Acceptance Conditions for ω-Languages

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Abstract. This paper investigates acceptance conditions for finite automata recognizing ω-regular languages. Their expressive power and their position *w.r.t.* the Borel hierarchy is also studied. The full characterization for the conditions $(ninf, \sqcap)$, $(ninf, \subseteq)$ and $(ninf, =)$ is given. The final section provides a partial characterization of $(fin, =)$.

Keywords: finite automata, acceptance conditions, ω-regular languages.

1 Introduction

Infinite words are widely used in formal specification and verification of non-terminating processes (e.g. web-servers, OS daemons, *etc.*) [4,3,13]. The overall state of the system is represented by an element of some finite alphabet. Hence runs of the systems can be conveniently represented as ω-words. Finite automata are often used to model the transitions of the system and their accepted language represents the set of admissible runs of the system under observation. Acceptance conditions on finite automata are therefore selectors of admissible runs. Main results and overall exposition about ω-languages can be found in [12,11,9].

Seminal studies about acceptance of infinite words by finite automata (FA) have been performed by Büchi while studying monadic second order theories [1]. According to Büchi an infinite word is accepted by an FA $\mathcal{A}$ if there exists a run of $\mathcal{A}$ which passes infinitely often through a set of accepting states. Later, Muller studied runs that pass through all elements of a given set of accepting states and visit them infinitely often [8]. Afterwards, several acceptance conditions appeared in a series of papers [2,5,7,10,6].

Clearly, the selection on runs operated by accepting conditions is also influenced by the structural properties of the FA under consideration: deterministic vs. non-deterministic, complete vs. non complete (see for instance [6]).

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In this work, we review the main acceptance conditions and we couple them with structural properties like determinism or completeness in the purpose of characterizing the relationships between the class of languages they induce. The Borel hierarchy is another important characterization of ω-rational languages and it is the basic skeleton of our study which helped to argue the placement of the other classes. Figure 1 illustrates the current state of art whilst Figure 2 summarizes the results provided by the present paper.

For lack of space, several proofs of lemmata will appear only in a journal version of this paper.

2 Notations and background

For any set A, $\mathrm{Card}\,(A)$ denotes the cardinality of A. Given a finite alphabet Σ, Σ^* and Σ^ω denote the set of all finite words and the set of all (mono) infinite words on Σ, respectively. As usual, $\epsilon \in \Sigma^*$ is the empty word. For any pair $u, v \in \Sigma^*$, uv is the concatenation of u with v.

A *language* is any set $L \subseteq \Sigma^*$. For any pair of languages L_1, L_2, $L_1L_2 = \{uv \in \Sigma^* : u \in L_1, v \in L_2\}$ is the concatenation of L_1 and L_2. For a language L, denote $L^0 = \{\epsilon\}$, $L^{n+1} = L^nL$ and $L^* = \bigcup_{n\in\mathbb{N}} L^n$ the Kleene star of L. The collection of *rational languages* is the smallest class of languages containing $\emptyset$, all sets $\{a\}$ (for $a \in \Sigma$) and which is closed by union, concatenation and Kleene star.

An *ω-language* is any subset $\mathcal{L}$ of Σ^ω. For a language L, the infinite extension of L is the ω-language

$$L^\omega = \left\{x \in \Sigma^\omega : \exists (u_i)_{i\in\mathbb{N}} \in (L \setminus \{\epsilon\})^{\mathbb{N}}, x = u_0u_1u_2\dots\right\} \enspace .$$

An ω-language $\mathcal{L}$ is *ω-rational* if there exist two families $\{L_i\}$ and $\{L'_i\}$ of rational languages such that $\mathcal{L} = \bigcup_{i=0}^{n} L'_i {L_i}^\omega$. Denote by RAT the set of all ω-rational languages.

A *finite state automaton (FA)* is a tuple $(\Sigma, Q, T, q_0, \mathcal{F})$ where Σ is a finite alphabet, Q a finite set of states, $T \subset Q \times \Sigma \times Q$ is the set of *transitions*, $q_0 \in Q$ is the *initial state* and $\mathcal{F} \subseteq \mathcal{P}(Q)$ collects the *accepting sets* of (accepting) states. A FA is a *deterministic* finite state automaton (DFA) if $\mathrm{Card}\,(\{q \in Q : (p, a, q) \in T\}) \leq 1$ for all $p \in Q$, $a \in \Sigma$. It is a *complete* finite state automaton (CFA) if $\mathrm{Card}\,(\{q \in Q : (p, a, q) \in T\}) \geq 1$ for all $p \in Q$, $a \in \Sigma$. We write $CDFA$ for a FA which is both deterministic and complete. An (infinite) *path* in $\mathcal{A} = (\Sigma, Q, T, q_0, \mathcal{F})$ is a sequence $(p_i, x_i, p_{i+1})_{i\in\mathbb{N}}$ such that $(p_i, x_i, p_{i+1}) \in T$ for all $i \in \mathbb{N}$. The (infinite) word $(x_i)_{i\in\mathbb{N}}$ is the *label* of the path p. A path is said to be *initial* if $p_0 = q_0$.

