Clique cover and graph separation: New incompressibility results

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Abstract

The field of kernelization studies polynomial-time preprocessing routines for hard problems in the framework of parameterized complexity. Although a framework for proving kernelization lower bounds has been discovered in 2008 and successfully applied multiple times over the last three years, establishing kernelization complexity of many important problems remains open. In this paper we show that, unless $NP \subseteq coNP/poly$ and the polynomial hierarchy collapses up to its third level, the following parameterized problems do not admit a polynomial-time preprocessing algorithm that reduces the size of an instance to polynomial in the parameter:

- EDGE CLIQUE COVER, parameterized by the number of cliques,
- DIRECTED EDGE/VERTEX MULTIWAY CUT, parameterized by the size of the cutset, even in the case of two
 terminals.
- EDGE/VERTEX MULTICUT, parameterized by the size of the cutset, and
- k-WAY CUT, parameterized by the size of the cutset.

The existence of a polynomial kernelization for EDGE CLIQUE COVER was a seasoned veteran in open problem sessions. Furthermore, our results complement very recent developments in designing parameterized algorithms for cut problems by Marx and Razgon [STOC'11], Bousquet et al. [STOC'11], Kawarabayashi and Thorup [FOCS'11] and Chitnis et al. [SODA'12].

1 Introduction

In order to cope with the NP-hardness of many natural combinatorial problems, various algorithmic paradigms such as brute-force, approximation, or heuristics are applied. However, while the paradigms are quite different, there is a commonly used opening move of first applying polynomial-time preprocessing routines, before making sacrifices in either exactness or runtime. The aim of the field of kernelization is to provide a rigorous mathematical framework for analyzing such preprocessing algorithms. One of its core features is to provide quantitative performance guarantees for preprocessing via the framework of parameterized complexity, a feature easily seen to be infeasible in classical complexity (cf. [48]).

In the framework of parameterized complexity an instance x of a parameterized problem comes with an integer parameter k. A *kernelization algorithm* (*kernel* for short) is a polynomial time preprocessing routine that reduces the input instance x with parameter k to an equivalent instance of size bounded by g(k) for some computable function g. If g is small, after preprocessing even an exponential-time brute-force algorithm might be feasible. Therefore small kernels, with g being linear or polynomial, are of big interest.

Although polynomial kernels for a wide range of problems have been developed for the last few decades (e.g., [1, 8, 14, 23, 35, 67, 76]; see also the surveys of Guo and Niedermeier [47] and Bodlaender [6]), a framework for proving kernelization lower bounds was discovered only three years ago by Bodlaender et al. [7], with the backbone theorem

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proven by Fortnow and Santhanam [36]. The crux of the framework is the following idea of a composition. Assume we are able to combine in polynomial time an arbitrary number of instances x_1, x_2, \ldots, x_t of an NP-complete problem L into a single instance (x,k) of a parameterized problem $Q \in NP$ such that $(x,k) \in Q$ if and only if one of the instances x_i is in L, while k is bounded polynomially in $\max_i |x_i|$. If such a *composition* algorithm was pipelined with a polynomial kernel for the problem Q, we would obtain an OR-distillation of the NP-complete language L: the resulting instance is of size polynomial in $\max_i |x_i|$, possibly significantly smaller than t, but encodes a disjunction of all input instances x_i (i.e., an OR-distillation is a compression of the logical OR of the instances). As proven by Fortnow and Santhanam [36], existence of such an algorithm would imply NP \subseteq coNP/poly, which is known to cause a collapse of the polynomial hierarchy to its third level [15, 80].

The astute reader may have noticed that the above description of a composition is actually using the slightly newer notion of a cross-composition [9]. This generalization of the original lower bound framework will be the main ingredient of our proofs. The framework of kernelization lower bounds was also extended by Dell and van Melkebeek [30] to allow excluding kernels of particular exponent in the polynomial. Recently, Dell and Marx [29] and, independently, Hermelin and Wu [49] simplified this approach and applied it to various packing problems.

The aforementioned (cross-)composition algorithm is sometimes called an OR-composition, as opposed to an AND-composition, where we require that the output instance (x,k) is in Q if and only if all input instances belong to L. Various problems have been shown to be AND-compositional, with the most important example being the problem of determining whether an input graph has treewidth no larger than the parameter [7]. It is conjectured [7] that no NP-complete problem admits an AND-distillation, which would be a result of pipelining an AND-composition with a polynomial kernel. However, it is now a major open problem in the field of kernelization to support this claim with a proof based on a plausible complexity assumption.

Although the framework of kernelization lower bounds has been applied successfully multiple times over the last three years (e.g., [9, 10, 22, 24, 31, 34, 58, 59]), there are still many important problems where the existence of a polynomial kernel is widely open. The reason for this situation is that an application of the idea of a composition (or appropriate reductions, called *polynomial parameter transformations* [11]) is far from being automatic. To obtain a composition algorithm, usually one needs to carefully choose the starting language L (for example, the choice of the starting language is crucial for compositions of Dell and Marx [29], and the core idea of the composition algorithms for connectivity problems in degenerate graphs [24] is to use GRAPH MOTIF as a starting point) or invent sophisticated gadgets to merge the instances (for example, the colors and IDs technique introduced by Dom et al. [31] or the idea of an instance selector, used mainly for structural parameters [9, 10]).

Our results. The main contribution of this paper is a proof of non-existence of polynomial kernels for four important problems.

Theorem 1.1. Unless $NP \subseteq coNP/poly$, EDGE CLIQUE COVER, parameterized by the number of cliques, as well as MULTIWAY CUT, MULTICUT and k-WAY CUT, parameterized by the size of the cutset, do not admit polynomial kernelizations.

The common theme of our compositions is a very careful choice of starting problems. Not only do we select particular NP-complete problems, but we also restrict instances given as the input, to make them satisfy certain conditions that allow designing cross-compositions. Each time we constrain the set of input instances of an NP-complete problem we need to prove that the problem remains NP-complete. Even though this paper is about negative results, in our constructions we use intuition derived from the design of parameterized algorithms techniques, including iterative compression (in case of EDGE CLIQUE COVER) introduced by Reed et al. [74] and important separators (in case of MULTICUT) defined by Marx [64].

For the three cut problems listed in Theorem 1.1 our kernelization hardness results complement very recent developments in the design of algorithm parameterized by the size of the cutset [12, 21, 65, 77]. In the following we give some motivation and related work for each of the four problems.

Edge clique cover. In the EDGE CLIQUE COVER problem the goal is to cover the edges of an input graph G with at most k cliques all of which are subgraphs of G. This problem, NP-complete even in very restricted graph classes [17, 50, 68], is also known as COVERING BY CLIQUES (GT17), INTERSECTION GRAPH BASIS (GT59) [38]

and KEYWORD CONFLICT [56]. It has multiple applications in various areas in practice, such as computational geometry [3], applied statistics [43, 69], and compiler optimization [70]. In particular, EDGE CLIQUE COVER is equivalent to the problem of finding a representation of a graph G as an intersection model with at most k elements in the universe [33, 44, 75]. Therefore, an algorithm for EDGE CLIQUE COVER may be used to reveal a structure in a complex real-world network [45]. Due to its importance, the EDGE CLIQUE COVER problem was studied from various perspectives, including approximation upper and lower bounds [4, 62], heuristics [5, 43, 56, 57, 69, 70] and polynomial-time algorithms for special graph classes [50, 51, 63, 68].

From the point of view of parameterized complexity, EDGE CLIQUE COVER was extensively studied by Gramm et al. [42]. A simple kernelization algorithm is known that reduces the size of the graph to at most 2^k vertices; the best known fixed-parameter algorithm is a brute-force search on the 2^k -vertex kernel. The question of a polynomial kernel for EDGE CLIQUE COVER, probably first verbalized by Gramm et al. [42], was repeatedly asked in the parameterized complexity community, for example on the last Workshop on Kernels (WorKer, Vienna, 2011). We show that EDGE CLIQUE COVER is both AND- and OR-compositional (i.e., both an AND- and an OR-composition algorithm exist for some NP-complete input language L), thus the existence of a polynomial kernel would both cause a collapse of the polynomial hierarchy as well as violate the AND-conjecture. To the best of our knowledge, this is the first natural parameterized problem that is known to admit both an AND- and an OR-composition algorithm.

Multicut and directed multiway cut. With MULTICUT and DIRECTED MULTIWAY CUT we move on to the family of graph separation problems. The central problems of this area are two natural generalizations of the s-t cut problem, namely MULTIWAY CUT and MULTICUT. In the first problem we are given a graph G with designated terminals and we are to delete at most p edges (or vertices, depending on the variant) so that the terminals remain in different connected components. In the MULTICUT problem we consider a more general setting where the input graph contains terminal pairs and we need to separate all pairs of terminals.

As generalizations of the well-known s-t cut problem, MULTIWAY CUT and MULTICUT received a lot of attention in past decades. MULTIWAY CUT is NP-complete even for the case of three terminals [27], thus the same holds for MULTICUT with three terminal pairs. Both problems were intensively studied from the approximation perspective [16, 39, 40, 53, 66]. The graph separation problems became one of the most important subareas in parameterized complexity after Marx introduced the concept of important separators [64]. This technique turns out to be very robust, and is now a key ingredient in fixed-parameter algorithms for various problems such as variants of the FEEDBACK VERTEX SET problem [20, 26] or ALMOST 2-SAT [73]. A long line of research on MULTIWAY CUT in the parameterized setting include [19, 25, 46, 64, 71, 72, 79]; the current fastest algorithm runs in $O(2^p n^{O(1)})$ time [25]. It is not very hard to prove that MULTICUT, parameterized by both the number of terminals and the size of the cutset, is reducible to MULTIWAY CUT [64]. Fixed-parameter tractability of MULTICUT parameterized by the size of the cutset only, after being a big open problem for a few years, was finally resolved positively in 2010 [12, 65].

In directed graphs MULTIWAY CUT is NP-complete even for two terminals [40]. Very recently Chitnis et al. [21] showed that DIRECTED MULTIWAY CUT is fixed-parameter tractable. The directed version of MULTICUT, parameterized by the size of the cutset, is W[1]-hard [65] (i.e., an existence of a fixed-parameter algorithm is unlikely). The parameterized complexity of DIRECTED MULTICUT with fixed number of terminal pairs or with the number of terminal pairs as an additional parameter remains open.

