x_T(0) = x_T(T)$ , with  $x_T$  going away to infinity when  $T \rightarrow 2\pi\omega^{-1}$ , which is the period of the linearized system at 0.

**Table 1.** This is the example table taken out of *The T<sub>E</sub>Xbook*, p. 246

Year	World population
8000 B.C.	5,000,000
50 A.D.	200,000,000
1650 A.D.	500,000,000
1945 A.D.	2,300,000,000
1980 A.D.	4,400,000,000

**Theorem 1 (Ghoussoub-Preiss).** *Assume  $H(t, x)$  is  $(0, \varepsilon)$ -subquadratic at infinity for all  $\varepsilon > 0$ , and  $T$ -periodic in  $t$*

$$H(t, \cdot) \quad \text{is convex} \quad \forall t \quad (23)$$

$$H(\cdot, x) \quad \text{is } T\text{-periodic} \quad \forall x \quad (24)$$

$$H(t, x) \geq n(\|x\|) \quad \text{with } n(s)s^{-1} \rightarrow \infty \quad \text{as } s \rightarrow \infty \quad (25)$$

$$\forall \varepsilon > 0, \quad \exists c : H(t, x) \leq \frac{\varepsilon}{2} \|x\|^2 + c. \quad (26)$$

Assume also that  $H$  is  $C^2$ , and  $H''(t, x)$  is positive definite everywhere. Then there is a sequence  $x_k, k \in \mathbb{N}$ , of  $kT$ -periodic solutions of the system

$$\dot{x} = JH'(t, x) \quad (27)$$

such that, for every  $k \in \mathbb{N}$ , there is some  $p_o \in \mathbb{N}$  with:

$$p \geq p_o \Rightarrow x_{pk} \neq x_k. \quad (28)$$

□

*Example 1* (External forcing). Consider the system:

$$\dot{x} = JH'(x) + f(t) \quad (29)$$

where the Hamiltonian  $H$  is  $(0, b_\infty)$ -subquadratic, and the forcing term is a distribution on the circle:

$$f = \frac{d}{dt}F + f_o \quad \text{with } F \in L^2(\mathbb{R}/T\mathbb{Z}; \mathbb{R}^{2n}), \quad (30)$$

where  $f_o := T^{-1} \int_0^T f(t) dt$ . For instance,

$$f(t) = \sum_{k \in \mathbb{N}} \delta_k \xi, \quad (31)$$

where  $\delta_k$  is the Dirac mass at  $t = k$  and  $\xi \in \mathbb{R}^{2n}$  is a constant, fits the prescription. This means that the system  $\dot{x} = JH'(x)$  is being excited by a series of identical shocks at interval  $T$ .

**Definition 1.** Let  $A_\infty(t)$  and  $B_\infty(t)$  be symmetric operators in  $\mathbb{R}^{2n}$ , depending continuously on  $t \in [0, T]$ , such that  $A_\infty(t) \leq B_\infty(t)$  for all  $t$ .

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$$\exists c \in \mathbb{R} : \quad H(t, x) \leq \frac{1}{2} (B_\infty(t)x, x) + c \quad \forall x. \quad (35)$$

If  $A_\infty(t) = a_\infty I$  and  $B_\infty(t) = b_\infty I$ , with  $a_\infty \leq b_\infty \in \mathbb{R}$ , we shall say that  $H$  is  $(a_\infty, b_\infty)$ -subquadratic at infinity. As an example, the function  $\|x\|^\alpha$ , with  $1 \leq \alpha < 2$ , is  $(0, \varepsilon)$ -subquadratic at infinity for every  $\varepsilon > 0$ . Similarly, the Hamiltonian

$$H(t, x) = \frac{1}{2} k \|k\|^2 + \|x\|^\alpha \quad (36)$$

is  $(k, k + \varepsilon)$ -subquadratic for every  $\varepsilon > 0$ . Note that, if  $k < 0$ , it is not convex.