Definition 1. *Let $\mathcal{A} = (\Sigma, Q, T, q_0, \mathcal{F})$ and $p = (p_i, x_i, q_i)_{i\in\mathbb{N}}$ be an automaton and an infinite path in $\mathcal{A}$. The sets*

- $run_{\mathcal{A}}(p) := \{q \in Q : \exists i > 0, p_i = q\}$
- $inf_{\mathcal{A}}(p) := \{q \in Q : \forall i > 0, \exists j \geq i, p_j = q\}$

- $fin_{\mathcal{A}}(p) := run(p) \setminus inf(p)$
- $ninf_{\mathcal{A}}(p) := Q \setminus inf(p)$

contain the states appearing at least one time, infinitely many times, finitely many times but at least once, and finitely many times or never *in p, respectively.*

An *acceptance condition* is a subset of all the initial infinite paths. The paths inside such a subset are called *accepting paths*. Let $\mathcal{A}$ and $cond_{\mathcal{A}}$ be a FA and an acceptance condition for $\mathcal{A}$, respectively. A word w is *accepted* by $\mathcal{A}$ if and only if it is the label of some accepting path. We denote by $\mathcal{L}_{\mathcal{A}}^{cond_{\mathcal{A}}}$ the *language accepted by* $\mathcal{A}$ *under the acceptance condition* $cond_{\mathcal{A}}$, *i.e.*, the set of all words accepted by $\mathcal{A}$ under the acceptance condition $cond_{\mathcal{A}}$.

Let $\sqcap$ be the relation such that for all sets A and B, $A \sqcap B$ if and only if $A \cap B \neq \emptyset$.

In the sequel, we will consider acceptance conditions derived by pairs $(c, \mathbf{R}) \in \{run, inf, fin, ninf\} \times \{\sqcap, \subseteq, =\}$. A pair $cond = (c, \mathbf{R})$ defines an acceptance condition $cond_{\mathcal{A}} = (c_{\mathcal{A}}, \mathbf{R})$ on an automaton $\mathcal{A} = (\Sigma, Q, T, i, \mathcal{F})$ as follows: an initial path $p = (p_i, a_i, p_{i+1})_{i \in \mathbb{N}}$ is accepting if and only if there exists a set $F \in \mathcal{F}$ such that $c_{\mathcal{A}}(p)\ \mathbf{R}\ F$. Moreover, when not explicitly indicated, all automata will be defined over the same finite alphabet Σ.

Definition 2. *For any pair* $cond = (c, \boldsymbol{R}) \in \{run, inf, fin, ninf\} \times \{\sqcap, \subseteq, =\}$, *the following sets*

- $FA(cond) = \left\{\mathcal{L}_{\mathcal{A}}^{cond_{\mathcal{A}}},\ \mathcal{A} \text{ is a } FA\right\}$
- $DFA(cond) = \left\{\mathcal{L}_{\mathcal{A}}^{cond_{\mathcal{A}}},\ \mathcal{A} \text{ is a } DFA\right\}$
- $CFA(cond) = \left\{\mathcal{L}_{\mathcal{A}}^{cond_{\mathcal{A}}},\ \mathcal{A} \text{ is a } CFA\right\}$
- $CDFA(cond) = \left\{\mathcal{L}_{\mathcal{A}}^{cond_{\mathcal{A}}},\ \mathcal{A} \text{ is a } CDFA\right\}$

are the classes of languages accepted by FA, DFA, CFA, *and* $CDFA$, *respectively, under the acceptance condition derived by cond.*

Some of the acceptance conditions derived by pairs $(c, \mathbf{R})$ have been studied in the literature as summarized in the following table.

	$\sqcap$	$\subseteq$	$=$
run	Landweber [5]	Hartmanis and Stearns [2]	Staiger and Wagner [10]
inf	Büchi [1]	Landweber [5]	Muller [8]
fin	Litovski and Staiger [6]		THIS PAPER**
$ninf$	THIS PAPER*	THIS PAPER*	THIS PAPER

* These conditions have been already investigated in [7] but only in the case of complete automata with a unique set of accepting states.
** Only FA and CFA are considered here. For DFA and $CDFA$ the question is still open.

For Σ equipped with discrete topology and Σ^ω with the induced product topology, let F, G, F_σ and G_δ be the collections of all closed sets, open sets, countable unions of closed set and countable intersections of open sets, respectively. For any pair A, B of collections of sets, denote by $\mathcal{B}(A)$, $A \varDelta B$, and A^R the boolean closure of A, the set $\{U \cap V : U \in A, V \in B\}$ and the set $A \cap RAT$, respectively. These, indeed, are the lower classes of the Borel hierarchy, for more on this subject we refer the reader to [14] or [9], for instance.

Figure 1 illustrates the known hierarchy of languages classes (arrows represents strict inclusions).

Let X and Y be two sets, $pr_1 : (X \times Y)^\omega \to X^\omega$ denotes the projection of words in $(X \times Y)^\omega$ on the first set, *i.e.* $pr_1((x_i, y_i)_{i\in\mathbb{N}}) = (x_i)_{i\in\mathbb{N}}$.

Lemma 3 (Staiger [11, Projection lemma]).
Let $cond \in \{run, inf, fin, ninf\} \times \{\sqcap, \subseteq, =\}$.

1. *Let* X, Y *be two finite alphabets and* $\mathcal{L} \subseteq (X \times Y)^\omega$. $\mathcal{L} \in FA(cond)$ *implies* $pr_1(\mathcal{L}) \in FA(cond)^\S$.
2. *Let* X *be a finite alphabet and* $\mathcal{L} \subseteq X^\omega$. $\mathcal{L} \in FA(cond)^\S$ *implies there exist a finite alphabet* Y *and a language* $\mathcal{L}' \subseteq (X \times Y)^\omega$ *such that* $\mathcal{L}' \in DFA(cond)^\S$ *and* $pr_1(\mathcal{L}') = \mathcal{L}$.

3 The accepting conditions $\mathbb{A}$ and $\mathbb{A}'$ and the Borel hierarchy

In [7], Moriya and Yamasaki introduced two more acceptance conditions, namely $\mathbb{A}$ and $\mathbb{A}'$, and they compared them to the Borel hierarchy for the case of CFA and CDFA having a unique set of accepting states. In this section, those results are generalized to FA and DFA and to any set of sets of accepting states.