Although the picture of the fixed-parameter tractability of the graph separation problems becomes more and more complete, very little is known about polynomial kernelization. Very recently, Kratsch and Wahlström came up with a genuine application of matroid theory to graph separation problems. They were able to obtain randomized polynomial kernels for ODD CYCLE TRANSVERSAL [61], ALMOST 2-SAT, and MULTIWAY CUT and MULTICUT restricted to a bounded number of terminals, among others [60]. We are not aware of any other results on kernelization of the graph separation problems.

We prove that DIRECTED MULTIWAY CUT, even in the case of two terminals, as well as MULTICUT, parameterized by the size of the cutset, are OR-compositional, thus a polynomial kernel for any of these two problems would cause a collapse of the polynomial hierarchy. In fact, we give two OR-composition algorithms for MULTICUT: the constructions are very different and the presented gadgets may inspire other researchers in showing lower bounds for similar problems.

The k-way cut problem. The last part of this work is devoted to another generalization of the s-t cut problem, but of a bit different flavor. The k-WAY CUT problem is defined as follows: given an undirected graph G and integers k and s, remove at most s edges from G to obtain a graph with at least k connected components. This problem has applications in numerous areas of computer science, such as finding cutting planes for the traveling salesman problem, clustering-related settings (e.g., VLSI design) or network reliability [13]. In general, k-WAY CUT is NP-complete [41] but solvable in polynomial time for fixed k: a long line of research [41, 52, 54, 77] led to a deterministic algorithm running in time $O(mn^{2k-2})$. The dependency on k in the exponent is probably unavoidable: from the parameterized perspective, the k-WAY CUT problem parameterized by k is W[1]-hard [32]. Moreover, the node-deletion variant is also W[1]-hard when parameterized by s [64]. Somewhat surprisingly, in 2011 Kawarabayashi and Thorup presented a fixed-parameter algorithm for (edge-deletion) k-WAY CUT parameterized by s [55]. In this paper we complete the parameterized picture of the edge-deletion k-WAY CUT problem parameterized by s by showing that it is OR-compositional and, therefore, a polynomial kernelization algorithm is unlikely to exist.

Organization of the paper. We give some notation and formally introduce the composition framework in Section 2. In subsequent sections we show compositions for the aforementioned four problems: we consider EDGE CLIQUE COVER in Section 3, DIRECTED MULTIWAY CUT in Section 4, MULTICUT in Section 5 and Section 6 and k-WAY CUT in Section 7. Section 8 concludes the paper.

Acknowledgements. We would like to thank Jakub Onufry Wojtaszczyk for some early discussions on the kernelization of the graph separation problems.

2 Preliminaries

Notation. We use standard graph notation. For a graph G, by V(G) and E(G) we denote its vertex and edge set (or arc set in case of directed graphs), respectively. For $v \in V(G)$, its neighborhood $N_G(v)$ is defined by $N_G(v) = \{u : uv \in E(G)\}$, and $N_G[v] = N_G(v) \cup \{v\}$ is the closed neighborhood of v. We extend this notation to subsets of vertices: $N_G[X] = \bigcup_{v \in X} N_G[v]$ and $N_G(X) = N_G[X] \setminus X$. For $X \subseteq V(G)$ by $\delta_G(X)$ we denote the set of edges in G with one endpoint in X and the other in $V(G) \setminus X$. For simplicity for a single vertex v we let $\delta(v) = \delta(\{v\})$. We omit the subscripts if no confusion is possible. For a set $X \subseteq V(G)$ by G[X] we denote the subgraph of G induced by G0. For a set G1 of vertices or edges of G2, by G3, we denote the graph with the vertices or edges of G3 removed; in case of a vertex removal, we remove also all its incident edges. For sets G3, G4, the set G4, G5 contains all edges of G5 that have one endpoint in G5 and the second endpoint in G5. In particular, G5, we denote its vertex and edge set G6, by G6, by G7, we denote the subgraph of G6 that have one endpoint in G8 and the second endpoint in G9. In particular, G5, we denote its vertex and edge set G6, by G6, by G7, we denote the subgraph of G8 and ends in G8.

For two disjoint vertex sets S, T by an S-T cut we denote any set of edges, which removal ensures that there is no path from a vertex in S to a vertex in T in the considered graph. By minimum S-T cut we denote an S-T cut of minimum cardinality.

Parameterized complexity. In the parameterized complexity setting, an instance comes with an integer parameter k — formally, a parameterized problem Q is a subset of $\Sigma^* \times \mathbb{N}$ for some finite alphabet Σ . We say that the problem is *fixed parameter tractable* (*FPT*) if there exists an algorithm solving any instance (x,k) in time $f(k)\operatorname{poly}(|x|)$ for some (usually exponential) computable function f. It is known that a problem is FPT iff it is kernelizable: a kernelization algorithm for a problem Q takes an instance (x,k) and in time polynomial in |x|+k produces an equivalent instance (x',k') (i.e., $(x,k)\in Q$ iff $(x',k')\in Q$) such that $|x'|+k'\leq g(k)$ for some computable function g. The function g is the *size of the kernel*, and if it is polynomial, we say that Q admits a polynomial kernel.

Kernelization lower bounds framework. We use the cross-composition technique introduced by Bodlaender et al. [9] which builds upon Bodlaender et al. [7] and Fortnow and Santhanam [36].

Definition 2.1 (Polynomial equivalence relation [9]). An equivalence relation \Re on Σ^* is called a *polynomial equivalence relation* if (1) there is an algorithm that given two strings $x, y \in \Sigma^*$ decides whether $\Re(x, y)$ in $(|x| + |y|)^{O(1)}$

time; (2) for any finite set $S \subseteq \Sigma^*$ the equivalence relation \mathcal{R} partitions the elements of S into at most $(\max_{x \in S} |x|)^{O(1)}$ classes.

Definition 2.2 (Cross-composition [9]). Let $L \subseteq \Sigma^*$ and let $Q \subseteq \Sigma^* \times \mathbb{N}$ be a parameterized problem. We say that L cross-composes into Q if there is a polynomial equivalence relation \mathbb{R} and an algorithm which, given t strings $x_1, x_2, \ldots x_t$ belonging to the same equivalence class of \mathbb{R} , computes an instance $(x^*, k^*) \in \Sigma^* \times \mathbb{N}$ in time polynomial in $\sum_{i=1}^t |x_i|$ such that (1) $(x^*, k^*) \in Q$ iff $x_i \in L$ for **some** $1 \le i \le t$; (2) k^* is bounded polynomially in $\max_{i=1}^t |x_i| + \log t$.

Theorem 2.3 ([9], Theorem 9). If $L \subseteq \Sigma^*$ is NP-hard under Karp reductions and L cross-composes into the parameterized problem Q that has a polynomial kernel, then NP \subseteq coNP/poly.

Behind Theorem 2.3 stands the following result of Fortnow and Santhanam [36].

Definition 2.4 ([7]). A distillation algorithm for a problem $L \subseteq \Sigma^*$ into a set $L' \subseteq \Sigma^*$ is a polynomial-time algorithm that given t strings x_1, x_2, \ldots, x_t outputs a string $y \in \Sigma^*$ such that (1) $y \in L'$ iff $x_i \in L$ for some $1 \le i \le t$; (2) |y| is bounded polynomially in $\max_{i=1}^t |x_i|$.

Theorem 2.5 ([36], Theorem 1.2). *An NP-complete language does not admit a distillation algorithm into an arbitrary set unless NP* \subseteq *coNP/poly.*

By replacing the OR operation in Definition 2.4 by the AND operation we obtain the AND-conjecture.

Conjecture 2.6 (AND-conjecture [7]). A coNP-complete language does not admit a distillation algorithm into itself. This conjecture motivates us to define the AND variant of a cross-composition algorithm.

Definition 2.7 (AND-cross-composition). Let $L\subseteq \Sigma^*$ and let $Q\subseteq \Sigma^*\times \mathbb{N}$ be a parameterized problem. We say that L *AND-cross-composes* into Q if there is a polynomial equivalence relation \mathbb{R} and an algorithm which, given t strings $x_1, x_2, \ldots x_t$ belonging to the same equivalence class of \mathbb{R} , computes an instance $(x^*, k^*) \in \Sigma^* \times \mathbb{N}$ in time polynomial in $\sum_{i=1}^t |x_i|$ such that (1) $(x^*, k^*) \in Q$ iff $x_i \in L$ for **each** $1 \le i \le t$; (2) k^* is bounded polynomially in $\max_{i=1}^t |x_i| + \log t$.

Following the lines of the proof of Theorem 9 in [9] we obtain the following result (for sake of completeness we include the formal proof).

Theorem 2.8. If $L \subseteq \Sigma^*$ is NP-complete under Karp reductions and L AND-cross-composes into a parameterized problem Q, which has a polynomial kernel and whose unparameterized variant (i.e., with the parameter appended to the instance in unary) is in NP, then the AND-conjecture fails.

Proof. In this proof we closely follow the lines of the proof of Theorem 9 of [9].

We show how the assumptions of the theorem lead to a distillation algorithm for the coNP-complete language $\bar{L} = \Sigma^* \setminus L$. Let $x_1, x_2, \dots, x_t \in \Sigma^*$ and $m = \max_{i=1}^t |x_i|$. First note that if $t > (|\Sigma| + 1)^m$, then there are duplicates in the input instances and we may remove them. Thus for the rest of the proof we assume that $t \leq (|\Sigma| + 1)^m$, in particular $\log t = O(m)$.

Using the polynomial equivalence relation \mathcal{R} assumed in the definition of the AND-cross-composition, in polynomial time we partition the strings x_i into r equivalence classes X_1, X_2, \ldots, X_r . Note that r is bounded polynomially in m.

For each class X_j we apply the assumed AND-cross-composition on the strings in X_j , obtaining an instance (y_j, k_j) of the parameterized problem Q. We then apply the assumed kernelization algorithm to the instance (y_j, k_j) , obtaining (y_j', k_j') . We note that $|y_j'| + k_j'$ is bounded polynomially in k_j , which is bounded polynomially in m. We infer that the total size of all instances (y_j', k_j') for $1 \le j \le r$ is bounded polynomially in m.

As the unparameterized version of \check{Q} is in NP, we may transform each instance (y'_j,k'_j) into a boolean formula ϕ_j of size polynomial in m such that ϕ_j is satisfiable iff $(y'_j,k'_j)\in Q$. Let $\phi=\bigwedge_{j=1}^r\phi_j$. Note that $|\phi|$ is bounded polynomially in m and ϕ is satisfiable if and only if $x_i\in L$ for all $1\leq i\leq t$. As L is NP-complete, in polynomial time we may transform ϕ into an equivalent instance x of the language L, where |x| is bounded polynomially in m. We conclude by noting that we obtained a distillation algorithm for \bar{L} : $x\in \bar{L}$ iff $x_i\notin L$ for some $1\leq i\leq t$, that is, $x_i\in \bar{L}$.

