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Foundations of Geometric Algebra Computing

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To 150 years of Hermann G. Grassmann's
Ausdehnungslehre of 1862

Foreword

If I have seen further it is by standing on the shoulders of giants.
– I. Newton, 15 Feb 1676.

Newton's well-known quote to Hooke finds no better illustration than in the development of what we call Geometric Algebra. The primary set of shoulders in the Geometric Algebra epic belong to Hermann Günther Grassmann, who epitomized the mathematical mind, at least in process, if not in expositional style. He systematically created algebras for geometric concepts by listing essential elements and operations and then reducing them to minimal axiomatic schemes. In his book *Die Lineale Ausdehnungslehre* he laid the foundation for scores of mathematical systems bearing names like projective and exterior geometry and Quaternion, Clifford, Gibbs, Cartan, and Boolean algebra. Unfortunately, his book was densely written with many religious allusions and it shifted without warning between projective and Euclidean spaces.

One of the fundamental ideas of Grassmann was that of different grade (intrinsic dimensions) elements; thus, in addition to scalar and vector elements, with which we are well familiarized today, he would have included bi-vectors, tri-vectors etc., not as combinations of vectors like the cross-product, but as axiomatic elements. His book introduced two fundamental operations between elements, the inner and outer products. One was simply a grade lowering and the other a grade elevating operation between elements. Between vectors, these operations became the familiar dot product (or projection) and the wedge product (or plane through the vectors). The import of these ideas becomes apparent when one considers what else can one say about the relation of two vectors? The grade lowering and grade elevating operators underpin any comprehensive vector algebra.

It was William Kingdon Clifford, who then recognized the sublime value of joining these two operations, inner and outer, into one product via a new algebraic "addition" of the two products. The geometric product of Clifford was cast as the fundamental operation with inner and outer products defined in terms of it. Its subtlety lies in the fact that it is not a binary operation as we are used to thinking of

additions; it does not replace two elements with a third "sum" of the same type of element. Instead, it is a prepositional operation that only makes sense in the overall algebraic structure. It is rather like adding real and imaginary parts in complex analysis.

The subsequent evolution of "Clifford" Algebra became a candidate for the mathematical model of nineteenth-century physics, along with the quaternions of Hamilton. The winner of this intellectual contest, however, was eventually another Grassmannian derivative, the vector algebra proposed by Josiah Gibbs, a thermodynamicist. Gibbs' focus was three dimensions, appropriate to his area of research. As such he considered scalars and vectors to be sufficient; the bi-vector could be represented by the cross product, another vector. Hence there was no need for bi-vectors, and certainly not tri-vectors, nor anything higher. Two elements, the scalar and the vector, seem to satisfy Occam's razor. This may well be true, but only for three dimensions. As physics evolved in higher dimensions, the inadequacies of the cross product have become apparent. Pauli and Dirac invented their own versions of spinors, for example, a concept which has since been shown to be fundamental in any comprehensive formulation of an axiomatic physics system.

Our current state of babel has different mathematical systems, which require sophisticated methods of translation between them. They are often so common to us now that we do not see the clumsiness of, say, rewriting complex analysis in terms of 2D vectors, or 3D vectors in terms of quaternions to rotate and rewriting back. Similar situations exist between exterior algebra, Pauli and linear algebras, and the list goes on. A single, comprehensive algebra is both pedagogically and operationally "a consummation devoutly to be wish'd". That is the goal of Geometric Algebra.

In the last 50 years, Clifford Algebra found a strong proponent in physicist David Hestenes, who used it as the basis to describe electron theory and celestial mechanics among other very successful applications. His version of the Geometric Algebra was developed with a strong emphasis on the geometrically intuitive aspects of the algebra.

While Geometric Algebra is generally acknowledged to be a compelling and comprehensive system of mathematics (and it is beginning to find traction in many application areas) one major obstacle exists to its broader adoption, which is the very practical one. How do I compute with it? Without a Maple- or Mathematica-like facility, its usability is vastly limited in today's modern research or engineering environment. The way forward is obvious and a number of researchers have addressed this problem with computer algebraic systems based on Geometric Algebra. Two of the most current and popular are CLUCalc and Gaalop, as described in this book.

Experience with these methods and the innate characteristics of Geometric Algebra now point to the next logical step in the evolution. It is the need to use modern parallel architecture to accelerate Geometric Algebra. We can not only increase the range of realizable applications through speed and efficiency, but it provides unique and valuable insights into the algebra itself. This is the natural

evolution and critical path to bringing this richer, more comprehensive system of mathematics, this Geometric Algebra, to the collective scientific consciousness.

This book by Hildenbrand is, in my opinion, the next, necessary and joyful twist in this elegant evolutionary thread stretching back to Grassmann and beyond. He gives a highly readable account of the development of Geometric Algebra. He is able to cook the subject matter like a good meal, and then, pulling all the pieces together, feeds us a very compelling solution for the next steps in creating the most advanced environment for learning, applying and enjoying the beauty of Geometric Algebra.

Bon appetit

Thuwal, Kingdom of Saudi Arabia Prof. Alyn Rockwood
May 2012

Preface

Hermann G. Grassmann's *Ausdehnungslehre* of 1862 laid the foundations for Geometric Algebra as a mathematical language combining geometry and algebra. A hundred and fifty years later, this book is intended to lay the foundations for the widespread use of this mathematical system on various computing platforms.

Seventeen years after Grassmann's first more philosophically written *Ausdehnungslehre* of 1844, he admitted in the preface of his mathematical version of 1862, "I remain completely confident that the labor I have expended on the science presented here and which has demanded a significant part of my life, as well as the most strenuous application of my powers, will not be lost. It is true that I am aware that the form which I have given the science is imperfect and must be imperfect. But I know and feel obliged to state (though I run the risk of seeming arrogant) that even if this work should again remain unused for another seventeen years or even longer, without entering into the actual development of science, still that time will come when it will be brought forth from the dust of oblivion and when ideas now dormant will bring forth fruit." And he went on to say, "there will come a time when these ideas, perhaps in a new form, will arise anew and will enter into a living communication with contemporary developments."

The form that we give to Geometric Algebra in this book has the power to lead easily from the geometric intuition of solving an engineering application to its efficient implementation on current and future computing platforms. We show how easy it is to develop new algorithms in areas such as computer graphics, robotics, computer animation, and computer simulation. Owing to its geometric intuitiveness, compactness, and simplicity, algorithms based on Geometric Algebra can lead to enhanced quality, a reduction in development time and solutions that are more easily understandable and maintainable. Often, a clear structure and greater elegance result in lower runtime performance. However, based on our computing technology, Geometric Algebra implementations can even be faster and more robust than conventional ones.

I really do hope that this book can support the widespread use of Geometric Algebra Computing technology in many engineering fields.

Darmstadt, Germany Dr. Dietmar Hildenbrand
May 2012

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