Definition 4. *Given an FA* $\mathcal{A} = (\Sigma, Q, T, q_0, \mathcal{F})$*, the acceptance condition* $\mathbb{A}$ *(resp.,* $\mathbb{A}'$*) on* $\mathcal{A}$ *is defined as follows: an initial path* p *is accepting under* $\mathbb{A}$ *(resp.,* $\mathbb{A}'$*) if and only if there exists a set* $F \in \mathcal{F}$ *such that* $F \subseteq run_{\mathcal{A}}(p)$ *(resp.,* $F \not\subseteq run_{\mathcal{A}}(p)$*).*

Lemma 5.

1. $FA(\mathbb{A}) \subseteq FA(run, \sqcap)$,
2. $DFA(\mathbb{A}) \subseteq DFA(run, \sqcap)$,
3. $CFA(\mathbb{A}) \subseteq CFA(run, \sqcap)$,
4. $CDFA(\mathbb{A}) \subseteq CDFA(run, \sqcap)$.

§ Remark that in the case 1. the languages belonging to $FA(cond)$ are defined over the alphabet X and not $X \times Y$. Similarly, in the case 2. the languages belonging to $FA(cond)$ are defined over X and those belonging to $DFA(cond)$ are defined over $X \times Y$.

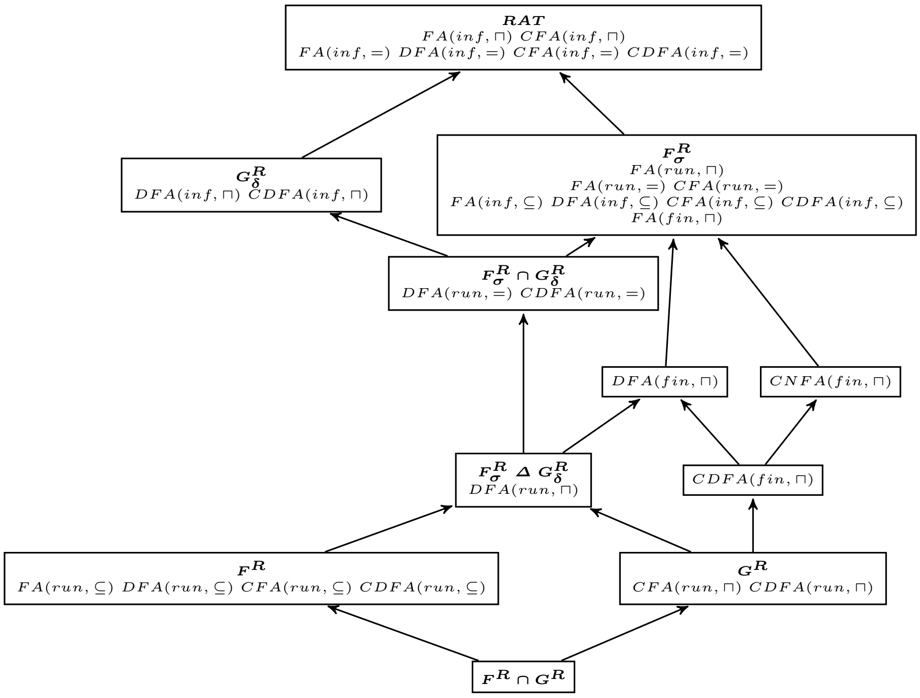

Fig. 1. Currently known relations between classes of ω-languages recognized by FA according to the considered acceptance conditions and structural properties like determinism or completeness. Classes of the Borel hierarchy are typeset in bold. Arrows mean strict inclusion. Classes in the same box coincide.

Lemma 6.

1. $FA(run,\sqcap) \subseteq FA(\mathbb{A})$,
2. $DFA(run,\sqcap) \subseteq DFA(\mathbb{A})$,
3. $CFA(run,\sqcap) \subseteq CFA(\mathbb{A})$,
4. $CDFA(run,\sqcap) \subseteq CDFA(\mathbb{A})$.

Lemma 7.

1. $FA(\mathbb{A}') \subseteq FA(run,\subseteq)$,
2. $DFA(\mathbb{A}') \subseteq DFA(run,\subseteq)$,
3. $CFA(\mathbb{A}') \subseteq CFA(run,\subseteq)$,
4. $CDFA(\mathbb{A}') \subseteq CDFA(run,\subseteq)$.

Lemma 8.

1. $FA(run,\subseteq) \subseteq FA(\mathbb{A}')$,
2. $DFA(run,\subseteq) \subseteq DFA(\mathbb{A}')$,
3. $CFA(run,\subseteq) \subseteq CFA(\mathbb{A}')$,
4. $CDFA(run,\subseteq) \subseteq CDFA(\mathbb{A}')$.

Proof. Let $cond = (run, \subseteq)$. We are going to show that for any $FA\ \mathcal{A} = (\Sigma, Q, T, q_0, \mathcal{F})$ there exists an automaton $\mathcal{A}'$ under the accepting condition $\mathbb{A}'$ such that $\mathcal{L}_{\mathcal{A}'}^{\mathbb{A}'} = \mathcal{L}_{\mathcal{A}}^{cond_{\mathcal{A}}}$ and $\mathcal{A}'$ is deterministic (resp. complete) if $\mathcal{A}$ is deterministic (resp. complete). Define the automaton $\mathcal{A}' = (\Sigma, Q', T', (q_0, \emptyset), \mathcal{F}')$ where $Q' = (Q \times \mathcal{P}(Q)) \cup \{\bot\}$, $\mathcal{F}' = \{\{\bot\}\}$, and

$$
\begin{aligned}
T' = &\{((p,S), a, (q, S \cup \{q\})) : (p,a,q) \in T, S \in \mathcal{P}(Q), \exists F \in \mathcal{F}, S \cup \{q\} \subseteq F\} \\
&\bigcup \{((p,S), a, \bot) : S \in \mathcal{P}(Q), \exists q \in Q, (p,a,q) \in T, \forall F \in \mathcal{F}, S \cup \{q\} \nsubseteq F\} \\
&\bigcup \{(\bot, a, \bot) : a \in \Sigma\}
\end{aligned}
$$

