Implementing Computations in Automaton (Semi)groups

Ines Klimann, Jean Mairesse, and Matthieu Picantin

Univ Paris Diderot, Sorbonne Paris Cité, LIAFA, UMR 7089 CNRS, Paris, France {klimann,mairesse,picantin}@liafa.univ-paris-diderot.fr

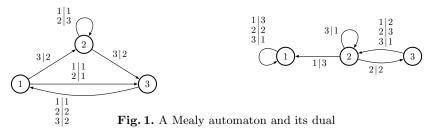
Abstract. We consider the growth, order, and finiteness problems for automaton (semi)groups. We propose new implementations and compare them with the existing ones. As a result of extensive experimentations, we propose some conjectures on the order of finite automaton (semi)groups.

Keywords: automaton (semi)groups, growth, order, finiteness, minimization

1 Introduction

Automaton (semi)groups — short for semigroups generated by Mealy automata or groups generated by invertible Mealy automata — were formally introduced a half century ago (for details, see [10,7] and references therein). Over the years, important results have started revealing their full potential. For instance, the article [9] constructs simple Mealy automata generating infinite torsion groups and so contributes to the Burnside problem, and, the article [5] produces Mealy automata generating the first examples of (semi)groups with intermediate growth and so answers the Milnor problem.

The classical decision problems have been investigated for such (semi)groups. The word problem is solvable using standard minimization techniques, while the conjugacy problem is undecidable [16]. Here we concentrate on the problems related to growth, order, and finiteness.



To illustrate, consider the two Mealy automata of Fig. 1. They are dual, that is, they can be obtained one from the other by exchanging the roles of stateset and alphabet. A (semi)group is associated in a natural way with each automaton

(formally defined below). The two Mealy automata of Fig. 1 are associated with finite (semi)groups. Their orders are respectively: on the left a semigroup of order 238, on the right a group of order 1 494 186 269 970 473 680 896 = $2^{64} \cdot 3^4 \approx 1.5 \times 10^{21}$.

Several points are illustrated by this example:

- An automaton and its dual generate (semi)groups which are either both finite or both infinite (see [12,2]).
- The order of a finite automaton (semi)group can be amazingly large. It makes
 a priori difficult to decide whether an automaton (semi)group is finite or not.
 Actually, the decidability of this question is open (see [10,2]).
- The order of the (semi)groups generated by a Mealy automaton and its dual can be strikingly different. It suggests to work with both automata together.

The contributions of the present paper are three-fold:

- We propose new implementations (in GAP [8]) of classical algorithms for the computation of the growth function; the computation of the order (if finite); the semidecision procedure for the finiteness.
- We compare the new implementations with the existing ones. Indeed, there exist two GAP packages dedicated to Mealy automata and their associated (semi)groups: FR by Bartholdi [4] and automgrp by Muntyan and Savchuk [11]
- We realize systematic experimentations on small Mealy automata as well as randomly chosen large Mealy automata. These serve as testbeds to some conjectures on the growth types of the associated (semi)groups, as well as on the order of a (semi)group.

The structure of the paper is the following. In Section 2, we present basic notions on Mealy automata and automaton (semi)groups. In Section 3, we give new implementations and compare them with the existing ones. Section 4 is dedicated to experimentations and to the resulting conjectures.

2 Automaton (Semi)groups

2.1 Mealy Automaton

If one forgets initial and final states, a *(finite, deterministic, and complete) automaton* \mathcal{A} is a triple $(A, \Sigma, \delta = (\delta_i : A \to A)_{i \in \Sigma})$, where the set of states A and the alphabet Σ are non-empty finite sets, and where the δ_i 's are functions.

A *Mealy automaton* is a quadruple

$$(A, \Sigma, \delta = (\delta_i : A \to A)_{i \in \Sigma}, \rho = (\rho_x : \Sigma \to \Sigma)_{x \in A}),$$

such that both (A, Σ, δ) and (Σ, A, ρ) are automata. In other terms, a Mealy automaton is a letter-to-letter transducer with the same input and output alphabets. The transitions of a Mealy automaton are

$$x \xrightarrow{i|\rho_x(i)} \delta_i(x)$$
.