Observe that any polynomial equivalence relation is defined on all words over the alphabet Σ and for this reason whenever we define a cross-composition, we should also define how the relation behaves on words that do not represent instances of the problem. In all our constructions the defined relation puts all malformed instances into one equivalence class, and the corresponding cross-composition outputs a trivial NO-instance, given a sequence of malformed instances. Thus, in the rest of this paper, we silently ignore the existence of malformed instances.

3 Clique Cover

EDGE CLIQUE COVER

Input: An undirected graph G and an integer k.

Task: Does there exist a set of k subgraphs of G, such that each subgraph is a clique and each edge of G is contained in at least one of these subgraphs?

In this section we present both the cross-composition and the AND-cross-composition of EDGE CLIQUE COVER parameterized by k. We start with the AND-cross-composition since the construction we present is also used in the cross-composition.

3.1 AND-cross-composition

Theorem 3.1. EDGE CLIQUE COVER AND-cross-composes to EDGE CLIQUE COVER parameterized by k.

Proof. For the equivalence relation \mathcal{R} we take a relation that puts two instances (G_1,k_1) , (G_2,k_2) of EDGE CLIQUE COVER are in the same equivalence class iff $k_1=k_2$ and the number of vertices in G_1 is equal to the number of vertices in G_2 . Therefore, in the rest of the proof we assume that we are given a sequence $(G_i,k)_{i=0}^{t-1}$ of EDGE CLIQUE COVER instances that are in the same equivalence class of \mathcal{R} (to avoid confusion we number everything starting from zero in this proof). Let n be the number of vertices in each of the instances. W.l.o.g. we assume that $n=2^{h_n}$ for a positive integer h_n , since otherwise we may add isolated vertices to each instance. Moreover, we assume that $t=2^{h_t}$ for some positive integer h_t , since we may copy some instance if needed, while increasing the number of instances at most two times.

Now we construct an instance (G^*, k^*) , where k^* is polynomial in $n+k+h_t$. Initially as G^* we take a disjoint union of graphs G_i for $i=0,\ldots,t-1$ with added edges between every pair of vertices from G_a and G_b for $a \neq b$. Next, in order to cover all the edges between different instances with few cliques we introduce the following construction. Let us assume that the vertex set of G_i is $V_i = \{v_0^i, \ldots, v_{n-1}^i\}$. For each $0 \leq a < n$, for each $0 \leq b < n$ and for each $0 \leq r < h_t$ we add to G^* a vertex w(a,b,r) which is adjacent to exactly one vertex in each V_i , that is v_j^i where $j = (a+b\lfloor \frac{i}{2^r} \rfloor) \mod n$. By W we denote the set of all added vertices w(a,b,r). As the new parameter k^* we set $k^* = |W| + k = n^2 h_t + k$. Note that W is an independent set in G^* and, moreover, each vertex in W is non-isolated.

Let us assume that for each $i=0,\ldots,t-1$ the instance (G_i,k) is a YES-instance. To show that (G^*,k^*) is a YES-instance we create a set ${\mathfrak C}$ of k^* cliques. We split all the edges of G^* into the following groups: (i) edges incident to vertices of W, (ii) edges between two different graphs G_i,G_j and (iii) edges in each graph G_i . For each vertex $w\in W$ we add to ${\mathfrak C}$ the subgraph $G^*[N[w]]$, which is a clique since every two vertices from two different graphs G_i,G_j are adjacent. Moreover, let ${\mathfrak C}_i=\{C_0^i,\ldots,C_{k-1}^i\}$ be any solution for the instance (G_i,k) . For each $\ell=0,\ldots,k-1$ we add to ${\mathfrak C}$ a clique $G^*\left[\bigcup_{i=0}^{t-1}C_\ell^i\right]$. Clearly all the edges mentioned in (i) and (iii) are covered. Consider any two vertices $v_x^i\in V_i$ and $v_y^j\in V_j$ for i< j. Let r be the greatest integer such that (j-i) is divisible by 2^r . Note that $0\leq r< h_t$ and $z=\lfloor\frac{j}{2^r}\rfloor-\lfloor\frac{i}{2^r}\rfloor\equiv 1\pmod{2}$ since otherwise (j-i) would be divisible by 2^{r+1} . Consequently, there exists $0\leq b< n$ satisfying the congruence $bz\equiv y-x\pmod{n}$, since the greatest common

divisor of z and n is equal to one (recall that n is a power of 2). Therefore, when we set $a = y - b \lfloor \frac{j}{2r} \rfloor$ we obtain

$$a + b \left\lfloor \frac{i}{2^r} \right\rfloor \equiv b \left(\left\lfloor \frac{i}{2^r} \right\rfloor - \left\lfloor \frac{j}{2^r} \right\rfloor \right) + y \equiv y - bz \equiv x \pmod{n}$$
$$a + b \left\lfloor \frac{j}{2^r} \right\rfloor \equiv y \pmod{n}$$

and both v_x^i, v_y^j belong to the clique of \mathcal{C} containing the vertex w(a, b, r).

Now let us assume that (G^*, k^*) is a YES-instance and let \mathcal{C} be a set of at most k^* cliques in G^* that cover every edge in G^* . We define $\mathcal{C}' \subseteq \mathcal{C}$ as the set of these cliques in \mathcal{C} which contain at least two vertices from some set V_i . Since W is an independent set in G^* , edges incident to two different vertices in W need to be covered by two different cliques in \mathcal{C} . Moreover, no clique in \mathcal{C}' contains a vertex from W, because each vertex in W is incident to exactly one vertex in each V_i . Therefore, $|\mathcal{C}'| \leq |\mathcal{C}| - |W| \leq k$ and a set $\mathcal{C}_i = \{X \cap V_i : X \in \mathcal{C}'\}$ for $i = 0, \ldots, t-1$ is a solution for (G_i, k) , as no clique in $\mathcal{C} \setminus \mathcal{C}'$ covers an edge between two vertices in V_i for any $i = 0, \ldots, t-1$. Hence each instance (G_i, k) is a YES-instance.

As a consequence, by Theorem 2.8 we obtain the following result.

Corollary 3.2. There is no polynomial kernel for the EDGE CLIQUE COVER problem parameterized by k unless the AND-conjecture fails.

3.2 Cross-composition

In this section we show cross-composition to EDGE CLIQUE COVER, which we obtain by extending the AND-cross-composition gadgets from the previous section.

COMPRESSION CLIQUE COVER

Input: An undirected graph G, an integer k and a set \mathcal{C} of k+1 cliques in G covering all edges of G.

Task: Does there exist a set of k subgraphs of G, such that each subgraph is a clique and each edge of G is contained in at least one of the subgraphs?

Lemma 3.3. COMPRESSION CLIQUE COVER is NP-complete with respect to Karp's reductions.

Proof. Clearly COMPRESSION CLIQUE COVER is in NP.

To prove that COMPRESSION CLIQUE COVER is NP-hard we show a reduction from 3-COLOURING of 4-regular planar graphs, which is NP-hard by [28]. Let a 4-regular planar graph G=(V,E) be an instance of 3-COLOURING. By Brooks theorem we know that G is 4-colourable (since by planarity, G has no connected component isomorphic to K_5) and we may find 4-colouring of G in polynomial time [78]. Let $\bar{G}=(V,\bar{E})$ be the complement of G, that is an edge e is in \bar{E} iff e does not belong to E. To construct the graph G' as the set of vertices we take two copies of V, namely $V_1=\{v_1:v\in V\},\,V_2=\{v_2:v\in V\}$. For each edge $uv\in\bar{E}$ we add to G' four vertices $w_{uv}^{p,q}$ for $1\leq p,q\leq 2$ and edges $w_{uv}^{p,q}u_p,\,w_{uv}^{p,q}v_q,\,u_pv_q$. By W we denote the set of all vertices $w_{uv}^{p,q}$ in G'. Finally, for each $v\in V$ we add to G' an edge v_1v_2 and set $k=|W|+3=4|\bar{E}|+3$.

In order to make (G',k) a proper instance of COMPRESSION CLIQUE COVER we need also to construct a set $\mathbb C$ of k+1 cliques covering all edges of G'. Observe that to cover edges incident to vertices of W we need at least |W| cliques since W is an independent set in G'. Moreover, for each $w \in W$ the set $N_{G'}[w]$ is a clique in G'; hence w.l.o.g. any set of cliques covering all edges of G' contains |W| cliques of the form $N_{G'}[w]$ for $w \in W$ and those |W| cliques cover all the edges of G' except for $E' = \{v_1v_2 : v \in V\}$. Note that to cover two different edges $u_1u_2, v_1v_2 \in E'$ we need u_1 and v_2 to be adjacent in G', that is, non-adjacent in G. Hence covering E' with I cliques is equivalent to colouring G in I colours. Since G is 4-colourable in an efficient way, we can construct a set $\mathbb C$ of k+1 cliques covering G' obtaining an instance of COMPRESSION CLIQUE COVER, which is a YES-instance iff G is 3-colourable.

Now the goal is to adjust the construction from the proof of Theorem 3.1 in order to obtain a classical cross-composition of COMPRESSION CLIQUE COVER into EDGE CLIQUE COVER. Observe that we cannot easily relax the assumption that the clique cover of size k+1 is given in the input to just promising its existence, as the composition

algorithm needs to be able to distinguish malformed instances from well-formed in the first place, which would not be the case unless P = NP. Moreover, the COMPRESSION CLIQUE COVER problem is trivially NP-hard with respect to Turing reductions; however, in order to make the composition work we need NP-completeness in Karp's sense.

Theorem 3.4. Compression Clique Cover cross-composes to Edge Clique Cover parameterized by k.

Proof. We define the polynomial equivalence relation \mathcal{R} in exactly the same way as in the proof of Theorem 3.1, that is we group instances according to their number of vertices and the value of k. Thus in the rest of the proof we assume we are given a sequence $(G_i, k, \mathcal{C}_i)_{i=0}^{t-1}$ of COMPRESSION CLIQUE COVER instances that are in the same equivalence class of \mathcal{R} . As in the proof of Theorem 3.1 we let n be the number of vertices in each of the instances and we assume $n=2^{h_n}$ and $t=2^{h_t}$.