Then, $\mathcal{A}'$ is deterministic (resp. complete) if $\mathcal{A}$ is deterministic (resp. complete). Moreover, $x \in \mathcal{L}_{\mathcal{A}}^{cond_{\mathcal{A}}}$ if and only if there exists an initial path p in $\mathcal{A}$ with label x and a set $F \in \mathcal{F}$ such that $run_{\mathcal{A}}(p) \subseteq F$ iff there exists an initial path p' in $\mathcal{A}'$ with label x such that $p'_n \neq \bot$ for all $n \in \mathbb{N}$, *i.e.*, iff $x \in \mathcal{L}_{\mathcal{A}'}^{\mathbb{A}'}$. □

The following result places the classes of langages characterized by $\mathbb{A}$ and $\mathbb{A}'$ *w.r.t.* the Borel hierarchy.

Theorem 9.
1. $CDFA(\mathbb{A}) = CFA(\mathbb{A}) = G^R$
2. $DFA(\mathbb{A}) = F_\sigma^R\ \Delta\ G_\delta^R$
3. $FA(\mathbb{A}) = F_\sigma^R$
4. $CDFA(\mathbb{A}') = DFA(\mathbb{A}') = CFA(\mathbb{A}') = FA(\mathbb{A}') = F^R$

Proof. It is a consequence of Lemmata 5, 6, 7 and 8, and the known results (see Figure 1) on the classes of languages accepted by FA, DFA, CFA, and $CDFA$ under the acceptance conditions derived by $(run, \sqcap)$ and $(run, \subseteq)$. □

Remark 10. Languages in $CDFA(\mathbb{A})$ (resp. $CDFA(\mathbb{A}')$ are unions of languages in the class $\mathbb{A}$ (resp. $\mathbb{A}'$) of [7]. This class equals G^R (resp. F^R) and is closed under union operation. These facts already prove $CDFA(\mathbb{A}) = G^R$ (resp. $CDFA(\mathbb{A}') = F^R$).

4 The accepting conditions $(ninf, \sqcap)$ and $(ninf, \subseteq)$.

In [6], Litovsky and Staiger studied the class of languages accepted by FA under the acceptance condition $(fin, \sqcap)$ *w.r.t.* which a path is successful if it visits an accepting state finitely many times but at least once. It is natural to study the expressivity of the similar accepting condition for which a path is successful if it visits an accepting state finitely many times or never: $(ninf, \sqcap)$. The expressivity of $(ninf, \subseteq)$ is also analized and compared with the previous ones to complete the picture in Figure 1. As a first step, we analyze two more acceptance conditions proposed by Moriya and Yamasaki [7]: $\mathbb{L}$ which represents the situation of a non-terminating process forced to pass through a finite set of "safe" states infinitely often and $\mathbb{L}'$ which is the negation of $\mathbb{L}$. Lemma 12 proves that $\mathbb{L}$ is equivalent to $(ninf, \sqcap)$ and $\mathbb{L}'$ to $(ninf, \subseteq)$. Moreover, the results of [7] are extended to any type of FA with any number of sets of accepting states.

Definition 11. *Given an FA $\mathcal{A} = (\Sigma, Q, T, q_0, \mathcal{F})$, the acceptance condition $\mathbb{L}$ (resp., $\mathbb{L}'$) on $\mathcal{A}$ is defined as follows: an initial path p is accepting under $\mathbb{L}$ (resp., $\mathbb{L}'$) if and only if there exists a set $F \in \mathcal{F}$ such that $F \subseteq inf_{\mathcal{A}}(p)$ (resp., $F \not\subseteq inf_{\mathcal{A}}(p)$).*

Lemma 12. *$\mathbb{L}$ and $(ninf, \subseteq)$ (resp., $\mathbb{L}'$ and $(ninf, \sqcap)$) define the same classes of languages.*

Remark that any FA can be completed with a sink state without changing the language accepted under $\mathbb{L}$. Therefore, the following claim is true.

Lemma 13. *$FA(\mathbb{L}) = CFA(\mathbb{L})$ and $DFA(\mathbb{L}) = CDFA(\mathbb{L})$.*

Proposition 14. *$CDFA(inf, \sqcap) \subseteq CDFA(\mathbb{L})$ and $CFA(inf, \sqcap) \subseteq CFA(\mathbb{L})$.*

Proof. For any $CDFA$ (resp., CFA) $\mathcal{A} = (\Sigma, Q, T, q_0, \mathcal{F})$, define the $CDFA$ (resp., CFA) $\mathcal{A}' = (\Sigma, Q, T, q_0, \mathcal{F}')$ where $\mathcal{F}' = \{\{q\} : \exists F \in \mathcal{F}, q \in F\}$. Then, it follows that $\mathcal{L}_{\mathcal{A}}^{(inf,\sqcap)_{\mathcal{A}}} = \mathcal{L}_{\mathcal{A}'}^{\mathbb{L}}$ and this concludes the proof. $\square$

Proposition 15. *$CDFA(\mathbb{L}) \subseteq CDFA(inf, \sqcap)$*

Proof. For any $CDFA$ $\mathcal{A} = (\Sigma, Q, T, q_0, \mathcal{F})$ and any $q \in Q$, define the $CDFA$ $\mathcal{A}_q = (\Sigma, Q, T, q_0, \{\{q\}\})$. By determinism of $\mathcal{A}$, it holds that

$$\mathcal{L}_{\mathcal{A}}^{\mathbb{L}} = \bigcup_{F \in \mathcal{F}} \bigcap_{q \in F} \mathcal{L}_{\mathcal{A}_q}^{(inf,\sqcap)_{\mathcal{A}_q}} \quad .$$