The graphical representation of a Mealy automaton is standard, see Fig. 1.

The notation $x \xrightarrow{\mathbf{u}|\mathbf{v}} y$ with $\mathbf{u} = u_1 \cdots u_n$, $\mathbf{v} = v_1 \cdots v_n$ is a shorthand for the existence of a path $x \xrightarrow{u_1|v_1} x_1 \xrightarrow{u_2|v_2} x_2 \longrightarrow \cdots \longrightarrow x_{n-1} \xrightarrow{u_n|v_n} y$ in \mathcal{A} .

In a Mealy automaton $(A, \Sigma, \delta, \rho)$, the sets A and Σ play dual roles. So we may consider the dual (Mealy) automaton defined by $\mathfrak{d}(A) = (\Sigma, A, \rho, \delta)$, that is:

$$i \xrightarrow{x|y} j \in \mathfrak{d}(\mathcal{A}) \iff x \xrightarrow{i|j} y \in \mathcal{A}.$$

It is pertinent to consider a Mealy automaton and its dual together, that is to work with the pair $\{A, \mathfrak{d}(A)\}$, see an example in Fig. 1.

Let $\mathcal{A}=(A,\Sigma,\delta,\rho)$ and $\mathcal{B}=(B,\Sigma,\gamma,\pi)$ be two Mealy automata acting on the same alphabet; their *product* $\mathcal{A}\times\mathcal{B}$ is defined as the Mealy automaton with stateset $A\times B$, alphabet Σ , and transitions:

$$xy \xrightarrow{i|\pi_y(\rho_x(i))} \delta_i(x)\gamma_{\rho_x(i)}(y)$$
.

2.2 Generating (Semi)groups

Let $\mathcal{A} = (A, \Sigma, \delta, \rho)$ be a Mealy automaton. We view \mathcal{A} as an automaton with an input and an output tape, thus defining mappings from input words over Σ to output words over Σ . Formally, for $x \in A$, the map $\rho_x : \Sigma^* \to \Sigma^*$, extending $\rho_x : \Sigma \to \Sigma$, is defined by:

$$\rho_x(\mathbf{u}) = \mathbf{v} \quad \text{if} \quad \exists y, \ x \xrightarrow{\mathbf{u} \mid \mathbf{v}} y.$$

By convention, the image of the empty word is itself. The mapping ρ_x is length-preserving and prefix-preserving. It satisfies

$$\forall u \in \Sigma, \ \forall \mathbf{v} \in \Sigma^*, \qquad \rho_x(u\mathbf{v}) = \rho_x(u)\rho_{\delta_v(x)}(\mathbf{v}).$$

We say that ρ_x is the production function associated with (\mathcal{A}, x) . For $\mathbf{x} = x_1 \cdots x_n \in A^n$ with n > 0, set $\rho_{\mathbf{x}} : \Sigma^* \to \Sigma^*, \rho_{\mathbf{x}} = \rho_{x_n} \circ \cdots \circ \rho_{x_1}$. Denote dually by $\delta_i : A^* \to A^*, i \in \Sigma$, the production mappings associated with the dual Mealy automaton $\mathfrak{d}(\mathcal{A})$. For $\mathbf{v} = v_1 \cdots v_n \in \Sigma^n$ with n > 0, set $\delta_{\mathbf{v}} : A^* \to A^*, \ \delta_{\mathbf{v}} = \delta_{v_n} \circ \cdots \circ \delta_{v_1}$.

Definition 1. Consider a Mealy automaton A. The semigroup of mappings from Σ^* to Σ^* generated by $\rho_x, x \in A$, is called the semigroup generated by A and is denoted by $\langle A \rangle_+$. A semigroup G is an automaton semigroup if there exists a Mealy automaton A such that $G = \langle A \rangle_+$.