Before we proceed to the proof let us give some intuition on what follows. We would like to use the construction from Theorem 3.1 and extend it by adding exactly h_t gadgets. We show that any solution w.l.o.g. behaves in only one of two possible ways in every gadget. Intuitively, each choice for the j-th gadget relaxes the constraint of using only k cliques for half of the instances. That is, choosing behaviour b, for b=0,1, allows using k+1 cliques, which are always sufficient by solution C_i given as a part of the input, for all instances with the j-th bit of the instance number equal to b. Hence there is exactly one instance which is not relaxed by any of the h_t gadgets, so intuitively the gadgets may be viewed as an instance selector from t instances.

Construction We create the instance of clique cover (G^*, k^*) as in the proof of Theorem 3.1. To obtain an instance (G', k') we set G' as G^* and for each $j=1,\ldots,h_t$ we add to G' a gadget D_j containing exactly 6 vertices $V(D_j)=\{d^L_{j,1},d^L_{j,2},d^L_{j,3},d^R_{j,1},d^R_{j,2},d^R_{j,3}\}$ and 12 edges $\binom{V(D_j)}{2}\setminus\{d^L_{j,r}d^R_{j,r}:1\leq r\leq 3\}$. In other words, D_j is a clique with a perfect matching removed (see Fig. 3.2). Let V_j^L be the union of all sets $V_i=\{v_a^i:0\leq a< n\}$ (recall that $V_i=V(G_i)$ is the set of vertices of the i-th instance) such that the j-th bit of the number i written in binary is equal to zero, whereas similarly V_j^R is the set of vertices of all instances having the j-th bit of their number equal to one. We make each vertex of $L_j=\{d^L_{j,r}:1\leq r\leq 3\}$ adjacent to each vertex of V_j^L and we make each vertex of $R_j=\{d^L_{j,r}:1\leq r\leq 3\}$ adjacent to each vertex of V_j^R in G'. Finally, in order to allow easy coverage of the edges between $V(D_j)$ and $V_j^L\cup V_j^R$, for each $0\leq a< n$, $1\leq r\leq 3$, and $Z\in\{L,R\}$ we add to G' a vertex s(a,r,Z) adjacent to each vertex in $\{v_a^i\in V_j^Z:0\leq i< t\}\cup\{d^Z_{j,r}\}$. Let S be the set of all added vertices s(a,r,Z). As the parameter we set $k'=k^*+|S|+4h_t=n^2h_t+6nh_t+4h_t+k$.

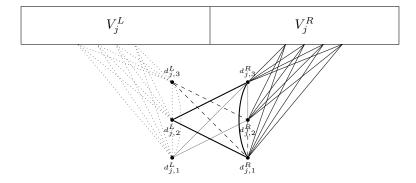


Figure 1: The gadget D_j set to relax the left part. Different styles of picturing the edges (thick, dotted, dashed or grayed) indicate, to which of the four cliques constructed for D_j the edge belongs to.

Analysis We split all the edges of G^* into the following groups:

- (i) edges incident to vertices of $W \cup S$ (recall that W is a set defined as in the proof of Theorem 3.1),
- (ii) edges between two different graphs G_i, G_j ,

- (iii) edges in each graph G_i ,
- (iv) edges within each gadget D_j .

First let us assume that for some $0 \le i_0 < t$ the instance (G_i, k, \mathcal{C}_i) of COMPRESSION CLIQUE COVER is a YES-instance. We construct a set of cliques \mathcal{C} . For each vertex $x \in W \cup S$ we add to \mathcal{C} a clique $N_{G'}[x]$. Hence by using $|W| + |S| = n^2 h_t + 6nh_t$ cliques we cover all edges of (i) and (ii) (by the same analysis as in the proof of Theorem 3.1). Let us assume that each set \mathcal{C}_i is of the form $\mathcal{C}_i = \{C_0^i, \dots, C_k^i\}$ and also let us somewhat abuse the notation and assume that the k cliques $C_0^{i_0}, \dots, C_{k-1}^{i_0}$ form a solution for the YES-instances (G_{i_0}, k) . Let $(b_0b_1 \dots b_{h_t})_2$ be the binary representation of i_0 . Set $Z_j = L, Z_j' = R$ iff b_j equals one and $Z_j = R, Z_j' = L$ otherwise, for $j = 0, \dots, h_t - 1$. For each $j = 0, \dots, h_t - 1$ we add to \mathcal{C} exactly 4 cliques $\left\{d_{j,1}^{Z_j}, d_{j,2}^{Z_j'}, d_{j,3}^{Z_j'}\right\}, \left\{d_{j,2}^{Z_j}, d_{j,1}^{Z_j'}, d_{j,3}^{Z_j'}\right\}, \left\{d_{j,3}^{Z_j}, d_{j,1}^{Z_j'}, d_{j,2}^{Z_j'}\right\}, \left\{d_{j,3}^{Z_j}, d_{j,1}^{Z_j'}, d_{j,3}^{Z_j'}\right\} \cup \left(\bigcup_{i=0}^{t-1} \left(C_k^i \cap V_j^{Z_j}\right)\right)$ (see Fig. 3.2). It is easy to verify that the 4 added sets are indeed cliques in G' and that they cover edges of (iv). Note that the last of the four cliques contains the last clique of the solution \mathcal{C}_i for each instance i that has j-th bit equal to b_j . Consequently, some of the edges of (iii) are covered. We add exactly k more cliques to \mathcal{C} , that is for each $\ell = 0, \dots, k-1$ we add to \mathcal{C} a clique $\bigcup_{i=0}^{t-1} C_i^i$. Since for each $i = 0, \dots, t-1$ such that $i \neq i_0$ there is a clique in \mathcal{C} containing C_k^i and the first k cliques of \mathcal{C}_{i_0} form a cover of G_i , we infer that all edges of (iii) are covered and, therefore, (G', k') is a YES-instances of EDGE CLIQUE COVER.

In the other direction, assume that (G',k') is a YES-instance of EDGE CLIQUE COVER and let $\mathcal C$ be any solution containing k' cliques. Since $W \cup S$ is an independent set in G' and each vertex in $W \cup S$ is not isolated, we infer that there are at least |W| + |S| cliques in $\mathcal C$ containing a vertex of $W \cup S$. Let $\mathcal C' \subseteq \mathcal C$ be the set of at most k' - |W| - |S| cliques of $\mathcal C$ which have empty intersection with $W \cup S$. Each vertex in $W \cup S$ is adjacent to exactly one vertex in each V_i for $0 \le i < t$ and at most one vertex in $V(D_j)$ for $0 \le j < h_t$, therefore cliques of $\mathcal C'$ cover all edges of (iii) and (iv). Moreover, $|\mathcal C'| \le k' - |W| - |S| = 4h_t + k$. We use the following lemma which we prove afterwards.

Lemma 3.5. One can modify the set C' maintaining coverage of edges of (iii) and (iv) and not incrementing its size, while at the same time for each $j = 0, ..., h_t - 1$ obeying the following conditions:

- (a) C' contains exactly 3 cliques containing both a vertex of L_i and R_j (recall that $L_i \cup R_i = V(D_i)$),
- (b) C' contains exactly 1 clique C having exactly one of the two intersections $C \cap L_i, C \cap R_i$ non-empty.

Before we prove Lemma 3.5 let us finish the proof of Theorem 3.4 assuming that Lemma 3.5 holds. Since no two vertices from different gadgets D_{j_1}, D_{j_2} are adjacent, we infer that \mathcal{C}' contains exactly $4h_t$ cliques containing a vertex from some D_j for $0 \leq j < h_t$. For each $j = 0, \ldots, h_t - 1$, let C be the clique from (b) of Lemma 3.5. If $C \cap L_j \neq \emptyset$, we take $I_j \subseteq \{0, \ldots, t-1\}$ to be the set of instance numbers that have the j-th bit equal to one, whereas if $C \cap R_j \neq \emptyset$, then as I_j we take all the instance numbers that have the j-th bit equal to zero. Observe that $\bigcap_{j=0}^{h_t-1} I_j$ contains exactly one element and denote it by i_0 . By Lemma 3.5 every clique from \mathcal{C}' , that contains a vertex of $V(D_j)$ for any $j = 0, \ldots, h_t - 1$, has to be disjoint with V_{i_0} . Indeed, cliques containing vertices from $V(D_j)$ satisfy (a) or (b) from Lemma 3.5; a clique from (a) contains both vertices of L_j and R_j and no vertex of V_{i_0} is incident to both V_{i_0} and V_{i_0} is incident to both V_{i_0} . Therefore, edges of V_{i_0} are covered by $V_{i_0} = 1$ consequently, $V_{i_0} = 1$

Proof of Lemma 3.5. Let j be any index for which the lemma does not hold, i.e., the cliques containing vertices from $V(D_j)$ do not behave as in the lemma statement. We refine the set \mathcal{C}' repairing its behaviour on gadget D_j and not spoiling the behaviour on other gadgets. By applying this reasoning to all the gadgets that need repairing, we prove the lemma.

Let us denote by \mathcal{H}_j the set of cliques from \mathcal{C}' that contain a vertex from $V(D_j)$, while let $\mathcal{I}_j \subseteq \mathcal{H}_j$ be the set of these cliques from \mathcal{H}_j , which have nonempty intersection with both L_j and R_j . The goal is to obtain a situation, when $|\mathcal{H}_j| = 4$ and $|\mathcal{I}_j| = 3$ for every j. During refining the set \mathcal{C}' we will change only the set \mathcal{H}_j . As there are no edges between the gadgets, the sets \mathcal{H}_j are always disjoint, so our repairs do not spoil the behaviour of \mathcal{C}' on other gadgets.

Let $C \in \mathcal{H}_j$. Observe that $|C \cap (L_j \cup R_j)| \leq 3$, since C contains at most one of the vertices $D_{j,p}^L$, $D_{j,p}^R$ for p = 1, 2, 3. Therefore, each clique in \mathcal{H}_j covers at most 3 out of 12 edges of D_j and, consequently, $|\mathcal{H}_j| \geq 4$. Consider two cases.