*Notes and Comments.* The first results on subharmonics were obtained by Rabinowitz in [5], who showed the existence of infinitely many subharmonics both in the subquadratic and superquadratic case, with suitable growth conditions on  $H'$ . Again the duality approach enabled Clarke and Ekeland in [2] to treat the same problem in the convex-subquadratic case, with growth conditions on  $H$  only.

Recently, Michalek and Tarantello (see [3] and [4]) have obtained lower bound on the number of subharmonics of period  $kT$ , based on symmetry considerations and on pinching estimates, as in Sect. 5.2 of this article.

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# Hamiltonian Mechanics2

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**Abstract.** The abstract should summarize the contents of the paper using at least 70 and at most 150 words. It will be set in 9-point font size and be inset 1.0 cm from the right and left margins. There will be two blank lines before and after the Abstract. . . .

**Keywords:** graph transformations, convex geometry, lattice computations, convex polygons, triangulations, discrete geometry

## 1 Fixed-Period Problems: The Sublinear Case

With this chapter, the preliminaries are over, and we begin the search for periodic solutions to Hamiltonian systems. All this will be done in the convex case; that is, we shall study the boundary-value problem

$$\begin{aligned}\dot{x} &= JH'(t, x) \\ x(0) &= x(T)\end{aligned}$$

with  $H(t, \cdot)$  a convex function of  $x$ , going to  $+\infty$  when  $\|x\| \rightarrow \infty$ .

### 1.1 Autonomous Systems

In this section, we will consider the case when the Hamiltonian  $H(x)$  is autonomous. For the sake of simplicity, we shall also assume that it is  $C^1$ .

We shall first consider the question of nontriviality, within the general framework of  $(A_\infty, B_\infty)$ -subquadratic Hamiltonians. In the second subsection, we shall look into the special case when  $H$  is  $(0, b_\infty)$ -subquadratic, and we shall try to derive additional information.

**The General Case: Nontriviality.** We assume that  $H$  is  $(A_\infty, B_\infty)$ -subquadratic at infinity, for some constant symmetric matrices  $A_\infty$  and  $B_\infty$ , with  $B_\infty - A_\infty$  positive definite. Set:

$$\gamma := \text{smallest eigenvalue of } B_\infty - A_\infty \tag{1}$$

$$\lambda := \text{largest negative eigenvalue of } J \frac{d}{dt} + A_\infty . \tag{2}$$

Theorem 21 tells us that if  $\lambda + \gamma < 0$ , the boundary-value problem:

$$\begin{aligned} \dot{x} &= JH'(x) \\ x(0) &= x(T) \end{aligned} \quad (3)$$

has at least one solution  $\bar{x}$ , which is found by minimizing the dual action functional:

$$\psi(u) = \int_0^T \left[ \frac{1}{2} (A_o^{-1}u, u) + N^*(-u) \right] dt \quad (4)$$

on the range of  $\Lambda$ , which is a subspace  $R(\Lambda)_L^2$  with finite codimension. Here

$$N(x) := H(x) - \frac{1}{2} (A_\infty x, x) \quad (5)$$

is a convex function, and

$$N(x) \leq \frac{1}{2} ((B_\infty - A_\infty)x, x) + c \quad \forall x. \quad (6)$$

**Proposition 1.** *Assume  $H'(0) = 0$  and  $H(0) = 0$ . Set:*

$$\delta := \liminf_{x \rightarrow 0} 2N(x) \|x\|^{-2}. \quad (7)$$

*If  $\gamma < -\lambda < \delta$ , the solution  $\bar{u}$  is non-zero:*

$$\bar{x}(t) \neq 0 \quad \forall t. \quad (8)$$

*Proof.* Condition (7) means that, for every  $\delta' > \delta$ , there is some  $\varepsilon > 0$  such that

$$\|x\| \leq \varepsilon \Rightarrow N(x) \leq \frac{\delta'}{2} \|x\|^2. \quad (9)$$