A Mealy automaton $\mathcal{A}=(A,\Sigma,\delta,\rho)$ is invertible if all the mappings $\rho_x:\Sigma\to\Sigma$ are permutations. Then the production functions $\rho_x:\Sigma^*\to\Sigma^*$ are invertible.

Definition 2. Let $\mathcal{A} = (A, \Sigma, \delta, \rho)$ be invertible. The group generated by \mathcal{A} is the group generated by the mappings $\rho_x : \Sigma^* \to \Sigma^*$, $x \in A$. It is denoted by $\langle \mathcal{A} \rangle$.

Let $\mathcal{A} = (A, \Sigma, \delta, \rho)$ be an invertible Mealy automaton. Its *inverse* is the Mealy automaton \mathcal{A}^{-1} with stateset $A^{-1} = \{x^{-1}, x \in A\}$ and set of transitions

$$x^{-1} \xrightarrow{j|i} y^{-1} \in \mathcal{A}^{-1} \quad \Longleftrightarrow \quad x \xrightarrow{i|j} y \in \mathcal{A} .$$

A Mealy automaton is reversible if its dual is invertible. A Mealy automaton \mathcal{A} is bireversible if both \mathcal{A} and \mathcal{A}^{-1} are invertible and reversible.

Theorem 1 ([2,12,13]). The (semi)group generated by a Mealy automaton is finite if and only if the (semi)group generated by its dual is finite.

2.3 Minimization and the Word Problem

Let $\mathcal{A} = (A, \Sigma, \delta, \rho)$ be a Mealy automaton. The Nerode equivalence on A is the limit of the sequence of increasingly finer equivalences (\equiv_k) recursively defined by:

$$\forall x, y \in A,$$
 $x \equiv_0 y \iff \rho_x = \rho_y,$ $\forall k \geqslant 0, x \equiv_{k+1} y \iff x \equiv_k y \text{ and } \forall i \in \Sigma, \ \delta_i(x) \equiv_k \delta_i(y).$

Since the set A is finite, this sequence is ultimately constant; moreover if two consecutive equivalences are equal, the sequence remains constant from this point. The limit is therefore computable. For every element x in A, we denote by [x] the class of x w.r.t. the Nerode equivalence.

Definition 3. Let $\mathcal{A} = (A, \Sigma, \delta, \rho)$ be a Mealy automaton and let Ξ be the Nerode equivalence on A. The minimization of \mathcal{A} is the Mealy automaton $\mathfrak{m}(\mathcal{A}) = (A/\Xi, \Sigma, \tilde{\delta}, \tilde{\rho})$, where for every (x, i) in $A \times \Sigma$, $\tilde{\delta}_i([x]) = [\delta_i(x)]$ and $\tilde{\rho}_{[x]} = \rho_x$.

This definition is consistent with the standard minimization of "deterministic finite automata" where instead of considering the mappings $(\rho_x : \Sigma \to \Sigma)_x$, the computation is initiated by the separation between terminal and non-terminal states. Using Hopcroft algorithm, the time complexity of minization is $\mathcal{O}(\Sigma A \log A)$, see [1].

By construction, a Mealy automaton and its minimization generate the same semigroup. Indeed, two states of a Mealy automaton belong to the same class w.r.t the Nerode equivalence if and only if they represent the same element in the generated (semi)group.

Consider the word problem:

Input: a Mealy automaton
$$(A, \Sigma, \delta, \rho)$$
; $\mathbf{x}, \mathbf{y} \in A^*$. Question: $(\rho_{\mathbf{x}} : \Sigma^* \to \Sigma^*) = (\rho_{\mathbf{y}} : \Sigma^* \to \Sigma^*)$?

The word problem is solvable by extending the above minimization procedure. FR uses this approach, while automgrp uses a method based on the wreath recursion [7].

3 Fully Exploiting the Minimization

Consider the following problems for the (semi)group given by a Mealy automaton: compute the growth function, compute the order (if finite), detect the finiteness. The packages FR and automgrp provide implementations of the three problems. Here we propose new implementations based on a simple idea which fully uses the automaton structure.