First assume that $\mathcal{H}_j \setminus \mathbb{J}_j \neq \emptyset$, i.e., there exists a clique $C_0 \in \mathcal{C}'$ which has an element of $L_j \cup R_j$, but has exactly one of the two intersections $C_0 \cap L_j$, $C_0 \cap R_j$ non-empty. By symmetry assume that $(C_0 \cap (L_j \cup R_j)) \subseteq L_j$. Note that $C_0 \cup L_j$ also forms a clique, hence w.l.o.g. we may assume that $L_j \subseteq C_0$. We know that $|\mathbb{J}_j| \geq 3$, since \mathbb{C}' covers all the 6 edges of $E(L_j, R_j)$, whereas each clique in \mathbb{C}' covers at most two of them. Note that each clique from \mathbb{J}_j is entirely contained in D_j , since there is no vertex outside of D_j which is adjacent to both a vertex of L_j and a vertex of R_j . Therefore, we may substitute the whole \mathbb{J}_j with just 3 cliques:

$$C_{1} := \{d_{j,1}^{L}, d_{j,2}^{R}, d_{j,3}^{R}\},$$

$$C_{2} := \{d_{j,2}^{L}, d_{j,1}^{R}, d_{j,3}^{R}\},$$

$$C_{3} := \{d_{j,3}^{L}, d_{j,1}^{R}, d_{j,2}^{R}\},$$

maintaining the property that \mathcal{C}' covers all the edges. Observe that cliques C_0, C_1, C_2, C_3 already cover all the edges in D_j . Finally, after this modification for any $C \in \mathcal{C}'$ such that $C \notin \{C_0, C_1, C_2, C_3\}$ we set $C := C \setminus (L_j \cup R_j)$ and satisfy both constraints (a) and (b) of the lemma for this particular j.

Now assume that for $\mathfrak{I}_j=\mathfrak{H}_j$. Similarly as in the previous case, for each clique $C\in\mathfrak{I}_j$ we have $C\subseteq(L_j\cup R_j)$. As $|\mathfrak{I}_j|=|\mathfrak{H}_j|\geq 4$, we may substitute the whole $\mathfrak{I}_j=\mathfrak{H}_j$ with just 4 cliques:

$$\begin{split} C_0 &:= \{d_{j,1}^L, d_{j,2}^L, d_{j,3}^L\}\,, \\ C_1 &:= \{d_{j,1}^L, d_{j,2}^R, d_{j,3}^R\}\,, \\ C_2 &:= \{d_{j,2}^L, d_{j,1}^R, d_{j,3}^R\}\,, \\ C_3 &:= \{d_{j,3}^L, d_{j,1}^R, d_{j,2}^R\}\,, \end{split}$$

out of which exactly one is not contained in the new \mathcal{I}_j . As these cliques cover all the edges of D_j and every removed clique was entirely contained in D_j , all the edges of G' are still covered. In this way we make the modified set \mathcal{C}' satisfy both constraints (a) and (b) for the considered value of j.

Corollary 3.6. There is no polynomial kernel for the EDGE CLIQUE COVER problem parameterized by k unless $NP \subseteq coNP/poly$.

4 Directed Multiway Cut

In the DIRECTED MULTIWAY CUT problem we want to disconnect every pair of terminals in a directed graph. The problem was previously studied in the following two versions.

DIRECTED EDGE MULTIWAY CUT

Input: A directed graph G = (V, A), a set of terminals $T \subseteq V$ and an integer p.

Task: Does there exist a set S of at most p arcs in A, such that in $G \setminus S$ there is no path between any pair of terminals in T?

DIRECTED VERTEX MULTIWAY CUT

Input: A directed graph G = (V, A), a set of terminals $T \subseteq V$, a set of forbidden vertices $V^{\infty} \supseteq T$ and an integer p. **Task:** Does there exist a set S of at most p vertices in $V \setminus V^{\infty}$, such that in $G \setminus S$ there is no path between any pair of terminals in T?

As a side note, observe that by replacing each vertex of $V^{\infty} \setminus T$ with a p+1-clique (i.e., a graph on p+1 vertices pairwise connected by arcs in both directions), one can reduce the above DIRECTED VERTEX MULTIWAY CUT version to a version, where the solution is allowed to remove any nonterminal vertex. Moreover, it is well known, that given an instance I of one of the two problems above, one can in polynomial time create an equivalent instance I' of the

other problem, where both the number of terminals and the value of *p* remain unchanged (e.g. see [21]). Therefore we show cross-composition to DIRECTED VERTEX MULTIWAY CUT and as a corollary we prove that DIRECTED EDGE MULTIWAY CUT also does not admit a polynomial kernel. The starting point is the following restricted variant of DIRECTED VERTEX MULTIWAY CUT, which we prove to be NP-complete with respect to Karp reductions.

PROMISED DIRECTED VERTEX MULTIWAY CUT

Input: A directed graph G = (V, A), two terminals $T = \{s_1, s_2\}$, a set of forbidden vertices $V^{\infty} \supseteq T$ and an integer p. Moreover, after removing any set of at most p/2 vertices of $V \setminus V^{\infty}$, both an s_1s_2 -path and an s_2s_1 -path remain. **Task:** Does there exist a set S of at most p vertices in $V \setminus V^{\infty}$, such that in $G \setminus S$ there is no s_1s_2 -path nor s_2s_1 -path?

The assumption that any set of size at most p/2 can not hit all the paths from s_1 to s_2 (and similarly from s_2 to s_1) will help us in constructing cross-composition.

Lemma 4.1. PROMISED DIRECTED VERTEX MULTIWAY CUT is NP-complete with respect to Karp's reductions.

Proof. Note that in order to show that the problem is in NP we need to argue that we can verify the condition concerning removal of p/2 vertices. However, this can be checked by a polynomial-time algorithm computing min s_1-s_2 cut and min s_2-s_1 cut. If any of those cuts is of size at most p/2, then the instance is not a proper instance of PROMISED DIRECTED VERTEX MULTIWAY CUT.

To prove that the problem is NP-hard we use the NP-completeness result of Garg et al. [40] for DIRECTED VERTEX MULTIWAY CUT with two terminals. Consider an instance $I=(G,T=\{s_1,s_2\},V^\infty,p)$ of DIRECTED VERTEX MULTIWAY CUT. As the graph G' we take G with z=p+1 vertices $\{u_1,\ldots,u_z\}$ added. In G' for $i=1,\ldots,z$ we add the following four arcs $\{(s_1,u_i),(u_i,s_1),(u_i,s_2),(s_2,u_i)\}$. Let $I'=(G',T,V^\infty,p+z)$ be an instance of PROMISED DIRECTED VERTEX MULTIWAY CUT. Since after removal of less than z vertices in G' at least one vertex u_i remains, we infer that I' is indeed a PROMISED DIRECTED VERTEX MULTIWAY CUT instance. To prove that I is a YES-instance iff I' is a YES-instance it is enough to observe that any solution in I' contains all the vertices $\{u_1,\ldots,u_z\}$.

Equipped with the PROMISED DIRECTED VERTEX MULTIWAY CUT problem definition, we are ready to show a cross-composition into DIRECTED VERTEX MULTIWAY CUT parameterized by p.

Theorem 4.2. PROMISED DIRECTED VERTEX MULTIWAY CUT *cross-composes into* DIRECTED VERTEX MULTIWAY CUT *with two terminals, parameterized by the size of the cutset p.*

Proof. For the equivalence relation \mathcal{R} , we take a relation that groups the input instances according to the value of p. Formally $(G_i, T_i, V_i^{\infty}, p_i)$ and $(G_j, T_j, V_j^{\infty}, p_j)$ are in the same equivalence class in \mathcal{R} iff $p_i = p_j$. Therefore, we assume that we are given a sequence $I_i = (G_i, T_i = \{s_1^i, s_2^i\}, V_i^{\infty}, p)_{i=1}^t$ of PROMISED DIRECTED VERTEX MULTIWAY CUT instances that are in the same equivalence class of \mathcal{R} .

As the graph G' we take disjoint union of all the graphs G_i . Moreover for each $i=1,\ldots,t-1$, in G' we identify the vertices s_2^i and s_1^{i+1} . Let $I'=(G',\{s_1^1,s_2^t\},\bigcup_{i=1}^tV_i^\infty,p)$ be an instance of DIRECTED VERTEX MULTIWAY Cut. Note that $\bigcup_{i=1}^tV_i^\infty$ contains both terminals from all input instances.

Let us assume that there exists $1 \leq i_0 \leq t$ such that I_{i_0} is a YES-instance of PROMISED DIRECTED VERTEX MULTIWAY CUT, and let $S \subseteq V(G_i) \setminus V_i^{\infty}$ be any solution for I_{i_0} . Since any $s_1^1 s_2^t$ -path and any $s_2^t s_1^t$ -path in G' goes through both $s_1^{i_0}$ and $s_2^{i_0}$, we observe that $G' \setminus S$ is a solution for I' and, consequently, I' is a YES-instance.

In the other direction, let us assume that I' is a YES-instance. Let $S \subseteq V(G) \setminus \bigcup_{i=1}^t V_i^{\infty}$ by any solution for I'. Observe that if the set S contains at most p/2 vertices of $V(G_i) \setminus V_i^{\infty}$ for some $1 \le i \le t$, then $S \setminus V(G_i)$ is also a solution for I', since after removing at most p/2 vertices of $V(G_i)$ there is still a path both from s_1^i to s_2^i and from s_2^i to s_1^i . Because $|S| \le p$, we infer that w.l.o.g. S contains only vertices of a single set $V(G_{i_0})$ for some $1 \le i_0 \le t$. Therefore, I_{i_0} is a YES-instance.

The equivalence of DIRECTED VERTEX MULTIWAY CUT and DIRECTED EDGE MULTIWAY CUT together with Theorem 2.3 give us the following corollary.

Corollary 4.3. Both DIRECTED VERTEX MULTIWAY CUT and DIRECTED EDGE MULTIWAY CUT do not admit a polynomial kernel when parameterized by p unless $NP \subseteq coNP/poly$, even in the case of two terminals.

5 Multicut

In this section we prove that both the edge and vertex versions of the MULTICUT problem do not admit a polynomial kernel, when parameterized by the size of the cutset.

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EDGE (VERTEX) MULTICUT
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Input: An undirected graph G = (V, E), a set of pairs of terminals $\mathfrak{T} = \{(s_1, t_1), \dots, (s_k, t_k)\}$ and an integer p.

Task: Does there exists a set $S \subseteq E$ ($S \subseteq V$) such that no connected component of $G \setminus S$ contains both vertices s_i and t_i , for some $1 \le i \le k$?

It is known that the vertex version of the MULTICUT problem is at least as hard as the edge version.

Lemma 5.1 (folklore). There is a polynomial time algorithm, which given an instance $I = (G, \mathfrak{T}, p)$ of EDGE MULTICUT produces an instance $I' = (G', \mathfrak{T}', p)$ of VERTEX MULTICUT, such that I is a YES-instance iff I' is a YES-instance.

In order to show a cross-composition into the MULTICUT problem parameterized by p we consider the following restricted variant of the MULTIWAY CUT problem with three terminals.

MULTIWAY CUT

Input: An undirected graph G = (V, E), a set of three terminals $T = \{s_1, s_2, s_3\} \subseteq V$ and an integer p.

Task: Does there exist a set S of at most p edges in E, such that in $G \setminus S$ there is no path between any pair of terminals in T?