Since $CDFA(inf, \sqcap)$ is stable by finite union and finite intersection [1], there exists a $CDFA$ $\mathcal{A}'$ such that $\mathcal{L}_{\mathcal{A}}^{\mathbb{L}} = \mathcal{L}_{\mathcal{A}'}^{(inf,\sqcap)_{\mathcal{A}'}}$. Hence, $CDFA(\mathbb{L}) \subseteq CDFA(inf, \sqcap)$. $\square$

Proposition 16. *$CFA(\mathbb{L}) \subseteq CFA(inf, =)$.*

Proof. For any CFA $\mathcal{A} = (\Sigma, Q, T, q_0, \mathcal{F})$ define $\mathcal{A}' = (\Sigma, Q, T, q_0, \mathcal{F}')$, where $\mathcal{F}' = \{S \in \mathcal{P}(Q) : \exists F \in \mathcal{F}, F \subseteq S\}$. Then, $\mathcal{A}'$ is complete and $\mathcal{L}_{\mathcal{A}}^{\mathbb{L}} = \mathcal{L}_{\mathcal{A}'}^{(inf,=)_{\mathcal{A}'}}$. Hence, the thesis is true. $\square$

Theorem 17. *The following equalities hold.*

(1) $CDFA(ninf, \subseteq) = DFA(ninf, \subseteq) = G_\delta^R$
(2) $CFA(ninf, \subseteq) = FA(ninf, \subseteq) = RAT$

Proof. Equality (1) follows from Lemmata 12 and 13, Proposition 15 and 14 and the known fact that $DFA(inf, \sqcap) = CDFA(inf, \sqcap) = G_\delta^R$, while equality (2) from Lemmata 12 and 13, Proposition 14 and 16 and the known fact that $CFA(inf, \sqcap) = CFA(inf, =) = RAT$. $\square$

Lemma 18. *For any automaton $\mathcal{A} = (\Sigma, Q, T, q_0, \mathcal{F})$ there exists an automaton $\mathcal{A}' = (\Sigma', Q', T', q_0', \mathcal{F}')$ such that $\mathcal{F}' = \{\{q'\}\}$ for some $q' \in Q'$, $\mathcal{L}_{\mathcal{A}}^{\mathbb{L}'} = \mathcal{L}_{\mathcal{A}'}^{\mathbb{L}'}$, and $\mathcal{A}'$ is deterministic (resp. complete) if $\mathcal{A}$ is deterministic (resp. complete).*

Proof. If either $\mathcal{F} = \{\}$ or $\mathcal{F} = \{\emptyset\}$ then the automaton $\mathcal{A}'$ defined by $\Sigma' = \Sigma$, $Q' = \{\perp\}$, $T' = \{(\perp, a, \perp) : a \in \Sigma\}$, $q_0' = q_0$, and $\mathcal{F}' = \{\{\perp\}\})$ verifies the statement of the Lemma. Otherwise, set $F = \bigcup_{X \in \mathcal{F}} X$, choose any $f \in F$, and define the automaton $\mathcal{A}'$ by $\Sigma' = \Sigma$, $Q' = Q \times \mathcal{P}(F)$, $q_0' = (q_0, \emptyset)$, $\mathcal{F}' = \{\{(f, F)\}\}$, and

$$T' = \{((p,S), a, (q, (S \cup \{q\}) \cap F)) : (p,a,q) \in T, (p,S) \neq (f,F)\} \\ \bigcup \{((f,F), a, (q, \emptyset)) : (f,a,q) \in T\}$$

Then, $\mathcal{A}'$ is deterministic (resp., complete) if $\mathcal{A}$ is deterministic (resp., complete). Moreover, $\mathcal{L}^{\mathbb{L}'}_{\mathcal{A}} \subseteq \mathcal{L}^{\mathbb{L}'}_{\mathcal{A}'}$. Indeed, if $x \in \mathcal{L}^{\mathbb{L}'}_{\mathcal{A}}$, there exist an initial path $p = (p_i, x_i, p_{i+1})_{i \in \mathbb{N}}$ in $\mathcal{A}$ with label x, a set $X \in \mathcal{F}$, and a state $s \in X$ such that $s \notin inf(p)$. Consider the path $p' = ((p_i, S_i), x_i, (p_{i+1}, S_{i+1}))_{i \in \mathbb{N}}$ where $S_0 = \emptyset$ and $S_{i+1} = (S_i \cup \{q_i\}) \cap F$ if $(p_i, S_i) \neq (f, F)$, $\emptyset$ otherwise. Then, p' is an initial path in $\mathcal{A}'$ with label x in which the state (f, F) appears finitely often in p' since s appears finitely often in p. Hence, $x \in \mathcal{L}^{\mathbb{L}'}_{\mathcal{A}'}$. Finally, the implication $\mathcal{L}^{\mathbb{L}'}_{\mathcal{A}'} \subseteq \mathcal{L}^{\mathbb{L}'}_{\mathcal{A}}$ is also true.

The following series of Lemmata is useful to prove strict inclusions between the the considered language classes.