3.1 Growth

Consider a Mealy automaton $\mathcal{A} = (A, \Sigma, \delta, \rho)$ and an element $\mathbf{x} \in A^*$. The *length* of $\rho_{\mathbf{x}}$, denoted by $|\rho_{\mathbf{x}}|$, is defined as follows:

$$|\rho_{\mathbf{x}}| = \min\{n \mid \exists \mathbf{y} \in A^n, \, \rho_{\mathbf{x}} = \rho_{\mathbf{y}}\}.$$

The growth series of A is the formal power series given by

$$\sum_{g \in \langle \mathcal{A} \rangle_+} t^{|g|} = \sum_{n \in \mathbb{N}} \# \{g \in \langle \mathcal{A} \rangle_+ \, ; \, |g| = n \} \; t^n \; .$$

In words, the growth series enumerates the semigroup elements according to their length. This is an instanciation of the notion of spherical growth series for a finitely generated semigroup. Observe that the series is a polynomial if and only if the semigroup is finite.

Using the Generic Algorithm. Since the word problem is solvable, it is possible to compute an arbitrary but finite number of coefficients of the growth series. Indeed for each n, generate the set of elements of length n by multiplying elements of length n-1 with generators and detecting-deleting duplicated elements by solving the word problem. The functions Growth from automorp and WordGrowth from FR both follow this pattern. Therefore the structure of the underlying Mealy automaton is used only to get a solution to the word problem (in fact, both Growth and WordGrowth are generic, in the sense that they are applicable for any (semi)group with an implemented solution to the word problem).

New Implementation. We propose a new implementation based on a simple observation. Knowing the elements of length n-1, Nerode minimization can be used in a global manner to obtain simultaneously the elements of length n. Concretely, with each integer $n \geq 1$ is associated a new Mealy automaton \mathcal{A}_n defined recursively as follows:

$$A_n = \mathfrak{m}(A_{n-1} \times \mathfrak{m}(A))$$
 and $A_1 = \mathfrak{m}(A)$.

Here, we assume, without real loss of generality, that the identity element is one of the generators (otherwise simply add a new state to the Mealy automaton coding the identity). This way, the elements of \mathcal{A}_n are exactly the elements of length at most n.

```
AutomatonGrowth := function(arg)
local aut, radius, growth, sph, curr, next, r;
aut:=arg[1];
              # Mealy automaton
if Length(arg)>1 then radius:=arg[2];
                 else radius:=infinity;
fi;
r := 0;
           curr := TrivialMealyMachine([1]);
next := Minimized(aut);
aut := Minimized(next+TrivialMealyMachine(Alphabet(aut)));
sph := aut!.nrstates - 1; # number of non-trivial states
growth := [next!.nrstates-sph];
while sph>0 and r<radius
do Add(growth,sph);
    r := r+1;
                 curr := next;
    next := Minimized(next*aut);
    sph := next!.nrstates-curr!.nrstates;
od;
return growth;
end;
```

Note that AutomatonGrowth(aut) computes the growth of the semigroup $\langle \text{aut} \rangle_+$, while AutomatonGrowth(aut+aut^-1) computes the growth of the group $\langle \text{aut} \rangle$.

Experimental Results. First we run AutomatonGrowth and FR's WordGrowth on the Grigorchuk automaton, a famous Mealy automaton generating an infinite group. For radius 10, AutomatonGrowth is much faster, 76 ms as opposed to 9912 ms¹. The explanation is simple: WordGrowth calls the minimization procedure 57577 times while AutomatonGrowth calls it only 12 times. Here are the details.

```
gap> aut := GrigorchukMachine;; radius:= 10;;
gap> ProfileFunctions([Minimized]);
gap> WordGrowth(SCSemigroupNC(aut), radius); time;
[ 1, 4, 6, 12, 17, 28, 40, 68, 95, 156, 216 ]
9912
gap> DisplayProfile();
  count self/ms chld/ms function
            7712
                        0 Minimized
  57577
            7712
                           TOTAL
gap> ProfileFunctions([Minimized]);
gap > AutomatonGrowth(aut, radius); time;
[ 1, 4, 6, 12, 17, 28, 40, 68, 95, 156, 216 ]
gap> DisplayProfile();
  count self/ms chld/ms
                           function
     12
              72
                        0
                           Minimized
              72
                           TOTAL
```

¹ All timings displayed in this paper have been obtained on an Intel Core 2 Duo computer with clock speed 3,06 GHz.