PROMISED MULTIWAY CUT

Input: An undirected graph G = (V, E), a set of three terminals $T = \{s_1, s_2, s_3\} \subseteq V$ and an integer p. An instance satisfies: (i) $\deg(s_1) = \deg(s_2) = \deg(s_3) = d > 0$, (ii) for each j = 1, 2, 3 and any non-empty set $X \subseteq V \setminus T$ we have $|\delta(X \cup \{s_j\})| > d$, and (iii) $d \le p < 2d$.

Task: Does there exist a set S of at most p edges in E, such that in $G \setminus S$ there is no path between any pair of terminals in T?

Condition (i) ensures that degrees of all the terminals are equal, whereas condition (ii) guarantees that the set of edges incident to a terminal s_j is the only minimum size s_j – $(T \setminus \{s_j\})$ cut. Having both (i) and (ii), condition (iii) verifies whether an instance is not a trivially YES- or NO-instance, because by (i) and (ii) there is no solution of size less than d and removing all the edges incident to two terminals always gives a solution of size at most 2d.

Lemma 5.2. PROMISED MULTIWAY CUT is NP-complete with respect to Karp's reductions.

Proof. To prove the lemma we may observe that the first NP-hardness reduction to the MULTIWAY CUT problem by Dahlhaus et al. [27] in fact yields a PROMISED MULTIWAY CUT instance. For sake of completeness, we present here how to reduce an arbitrary instance of the MULTIWAY CUT problem with three terminals to a PROMISED MULTIWAY CUT instance.

Let $I=(G,T=\{s_1,s_2,s_3\},p)$ be an instance of MULTIWAY CUT. As observed by Marx [64], we can assume that for each terminal s_i the cut $\delta(s_i)$ is the only minimum cardinality s_i – $(T\setminus\{s_i\})$ cut, since otherwise w.l.o.g. we may contract some edge incident to s_i obtaining a smaller equivalent instance. Therefore condition (ii) would be satisfied if only degrees of terminals were equal. Let G_1, G_2, G_3 be three copies of the graph G, where terminals in the i-th copy are denoted by $T_i=\{s_1^i,s_2^i,s_3^i\}$. Construct a graph G' as a disjoint union of G_1, G_2 and G_3 . Next in G' we identify vertices $\{s_1^1,s_2^2,s_3^3\}$ into a single vertex s_1' , similarly identify vertices $\{s_2^1,s_3^2,s_3^3\}$ into a single vertex s_2' , and finally identify vertices $\{s_3^1,s_1^2,s_1^3\}$ into a single vertex s_3' . Let $I'=(G',T'=\{s_1',s_2',s_3'\},p'=3p\}$. Observe that due to the performed identification I' is a YES-instance of MULTIWAY CUT iff I is a YES-instance of MULTIWAY CUT. Therefore, to finish the reduction it suffices to argue that I' satisfies (i), (ii) and (iii).

Let $d = \sum_{i=1}^{3} \deg_G(s_i)$. Note that in G' for each i = 1, 2, 3 we have $\deg_{G'}(s_i') = d$, hence condition (i) is satisfied. Observe that if there exists $1 \le j \le 3$ and $s_j' - (T' \setminus \{s_j'\})$ cut in G' of size at most d, which is different from $\delta(s_j')$, then there exists $1 \le r \le 3$ and $s_r - (T \setminus \{s_r\})$ cut in G of size at most $\deg_G(s_r)$ which is different from $\delta_G(s_r)$, a contradiction. Hence condition (ii) is satisfied. Unfortunately, it is possible that $p' \le d$ or $p' \ge 2d$.

However, if $p' \geq 2d$ then clearly I' is a YES-instance (we can remove edges incident to two terminals), and hence I is a YES-instance. On the other hand if p' < d, then I' (and consequently I) is a NO-instance, since any $s'_1 - \{s'_2, s'_3\}$ cut has size at least d. Therefore, if condition (iii) is not satisfied, then in polynomial time we can compute the answer for the instance I, and as the instance I' we set a trivial YES- or NO-instance.

Theorem 5.3. Promised Multiway Cut cross-composes into Edge Multicut parameterized the size of the cutset p.

Proof. For the equivalence relation \mathcal{R} , we take a relation where all well-formed instances are grouped according to the values of p and d. Formally, (G_i, T_i, p_i) and (G_j, T_j, p_j) are in the same equivalence class in \mathcal{R} iff $p_i = p_j$ and the degree of each terminal in G_i equals the degree of each terminal in G_j . Therefore, we assume that we are given a sequence $I_i = (G_i, T_i = \{s_1^i, s_2^i, s_3^i\}, p)_{i=0}^{t-1}$ of PROMISED MULTIWAY CUT instances that are in the same equivalence class of \mathcal{R} (note that we number instances starting from 0). Let d be the degree of each terminal in each of the instances. W.l.o.g. we assume that $t \geq 5$ is an odd integer, since we may copy some instances if needed, and let h = (t-1)/2.

Construction Let G' be the disjoint union of all graphs G_i for $i=0,\ldots,t-1$. For each $i=0,\ldots,t-1$ we add d parallel edges between vertices s_2^i and $s_1^{(i+1) \bmod t}$. To the set ${\mathfrak T}$ we add exactly t pairs, that is for each $i=0,\ldots,t-1$ we add to ${\mathfrak T}$ the pair $(s_i'=s_3^i,t_i'=s_3^{(i+h) \bmod t})$. We set p'=p+d and $I'=(G',{\mathfrak T},p')$ is the constructed EDGE MULTICUT instance. Note that in order to avoid using parallel edges it is enough to subdivide them.

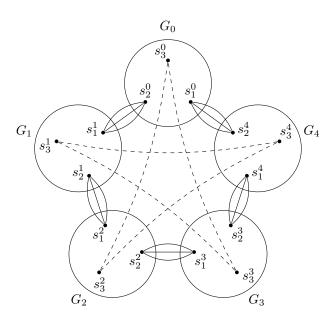


Figure 2: Construction of the graph G' for t=5 and d=3. Dashed edges represent pairs of vertices in \mathfrak{T} .

Analysis First assume that there exists $0 \le i_0 < t$ such that I_{i_0} is a YES-instance of PROMISED MULTIWAY CUT and let $S \subseteq E(G_{i_0})$ be any solution for I_{i_0} . Let S_1 be the set of edges in G' between $s_2^{(i_0+h) \bmod t}$ and $s_1^{(i_0+h+1) \bmod t}$ (see Fig. 2). We prove that $S' = S \cup S_1$ is a solution for I'. Observe that $|S'| = |S| + |S_1| \le p + d$. Consider any pair $(s',t') \in \mathcal{T}$ such that $s',t' \ne s_3^{i_0}$. Note that in $G' \setminus S'$ there is neither an $s_1^{i_0}s_2^{i_0}$ -path, nor an $s_2^{(i_0+h) \bmod t}s_1^{(i_0+h+1) \bmod t}$ -path. Therefore, there is no s't'-path $G' \setminus S'$. Moreover, in $G' \setminus S'$ there is neither an $s_3^{i_0}s_1^{i_0}$ -path, nor an $s_3^{i_0}s_2^{i_0}$ -path. Consequently, for each $(s',t') \in \mathcal{T}$, where $s' = s_3^{i_0}$ or $t' = s_3^{i_0}$, there is no s't'-path in $G' \setminus S'$, so I' is a YES-instance of EDGE MULTICUT.

Now assume that I' is a YES-instance and our goal is to show that for some $0 \le i < t$ the instance I_i is a YES-instance. Let $E_i = E(G_i)$ and let E'_i be the set of edges between s_2^i and $s_1^{(i+1) \bmod t}$ in G'. Let $S' \subseteq E(G')$ be any solution for I'. Note that if for some E'_i , where $0 \le i < t$, the set S' contains less than d edges from the set E'_i , then $S' \setminus E'_i$ is also a solution for I'. By conditions (i) and (ii) of the PROMISED MULTIWAY CUT problem definition we have the following: if for some $0 \le i < t$ the set S' contains less than d edges from the set E_i , then $S' \setminus E_i$ is also a solution for I'. Indeed, if S' contains less than d edges from E_i , then in the graph $G' \setminus E_i$ all the vertices s_1^i , s_2^i , s_3^i are in the same connected component, since otherwise for some $a \in T_i$ there would be an $a-(T_i \setminus \{a\})$ cut in G_i of size smaller than d. Let us recall that $|S'| \le p' = p + d < 3d$. Therefore, w.l.o.g. we may assume that the set S' has non-empty intersection with at most two sets from the set $E = \{E_0, \dots, E_{t-1}, E'_0, \dots, E'_{t-1}\}$. Moreover we assume that if S' has non-empty intersection with some set from E, then this intersection is of size at least d.

Case 1. Consider the case, when S' has an empty intersection with each of the sets E_i for $0 \le i < t$. Since $|E_i'| = d$ and $p' \ge 2d$, w.l.o.g. S' has a non-empty intersection with exactly two sets E_{i_0}' , E_{i_1}' for $i_0 \ne i_1$. Since t is odd, in the graph $G' \setminus S'$ either there is an $s_3^{i_0} s_3^{(i_0-h)\pmod{t}}$ -path or an $s_3^{(i_0+1)\pmod{t}} s_3^{(i_0+1+h)\pmod{t}}$ -path. Hence a contradiction.

Case 2. Next assume that S' has a non-empty intersection with some set E_{i_0} for $0 \le i_0 < t$. By symmetry w.l.o.g. we may assume that $i_0 = 0$. Since the set S' hits all the $s_3^1 s_3^{h+1}$ -paths as well as all the $s_3^h s_3^{t-1}$ -paths in the graph G', we infer that S' has non-empty intersection with exactly one of the sets E_h , E'_h , E_{h+1} .

we infer that S' has non-empty intersection with exactly one of the sets E_h , E_h' , E_{h+1} .

Case 2.1. In this case we assume that S' has a non-empty intersection with E_h' . Since S' hits all $s_3^0 s_3^h$ -paths in G', in $G_0 \setminus S'$ there is no $s_3^0 s_2^0$ -path. Similarly, since S' hits all $s_3^0 s_3^{h+1}$ -paths in G', in $G_0 \setminus S'$ there is no $s_3^0 s_3^0$ -path. Moreover, since S' hits all $s_3^{t-1} s_3^{h-1}$ -paths in G', in $G_0 \setminus S'$ there is no $s_1^0 s_2^0$ -path. Since $|S'| \leq p' = p + d$ and $|S' \cap E_h'| = d$, we infer that $|S' \cap E_0| \leq p$, and, consequently, I_0 is a YES-instance.