Lemma 19 (Moriya and Yamasaki [7]). *$\mathcal{L} = (a+b)^* a^\omega \in CDFA(\mathbb{L}')$.*

Lemma 20. *$ab^*a(a+b)^\omega \in DFA(\mathbb{L}') \setminus CFA(\mathbb{L}')$.*

Lemma 21. *$b^*ab^*a(a+b)^\omega \notin FA(\mathbb{L}')$.*

Lemma 22. *$(a+b)^*ba^\omega \in CFA(\mathbb{L}') \setminus DFA(\mathbb{L}')$.*

Proposition 23. *$FA(\mathbb{L}') \subsetneq F^R_\sigma$*

Proof. For any FA $\mathcal{A} = (\Sigma, Q, T, q_0, \mathcal{F})$, by Lemma 18 we can assume that $\mathcal{F} = \{\{f\}\}$. Define the FA $\mathcal{A}' = (\Sigma, Q, T, q_0, \{Q \setminus \{f\}\})$. Then, $\mathcal{L}^{\mathbb{L}'}_{\mathcal{A}} = \mathcal{L}^{(inf, \subseteq)_{\mathcal{A}'}}_{\mathcal{A}'}$ and, so, $FA(\mathbb{L}') \subseteq FA(inf, \subseteq)$. Moreover, by the know fact $FA(inf, \subseteq) = F^R_\sigma$, we obtain that $\mathcal{L}^{(inf, \subseteq)_{\mathcal{A}'}}_{\mathcal{A}'} \in F^R_\sigma$. Lemma 21 gives the strict inclusion. □

Proposition 24. *$DFA(\mathbb{L}')$ and $CFA(\mathbb{L}')$ are incomparable.*

Proof. It is an immediate consequence of Lemmata 20 and 22.

Proposition 25. *The following statements are true.*

(1) $FA(\mathbb{L}')$ and G^R_δ are incomparable.
(2) $FA(\mathbb{L}')$ and G^R are incomparable.

Proof. By Lemma 19, $(a+b)^* a^\omega \in CDFA(\mathbb{L}') \setminus G^R_\delta$ and, by Lemma 21, $b^*ab^*a(a+b)^\omega \in G^R \setminus FA(\mathbb{L}')$. To conclude, recall that $G^R \subseteq G^R_\delta$. □

Proposition 26. *$CDFA(\mathbb{L}')$ and $DFA(fin, \sqcap)$ are incomparable.*

Proof. By Proposition 25 and by the known fact $G^R \subseteq DFA(fin, \sqcap)$, it follows that $DFA(fin, \sqcap) \not\subseteq CDFA(\mathbb{L}')$. Furthermore, it has been shown in [6] that $CDFA(\mathbb{L}') \not\subseteq DFA(fin, \sqcap)$. □

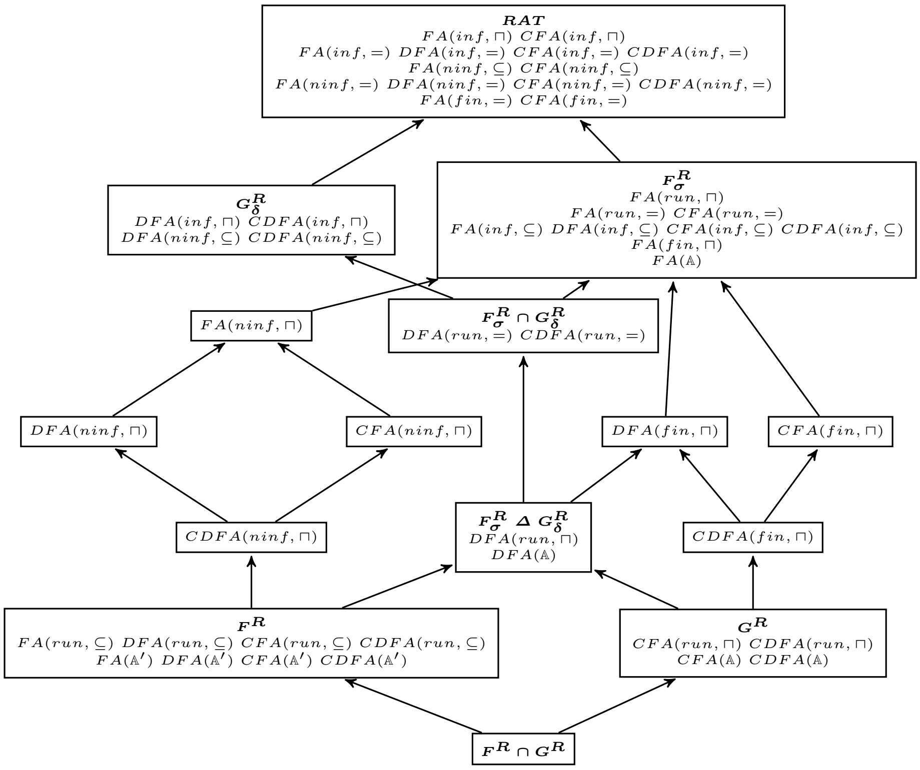

Fig. 2. The completion of Figure 1 with the results in the paper. Classes of the Borel hierarchy are typeset in bold. Arrows mean strict inclusion. Classes in the same box coincide.

5 Towards a characterization of $(fin,=)$ and $(fin,\subseteq)$.

In this section we start studying the conditions $(fin,=)$ and $(fin,\subseteq)$. Concerning $(fin,=)$, Theorem 34 tells us that, in the non-deterministic case, the class of recognized languages coincides with RAT. In the deterministic case, either it again coincides with RAT or it defines a completely new class (Proposition 35).

Intuitively, any class of ω-languages defined using a MSO definable accepting condition should be included in RAT. A formal proof for this statement is still unknown. Anyway, we now prove this statement for the particular cases investigated so far.

Proposition 27. *The following equality holds for* $(ninf,=)$*:*

$$CDFA(ninf,=) = DFA(ninf,=) = CFA(ninf,=) = FA(ninf,=) = RAT$$

Proof. For any FA $\mathcal{A} = (\Sigma, Q, T, q_0, \mathcal{F})$, let $\mathcal{A}' = (\Sigma, Q, T, q_0, \{Q \setminus F : F \in \mathcal{F}\})$. Clearly, $\mathcal{A}'$ is deterministic (resp. complete) if $\mathcal{A}$ is deterministic (resp. complete).