It is an exercise in convex analysis, into which we shall not go, to show that this implies that there is an  $\eta > 0$  such that

$$f \|x\| \leq \eta \Rightarrow N^*(y) \leq \frac{1}{2\delta'} \|y\|^2. \quad (10)$$

**Fig. 1.** This is the caption of the figure displaying a white eagle and a white horse on a snow field

Since  $u_1$  is a smooth function, we will have  $\|hu_1\|_\infty \leq \eta$  for  $h$  small enough, and inequality (10) will hold, yielding thereby:

$$\psi(hu_1) \leq \frac{h^2}{2} \frac{1}{\lambda} \|u_1\|_2^2 + \frac{h^2}{2} \frac{1}{\delta'} \|u_1\|^2 . \quad (11)$$

If we choose  $\delta'$  close enough to  $\delta$ , the quantity  $(\frac{1}{\lambda} + \frac{1}{\delta'})$  will be negative, and we end up with

$$\psi(hu_1) < 0 \quad \text{for } h \neq 0 \text{ small} . \quad (12)$$

On the other hand, we check directly that  $\psi(0) = 0$ . This shows that 0 cannot be a minimizer of  $\psi$ , not even a local one. So  $\bar{u} \neq 0$  and  $\bar{u} \neq \Lambda_o^{-1}(0) = 0$ .  $\square$

**Corollary 1.** *Assume  $H$  is  $C^2$  and  $(a_\infty, b_\infty)$ -subquadratic at infinity. Let  $\xi_1, \dots, \xi_N$  be the equilibria, that is, the solutions of  $H'(\xi) = 0$ . Denote by  $\omega_k$  the smallest eigenvalue of  $H''(\xi_k)$ , and set:*

$$\omega := \text{Min} \{ \omega_1, \dots, \omega_k \} . \quad (13)$$

If:

$$\frac{T}{2\pi} b_\infty < -E \left[ -\frac{T}{2\pi} a_\infty \right] < \frac{T}{2\pi} \omega \quad (14)$$

then minimization of  $\psi$  yields a non-constant  $T$ -periodic solution  $\bar{x}$ .

We recall once more that by the integer part  $E[\alpha]$  of  $\alpha \in \mathbb{R}$ , we mean the  $a \in \mathbb{Z}$  such that  $a < \alpha \leq a + 1$ . For instance, if we take  $a_\infty = 0$ , Corollary 2 tells us that  $\bar{x}$  exists and is non-constant provided that:

$$\frac{T}{2\pi} b_\infty < 1 < \frac{T}{2\pi} \quad (15)$$

or

$$T \in \left( \frac{2\pi}{\omega}, \frac{2\pi}{b_\infty} \right) . \quad (16)$$

*Proof.* The spectrum of  $\Lambda$  is  $\frac{2\pi}{T}\mathbb{Z} + a_\infty$ . The largest negative eigenvalue  $\lambda$  is given by  $\frac{2\pi}{T}k_o + a_\infty$ , where

$$\frac{2\pi}{T}k_o + a_\infty < 0 \leq \frac{2\pi}{T}(k_o + 1) + a_\infty . \quad (17)$$

Hence:

$$k_o = E \left[ -\frac{T}{2\pi} a_\infty \right] . \quad (18)$$

The condition  $\gamma < -\lambda < \delta$  now becomes:

$$b_\infty - a_\infty < -\frac{2\pi}{T}k_o - a_\infty < \omega - a_\infty \quad (19)$$

which is precisely condition (14).  $\square$

**Lemma 1.** *Assume that  $H$  is  $C^2$  on  $\mathbb{R}^{2n} \setminus \{0\}$  and that  $H''(x)$  is non-degenerate for any  $x \neq 0$ . Then any local minimizer  $\tilde{x}$  of  $\psi$  has minimal period  $T$ .*

*Proof.* We know that  $\tilde{x}$ , or  $\tilde{x} + \xi$  for some constant  $\xi \in \mathbb{R}^{2n}$ , is a  $T$ -periodic solution of the Hamiltonian system:

$$\dot{x} = JH'(x) . \quad (20)$$

There is no loss of generality in taking  $\xi = 0$ . So  $\psi(x) \geq \psi(\tilde{x})$  for all  $\tilde{x}$  in some neighbourhood of  $x$  in  $W^{1,2}(\mathbb{R}/T\mathbb{Z}; \mathbb{R}^{2n})$ .