Case 2.2. Since S' has a non-empty intersection with one of the sets E_h , E_{h+1} , by symmetry we assume that $S' \cap E_h \neq \emptyset$. Recall that $t \geq 5$, and hence h > 1. Since S' hits all $s_3^1 s_3^{h+1}$ -paths and all $s_3^1 s_3^{t+1-h}$ -paths in G', we infer that in $G_0 \setminus S'$ there is no $s_1^0 s_2^0$ -path and in $G_h \setminus S'$ there is no $s_1^h s_2^h$ -path. Moreover, S' hits all $s_3^0 s_3^{h+1}$ -paths and all $s_3^h s_3^{t-1}$ -paths in G'; therefore, in $G_0 \setminus S'$ there is no $s_3^0 s_1^0$ -path and in $G_h \setminus S'$ there is no $s_3^h s_2^h$ -path. Finally since S' hits all $s_3^0 s_3^h$ -paths in G', either in $G_0 \setminus S'$ there is no $s_3^0 s_2^0$ -path, or in $G_h \setminus S'$ there is no $s_3^h s_1^h$ -path. Since $|S'| \leq p + d$, $|S' \cap E_0| \geq d$ and $|S' \cap E_h| \geq d$, we infer that $|S' \cap E_0| \leq p$ and $|S' \cap E_h| \leq p$. Consequently, either I_0 or I_h is a YES-instance, which finishes the proof of Theorem 5.3.

6 Alternative cross-composition of Multicut

In this section we present an alternative proof of cross-composition to EDGE MULTICUT parameterized by the size of the cutset. Despite the fact that the cross-composition presented in this section is more involved comparing to the one presented in Section 5, we find the ideas used here more general, which might be helpful in designing future cross-compositions for other problems.

Theorem 6.1. PROMISED MULTIWAY CUT cross-composes into EDGE MULTICUT parameterized the size of the cutset p.

Proof. We start by defining a relation $\mathcal R$ on PROMISED MULTIWAY CUT instances, which groups instances according to the size of the cutset p. Formally (G,T,p) is in relation $\mathcal R$ with (G',T',p') iff p=p'. Clearly, $\mathcal R$ is a polynomial equivalence relation. Hence we assume that we are given $t\geq 1$ instances $I_i=(G_i,T_i=\{s_1^i,s_2^i,s_3^i\},p)$, for $1\leq i\leq t$, of the PROMISED MULTIWAY CUT problem (note that we number instances starting from 1).

Construction. Let M=p+1 and $M_{\infty}=6M+p+1$. In our construction each edge of EDGE MULTICUT instance will have one of three possible weights $\{1,M,M_{\infty}\}$. We can implement those weights by putting 1,M or M_{∞} parallel edges and subdividing them to obtain a simple graph (note that both M and M_{∞} are polynomially bounded in p). Initially as the graph G' we take a disjoint union of two cycles C_1, C_2 , each containing exactly 3(t+1) vertices. To simplify presentation, each vertex on each of the two cycles has two different names, that is $C_j=(x_0^j,x_1^j,\ldots,x_t^j,y_1^j,y_2^j,\ldots,y_t^j,y_0^j,z_0^j,z_1^j,\ldots,z_t^j,x_0^j)$ for j=1,2, and at the same time $C_j=(c_0^j,\ldots,c_{3t+2}^j,c_0^j)$, where $c_0^j=x_0^j$ (see Fig. 3); note the position of y_0^j between y_t^j and z_0^j in favor of a uniform adjacency to the instances later. We set weights of each edge on the cycle as M, except for three edges $z_t^jx_0^j, x_t^jy_1^j, y_0^jz_0^j$, which have weight M_{∞} .

For each $i=1,\ldots,t$ we add to G' the graph G_i (with edges of weight 1), and connected s_1^i with y_i^1 by an edge of weight M_{∞} , and also add an edge of weight M_{∞} between s_2^i and x_i^2 (see Fig. 3). To the set $\mathcal T$ we add the following pairs:

- 1. for each j=1,2 and $i=0,\ldots,3t+2$, add to $\mathfrak T$ the pair $(c_i^j,c_{(i+t+1) \bmod 3t+3}^j)$,
- 2. for each $i=1,\ldots,t$ add to \mathcal{T} every pair of vertices from the set $\{s_3^i,x_i^1,y_i^2\}$.

Finally as the target cutset size we set p' = 6M + p, which is polynomially bounded in p. Our constructed instance of EDGE MULTICUT is $I' = (G', \Upsilon', p')$.

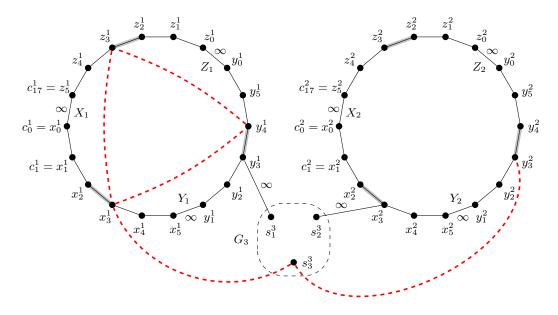


Figure 3: Construction of the graph G' for t=5. Dashed edges represent some of the pairs in the set \mathcal{T} , in particular the pair $(x_3^1,y_3^2)\in\mathcal{T}$, but it is not depicted to simplify the figure. Gray-marked edges belong to the set S' constructed in the proof of Theorem 6.1.

Analysis. First let us assume, that there is an index $1 \leq i_0 \leq t$, such that I_{i_0} is a YES-instance, and let S by any solution for I_{i_0} . We show that I' is also a YES-instance. Let S' := S and add to S' the 6 edges $c^j_{i_0-1+r(t+1)}c^j_{i_0+r(t+1)}$ for j=1,2 and r=0,1,2 (see gray-marked edges in Fig. 3). Note that each of those 6 edges has weight exactly M. We claim that S' is a solution for I'. First, observe that the total weight of edges in S' is at most 6M+p=p'. Let us analyze how connected components in $G'\setminus S'$ look like. For $1\leq j\leq 2$, let X_j,Y_j,Z_j be the connected components of $G'\setminus S'$ (not necessarily pairwise different), containing the edges $z^j_t x^j_0, x^j_t y^j_1$, and $y^j_0 z^j_0$ respectively. Note that those are the edges of weight M_∞ and they do not belong to S'. Next consider each graph G_i , for $i=1,\ldots,t$, and see which parts of the cycles C_1, C_2 it connects. For $1\leq i< i_0$, the graph G_i connects Y_1 with X_2 . For $i_0< i\leq t$, the graph G_i connects Z_1 with Y_2 . Finally the graph $G_{i_0}\setminus S'$ does not connect any connected components, since in G_i and G_i each of the three terminals of G_i is in a different connected component. Therefore we know that for each G_i each of the three terminals of G_i is in a different connected components G_i . On are pairwise different. We analyze all the pairs in the set G_i and argue that the two elements of each pair are in different connected components in G_i of G_i . Consider any G_i are pairwise different, G_i be the connected components G_i . So consider any G_i are pairwise different, G_i are pairwise di

and hence for $i=0,\ldots,3t+2$ the vertex c_i^j is in a different connected component that $c_{(i+t+1) \bmod 3t+3}^j$ in $G'\setminus S'$. Therefore it is enough to analyze pairs $(s_3^i,x_i^1),(s_3^i,y_i^2),(x_i^1,y_i^2)\in \mathfrak{T}$ for $i=1,\ldots,t$.

- 1. For $1 \le i < i_0$, we have $s_3^i \in Y_1 = X_2$, while $x_i^1 \in X_1$ and $y_i^2 \in Y_2 = Z_1$.
- 2. For $i_0 < i \le t$, we have $s_3^i \in Z_1 = Y_2$, while $x_i^1 \in Y_1 = X_2$ and $y_i^2 \in Z_2$.
- 3. For $i=i_0$, the vertex s_3^i is in different connected component than all the vertices in the cycles C_1 , C_2 , since in $G_{i_0}\setminus S'$ there are no $s_3^is_1^i$ -paths, nor $s_3^is_2^i$ -paths. At the same time $x_1^1\in Y_1=X_2$ and $y_i^2\in Y_2=Z_1$.

Consequently for each pair $(s,t) \in \mathcal{T}$, vertices s and t are in different connected components of $G' \setminus S'$, therefore I' is a YES-instance.

In the other direction, let us assume that I' is a YES-instance and let S' be any solution for I'. Observe that out of each t+1 consecutive edges on a cycle C_j , for j=1,2, the set S' has to contain at least one edge, since otherwise there would be a pair of vertices $(c_i^j, c_{(i+1+1) \bmod 3t+3}^j) \in \mathfrak{T}$ belonging to the same connected component of $G' \setminus S'$. However, $|C_j| = 3(t+1)$ and therefore the set S' contains at least three edges of C_j . Moreover 7M > p', hence S' contains exactly three equidistant edges out of each cycle C_j , since otherwise there would be t+1 consecutive edges not belonging to S'. Since $M_\infty > p'$ there are exactly t layouts of three equidistant edges in each C_j which do not contain any edge of weight M_∞ . Observe that because of the way we labeled vertices on each cycle, there exists an index i_j , such that x_i^j and y_i^j are in the same connected component of $G' \setminus S'$, namely $1 \le i_j \le t$, such that S' contains the edge $x_{i_j-1}^j x_{i_j}^j$.

We have to consider two cases, either $i_1 = i_2$ or $i_1 \neq i_2$. Let $S'' \subseteq S'$ be the subset of edges of S' of weight 1. Note that $|S''| \leq p$.

Case $\mathbf{1}$ $(i_1 \neq i_2)$. Since $(s_3^{i_1}, x_{i_1}^1) \in \mathcal{T}$, and by the fact that $x_{i_1}^1$ and $y_{i_1}^1$ are in the same connected component of $G' \setminus S'$, there is no $s_3^{i_1} s_1^{i_1}$ -path in $G_i \setminus S''$. By properties (i), (ii), (iii) of PROMISED MULTIWAY CUT problem definition, we infer $|S'' \cap E(G_{i_1})| > p/2$, since for $d = \deg_{G_i}(s_1^{i_1})$ we have $|S'' \cap E(G_{i_1})| \geq d$ and 2d > p. Analogously, since $(s_3^{i_2}, y_{i_2}^2) \in \mathcal{T}$, we have $|S'' \cap E(G_{i_2})| > p/2$, but then |S''| > p, a contradiction.