It is not difficult to see that $\mathcal{L}_{\mathcal{A}}^{(ninf,=)_{\mathcal{A}}} = \mathcal{L}_{\mathcal{A}'}^{(inf,=)_{\mathcal{A}'}}$ and $\mathcal{L}_{\mathcal{A}}^{(inf,=)_{\mathcal{A}}} = \mathcal{L}_{\mathcal{A}'}^{(ninf,=)_{\mathcal{A}'}}$. Hence, it holds that $FA(ninf,=) = FA(inf,=)$, $DFA(ninf,=) = DFA(inf,=)$, $CFA(ninf,=) = CFA(inf,=)$, and $CDFA(ninf,=) = CDFA(inf,=)$. The known results on the language classes regarding $(inf,=)$ conclude the proofs. □

Proposition 28. *The following equalities hold for $(fin,\subseteq)$ and $(fin,=)$:*

$$DFA(fin,\subseteq) = CDFA(fin,\subseteq) \text{ and } FA(fin,\subseteq) = CFA(fin,\subseteq),$$
$$DFA(fin,=) = CDFA(fin,=) \text{ and } FA(fin,=) = CFA(fin,=).$$

Proof. For any FA $\mathcal{A} = (\Sigma, Q, T, q_0, \mathcal{F})$, let $\mathcal{A}' = (\Sigma, Q \cup \{\bot, \bot'\}, T', q_0, \mathcal{F})$ where

$$T' = T \cup \{(p,a,\bot) : p \in Q, a \in \Sigma, \forall q \in Q, (p,a,q) \notin T\} \cup \{(\bot, a, \bot') : a \in \Sigma\} \cup \{(\bot', a, \bot') : a \in \Sigma\}$$

The FA $\mathcal{A}'$ is complete. Moreover, $\mathcal{A}'$ is a DFA if and only if $\mathcal{A}$ is a DFA. Furthermore, under both the conditions $(fin,\subseteq)$ and $(fin,=)$, every accepting path in $\mathcal{A}$ is still an accepting path in $\mathcal{A}'$, and if p is an initial path in $\mathcal{A}'$ which is not a path in $\mathcal{A}$, then $\bot \in fin(p)$. Since $\forall F \in \mathcal{F}, \bot \notin F$, the path p is non accepting in $\mathcal{A}'$. Therefore, $\mathcal{L}_{\mathcal{A}}^{(fin,\subseteq)_{\mathcal{A}}} = \mathcal{L}_{\mathcal{A}'}^{(fin,\subseteq)_{\mathcal{A}'}}$ and $\mathcal{L}_{\mathcal{A}}^{(fin,=)_{\mathcal{A}}} = \mathcal{L}_{\mathcal{A}'}^{(fin,=)_{\mathcal{A}'}}$ and this concludes the proof.

Proposition 29 (Staiger [11]).
$CDFA(fin,\subseteq) \subseteq CDFA(fin,=)$ and $CFA(fin,\subseteq) \subseteq CFA(fin,=)$.

Proposition 30 (Staiger [11]).
$FA(fin,\sqcap) \subseteq FA(fin,=)$ and $DFA(fin,\sqcap) \subseteq DFA(fin,=)$.

Lemma 31. *$RAT \subseteq FA(fin,=)$.*

Proof. We are going to show that $FA(inf,\sqcap) \subseteq FA(fin,=)$, *i.e.*, for any FA $\mathcal{A} = (\Sigma, Q, T, q_0, \mathcal{F})$ there exists a FA $\mathcal{A}'$ such that $\mathcal{L}_{\mathcal{A}}^{(inf,\sqcap)_{\mathcal{A}}} = \mathcal{L}_{\mathcal{A}'}^{(fin,=)_{\mathcal{A}'}}$. The known fact that $RAT = FA(inf,\sqcap)$ concludes the proof.

Let $\mathcal{A}' = (\Sigma, Q \cup Q \times Q, T', q_0, \mathcal{F}')$ where

$$T' = T \cup \{(p,a,(q,p)) : (p,a,q) \in T\} \cup \{((p_1,p_2),a,q) : (p_1,a,q) \in T, p_2 \in Q\}$$

and $\mathcal{F}' = \{F \setminus \{p_2\} \cup \{(p_1,p_2)\} : p_1 \in Q, F \in \mathcal{P}(Q), \exists X \in \mathcal{F}, p_2 \in X\}$.

We prove that $\mathcal{L}_{\mathcal{A}}^{(inf,\sqcap)_{\mathcal{A}}} \subseteq \mathcal{L}_{\mathcal{A}'}^{(fin,=)_{\mathcal{A}'}}$. Let $x \in \mathcal{L}_{\mathcal{A}}^{(inf,\sqcap)_{\mathcal{A}}}$. There exists a path $p = (p_i, x_i, p_{i+1})_{i\in\mathbb{N}}$ in $\mathcal{A}$, a state $q \in Q$ and a set $F \in \mathcal{F}$ such that $q \in F$ and $q = p_i$ for infinitely many $i \in \mathbb{N}$. Let $n > 0$ be such that $p_n = q$ and let $p' = (p'_i, x_i, p'_{i+1})_{i\in\mathbb{N}}$ be the initial path in $\mathcal{A}'$ defined by $\forall i \neq n+1, p'_i = p_i$ and $p'_{n+1} = (p_{n+1}, q)$. As $q \notin fin(p')$, $fin(p') = (fin(p') \cap Q) \setminus \{q\} \cup \{(p_{n+1}, q)\} \in \mathcal{F}'$. Hence, $x \in \mathcal{L}_{\mathcal{A}'}^{(fin,=)_{\mathcal{A}'}}$.