Now we compare the running times of the implementations for the computation of the first terms of the growth series for all 335 bireversible 3-letter 3-state Mealy automata (up to equivalence). In Tab. 1, some computations with FR's WordGrowth or with automgrp's Growth could not be completed in reasonable time for radius 7.

Table 1. Average time (in ms)

radius	1	2	3	4	5	6	7
FR's WordGrowth	3.4	29.0	555.0	8616.5	131 091.4	2530170.3	?
automgrp's Growth	0.7	2.8	16.9	158.9	1 909.0	22952.8	?
AutomatonGrowth	0.6	1.8	5.9	28.9	187.3	1005.9	7131.4

3.2 Order of the (Semi)group

Although the finiteness problem is still open, some semidecision procedures enable to find the order of an expected finite (semi)group. FR and automgrp use orthogonal approaches. Our new implementation refines the one of FR and remains orthogonal to the one of automgrp.

automgrp's Implementation. The GAP package automgrp provides the function LevelOfFaithfulAction, which allows to compute—very efficiently in some cases—the order of the generated group. The principle is the following. Let $\mathcal{A} = (A, \Sigma, \delta, \rho)$ be an invertible Mealy automaton and let G_k be the group generated by the restrictions of the production functions to Σ^k . If $\#G_k = \#G_{k+1}$ for some k, then $\langle \mathcal{A} \rangle$ is finite of order $\#G_k$. This function can be easily adapted to a non-invertible Mealy automaton.

Observe that LevelOfFaithfulAction cannot be used to compute the growth series. Indeed at each step a quotient of the (semi)group is computed. On the other hand LevelOfFaithfulAction is a good bypass strategy for the order computation. Furthermore, it takes advantage from the special ability of GAP to manipulate permutation groups.

FR's Implementation and the New Implementation. Any algorithm computing the growth series can be used to compute the order of the generated (semi)group if finite. It suffices to compute the growth series until finding a coefficient equal to zero. This is the approach followed by FR. Since we proposed, in the previous section, a new implementation to compute the growth series, we obtain as a byproduct a new procedure to compute the order. We call it AutomSGrOrder.

Experimental Results. The orthogonality of the two previous approaches can be simply illustrated by recalling the introductory example of Fig. 1. Neither FR's Order nor AutomSGrOrder are able to compute the order of the large group, while automgrp via LevelOfFaithfulAction succeeds in only 14338 ms. Conversely, AutomSGrOrder computes the order of the small semigroup in 17 ms, while an adaptation of LevelOfFaithfulAction (to non-invertible Mealy automata) takes 2193 ms.

3.3 Finiteness

There exist several criteria to detect the finiteness of an automaton (semi)group, see [2,3,6,14,15, ...]. But the decidability of the finiteness is still an open question. Each procedure to compute the order of a (semi)group yields a semidecision procedure for the finiteness problem. Both packages FR and automgrp apply a number of previously known criteria of (in)finiteness and then intend to conclude by ultimately using an order computation.

We propose an additional ingredient which uses minimization in a subtle way. Here, the semigroup to be tested is successively replaced by new ones which are finite if and only if the original one is finite. It is possible to incorporate this ingredient to get two new implementations, one in the spirit of FR and one in the spirit of automgrp. The new implementations are order of magnitudes better than the old ones. Both are useful since the fastest one depends on the cases.

3.3.1 mo-reduction of Mealy Automata and Finiteness

The mo-reduction was introduced in [2] to give a sufficient condition of finiteness. The new semidecision procedures start with this reduction.