But this index is precisely the index  $i_T(\tilde{x})$  of the  $T$ -periodic solution  $\tilde{x}$  over the interval  $(0, T)$ , as defined in Sect. 2.6. So

$$i_T(\tilde{x}) = 0 . \quad (21)$$

Now if  $\tilde{x}$  has a lower period,  $T/k$  say, we would have, by Corollary 31:

$$i_T(\tilde{x}) = i_{kT/k}(\tilde{x}) \geq ki_{T/k}(\tilde{x}) + k - 1 \geq k - 1 \geq 1 . \quad (22)$$

This would contradict (21), and thus cannot happen.  $\square$

*Notes and Comments.* The results in this section are a refined version of 1980; the minimality result of Proposition 14 was the first of its kind.

To understand the nontriviality conditions, such as the one in formula (16), one may think of a one-parameter family  $x_T$ ,  $T \in (2\pi\omega^{-1}, 2\pi b_\infty^{-1})$  of periodic solutions,  $x_T(0) = x_T(T)$ , with  $x_T$  going away to infinity when  $T \rightarrow 2\pi\omega^{-1}$ , which is the period of the linearized system at 0.

**Table 1.** This is the example table taken out of *The T<sub>E</sub>Xbook*, p. 246

Year	World population
8000 B.C.	5,000,000
50 A.D.	200,000,000
1650 A.D.	500,000,000
1945 A.D.	2,300,000,000
1980 A.D.	4,400,000,000

**Theorem 1 (Ghoussoub-Preiss).** *Assume  $H(t, x)$  is  $(0, \varepsilon)$ -subquadratic at infinity for all  $\varepsilon > 0$ , and  $T$ -periodic in  $t$*

$$H(t, \cdot) \quad \text{is convex} \quad \forall t \quad (23)$$

$$H(\cdot, x) \quad \text{is } T\text{-periodic} \quad \forall x \quad (24)$$

$$H(t, x) \geq n(\|x\|) \quad \text{with } n(s)s^{-1} \rightarrow \infty \quad \text{as } s \rightarrow \infty \quad (25)$$

$$\forall \varepsilon > 0, \quad \exists c : H(t, x) \leq \frac{\varepsilon}{2} \|x\|^2 + c. \quad (26)$$

Assume also that  $H$  is  $C^2$ , and  $H''(t, x)$  is positive definite everywhere. Then there is a sequence  $x_k, k \in \mathbb{N}$ , of  $kT$ -periodic solutions of the system

$$\dot{x} = JH'(t, x) \quad (27)$$

such that, for every  $k \in \mathbb{N}$ , there is some  $p_o \in \mathbb{N}$  with:

$$p \geq p_o \Rightarrow x_{pk} \neq x_k. \quad (28)$$

□

*Example 1* (External forcing). Consider the system:

$$\dot{x} = JH'(x) + f(t) \quad (29)$$

where the Hamiltonian  $H$  is  $(0, b_\infty)$ -subquadratic, and the forcing term is a distribution on the circle:

$$f = \frac{d}{dt}F + f_o \quad \text{with } F \in L^2(\mathbb{R}/T\mathbb{Z}; \mathbb{R}^{2n}), \quad (30)$$

where  $f_o := T^{-1} \int_o^T f(t) dt$ . For instance,

$$f(t) = \sum_{k \in \mathbb{N}} \delta_k \xi, \quad (31)$$

where  $\delta_k$  is the Dirac mass at  $t = k$  and  $\xi \in \mathbb{R}^{2n}$  is a constant, fits the prescription. This means that the system  $\dot{x} = JH'(x)$  is being excited by a series of identical shocks at interval  $T$ .

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