Case 2 $(i_1=i_2)$. Let $i_0=i_1=i_2$. By the definition of \mathfrak{T} , each pair of vertices of $\{s_3^{i_0},x_{i_0}^1,y_{i_0}^2\}$ belongs to different connected component of $G'\setminus S'$. Observe that $s_1^{i_0},y_{i_0}^1,x_{i_0}^1$ are in the same connected component of $G'\setminus S'$, since the edge $s_1^{i_0}y_{i_0}^1$ is of weight M_∞ , and for this reason it does not belong to S'. Similarly $s_2^{i_0},x_{i_0}^2,y_{i_0}^2$ are in the same connected component of $G'\setminus S'$. Therefore there is no path between any pair of vertices of T_{i_0} in T_{i_0} 0, T_{i_0} 1, and since T_{i_0} 2 we infer that T_{i_0} 3 a YES-instance.

7 k-Way Cut

In this section we study the following graph separation problem.

k-Way Cut

Input: An undirected connected graph G and integers k and s.

Task: Does there exist a set X of at most s edges in G such that $G \setminus X$ has at least k connected components?

The k-WAY CUT problem, parameterized by s, was proven to be fixed-parameter tractable by Kawarabayashi and Thorup [55]. The problem is W[1]-hard when parameterized by k [32], as well as when we allow vertex deletions instead of edge deletions, and parameterize by s [64].

Note that in the problem definition we assume that the input graph is connected and, therefore, for k > s+1 the input instances are trivial. However, if we are given an instance (G,k,s) where G has c > 1 connected components, we can easily reduce it to the connected version: we add to G a complete graph on s+2 vertices (so that no two vertices of the complete graph can be separated by a cut of size s), connect one vertex from each connected component of G to all vertices of the complete graph, and decrease k by c-1. Thus, by restricting ourselves to connected graphs G we do not make the problem easier.

The main result of this section is that k-WAY CUT, parameterized by s, does not admit a polynomial kernel (unless NP \subseteq coNP/poly). We show a cross-composition from the CLIQUE problem, well-known to be NP-complete.

CLIQUE

Input: An undirected graph G and an integer ℓ .

Task: Does G contain a clique on ℓ vertices as a subgraph?

Theorem 7.1. CLIQUE cross-composes to k-WAY CUT parameterized by s.

Proof. We start by defining a relation \mathcal{R} on CLIQUE input instances as follows: (G,ℓ) is in relation \mathcal{R} with (G',ℓ') if $\ell=\ell', |V(G)|=|V(G')|$ and |E(G)|=|E(G')|. Clearly, \mathcal{R} is a polynomial equivalence relation. Thus, in the designed cross-composition, we may assume that we are given t instances (G_i,ℓ) $(1 \le i \le t)$ of the CLIQUE problem and $|V(G_i)|=n, |E(G_i)|=m$ for all $1 \le i \le t$. Moreover, we assume that $m \ge {\ell \choose 2}$ and $1 < \ell \le n$, as otherwise all input instances are trivial.

We first consider a weighted version of the k-WAY CUT problem where each edge may have a positive integer weight and the cutset X needs to be of total weight at most s. The weights in our construction are polynomial in n and m. At the end we show how to reduce the weighted version to the unweighted one.

We start by defining $k = n - \ell + 1$, $w_1 = m$, $w_2 = m \binom{n}{2}$ and $s = w_2(n - \ell) + w_1 \left(\binom{n}{2} - \binom{\ell}{2} \right) + m - \binom{\ell}{2}$. Note that $s < w_2(n - \ell + 1)$ and $s < w_2(n - \ell) + w_1 \binom{n}{2} - \binom{\ell}{2} + 1$.

that $s < w_2(n-\ell+1)$ and $s < w_2(n-\ell) + w_1(\binom{n}{2} - \binom{\ell}{2} + 1)$. For each graph G_i , $1 \le i \le t$, we define a graph G_i' as a complete graph on n vertices with vertex set $V(G_i)$, where the edge uv has weight $w_1 + 1$ if $uv \in E(G_i)$ and weight w_1 otherwise. We construct a graph G as follows. We take a disjoint union of all graphs G_i' for $1 \le i \le t$, add a root vertex r and for each $1 \le i \le t$, $v \in V(G_i')$ we add an edge vv of weight vv.

Clearly G is connected, s is polynomial in n and m and the graph G can be constructed in polynomial time. We claim that (G, k, s) is a weighted k-WAY CUT YES-instance if and only if one of the input CLIQUE instances (G_i, ℓ) is a YES-instance.

First, assume that for some $1 \leq i \leq t$, the CLIQUE instance (G_i,ℓ) is a YES-instance. Let $C \subseteq V(G_i)$ be a witness: $|C| = \ell$ and $G_i[C]$ is a clique. Consider a set $X \subseteq E(G)$ containing all edges of G incident to $V(G_i') \setminus C$. Clearly, $G \setminus X$ contains $k = n - \ell + 1$ connected components: we have one large connected component with vertex set $(V(G) \setminus V(G_i')) \cup C$ and each of $n - \ell$ vertices of $V(G_i') \setminus C$ is an isolated vertex in $G \setminus X$. Let us now count the total weight of edges in X. X contains $n - \ell$ edges of weight w_2 that connect $V(G_i') \setminus C$ to the root r. Moreover, X contains $\binom{n}{2} - \binom{\ell}{2}$ edges of G_i' , of weight w_1 or $w_1 + 1$. Since $G_i[C]$ is a clique, only $m - \binom{\ell}{2}$ of the edges in X are of weight $w_1 + 1$. Thus the total weight of edges in X is equal to $w_2(n - \ell) + w_1\left(\binom{n}{2} - \binom{\ell}{2}\right) + m - \binom{\ell}{2} = s$.

In the other direction, let $X \subseteq E(G)$ be a solution to the k-WAY CUT instance (G,k,s). Let Z be the connected component of $G \setminus X$ that contains the root r. Let $Y \subseteq V(G)$ be the set of vertices that are not in Z. If $v \in Y, X$ contains the edge rv of weight w_2 . As $s < w_2(n-\ell+1)$, we have $|Y| \le n-\ell$. As $k=n-\ell+1$, we infer that $G \setminus X$ contains $n-\ell+1$ connected components: Z and $n-\ell$ isolated vertices. That is, $|Y|=n-\ell$ and all vertices in Y are isolated in $G \setminus X$. Note that X includes $n-\ell$ edges of weight w_2 that connect the root r with the vertices of Y.

The next step is to prove that all vertices of Y are contained in one of the graphs G_i' . To this end, let $a_i = |Y \cap V(G_i')|$ for $1 \le i \le t$. Note that $X \cap E(G_i')$ contains at least $\binom{a_i}{2} + a_i(n-a_i)$ edges of weight w_1 or $w_1 + 1$. Thus, the number of edges of weight w_1 or $w_1 + 1$ contained in X is at least:

$$\sum_{i=1}^{t} {a_i \choose 2} + a_i (n - a_i) = \left(n - \frac{1}{2}\right) \sum_{i=1}^{t} a_i - \frac{1}{2} \sum_{i=1}^{t} a_i^2 = (n - \ell) \left(n - \frac{1}{2}\right) - \frac{1}{2} \sum_{i=1}^{t} a_i^2$$

$$\geq (n - \ell) \left(n - \frac{1}{2}\right) - \frac{1}{2} \left(\sum_{i=1}^{t} a_i\right)^2 = (n - \ell) \left(n - \frac{1}{2}\right) - \frac{1}{2} (n - \ell)^2 = {n \choose 2} - {\ell \choose 2}.$$

As $s < w_2(n-\ell) + w_1\left(\binom{n}{2} - \binom{\ell}{2} + 1\right)$, we infer that the number of edges in X of weight w_1 or $w_1 + 1$ is exactly $\binom{n}{2} - \binom{\ell}{2}$. This is only possible if $\sum_{i=1}^t a_i^2 = (\sum_{i=1}^t a_i)^2$. As a_i are nonnegative integers, we infer that only one value a_i is positive.

Thus $Y \subseteq V(G_i')$ for some $1 \le i \le t$. Let $C = V(G_i) \setminus Y$. Note that $|C| = \ell$. The set X contains all $\binom{n}{2} - \binom{\ell}{2}$ edges of G_i' that are incident to Y. As the total weight of the edges of X is at most s, X contains at most $m - \binom{\ell}{2}$ edges of weight $w_1 + 1$. We infer that there are at least $\binom{\ell}{2}$ edges in the graph $G_i[C]$, $G_i[C]$ is a clique and (G_i, ℓ) is a YES-instance of the CLIQUE problem.

To finish the proof, we show how to reduce the weighted version of the k-WAY CUT problem to the unweighted one. We replace each vertex u with a complete graph H_u on s+2 vertices and for each edge uv of weight w we add to the graph w arbitrarily chosen edges between H_u and H_v (note that in our construction all weights are smaller than s). Note that this reduction preserves the connectivity of the graph G. Let X be a solution to the unweighted instance (G, k, s) constructed in this way. As no cut of size at most s can separate two vertices of H_u , each clique H_u is contained in one connected component of $G \setminus X$. Moreover, to separate H_u from H_v , X needs to include all w edges between H_u and H_w . Thus, the constructed unweighted instance is indeed equivalent to the weighted one. Note that in the presented cross-composition the edge weights were polynomial in n and m, so the presented reduction can be performed in polynomial time.

By applying Theorem 2.3 we obtain the following corollary.

Corollary 7.2. k-WAY CUT parameterized by s does not admit a polynomial kernel unless $NP \subseteq coNP/poly$.

8 Conclusion and open problems

We have shown that four important parameterized problems do not admit a kernelization algorithm with a polynomial guarantee on the output size unless $NP \subseteq coNP/poly$ and the polynomial hierarchy collapses. We would like to mention here a few open problems very closely related to our work.

- The 2^k -vertex kernel for EDGE CLIQUE COVER [42] is probably close to optimal. Currently the fastest fixed-parameter algorithm for EDGE CLIQUE COVER is a brute-force algorithm on the exponential kernel. Is this double-exponential dependency on k necessary?
- The OR-composition for DIRECTED MULTIWAY CUT in the case of two terminals excludes the existence of a polynomial kernel for most graph separation problems in directed graphs. There are two important cases not covered by this result: one is the MULTICUT problem in directed acyclic graphs, and the second is DIRECTED MULTIWAY CUT with deletable terminals. To the best of our knowledge, it is also open whether the first problem is fixed-parameter tractable.
- Both our OR-compositions for MULTICUT use a number of terminal pairs that is linear in the number of input instances. Is MULTICUT parameterized by both the size of the cutset and the number of terminal pairs similarly hard to kernelize?

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