Definition 4. A pair of dual Mealy automata is reduced if both automata are minimal. Recall that \mathfrak{m} (resp. \mathfrak{d}) is the operation of minimization (resp. dualization). The \mathfrak{md} -reduction of a Mealy automaton \mathcal{A} consists in minimizing the automaton or its dual until the resulting pair of dual Mealy automata is reduced.

The mo-reduction is well-defined: if both a Mealy automaton and its dual automaton are non-minimal, the reduction is confluent [2]. An example of mo-reduction is given in Fig. 2.

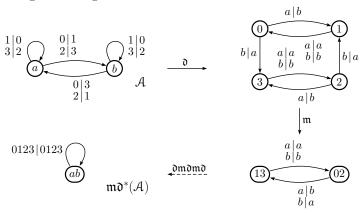


Fig. 2. The $\mathfrak{md}\text{-}\mathrm{reduction}$ of a pair of dual Mealy automata

The sequence of minimization-dualization can be arbitrarily long: the minimization of a Mealy automaton with a minimal dual can make the dual automaton non-minimal.

If \mathcal{A} is a Mealy automaton, we denote by $\mathfrak{md}^*(\mathcal{A})$ the corresponding Mealy automaton after \mathfrak{md} -reduction.

Theorem 2 ([2]). A Mealy automaton \mathcal{A} generates a finite (semi)group if and only if $\mathfrak{md}^*(\mathcal{A})$ generates a finite (semi)group.

This is the starting point of the new implementations. We use an additional fact. We can prune a Mealy automaton by deleting the states which are not accessible from a cycle. This does not change the finiteness or infiniteness of the generated (semi)group [3].

3.3.2 The New Implementations

The design of procedure IsFinite1 is consistent with the one of AutomatonGrowth. Hence IsFinite1 is much closer to FR than to automgrp. Here we propose a version that works with the automaton and its dual in parallel.

```
IsFinite1 := function (aut, limit)
local radius, dual, curr1, next1, curr2, next2;
radius := 0;
aut := MDReduced(Prune(aut));     dual := DualMachine(aut);
curr1 := MealyMachine([[1]],[()]); curr2 := curr1;
next1 := aut;
                next2 := dual;
while curr2!.nrstates <> next2!.nrstates and radius < limit
do radius := radius + 1;
                            curr1 := next1;
    next1 := Minimized(next1*aut);
    if curr1!.nrstates <>next1!.nrstates
    then curr2 := next2;
          next2 := Minimized(next2*dual);
    else return true;
    fi;
od;
if curr2!.nrstates = next2!.nrstates then return true; fi;
return fail;
end;
```

The procedure IsFinite2 is a refinement of automgrp's LevelOfFaithfulAction: the minimization is called on the dual and can be enhanced again to work in parallel on the Mealy automaton and its dual.

Experimental Results. Tab. 2 presents the average time to detect finiteness of (semi)groups generated by p-letter q-state invertible or reversible Mealy automata with $p+q \in \{5,6\}$. To get a fair comparison of the implementations, what is given is the minimum of the running times for an automaton and its dual (see Theorem 1).

Table 2. Average time (in ms) to detect finiteness of (semi)groups

	2-	- 3-		2- 4-			3- 3-				
FR	aut	Fin1	Fin2	FR	aut	Fin1	Fin2	FR	aut	Fin1	Fin2
0.68	0.81	0.49	0.49	36.36	1.79	0.52	0.62	1342.12	3.78	0.61	0.70

FR: FR's IsFinite; aut: automgrp's IsFinite; Fin1: IsFinite1; Fin2: IsFinite2

4 Conjectures

The efficiency of the new implementations enables to carry out extensive experimentations. We propose several conjectures supported by these experiments.

Recall the example given in the introduction. The (semi)groups generated by the Mealy automaton and its dual were strikingly different, with a very large one and a rather small one. This seems to be a general fact that we can state as an informal conjecture:

Whenever a Mealy automaton generates a finite (semi)group which is very large with respect to the number of states and letters of the automaton, then its dual generates a small one.