We now show that $\mathcal{L}_{\mathcal{A}'}^{(fin,=)_{\mathcal{A}'}} \subseteq \mathcal{L}_{\mathcal{A}}^{(inf,\sqcap)_{\mathcal{A}}}$. Let $x \in \mathcal{L}_{\mathcal{A}'}^{(fin,=)_{\mathcal{A}'}}$. There exists a path $p = (p_i, x_i, p_{i+1})_{i\in\mathbb{N}}$ in $\mathcal{A}'$, two states $q_1, q_2 \in Q$ and a set $F \in \mathcal{P}(Q)$ such that $\exists X \in \mathcal{F}$ with $q_2 \in X$ and $fin(p) = F \setminus \{q_2\} \cup \{(q_1,q_2)\}$. Let $p' = (p'_i, x_i, p'_{i+1})_{i\in\mathbb{N}}$ be the initial path in $\mathcal{A}$ defined by $\forall i \in \mathbb{N}, p'_i = p_i$ if $p_i \in Q$, $p'_i = a_i$ with $p_i = (a_i, b_i) \in Q \times Q$, otherwise. As $(q_1,q_2) \in fin(p)$, $q_2 \in run(p)$ but $q_2 \notin fin(p)$, then $q_2 \in inf(p) \subseteq inf(p')$. Hence, $x \in \mathcal{L}_{\mathcal{A}}^{(inf,\sqcap)_{\mathcal{A}}}$. □

Lemma 32. $DFA(fin, =) \subseteq RAT$.

Proof. For any DFA $\mathcal{A} = (\Sigma, Q, T, q_0, \mathcal{F})$, let $\mathcal{A}_S = (\Sigma, Q, T, q_0, \{S\})$ for any set $S \subseteq Q$. Then,

$$\mathcal{L}_{\mathcal{A}}^{(fin,=)} = \bigcup_{S \subseteq Q, S' \subseteq Q, S \setminus S' \in \mathcal{F}} \mathcal{L}_{\mathcal{A}_S}^{(run,=)} \setminus \mathcal{L}_{\mathcal{A}_{S'}}^{(inf,=)} \in RAT \ .$$

□

Corollary 33. $FA(fin, =) \subseteq RAT$.

Proof. Combine Lemmata 3 and 32. □

Theorem 34. $FA(fin, =) = RAT$.

Proof. Combine Lemmata 31 and Corollary 33. □

Proposition 35. $a(a^*b)^\omega + b(a+b)^*a^\omega \in CDFA(fin, =) \setminus (F_\sigma^R \cup G_\delta^R)$.

6 Conclusions

In this paper we have studied the expressivity power of acceptance condition for finite automata. Three new classes have been fully characterized. For a fourth one, partial results are given. In particular, $(ninf, \sqcap)$ provides four distinct new classes of languages (see the diamond in the left part of Figure 2), all other acceptance conditions considered tend to give (classes of) languages populating known classes.

Remark that some well-known acceptance conditions like Rabin, Strett or Parity conditions have not been taken in consideration in this work since it is known that they are equivalent to Muller's condition.

A first research direction, of course, consists in completing the characterisation of $(fin, =)$. The characterization of $(fin, \subseteq)$ is still open.

A further interesting research direction consists in studying the closure properties of the above new classes of languages and see if they cram the known classes or if they add new elements to Figure 2.

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References

1. Julius R. Büchi. Symposium on decision problems: On a decision method in restricted second order arithmetic. In Patrick Suppes Ernest Nagel and Alfred Tarski, editors, *Logic, Methodology and Philosophy of Science Proceeding of the 1960 International Congress*, volume 44 of *Studies in Logic and the Foundations of Mathematics*, pages 1 – 11. Elsevier, 1960.
2. Juris Hartmanis and Richard E. Stearns. Sets of numbers defined by finite automata. *American Mathematical Monthly*, 74:539–542, 1967.
3. Orna Kupferman and Moshe Y. Vardi. From complementation to certification. In Kurt Jensen and Andreas Podelski, editors, *10th TACAS*, volume 2988 of *Lecture Notes in Computer Science*, pages 591–606. Springer, 2004.
4. R. P. Kurshan. *Computer aided verification of coodinating process.* Princeton Univ. Press, 1994.
5. L. H. Landweber. Decision problems for omega-automata. *Mathematical Systems Theory*, 3(4):376–384, 1969.
6. Igor Litovsky and Ludwig Staiger. Finite acceptance of infinite words. *Theor. Comput. Sci.*, 174(1-2):1–21, 1997.
7. Tetsuo Moriya and Hideki Yamasaki. Accepting conditions for automata on ω-languages. *Theor. Comput. Sci.*, 61:137–147, 1988.
8. D. E. Muller. Infinite sequences and finite machines. In *Proceedings of the 1963 Proceedings of the Fourth Annual Symposium on Switching Circuit Theory and Logical Design*, SWCT '63, pages 3–16, Washington, DC, USA, 1963. IEEE Computer Society.
9. Dominique Perrin and Jean-Eric Pin. *Infinite words, automata, semigroups, logic and games*, volume 141 of *Pure and Applied Mathematics.* Elsevier, 2004.
10. L. Staiger and K. W. Wagner. Automatentheoretische und automatenfreie charakterisierungen topologischer klassen regulärer folgenmengen. *Elektronische Informationsverarbeitung und Kybernetik*, 10(7):379–392, 1974.
11. Ludwig Staiger. ω-languages. In *Handbook of formal languages*, volume 3, pages 339–387. 1997.
12. Wolfgang Thomas. Automata on infinite objects. In J. van Leeuwen, editor, *Handbook of Theoretical Computer Science*, volume B (Formal models and semantics), pages 135–191. Elsevier, 1990.
13. Moshe Y. Vardi. The Büchi complementation saga. In Wolfgang Thomas and Pascal Weil, editors, *STACS 2007*, volume 4393 of *Lecture Notes in Computer Science*, pages 12–22. Springer, 2007.
14. Klaus W. Wagner. On ω-regular sets. *Information and Control*, 43(2):123 – 177, 1979.