Observation: Any pair of finite (semi)groups can be generated by a pair of dual Mealy automata, see [2, Prop. 9]. The standard construction leads to automata whose sizes are related to the orders of the (semi)groups. Therefore it does not contradict the informal conjecture.

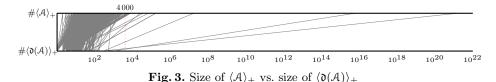


Fig. 3 illustrates this informal conjecture: for \mathcal{A} covering the set of all 3-letter 3-state invertible Mealy automata, the endpoints of each segment represent respectively the order of $\langle \mathcal{A} \rangle_+$ and of $\langle \mathfrak{d}(\mathcal{A}) \rangle_+$, for all pairs detected as being finite. To assess finiteness, the procedures IsFinite1 and IsFinite2 have been used. If the tested Mealy automaton and its dual were both found to have more than 4000 elements, the procedures were stopped, and the (semi)groups were supposed to be infinite. Based on the informal conjecture, we believe to have captured all finite groups. If true:

- There are 14 089 Mealy automata generating finite (semi)groups among the 233 339 invertible or reversible 3-letter 3-state Mealy automata;
- The group generated by Fig. 1-right is the largest finite group.

$$\rho_x = (1, 2, \dots, p)$$

$$\forall y \neq x, \ \rho_y = (1, 3, \dots, p)$$

$$2$$

$$2$$

$$2$$

$$2$$

$$3$$

$$4z \neq x, \ \rho_z = (1, 2)$$

$$4z \neq x, \ \rho_z = (1, 2)$$

4.1: among invertible automata: $\mathcal{M}_{p,q}$

4.2: among 2-letter invertible automata: $\mathcal{M}_{2,q}$

4.3: among 2-state invertible automata: $\mathcal{M}_{p,2}$

4.4: among bireversible automata: $\mathcal{B}_{p,q}$

Fig. 4. Automata conjectured to generate the largest finite automaton groups

Our next conjectures are concerned with the largest finite groups that can be generated by automata of a given size.

Consider the family of p-letter q-state Mealy automata $(\mathcal{M}_{p,q})_{p+q>5}$ displayed on Fig. 4.1 for p>2 and q>2, while the specializations for p=2 and q=2 are displayed on Fig. 4.2 and Fig. 4.3. The example of Fig. 1-right is $\mathcal{M}_{3,3}$.

Conjecture 1. The group $\langle \mathcal{M}_{p,q} \rangle$ is finite. Every p-letter q-state invertible Mealy automaton generates a group which is either infinite or has an order smaller than $\#\langle \mathcal{M}_{p,q} \rangle$.

If true, Conjecture 1 implies the decidability of the finiteness problem for automaton groups. Without entering into the details of the experimentations, we consider that Conj. 1 is reasonably well supported for p+q < 9. As for actually computing $\#\langle \mathcal{M}_{p,q} \rangle$, here are the only cases with q > 2 for which we succeeded:

$$\begin{split} \forall q,\, 4 \leq q \leq 8, & \# \langle \mathcal{M}_{2,q} \rangle = 2^{2^{q-1} + \frac{(q-2)(q-1)}{2} - 2} \,, \\ \# \langle \mathcal{M}_{3,3} \rangle = 2^{64} \cdot 3^4, & \# \langle \mathcal{M}_{3,4} \rangle = 2^{325} \cdot 3^{13}, & \# \langle \mathcal{M}_{4,3} \rangle = 2^{288} \cdot 3^{422} \,. \end{split}$$

These groups are indeed huge. Incidentally, the finiteness of $\langle \mathcal{M}_{p,q} \rangle$ is checked for p+q<11 and the informal conjecture is supported further by computing the order of the much smaller semigroups generated by the duals:

$\#\langle \mathfrak{d}(\mathfrak{M}_{p,q})\rangle_+$	2	3	4	5	6	7	8
2	-	-	219	1759	13 135	94 143	656831
3	-	238	1552	8 140	37786	162202	
4	89	1381	12309	87125	543061		
5	131	6056	67906	602656			
6	337	22399	302011				
7	351	74194					

Experimentally, the finite groups generated by bireversible Mealy automata seem to be much smaller. Consider the family of bireversible automata $(\mathcal{B}_{p,q})_{p,q}$ of Fig. 4.4. The group $\langle \mathcal{B}_{p,q} \rangle$ is isomorphic to \mathfrak{S}_p^q , while the group $\langle \mathfrak{d}(\mathcal{B}_{p,q}) \rangle$ is isomorphic to \mathbb{Z}_q . Again, the following is reasonably well supported for p+q<9:

Conjecture 2. Every p-letter q-state bireversible Mealy automaton generates a group which is either infinite or has an order smaller than $\#\langle \mathcal{B}_{p,q} \rangle = p!^q$.

Our last conjecture is of a different nature and deals with the structure of infinite automaton semigroups.

Conjecture 3. Every 2-state reversible Mealy automaton generates a semigroup which is either finite or free of rank 2.

The conjecture has been tested and seems correct for reversible 2-state Mealy automata up to 6 letters. In the experiments, a semigroup generated by a p-letter automaton is conjectured to be free if its growth series coincides with $(2t)^n$ up to radius $p^2/2$ and if its dual generates a seemingly infinite group.

References

- A.V. Aho, J.E. Hopcroft, and J.D. Ullman. The Design and Analysis of Computer Algorithms. Addison-Wesley, 1974.
- A. Akhavi, I. Klimann, S. Lombardy, J. Mairesse, and M. Picantin. On the finiteness problem for automaton (semi)groups. *Internat. J. Algebra Comput.*, (accepted), 2012. arXiv:cs.FL/1105.4725.
- 3. A. S. Antonenko. On transition functions of Mealy automata of finite growth. *Matematychni Studii.*, 29(1):3–17, 2008.
- 4. L. Bartholdi. FR Functionally recursive groups a GAP package, v.1.2.4.2, 2011.
- 5. L. Bartholdi, I. I. Reznykov, and V. I. Sushchanskiĭ. The smallest Mealy automaton of intermediate growth. *J. Algebra*, 295(2):387–414, 2006.
- I. V. Bondarenko, N. V. Bondarenko, S. N. Sidki, and F. R. Zapata. On the conjugacy problem for finite-state automorphisms of regular rooted trees. arXiv:math.GR/1011.2227.
- 7. A. J. Cain. Automaton semigroups. Theor. Comput. Sci., 410:5022-5038, 2009.
- 8. The GAP Group. GAP Groups, Algorithms, and Programming, v.4.4.12, 2008.
- 9. R. I. Grigorchuk. On Burnside's problem on periodic groups. Funktsional. Anal. i Prilozhen., 14(1):53–54, 1980.
- R. I. Grigorchuk, V. V. Nekrashevich, and V. I. Sushchanskii. Automata, dynamical systems, and groups. Tr. Mat. Inst. Steklova, 231:134–214, 2000.
- 11. Y. Muntyan and D. Savchuk. automgrp Automata Groups a GAP package, v.1.1.4.1, 2008.
- V. Nekrashevych. Self-similar groups, volume 117 of Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 2005.
- 13. D. M Savchuk and Y. Vorobets. Automata generating free products of groups of order 2. J. Algebra, 336(1):53–66, 2011.
- 14. S. N. Sidki. Automorphisms of one-rooted trees: growth, circuit structure, and acyclicity. J. Math. Sci. (New York), 100(1):1925–1943, 2000. Algebra, 12.
- 15. P. V. Silva and B. Steinberg. On a class of automata groups generalizing lamplighter groups. *Internat. J. Algebra Comput.*, 15(5-6):1213–1234, 2005.
- Z. Šunik and E. Ventura. The conjugacy problem is not solvable in automaton groups. 2010. arXiv:math.GR/1010.1993.